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APPLICATION OF THE KRYLOV-BOGOLUBOV METHOD TO SOLUTION OF THE STELLAR THREE-BODY PROBLEM

by Peter Musen

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ABSTRACT

The motion in a ternary system consisting of a central body, A, close companion B, and distant companion, C, is considered and the perturbations of B due to C are determined. The osculating orbit of B relative to A can be an eccentric ellipse, not extremely elongated (say, $e \lesssim 0.3$), and the orbit of C around the center of mass of A and B is a fixed ellipse which might have a large eccentricity, e'. The mutual inclination, I of the orbits of B and C can also be large. The possibility of solving the problem without limitation to small e' was indicated earlier by Brown. His basic small parameter is $\lambda = \left[m/(m_1 + m_2 + m_3)\right] (n'/n) \left(1 - e'^2\right)^{-3/2}$. The use of λ and the introduction of the true anomaly f' of C as the independent variable permit one to take the effect of e' into account in a closed form without developing series in powers of e'.

The solution pursued here is a development of perturbations into trigonometric series with arguments linear in f' and similar in form to the standard arguments of the lunar theory. To this end we consider the squares of λ, of the parallax, and of the mean eccentricity of B as small parameters.

The problem is solved by applying the Krylov-Bogolubov method to eliminate the significant short-period effects from the Milankovich differential equations for the general perturbations of the Laplacian vector $\vec{e}$ and the areal velocity $\vec{c}$. Thus, the elements which are affected only by the long period perturbations must then be established.

We conclude that for small eccentricity there are resonances at $I \approx 39°$ and at $I \approx 141°$, and that the proposed trigonometric series solution cannot be done outside the intervals $0° \leqq I_0 < \sim 39°$ and $180° \geqq I_0 > \sim 141°$. This fact gives rise to a discussion of the convergence and limits of applicability of the Delaunay series. Some of our results are compared with those of Delaunay. We conclude that the Krylov-Bogolubov method can improve the Delaunay theory because it leads to more compact series and their convergence can be investigated more easily. In addition, they apply to any orbital inclination up to the critical value.

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INTRODUCTION

The motion in a ternary system consisting of the central body, A, the close companion B, and the distant companion, C, is considered in the present paper. The masses of the components are m_1, m_2, and m_3, respectively.

We assume that the osculating orbit of B relative to A is an eccentric ellipse, not extremely elongated (say, $e \lessapprox 0.3$), and that the orbit of C around the center of masses of two close companions A and B is a fixed ellipse which might have a large eccentricity, e'. The mutual inclination I of the orbits of B and C can also be large. In the further exposition we shall discuss a limitation which must be imposed on I in order to avoid a resonance condition.

We have to determine the perturbations of B as caused by C. The problem so formulated is called the stellar three-body problem because the conditions described above exist in some triple star systems. The case of the lunar satellite, because neither e nor I must be small, evidently represents a special case of the stellar problem. The role of C is played by either the Earth or the Sun. If e, e', and I are small we have the classic lunar problem; thus the importance of the general stellar problem is evident, and attempts should be made to solve it in an analytic or semi-analytic way.

The works of Kozai and Kovalevsky (References 1 and 2, respectively) on the problem of motion of the lunar satellite deserve to be mentioned. They solved the problem by assuming that $e' = 0$. The possibility of solving the problem without this limitation was indicated by Brown in his work on the stellar three-body problem (Reference 3). The basic small parameter in Brown's theory is

$$\lambda = \frac{m_3}{m_1 + m_2 + m_3} \frac{n'}{n} \left(1 - e'^2\right)^{-3/2} ,$$

where n' is the mean motion of C and n is the mean motion of B. In the theory of Delaunay (Reference 4) the basic parameter is n'/n, but the use of λ, and also the introduction of the true anomaly f' of C as the indendent variable, permit one to take the effect of e' into account in a closed form without resorting to the development of series in powers of e'.

The form of the solution we are pursuing here is the development of perturbations into trigonometric series with the arguments linear in f' and similar in form to the standard arguments of the lunar theory. In order to achieve such a solution we consider the square of the mean eccentricity of B, λ, and the parallax as small parameters. This limitation is justified by the circumstance that the orbits of lunar satellites with large eccentricities will be highly unstable. In addition the fact that the Moon is a body and not a point, together with the assumption that the parallax is small, imposes a certain upper limit on the eccentricity of B. The mutual inclination I and e', however, are not presupposed to be small.

In keeping the orbit stable, the long-period perturbations are more important than the short-period ones. This fact is the foundation of every satellite theory which makes use of the method of variation of astronomical constants. The way to solve the problem is to eliminate gradually all of the significant short-period perturbations from the differential equations. Thus the elements which are affected only by the long-period perturbations must be established. The establishment of such elements and the formation of the differential equations governing their variations require the solution of a chain of linear partial differential equations of the first order. After the differential equations for the new elements are finally formed, we can integrate them using the Poincaré small parameter method (Reference 5), provided the eccentricity is not large.

The elimination of the short-period terms is facilitated if in the original differential equations the coefficients of the derivatives of the disturbing function are linear in the elements. For this reason the Laplacian vector $\vec{e}$ and the areal velocity $\vec{c}$ represent a convenient choice. Their general perturbations are governed by the differential equations of Milankovich (Reference 6). The vectorial products contained in these equations are the linear functions of $\vec{e}$ and $\vec{c}$. The inconvenience caused by the division by $\vec{e}$ in the equation for $d\vec{e}/dt$ disappears after the elimination of the short-period effects. These elements have a direct kinematic meaning and with their help we can easily visualize the motion of the orbit in space. They are not canonical and consequently it is convenient to apply the method of Krylov-Bogolubov (Reference 7) to solve the problem.

The differential equations for the elements affected only by the long-period perturbations can be written as a quasi-linear system. We can solve it either by developing the solutions into power series in a small parameter or by applying the method of successive approximations. In recent years we have applied the digital computer to the development of the semianalytic lunar theories of the Hansen type (Reference 8). The machine develops the disturbing function, forms the differential equations, and integrates them by the process of successive approximations. In this way we produced the theories of lunar satellites and of the xth satellite of Jupiter. As a consequence of this experiment it is suggested that one substitutes the numerical values of the constants of integration at the outset and then let the machine obtain the solution in the form of trigonometric series with purely numerical coefficients by means of successive approximations. At each approximation we shall obtain c_1, the mean motion of the longitude of the perigee, from a quadratic equation.

If the mean eccentricity ϵ is zero, then, as we shall see

$$c_1 = \frac{3}{2} \lambda \left(\sqrt{1 - \frac{5}{2} k^2} - \frac{1}{2} \sqrt{1 - k^2} \right) + O(\lambda^2 e^2) ,$$

where k is the sine of the mean inclination I_0 and e is the mean eccentricity. From this we conclude that for small eccentricity there is a resonance at $I_0 \sim 39°$ and also at $I_0 \sim 141°$. We also conclude that the development of perturbations of $\vec{e}$ and $\vec{c}$ into trigonometric series with the standard arguments of the lunar theory cannot be done outside the intervals $0° \leqq I_0 < \sim 39°$ and $180° \geqq I_0 > \sim 141°$. The existence of this critical inclination in the special case $e' = 0$ was pointed out by Kevorkian (Reference 9) and Kozai (Reference 10). The fact that it exists also for $e' \neq 0$ gives rise to the discussion about the convergence and limits of applicability of the series of Delaunay (Reference 4). In a further exposition we shall compare some of our results with the results of Delaunay.

BASIC DIFFERENTIAL EQUATIONS

Let $\vec{P}$ be the unit vector directed toward the osculating pericenter of B and $\vec{R}$ be the unit vector normal to the osculating orbit plane of B. For the components of $\vec{P}$ and $\vec{R}$ relative to an inertial system located in the orbit plane of C, we have a set of standard formulas:

$$\left.\begin{aligned} P_1 &= \cos^2 \frac{I}{2} \cos \pi + \sin^2 \frac{I}{2} \cos (\pi - 2\Omega), \\ P_2 &= \cos^2 \frac{I}{2} \sin \pi - \sin^2 \frac{I}{2} \sin (\pi - 2\Omega), \\ P_3 &= \sin (\pi - \Omega) \sin I ; \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} R_1 &= \sin \Omega \sin I , \\ R_2 &= - \cos \Omega \sin I , \\ R_3 &= \cos I. \end{aligned}\right\} \quad (2)$$

Then

$$\vec{e} = e\vec{P} , \quad (3)$$

$$\vec{c} = \sqrt{a(1 - e^2)}\, \vec{R} . \quad (4)$$

If the equation of motion of the satellite is taken in the normalized form,

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\vec{r}}{r^3} + \operatorname{grad} R, \tag{5}$$

then the equations of Milankovich have the form

$$\frac{d\vec{e}}{dt} = \vec{e} \times \frac{\partial R}{\partial \vec{c}} + \frac{1-e^2}{c^2}\,\vec{c} \times \frac{\partial R}{\partial \vec{e}} + n\,\frac{c^2}{e^2}\,\vec{e}\,\frac{\partial R}{\partial \ell}\,, \tag{6}$$

$$\frac{d\vec{c}}{dt} = \vec{c} \times \frac{\partial R}{\partial \vec{c}} + \vec{e} \times \frac{\partial R}{\partial \vec{e}}\,, \tag{7}$$

and

$$\frac{d\ell}{dt} = n - n\,\frac{c^2}{e^2}\,\vec{e} \cdot \frac{\partial R}{\partial \vec{e}}\,. \tag{8}$$

Decomposing $\vec{e}$, $\vec{c}$, $\partial R/\partial \vec{e}$, and $\partial R/\partial \vec{c}$ along the axes of the inertial system, with the x- and y-axes in the orbit plane of C, we have the system of Milankovich's scalar equations:

$$\frac{de_1}{dt} = \left(e_2 \frac{\partial R}{\partial c_3} - e_3 \frac{\partial R}{\partial c_2}\right) + \frac{1-e^2}{c^2}\left(c_2 \frac{\partial R}{\partial e_3} - c_3 \frac{\partial R}{\partial e_2}\right) + n\,\frac{c^2}{e^2}\,\frac{\partial R}{\partial \ell}\,e_1\,, \tag{9}$$

$$\frac{de_2}{dt} = \left(e_3 \frac{\partial R}{\partial c_1} - e_1 \frac{\partial R}{\partial c_3}\right) + \frac{1-e^2}{c^2}\left(c_3 \frac{\partial R}{\partial e_1} - c_1 \frac{\partial R}{\partial e_3}\right) + n\,\frac{c^2}{e^2}\,\frac{\partial R}{\partial \ell}\,e_2\,, \tag{10}$$

$$\frac{de_3}{dt} = \left(e_1 \frac{\partial R}{\partial c_2} - e_2 \frac{\partial R}{\partial c_1}\right) + \frac{1-e^2}{c^2}\left(c_1 \frac{\partial R}{\partial e_2} - c_2 \frac{\partial R}{\partial e_1}\right) + n\,\frac{c^2}{e^2}\,\frac{\partial R}{\partial \ell}\,e_3\,, \tag{11}$$

$$\frac{dc_1}{dt} = \left(c_2 \frac{\partial R}{\partial c_3} - c_3 \frac{\partial R}{\partial c_2}\right) + \left(e_2 \frac{\partial R}{\partial e_3} - e_3 \frac{\partial R}{\partial e_2}\right), \tag{12}$$

$$\frac{dc_2}{dt} = \left(c_3 \frac{\partial R}{\partial c_1} - c_1 \frac{\partial R}{\partial c_3}\right) + \left(e_3 \frac{\partial R}{\partial e_1} - e_1 \frac{\partial R}{\partial e_3}\right), \tag{13}$$

$$\frac{dc_3}{dt} = \left(c_1 \frac{\partial R}{\partial c_2} - c_2 \frac{\partial R}{\partial c_1}\right) + \left(e_1 \frac{\partial R}{\partial e_2} - e_2 \frac{\partial R}{\partial e_1}\right). \tag{14}$$

We shall make use of the complex elements

$$\begin{aligned} g_1 &= e_1 + i\,e_2, & h_1 &= -c_2 + i\,c_1, \\ g_2 &= e_1 - i\,e_2, & h_2 &= -c_2 - i\,c_1, \\ g_3 &= -i\,e_3. & h_3 &= c_3\,. \end{aligned}$$

$$\left(i = \sqrt{-1}\right)$$

instead of the real elements e_1, e_2, e_3, c_1, c_2, and c_3. Their introduction permits us to operate with power series instead of trigonometric series and to put the result in compact and symmetrical form. There is an affinity between the use of the complex coordinates in Hill's lunar theory and the use of the complex vectors in the present theory. From Equations 1 and 2 we have

$$\begin{aligned} g_1 &= e \cos^2 \frac{I}{2} \exp i\pi + e \sin^2 \frac{I}{2} \exp i(-\pi + 2\Omega) \\ &= e(P_1 + i P_2) , \end{aligned} \tag{15}$$

$$\begin{aligned} g_2 &= e \cos^2 \frac{I}{2} \exp(-i\pi) + e \sin^2 \frac{I}{2} \exp i(\pi - 2\Omega) \\ &= e(P_1 - i P_2) , \end{aligned} \tag{16}$$

$$\begin{aligned} g_3 &= \frac{1}{2} e \sin I [\exp i(-\pi + \Omega) - \exp - i(-\pi + \Omega)] \\ &= - i e P_3, \end{aligned} \tag{17}$$

$$h_1 = \sqrt{a(1 - e^2)} \sin I \exp i\Omega = \sqrt{a(1 - e^2)} (- R_2 + i R_1), \tag{18}$$

$$h_2 = \sqrt{a(1 - e^2)} \sin I \exp(-\Omega) = \sqrt{a(1 - e^2)} (- R_2 - i R_1), \tag{19}$$

and

$$h_3 = \sqrt{a(1 - e^2)} \cos I = \sqrt{a(1 - e^2)} R_3 . \tag{20}$$

We have from Equations 15 to 20:

$$2 \frac{\partial R}{\partial g_i} = \frac{\partial R}{\partial e_1} - i \frac{\partial R}{\partial e_2} , \tag{21}$$

$$2 \frac{\partial R}{\partial g_2} = \frac{\partial R}{\partial e_1} + i \frac{\partial R}{\partial e_2} , \tag{22}$$

$$\frac{\partial R}{\partial g_3} = + i \frac{\partial R}{\partial e_3} , \tag{23}$$

$$2 \frac{\partial R}{\partial h_1} = - \frac{\partial R}{\partial c_2} - i \frac{\partial R}{\partial c_1} , \tag{24}$$

$$2 \frac{\partial R}{\partial h_2} = - \frac{\partial R}{\partial c_2} + i \frac{\partial R}{\partial c_1} , \tag{25}$$

and

$$\frac{\partial R}{\partial h_3} = \frac{\partial R}{\partial c_3}. \tag{26}$$

Equations 9 to 14 in the new variables are:

$$\frac{dg_1}{d\tau} = \left(2g_3 \frac{\partial R}{\partial h_2} - g_1 \frac{\partial R}{\partial h_3}\right) + \frac{1 - e^2}{c^2} \left(2 h_3 \frac{\partial R}{\partial g_2} + h_1 \frac{\partial R}{\partial g_3}\right) + n \frac{c^2}{e^2} \frac{\partial R}{\partial \lambda} g_1, \tag{27}$$

$$\frac{dg_2}{d\tau} = \left(2g_3 \frac{\partial R}{\partial h_1} + g_2 \frac{\partial R}{\partial h_3}\right) + \frac{1 - e^2}{c^2} \left(- 2 h_3 \frac{\partial R}{\partial g_1} + h_2 \frac{\partial R}{\partial g_3}\right) + n \frac{c^2}{e^2} \frac{\partial R}{\partial \lambda} g_2, \tag{28}$$

$$\frac{dg_3}{d\tau} = \left(g_1 \frac{\partial R}{\partial h_1} + g_2 \frac{\partial R}{\partial h_2}\right) - \frac{1 - e^2}{c^2} \left(h_1 \frac{\partial R}{\partial g_1} + h_2 \frac{\partial R}{\partial g_2}\right) + n \frac{c^2}{e^2} \frac{\partial R}{\partial \lambda} g_3, \tag{29}$$

$$\frac{dh_1}{d\tau} = \left(2h_3 \frac{\partial R}{\partial h_2} - h_1 \frac{\partial R}{\partial h_3}\right) - \left(2g_3 \frac{\partial R}{\partial g_2} + g_1 \frac{\partial R}{\partial g_3}\right), \tag{30}$$

$$\frac{dh_2}{d\tau} = \left(- 2h_3 \frac{\partial R}{\partial h_1} + h_2 \frac{\partial R}{\partial h_3}\right) - \left(2g_3 \frac{\partial R}{\partial g_2} + g_2 \frac{\partial R}{\partial g_3}\right), \tag{31}$$

and

$$\frac{dh_3}{d\tau} = \left(h_1 \frac{\partial R}{\partial h_1} - h_2 \frac{\partial R}{\partial h_2}\right) + \left(g_1 \frac{\partial R}{\partial g_1} - g_2 \frac{\partial R}{\partial g_2}\right), \tag{32}$$

where we put

$$\left.\begin{aligned} \tau &= i\, t, \\ \lambda &= i\, \ell. \end{aligned}\right\} \tag{33}$$

This system of differential equations admits a first integral

$$g_1 h_2 - g_2 h_1 + 2 g_3 h_3 = 0, \tag{34}$$

which is equivalent to the statement that

$$\vec{P} \cdot \vec{R} = 0. \tag{35}$$

If we eliminate from R the short-period terms containing ℓ in the argument, then the new disturbing function [R] will contain only ℓ', π, and Ω in the argument. Thus [R] will yield only the long-period and secular effects in the elements.

The elimination of the short-period terms can be performed either by the method of Von Zeipel (Reference 11) or by the Krylov-Bogolubov method. In the treatment of the perturbations of the noncanonical elements the Krylov-Bogolubov method should be used.

In the problem of motion of a close satellite we can set, with an accuracy up to λ^3,

$$[R] = \frac{1}{2\pi} \int_0^{2\pi} R\, d\ell, \tag{36}$$

where the long-period terms having λ^4 as a factor are neglected. With the elimination of the effects depending upon ℓ the element a becomes invariable. Then it is more convenient to define h_1, h_2, and h_3 by

$$h_1 = \sqrt{1 - e^2}\,(-R_2 + i\,R_1), \tag{37}$$

$$h_2 = \sqrt{1 - e^2}\,(-R_2 - i\,R_1), \tag{38}$$

$$h_3 = \sqrt{1 - e^2}\,R_3, \tag{39}$$

than by Equations 18 to 20.

Now we set

$$\vec{c} = \sqrt{1 - e^2}\,\vec{R}. \tag{4'}$$

With these new definitions Equations 27 to 32 take the form:

$$\frac{dg_1}{d\tau} = 2\left(g_3 \frac{\partial\Omega}{\partial h_2} + h_3 \frac{\partial\Omega}{\partial g_2}\right) + \left(h_1 \frac{\partial\Omega}{\partial g_3} - g_1 \frac{\partial\Omega}{\partial h_3}\right), \tag{40}$$

$$\frac{dg_2}{d\tau} = 2\left(g_3 \frac{\partial\Omega}{\partial h_1} - h_3 \frac{\partial\Omega}{\partial g_1}\right) + \left(h_2 \frac{\partial\Omega}{\partial g_3} + g_2 \frac{\partial\Omega}{\partial h_3}\right), \tag{41}$$

$$\frac{dg_3}{d\tau} = \left(g_1 \frac{\partial\Omega}{\partial h_1} + g_2 \frac{\partial\Omega}{\partial h_2}\right) - \left(h_1 \frac{\partial\Omega}{\partial g_1} + h_2 \frac{\partial\Omega}{\partial g_2}\right), \tag{42}$$

$$\frac{dh_1}{d\tau} = 2\left(h_3 \frac{\partial\Omega}{\partial h_2} - g_3 \frac{\partial\Omega}{\partial g_2}\right) - \left(h_1 \frac{\partial\Omega}{\partial h_3} + g_1 \frac{\partial\Omega}{\partial g_3}\right), \tag{43}$$

$$\frac{dh_2}{d\tau} = -2\left(h_3 \frac{\partial\Omega}{\partial h_1} + g_3 \frac{\partial\Omega}{\partial g_1}\right) + \left(h_2 \frac{\partial\Omega}{\partial h_3} - g_2 \frac{\partial\Omega}{\partial g_3}\right), \tag{44}$$

and

$$\frac{dh_3}{d\tau} = \left(g_1 \frac{\partial\Omega}{\partial g_1} - g_2 \frac{\partial\Omega}{\partial g_2}\right) + \left(h_1 \frac{\partial\Omega}{\partial h_1} - h_2 \frac{\partial\Omega}{\partial h_2}\right), \tag{45}$$

where we put

$$\Omega = \frac{[R]}{\sqrt{a}}. \tag{46}$$

From

$$e^2 = g_1\,g_2 - g_3^2 \tag{47}$$

and

$$1 - e^2 = h_1\,h_2 + h_3^2, \tag{48}$$

we conclude that in addition to the integral 33 the system of Equations 40 to 45 admits also the integral

$$g_1 g_2 + h_1 h_2 - g_3^2 + h_3^2 = 1. \tag{49}$$

Because $g_2 = \bar{g}_1$ and $h_2 = \bar{h}_1$ we are not compelled to use all of Equations 40 to 45, but only four of them, say, Equations 40, 42, 43, and 45.

We consider the main problem in the theory of satellites: to determine the motion of the satellite under the influence of the disturbing function

$$R = \frac{m_3}{m_1 + m_2} \frac{r^2}{r'^3} P_2(\sigma), \tag{50}$$

where σ is the cosine of the angle between $\vec{r}$ and $\vec{r}'$. In the beginning we omit the parallactic terms of the first and higher orders. Their inclusion with accuracy up to λ^3 does not present any difficulty. It is necessary to emphasize that in any attempt to achieve a higher degree of accuracy it is meaningless to include the higher harmonics alone without taking the "cross actions" between the lower harmonics into consideration.

Taking into account Kepler's law,

$$\frac{m_1 + m_2 + m_3}{m_1 + m_2} = \left(\frac{n'}{n}\right)^2 \left(\frac{a'}{a}\right)^3,$$

we can write Equation 46 in the form

$$\Omega = \frac{m_3}{m_1 + m_2 + m_3} \left(\frac{n'}{n}\right)^2 a^{-3/2} \left(\frac{a'}{r'}\right)^3 \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 P_2(\sigma)\, d\ell. \tag{51}$$

If e' not small, then in treating the long-period effects it is better to use f', the true anomaly of the distant companion, as the independent variable instead of time. This idea was suggested by Brown (Reference 3) in his work on the stellar three-body problem.

In accordance with the form of the differential equation of motion of the satellite given in Equation 5, we have

$$r'^2 \frac{df'}{dt} = \sqrt{\frac{(m_1 + m_2 + m_3)\, a'(1 - e'^2)}{m_1 + m_2}}$$

or

$$r'^2 \frac{df'}{dt} = \frac{n'}{n} a^{-3/2} \left(1 - e'^2\right)^{-1/2} a'^2 . \tag{52}$$

Putting

$$\zeta = \exp i f'$$

and introducing the operator

$$D = \zeta \frac{d}{d\zeta} = \frac{d}{d(i f')} \tag{53}$$

analogous to the operator

$$\frac{d}{d(n - n')it}$$

of Hill's lunar theory, the system of Equations 40 to 45 becomes

$$Dg_1 = 2\left(g_3 \frac{\partial F}{\partial h_2} + h_3 \frac{\partial F}{\partial g_2}\right) + \left(h_1 \frac{\partial F}{\partial g_3} - g_1 \frac{\partial F}{\partial h_3}\right), \tag{54}$$

$$Dg_2 = 2\left(g_3 \frac{\partial F}{\partial h_1} - h_3 \frac{\partial F}{\partial g_1}\right) + \left(h_2 \frac{\partial F}{\partial g_3} + g_2 \frac{\partial F}{\partial h_3}\right), \tag{55}$$

$$Dg_3 = \left(g_1 \frac{\partial F}{\partial h_1} + g_2 \frac{\partial F}{\partial h_2}\right) - \left(h_1 \frac{\partial F}{\partial g_1} + h_2 \frac{\partial F}{\partial g_2}\right), \tag{56}$$

$$Dh_1 = 2\left(h_3 \frac{\partial F}{\partial h_2} - g_3 \frac{\partial F}{\partial g_2}\right) - \left(h_1 \frac{\partial F}{\partial h_3} + g_1 \frac{\partial F}{\partial g_3}\right), \tag{57}$$

$$Dh_2 = -2\left(h_3 \frac{\partial F}{\partial h_1} + g_3 \frac{\partial F}{\partial g_1}\right) + \left(h_2 \frac{\partial F}{\partial h_3} - g_2 \frac{\partial F}{\partial g_3}\right), \tag{58}$$

and

$$Dh_3 = \left(g_1 \frac{\partial F}{\partial g_1} - g_2 \frac{\partial F}{\partial g_2}\right) + \left(h_1 \frac{\partial F}{\partial h_1} - h_2 \frac{\partial F}{\partial h_2}\right), \tag{59}$$

where

$$F = \lambda(1 + e' \cos f') \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 P_2(\sigma)\, d\ell \tag{60}$$

and

$$\lambda = \frac{m_3}{m_1 + m_2 + m_3} \frac{n'}{n} \left(1 - e'^2\right)^{-3/2} \tag{61}$$

is Brown's parameter. The use of λ permits one to write the final results in a more compact form, thus avoiding a considerable portion of the development into the power series relative to e'.

In his earlier work (Reference 12) the author obtained the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^2 P_2(\sigma)\, d\ell = \frac{3}{4}\left[5(\vec{e} \cdot \vec{u}^0)^2 + (\vec{c} \times \vec{u}^0)^2 - e^2\right], \tag{62}$$

where $\vec{u}^0$ is the unit vector in the direction of $\vec{r}'$, and $\vec{e}$ and $\vec{c}$ are defined by means of Equations 3 and 4'. We have

$$\vec{e} \cdot \vec{u}^0 = e_1 \cos f' + e_2 \sin f' = \frac{1}{2}(g_1 \zeta^{-1} + g_2 \zeta), \tag{63}$$

and

$$\vec{c} \cdot \vec{u}^0 = \frac{i}{2}(h_2 \zeta - h_1 \zeta^{-1}). \tag{64}$$

Substituting these values into Equation 62 we obtain, after some easy vectorial transformations, taking Equations 47 and 48 into account;

$$F = \frac{3}{16}\lambda(1 + e' \cos f') \left[(5 g_1^2 + h_1^2) \zeta^{-2} + (6 g_1 g_2 + 2 h_1 h_2 + 4 g_3^2 + 4 h_3^2) + (5 g_2^2 + h_2^2) \zeta^2\right]. \tag{65}$$

Substituting this value of F into Equations 54 to 59 we obtain

$$Dg_1 = \frac{3}{4}\lambda(1 + e' \cos f')(\alpha_{1,0} + \alpha_{1,+2} \zeta^2), \tag{66}$$

$$Dg_2 = -\frac{3}{4}\lambda(1 + e' \cos f')(\beta_{1,-2} \zeta^{-2} + \beta_{1,0}), \tag{67}$$

$$Dg_3 = \frac{3}{4}\lambda(1 + e' \cos f')(\gamma_{1,-2} \zeta^{-2} + \gamma_{1,0} + \gamma_{1,+2} \zeta^2), \tag{68}$$

$$Dh_1 = \frac{3}{4}\lambda(1 + e' \cos f')(\alpha_{2,0} + \alpha_{2,+2} \zeta^2), \tag{69}$$

$$Dh_2 = -\frac{3}{4}\lambda(1 + e' \cos f')(\beta_{2,-2} \zeta^{-2} + \beta_{2,0}), \tag{70}$$

and

$$Dh_3 = \frac{3}{8}\lambda(1 + e' \cos f')(\gamma_{2,-2} \zeta^{-2} + \gamma_{2,+2} \zeta^2), \tag{71}$$

where we put

$$\alpha_{1,0} = 3 g_3 h_1 + g_1 h_3, \tag{72}$$

$$\alpha_{1,+2} = 5 g_2 h_3 + g_3 h_2, \tag{73}$$

$$\beta_{1,-2} = 5\, g_1 h_3 - g_3 h_1, \tag{74}$$

$$\beta_{1,0} = -3\, g_3 h_2 + g_2 h_3, \tag{75}$$

$$\gamma_{1,-2} = -2\, g_1\, h_1, \tag{76}$$

$$\gamma_{1,0} = -\, g_1 h_2 - g_2 h_1, \tag{77}$$

$$\gamma_{1,+2} = -\, 2 g_2 h_2, \tag{78}$$

$$\alpha_{2,0} = -\, 5 g_1 g_3 - h_1 h_3, \tag{79}$$

$$\alpha_{2,+2} = -\, 5 g_2 g_3 + h_2 h_3, \tag{80}$$

$$\beta_{2,-2} = 5 g_1 g_3 + h_1 h_3, \tag{81}$$

$$\beta_{2,0} = 5 g_2 g_3 - h_2 h_3, \tag{82}$$

$$\gamma_{2,-2} = 5 g_1^2 + h_1^2, \tag{83}$$

and

$$\gamma_{2,+2} = -\, 5 g_2^2 - h_2^2. \tag{84}$$

KRYLOV-BOGOLUBOV METHOD

The Krylov-Bogolubov method (Reference 7) has not yet achieved general recognition in lunar problems: A recent application of this method by Morrison (Reference 13) to the planar lunar problem deserves especially to be mentioned. We shall give here a short outline of the method of Krylov-Bogolubov from the viewpoint of its application to the stellar three-body problem. The differential equations we are considering here are of the form:

$$Dg_1 = \lambda G_1\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{85}$$

$$Dg_2 = \lambda G_2\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{86}$$

$$Dg_3 = \lambda G_3\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{87}$$

$$Dh_1 = \lambda H_1\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{88}$$

$$Dh_2 = \lambda H_2\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{89}$$

and

$$Dh_3 = \lambda H_3\, (g_1,\ g_2,\ g_3,\ h_1,\ h_2,\ h_3;\ \zeta;\ \lambda), \tag{90}$$

where G_1, G_2, G_3, H_1, H_2, and H_3 are the series of polynomials in ζ and ζ^{-1} with the coefficients being polynomials in g_1, g_2, g_3, h_1, h_2, and h_3.

The Krylov-Bogolubov method consists in finding a transformation

$$g_1 = g_1^* + \lambda a_1 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda), \tag{91}$$

$$g_2 = g_2^* + \lambda b_1 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda), \tag{92}$$

$$g_3 = g_3^* + \lambda c_1 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda), \tag{93}$$

$$h_1 = h_1^* + \lambda a_2 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda), \tag{94}$$

$$h_2 = h_2^* + \lambda b_2 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda), \tag{95}$$

and

$$h_3 = h_3^* + \lambda c_2 (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \zeta; \lambda) \tag{96}$$

so that the differential equations in the new variables do not contain ζ:

$$D g_1^* = \lambda G_1^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda), \tag{97}$$

$$D g_2^* = \lambda G_2^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda), \tag{98}$$

$$D g_3^* = \lambda G_3^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda), \tag{99}$$

$$D h_1^* = \lambda H_1^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda), \tag{100}$$

$$D h_2^* = \lambda H_2^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda), \tag{101}$$

and

$$D h_3^* = \lambda H_3^* (g_1^*, g_2^*, g_3^*, h_1^*, h_2^*, h_3^*; \lambda). \tag{102}$$

Let us introduce the operators

$$K = \zeta \frac{\partial}{\partial \zeta} \tag{103}$$

and

$$A = G_1^* \frac{\partial}{\partial g_1^*} + G_2^* \frac{\partial}{\partial g_2^*} + G_3^* \frac{\partial}{\partial g_3^*} + H_1^* \frac{\partial}{\partial h_1^*} + H_2^* \frac{\partial}{\partial h_2^*} + H_3^* \frac{\partial}{\partial h_3^*}. \tag{104}$$

Then the operator D as applied to Equations 91 to 96 can be written in the form

$$D = K + \lambda A. \tag{105}$$

It follows from Equations 91 to 105 that

$$D g_1 = \lambda G_1^* + \lambda K a_1 + \lambda^2 A\, a_1, \tag{106}$$

$$Dg_2 = \lambda G_2^* + \lambda K b_1 + \lambda^2 A b_1, \tag{107}$$

$$Dg_3 = \lambda G_3^* + \lambda K c_1 + \lambda^2 A c_1, \tag{108}$$

$$Dh_1 = \lambda H_1^* + \lambda K a_2 + \lambda^2 A a_2, \tag{109}$$

$$Dh_2 = \lambda H_2^* + \lambda K b_2 + \lambda^2 A b_2, \tag{110}$$

and

$$Dh_3 = \lambda H_3^* + \lambda K c_2 + \lambda^2 A c_2. \tag{111}$$

Introducing the shift operator

$$1 + T = \exp\left[\lambda\left(a_1 \frac{\partial}{\partial g_1^*} + b_1 \frac{\partial}{\partial g_2^*} + c_1 \frac{\partial}{\partial g_3^*} + a_2 \frac{\partial}{\partial h_1^*} + b_2 \frac{\partial}{\partial h_2^*} + c_2 \frac{\partial}{\partial h_3^*}\right)\right], \tag{112}$$

and combining Equations 85 to 90 and 106 to 111, we obtain

$$\lambda G_1^* + \lambda K a_1 + \lambda^2 A a_1 = (1 + T)\lambda G_1, \tag{113}$$

$$\lambda G_2^* + \lambda K b_1 + \lambda^2 A b_1 = (1 + T)\lambda G_2, \tag{114}$$

$$\lambda G_3^* + \lambda K c_1 + \lambda^2 A c_1 = (1 + T)\lambda G_3, \tag{115}$$

$$\lambda H_1^* + \lambda K a_2 + \lambda^2 A a_2 = (1 + T)\lambda H_1, \tag{116}$$

$$\lambda H_2^* + \lambda K b_2 + \lambda^2 A b_2 = (1 + T)\lambda H_2, \tag{117}$$

and

$$\lambda H_3^* + \lambda K c_2 + \lambda^2 A c_2 = (1 + T)\lambda H_3, \tag{118}$$

where in $G_1, G_2, \ldots$ the elements $g_1, g_2, \ldots$ are replaced by $g_1^*, g_2^*, \ldots$.

We seek the solutions for $a_1, b_1, \ldots c_2$, and $G_1^*, G_2^*, \ldots H_3^*$ in the form of power series in λ:

$$\lambda a_1 = \lambda a_{11} + \lambda^2 a_{12} + \cdots, \tag{119}$$

$$\lambda b_1 = \lambda b_{11} + \lambda^2 b_{12} + \cdots, \tag{120}$$

$$\lambda c_1 = \lambda c_{11} + \lambda^2 c_{12} + \cdots, \tag{121}$$

$$\lambda a_2 = \lambda a_{21} + \lambda^2 a_{22} + \cdots, \tag{122}$$

$$\lambda b_2 = \lambda b_{21} + \lambda^2 b_{22} + \cdots, \tag{123}$$

$$\lambda c_2 = \lambda c_{21} + \lambda^2 c_{22} + \cdots, \tag{124}$$

$$\lambda G_1^* = \lambda G_{11}^* + \lambda^2 G_{12}^* + \cdots, \tag{125}$$

$$\lambda G_2^* = \lambda G_{21}^* + \lambda^2 G_{22}^* + \cdots, \tag{126}$$

$$\lambda G_3^* = \lambda G_{31}^* + \lambda^2 G_{32}^* + \cdots, \tag{127}$$

$$\lambda H_1^* = \lambda H_{11}^* + \lambda^2 H_{12}^* + \cdots, \tag{128}$$

$$\lambda H_2^* = \lambda H_{21}^* + \lambda^2 H_{22}^* + \cdots, \tag{129}$$

and

$$\lambda H_3^* = \lambda H_{31}^* + \lambda^2 H_{32}^* + \cdots. \tag{130}$$

We have similar developments for the operators $1 + T$ and λA,

$$1 + T = 1 + \lambda T_1 + \lambda^2 T_2 + \cdots \quad , \tag{131}$$

the operators T_k are Faá de Bruno differential operators (Reference 14)

$$\lambda A = \lambda A_1 + \lambda^2 A_2 + \cdots, \tag{132}$$

where, from Equations 112 and 104, we deduce

$$T_1 = \left(a_{11} \frac{\partial}{\partial g_1^*} + b_{11} \frac{\partial}{\partial g_2^*} + c_{11} \frac{\partial}{\partial g_3^*}\right) + \left(a_{21} \frac{\partial}{\partial h_1^*} + b_{21} \frac{\partial}{\partial h_2^*} + c_{21} \frac{\partial}{\partial h_3^*}\right), \tag{133}$$

. . . .

$$A_1 = \left(G_{11}^* \frac{\partial}{\partial g_1^*} + G_{21}^* \frac{\partial}{\partial g_2^*} + G_{31}^* \frac{\partial}{\partial g_3^*}\right) + \left(H_{11}^* \frac{\partial}{\partial h_1^*} + H_{21}^* \frac{\partial}{\partial h_2^*} + H_{31}^* \frac{\partial}{\partial h_3^*}\right), \tag{134}$$

. . . .

From Equations 113 to 130 we obtain

$$\left.\begin{array}{ll} G_{11}^* + K a_{11} = G_1, & H_{11}^* + K a_{21} = H_1, \\ G_{21}^* + K b_{11} = G_2, & H_{21}^* + K b_{21} = H_2, \\ G_{31}^* + K c_{11} = G_3, & H_{31}^* + K c_{21} = H_3, \end{array}\right\} \tag{135}$$

$$\left.\begin{array}{ll} G_{12}^* + K a_{12} + A_1 a_{11} = T_1 G_1, & H_{12}^* + K a_{22} + A_1 a_{21} = T_1 H_1, \\ G_{22}^* + K b_{12} + A_1 b_{11} = T_1 G_2, & H_{22}^* + K b_{22} + A_1 b_{21} = T_1 H_2, \\ G_{32}^* + K c_{12} + A_1 c_{11} = T_1 G_3, & H_{32}^* + K c_{22} + A_1 c_{21} = T_1 H_3, \end{array}\right\} \tag{136}$$

. . . .

Let us make use of the Krylov-Bogolubov averaging operator M. In our case its role consists in removing the positive and the negative powers of ζ from the expression to which it is applied, thus reducing this expression to the term which is independent of ζ. In addition to M, it is convenient to introduce the operator P by extracting the nonzero powers of ζ from the given expression. It is useful to remark that

$$\left.\begin{aligned} MK &\equiv 0, \\ MP &\equiv PM \equiv 0, \end{aligned}\right\} \tag{137}$$

and

$$MK^{-1}P \equiv 0, \tag{138}$$

where K^{-1} designates the inverse operator of K, and the operators A, A_1, A_2, . . . , and M are commutative:

$$\left.\begin{aligned} MA &= AM, \\ MA_j &= A_j M, \qquad j = 1, 2, 3, \cdots \end{aligned}\right\} \tag{139}$$

From Equations 138 and 139 we have, in addition,

$$MAK^{-1}P = 0. \tag{140}$$

Taking Equation 137 into consideration we deduce from Equation 135:

$$\left.\begin{aligned} G^*_{11} &= MG_1, \qquad & H^*_{11} &= MH_1, \\ G^*_{21} &= MG_2, & H^*_{21} &= MH_2, \\ G^*_{31} &= MG_3, & H^*_{31} &= MH_3, \end{aligned}\right\} \tag{141}$$

$$\left.\begin{aligned} a_{11} &= K^{-1}PG_1, \qquad & a_{21} &= K^{-1}PH_1, \\ b_{11} &= K^{-1}PG_2, & b_{21} &= K^{-1}PH_2, \\ c_{11} &= K^{-1}PG_3, & c_{21} &= K^{-1}PH_3. \end{aligned}\right\} \tag{142}$$

From Equations 137, 138, 140, and 142 we obtain

$$\left.\begin{aligned} G^*_{12} &= MT_1 G_1, \qquad & H^*_{12} &= MT_1 H_1, \\ G^*_{22} &= MT_1 G_2, & H^*_{22} &= MT_1 H_2, \\ G^*_{32} &= MT_1 G_3, & H^*_{32} &= MT_1 H_3, \end{aligned}\right\} \tag{143}$$

and, as a consequence, we have

$$\left.\begin{aligned} a_{12} &= K^{-1} P(T_1 G_1 - A_1 a_{11}), & a_{22} &= K^{-1} P(T_1 H_1 - A_1 a_{21}),\\ b_{12} &= K^{-1} P(T_1 G_2 - A_1 b_{11}), & b_{22} &= K^{-1} P(T_1 H_2 - A_1 b_{21}),\\ c_{12} &= K^{-1} P(T_1 G_3 - A_1 c_{11}), & c_{22} &= K^{-1} P(T_1 H_3 - A_1 c_{21}),\end{aligned}\right\} \tag{144}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

This process is continued until all the significant terms containing ζ are eliminated.

The differential equations for the elements affected only by the long-period terms become

$$\left.\begin{aligned} Dg_1^* &= M(1 + T_1) G_1 + \cdots, & Dh_1^* &= M(1 + T_1) H_1 + \cdots,\\ Dg_2^* &= M(1 + T_1) G_2 + \cdots, & Dh_2^* &= M(1 + T_1) H_2 + \cdots,\\ Dg_3^* &= M(1 + T_1) G_3 + \cdots, & Dh_3^* &= M(1 + T_1) H_3 + \cdots.\end{aligned}\right\} \tag{145}$$

In further exposition we shall change the notations and shall write $g_1, g_2, \ldots$ instead of $g_1^*, g_2^*, \ldots$.

INTEGRATION OF DIFFERENTIAL EQUATIONS OF STELLAR THREE-BODY PROBLEM

From the Equations 66 through 71 we have:

$$\left.\begin{aligned} G_1 &= \frac{3}{4}(1 + e' \cos f')(\alpha_{1,0} + \alpha_{1,+2}\,\zeta^{+2}),\\ G_2 &= -\frac{3}{4}(1 + e' \cos f')(\beta_{1,-2}\,\zeta^{-2} + \beta_{1,0}),\\ G_3 &= \frac{3}{4}(1 + e' \cos f')(\gamma_{1,-2}\,\zeta^{-2} + \gamma_{1,0} + \gamma_{1,+2}\,\zeta^{+2}),\\ H_1 &= \frac{3}{4}(1 + e' \cos f')(\alpha_{2,0} + \alpha_{2,+2}\,\zeta^{+2}),\\ H_2 &= -\frac{3}{4}(1 + e' \cos f')(\beta_{2,-2}\,\zeta^{-2} + \beta_{2,0}),\\ H_3 &= \frac{3}{8}(1 + e' \cos f')(\gamma_{2,-2}\,\zeta^{-2} + \gamma_{2,+2}\,\zeta^{+2}).\end{aligned}\right\} \tag{146}$$

Making use of Equations 141, 142, and 146 we obtain:

$$
\left.
\begin{aligned}
G^*_{11} &= +\frac{3}{4}\alpha_{1,0}, \quad H^*_{11} = +\frac{3}{4}\alpha_{2,0} \quad ,\\
G^*_{21} &= -\frac{3}{4}\beta_{1,0}, \quad H^*_{21} = -\frac{3}{4}\beta_{2,0} \quad ,\\
G^*_{31} &= +\frac{3}{4}\gamma_{1,0}, \quad H^*_{31} = 0 \quad ,
\end{aligned}
\right\} \tag{147}
$$

and

$$
\left.
\begin{aligned}
a_{11} &= \frac{3}{8}\alpha_{1,+2}\,\zeta^2 + \frac{3}{8}e'\,(\alpha_{1,0}\,\tau_0 + \alpha_{1,+2}\,\tau_{+2}),\\
b_{11} &= \frac{3}{8}\beta_{1,-2}\,\zeta^{-2} - \frac{3}{8}e'\,(\beta_{1,-2}\,\tau_{-2} + \beta_{1,0}\,\tau_0),\\
c_{11} &= -\frac{3}{8}(\gamma_{1,-2}\,\zeta^{-2} - \gamma_{1,+2}\,\zeta^2) + \frac{3}{8}e'\,(\gamma_{1,-2}\,\tau_{-2} + \gamma_{1,0}\,\tau_0 + \gamma_{1,+2}\,\tau_{+2}) \quad ,\\
a_{21} &= \frac{3}{8}\alpha_{2,+2}\,\zeta^2 + \frac{3}{8}\,e'(\alpha_{2,0}\,\tau_0 + \alpha_{2,+2}\,\tau_{+2}),\\
b_{21} &= \frac{3}{8}\beta_{2,-2}\,\zeta^{-2} - \frac{3}{8}e'\,(\beta_{2,-2}\,\tau_{-2} + \beta_{2,0}\,\tau_0),\\
c_{21} &= -\frac{3}{16}\,(\gamma_{2,-2}\,\zeta^{-2} - \gamma_{2,+2}\,\zeta^2)\\
&\quad + \frac{3}{16}\,e'(\gamma_{2,-2}\,\tau_{-2} + \gamma_{2,+2}\,\tau_{+2}),
\end{aligned}
\right\} \tag{148}
$$

where we put

$$
\left.
\begin{aligned}
\tau_{-2} &= -\frac{1}{3}\,\zeta^{-3} - \zeta^{-1},\\
\tau_0 &= -\,\zeta^{-1} + \zeta,\\
\tau_{+2} &= +\,\zeta + \frac{1}{3}\,\zeta^3 \quad .
\end{aligned}
\right\} \tag{149}
$$

From Equations 133 and 145 to 148 and taking

$$M(\zeta^{-2}\tau_{-2}\cos f') = 0\,, \qquad M(\tau_{-2}\cos f') = -\frac{1}{2}, \qquad M(\zeta^{2}\tau_{-2}\cos f') = -\frac{2}{3},$$

$$M(\zeta^{-2}\tau_{0}\cos f') = \frac{1}{2}, \qquad M(\tau_{0}\cos f') = 0, \qquad M(\zeta^{2}\tau_{0}\cos f') = -\frac{1}{2},$$

$$M(\zeta^{-2}\tau_{+2}\cos f') = \frac{2}{3}, \qquad M(\tau_{+2}\cos f') = \frac{1}{2}, \qquad M(\zeta^{2}\tau_{+2}\cos f') = 0$$

into consideration, we obtain:

$$\begin{aligned} Dg_1 = {} & \frac{3}{4}\lambda\alpha_{1,0} + \frac{9}{64}\lambda^2\left(1+\frac{2}{3}e'^2\right)\left(-5g_2\gamma_{2,-2} + 2g_3\beta_{2,-2} - 2h_2\gamma_{1,-2} + 10h_3\beta_{1,-2}\right) \\ & + \frac{9}{64}\lambda^2 e'^2\left[+\frac{1}{2}g_1(\gamma_{2,+2} - \gamma_{2,-2}) + g_3(\beta_{2,0} + 3\alpha_{2,+2})\right. \\ & \left. + 3h_1(\gamma_{1,+2} - \gamma_{1,-2}) - h_2\gamma_{1,0} + h_3(\alpha_{1,+2} + 5\beta_{1,0})\right], \end{aligned} \tag{150}$$

$$\begin{aligned} Dg_2 = {} & -\frac{3}{4}\lambda\beta_{1,0} - \frac{9}{64}\lambda^2\left(1+\frac{2}{3}e'^2\right)\left(+5g_1\gamma_{2,+2} - 2g_3\alpha_{2,+2} - 2h_1\gamma_{1,+2} + 10h_3\alpha_{1,+2}\right) \\ & - \frac{9}{64}\lambda^2 e'^2\left[+\frac{1}{2}g_2(\gamma_{2,+2} - \gamma_{2,-2}) - g_3(\alpha_{2,0} + 3\beta_{2,-2})\right. \\ & \left. + 3h_2(\gamma_{1,-2} - \gamma_{1,+2}) - h_1\gamma_{1,0} + h_3(\beta_{1,-2} + 5\alpha_{1,0})\right], \end{aligned} \tag{151}$$

$$\begin{aligned} Dg_3 = {} & \frac{3}{4}\lambda\gamma_{1,0} - \frac{9}{16}\lambda^2\left(1+\frac{2}{3}e'^2\right)\left(g_1\alpha_{2,+2} + g_2\beta_{2,-2} + h_1\alpha_{1,+2} + h_2\beta_{1,-2}\right) \\ & - \frac{9}{64}e'^2\left[g_1(2\alpha_{2,0} + \beta_{2,-2}) + g_2(2\beta_{2,0} + \alpha_{2,+2})\right. \\ & \left. + h_1(2\alpha_{1,0} + \beta_{1,-2}) + h_2(2\beta_{1,0} + \alpha_{1,+2})\right], \end{aligned} \tag{152}$$

$$\begin{aligned} Dh_1 = {} & \frac{3}{4}\lambda\alpha_{2,0} + \frac{9}{64}\lambda^2\left(1+\frac{2}{3}e'^2\right)\left(+10g_2\gamma_{1,-2} - 10g_3\beta_{1,-2} - h_2\gamma_{2,-2} + 2h_3\beta_{2,-2}\right) \\ & + \frac{9}{64}\lambda^2 e'^2\left[+5g_1(\gamma_{1,-2} - \gamma_{1,+2}) + 5g_2\gamma_{1,0} - 5g_3(\beta_{1,0} + \alpha_{1,+2})\right. \\ & \left. + \frac{1}{2}h_1(\gamma_{2,-2} - \gamma_{2,+2}) + h_3(\beta_{2,0} - \alpha_{2,+2})\right], \end{aligned} \tag{153}$$

$$Dh_2 = -\frac{3}{4}\lambda\beta_{2,-2}$$

$$-\frac{9}{64}\lambda^2\left(1+\frac{2}{3}e'^2\right)(+10g_1\gamma_{1,+2}+10g_3\alpha_{1,+2}+h_1\gamma_{2,+2}+2h_3\alpha_{2,+2})$$

$$-\frac{9}{64}\lambda^2 e'^2\Big[+5g_2(\gamma_{1,+2}-\gamma_{1,-2})+5g_1\gamma_{1,0}+5g_3(\alpha_{1,0}+\beta_{1,-2})$$

$$+\frac{1}{2}h_2(\gamma_{2,-2}-\gamma_{2,+2})+h_3(\alpha_{2,0}-\beta_{2,-2})\Big], \tag{154}$$

$$Dh_3 = \frac{9}{64}e'^2\Big[5(g_1\alpha_{1,0}-g_2\beta_{1,0})+(h_1\alpha_{2,0}-h_2\beta_{2,0})\Big]. \tag{155}$$

Because $g_2 = \overline{g}_1$ and $h_2 = \overline{h}_1$, we are not obliged to use all of Equations 150 to 155. The choice of four, Equations 150, 152, 153, and 155, will suffice.

The formulas 15 to 20 suggest the substitution

$$\left.\begin{array}{lll} g_1 = \epsilon p_1, & g_2 = \epsilon p_2, & g_3 = \epsilon k p_3, \\ h_1 = k w_1, & h_2 = k w_2, & h_3 = w_3, \end{array}\right\} \tag{156}$$

Here p_1, p_2, p_3, w_1, w_2, and w_3 are the functions of ζ^{c_1}, ζ^{c_2} and ϵ^2, k^2; c_1 is the mean motion of the longitude of the perigee; and c_2 is the mean motion of the longitude of the ascending node. For the values of ϵ and k small enough, the quantities g_1, g_2, g_3, h_1, h_2, and h_3 are developable into series in $\epsilon\zeta^{c_1}$, $\epsilon\zeta^{c_1}$, $k\zeta^{c_2}$, and $k\zeta^{-c_2}$; both c_1 and c_2 are series in ϵ^2, k^2, e'^2, and α^2, where α is the parallactic factor. We consider here the case of a close satellite and do not discuss the influence of the parallactic terms. The theory presented here could be extended further by including the higher powers of λ and α.

The fact that the Moon is a body and not a point, together with the assumption that the parallactic factor is small, imposes a limitation on the upper value of ϵ. We shall consider the case when k is not small, but let us assume that ϵ is small or moderate, say $\epsilon \lessapprox 0.3$, or less. If we make use of the substitution (Equation 156) the differential equations become:

$$Dw_1 = -\frac{3}{4}\lambda w_1 w_3 + \frac{9}{64}\lambda^2\left(1+\frac{2}{3}e'^2\right)\left(2w_1 w_3^2 - k^2 w_1^2 w_2\right)$$

$$+\lambda\epsilon^2 A_1 + \overline{\lambda}^2 e'^2 B_1, \tag{157}$$

$$Dw_3 = \lambda^2 e'^2 B_3, \tag{158}$$

$$Dp_1 = \frac{3}{4}\lambda(p_1 w_3 + 3k^2 p_3 w_1)$$

$$+\frac{9}{64}\lambda^2\left(1+\frac{2}{3}\ e'^2\right)\left[50\, p_1 w_3^2 + k^2\left(-5\, p_2 w_1^2 - 8 p_3 w_1 w_3 + 4 p_1 w_1 w_2\right)\right]$$

$$+\lambda\epsilon^2 F_1 + \lambda\ e'^2\ G_1\ , \tag{159}$$

and

$$Dp_3 = -\frac{3}{4}\lambda(p_1 w_2 + p_2 w_1)$$

$$-\frac{27}{8}\ \lambda^2\left(1+\frac{2}{3}\ e'^2\right) w_3\ (p_1 w_2 + p_2 w_1) + \lambda\ e'^2\ Q_3\ , \tag{160}$$

where we put

$$A_1 = -\frac{15}{4}\, p_1 p_3$$

$$+\frac{9}{64}\,\lambda\left(1+\frac{2}{3}\ e'^2\right)\left[\left(-20\, p_1 p_2 w_1 - 5\, p_1^2 w_2 - 40\, p_1 p_3 w_3\right) + 10\, k^2 p_3^2 w_1\right], \tag{161}$$

$$B_1 = \frac{9}{64}\left[-2 w_2 w_3^2 + \frac{1}{2} k^2 w_1\left(w_1^2 + w_2^2\right)\right]$$

$$+\frac{9}{64}\,\epsilon^2\left[\left(-\frac{15}{2}\, p_1^2 w_1 - \frac{5}{2} p_2^2 w_1 + 5\, p_1 p_2 w_2 - 20\, p_2 p_3 w_3\right) + 10\, k^2 p_3^2 w_2\right] \tag{162}$$

$$B_3 = \frac{9}{64}\ k^2 w_3\left(w_2^2 - w_1^2\right) + \frac{45}{64}\,\epsilon^2 w_3\left(p_1^2 - p_2^2\right)$$

$$+\frac{45}{32}\,\epsilon^2 k^2\ (p_1 w_1 + p_2 w_2), \tag{163}$$

$$F_1 = \frac{9}{64}\left(1+\frac{2}{3}\ e'^2\right)\left(-25\, p_1^2 p_2 + 10\, k^2\, p_1 p_3^2\right), \tag{164}$$

$$G_1 = \frac{45}{32} p_2 w_3^2$$

$$+ \frac{9}{128} k^2 (+ 11\, p_1 w_1^2 - 24\, p_3 w_2 w_3 - 10\, p_2 w_1 w_2 + p_1 w_2^2), \tag{165}$$

and

$$Q_3 = - \frac{27}{32} w_3 (p_1 w_1 + p_2 w_2)$$

$$+ \frac{45}{64} \epsilon^2 p_3 (p_1^2 - p_2^2) - \frac{45}{64} k^2 p_3 (w_1^2 - w_2^2). \tag{166}$$

In the lunar theories of Delaunay, Hill, and Brown the motion of the node and of the perigee are developed into the power series in ϵ^2, k^2, e'^2 and α^2. However, the domain of covergence of these series is not given and their application to the cases of large inclinations and large eccentricities cannot be taken for granted. The choice of the small parameter λ and the appearance of the closed form factor $1 + (2/3)\ e'^2$ remove the necessity of development relative to e', as it has been found by Brown. In order to remedy the situation relative to ϵ^2 and k^2, we suggest here the numerical computation of c_1 and c_2 by successive approximation rather than development into power series. The numerical values of ϵ, k, e', and λ are substituted from the outset.

We have to rewrite Equations 157 to 160 and also to develop the equations for the determination of c_1 and c_2 in a form suitable for the application of the process of successive approximations. As a by-product we shall also establish the domain of convergence to the inclination.

The integrals 34 and 49 take the form

$$p_1 w_2 - p_2 w_1 + 2\, p_3 w_3 = 0, \tag{167}$$

$$\left(k^2 w_1 w_2 + w_3^2\right) + \epsilon^2 \left(p_1 p_2 - k^2 p_3^2\right) = 1. \tag{168}$$

The last equation is to be used for determining χ_0 at each cycle of the iterative process. The first equation can serve as a check of computations.

We have three groups of terms in the right side of Equations 157 to 160: the terms proportional to λ, those proportional to $\lambda^2\ (1 + (2/3)\ e'^2)$, and those proportional to $\lambda^2 e'^2$. The last two groups were produced by the mutual action of the short-period terms. The terms of the first two groups correspond to the terms included by Brown in the theory of the stellar three-body problem. They are the source of secular changes in the argument of the perigee and the longitude of the node, and in addition they produce the main long-period effects in the elements. In the perturbations of canonical elements of Delaunay, the first two groups give rise to the terms depending upon the mean argument of the perigee. The terms proportional to $\lambda^2 e'^2$ are omitted in Brown's theory. To the accuracy with which we are concerned, they have no influence on the mean motions of the arguments; they will produce in $\vec{e}$ and $\vec{c}$ only the long-period effects proportional to $\lambda\, e'^2$.

We have

$$Dw_1 = \lambda\nu_{11}\, w_1 + \lambda\, W_1, \tag{157'}$$

$$Dw_3 = \lambda^2\, e'^2\, W_3, \tag{158'}$$

$$D(\zeta^{-c_2}\, p_1) = \lambda\mu_{11}(\zeta^{-c_2}\, p_1)+\lambda^2\mu_{12}k^2(\zeta^{+c_2}\, p_2)+\lambda\mu_{13}k^2 p_3 + \lambda(\zeta^{-c_2}\, P_1), \tag{159'}$$

and

$$Dp_3 = \lambda\mu_{31}\left[(\zeta^{-c_2} p_1) + (\zeta^{+c_2}\, p_2)\right] + \lambda\, P_3, \tag{160'}$$

where we set

$$\nu_{11} = -\frac{3}{4}\chi_0 + \frac{9}{64}\lambda\left(1 + \frac{2}{3}e'^2\right)(2\chi_0^2 - k^2), \tag{161'}$$

$$\mu_{11} = \left(\frac{3}{4}\chi_0\right) + \frac{9}{32}\lambda\left(1 + \frac{2}{3}e'^2\right)(2\,5\,\chi_0^2 + 2k^2), \tag{162'}$$

$$\mu_{12} = -\frac{45}{64}\left(1 + \frac{2}{3}e'^2\right), \tag{163'}$$

$$\mu_{13} = \frac{9}{4} - \frac{9}{8}\lambda\left(1 + \frac{2}{3}e'^2\right)\chi_0, \quad \mu_{31} = -\frac{3}{4} - \frac{27}{8}\lambda\left(1 + \frac{2}{3}e'^2\right)\chi_0, \tag{164'}$$

$$W_1 = \epsilon^2 A_1 - \frac{9}{64}\lambda\left(1 + \frac{2}{3}e'^2\right)k^2 w_1 (w_1 w_2 - 1)$$

$$-\frac{3}{4} w_1 (w_3 - \chi_0) + \lambda\, e'^2\, B_1\,, \tag{165'}$$

$$W_3 = B_3\,, \tag{166'}$$

$$P_1 = +\frac{9}{4} k^2 (w_1 - \zeta^{+c_2})\, p_3$$

$$-\frac{9}{64}\lambda\left(1 + \frac{2}{3}e'^2\right)k^2\left[5(w_1^2 - \zeta^{+2c_2})\, p_2 + 8\chi_0 (w_1 - \zeta^{+c_2})\, p_3\right] + \epsilon^2 F_1$$

$$+\frac{3}{4} p_1 (w_3 - \chi_0) + \lambda\, e'^2\, G_1\,, \tag{167'}$$

and

$$P_3 = \mu_{31}\left[(w_2 - \zeta^{-c_2})\, p_1 + (w_1 - \zeta^{c_2})\, p_2\right] + \lambda\, e'^2\, Q_3 . \tag{168'}$$

Eliminating p_3 from Equation 159' by means of Equation 160', we obtain

$$D^2 (\zeta^{-c_2} p_1) = \lambda\, \mu_{11}\, D(\zeta^{-c_2} p_1) + \lambda^2 \mu_{12}\, k^2\, D(\zeta^{c_2}\ p_2)$$

$$+ \lambda^2 \mu_{13}\, \mu_{31}\, k^2 \left[(\zeta^{c_2}\ p_1) + (\zeta^{+c_2} p_2)\right] + \lambda^2 S_1 , \tag{159''}$$

where we set

$$\lambda^2 S_1 = \lambda^2 \mu_{13}\, k^2\, P_3 + \lambda\, D\, (\zeta^{-c_2}\ P_1) . \tag{159'''}$$

Equations 157', 158', 159'', and 160' are in a form suitable for the application of the method of successive approximations.

In order to describe the process of integration of Equation 159'' let us consider a typical term and a typical differential equation

$$D^2 (\zeta^{-c_2} p_1) = \lambda\, \mu_{11}\, D(\zeta^{-c_2} p_1) + \lambda^2 \mu_{12}\, k^2\, D(\zeta^{c_2}\ p_2)$$

$$+ \lambda^2 \mu_{13} \mu_{31} k^2 \left[(\zeta^{-c_2} p_1) + (\zeta^{c_2}\ p_2)\right] + \lambda^2 (N_1\, \zeta^{\alpha} + N_2\, \zeta^{-\alpha}) , \tag{169}$$

where $\alpha = (2\, k_1 + 1)\, c_1 + (2\, k_2 - 1)\, c_2$.

Using the substitution

$$\zeta^{-c_2}\ p_1 = K_1\, \zeta^{\alpha} + K_2\, \zeta^{-\alpha} ,$$

we obtain

$$\alpha^2 K_1 = \lambda\, \mu_{11}\, \alpha\, K_1 + \lambda^2 \mu_{12}\, k^2\, \alpha\, K_2 + \lambda^2\ \mu_{13}\, \mu_{31}\, k^2\, (K_1 + K_2) + \lambda^2 N_1 = 0 ,$$

$$\alpha^2 K_2 = -\lambda\, \mu_{11}\, \alpha\, K_2 - \lambda^2 \mu_{12}\, k^2\, \alpha\, K_1 + \lambda^2 \mu_{13}\, \mu_{31}\, k^2\, (K_1 + K_2) + \lambda^2 N_2 = 0 ,$$

or

$$\left.\begin{aligned} &(\alpha^2 - 2\, \lambda^2 k^2 \mu_{13}\, \mu_{31})\, (K_1 + K_2) - \alpha\, (\lambda\, \mu_{11} - \lambda^2 k^2 \mu_{12})\, (K_1 - K_2) = \lambda^2 (N_2 + N_1) , \\ &\alpha(\lambda \mu_{11} + \lambda^2 k^2 \mu_{12})\ (K_1 + K_2) - \alpha^2 (K_1 - K_2) = \lambda^2 (N_2 - N_1). \end{aligned}\right\} \tag{170}$$

Setting

$$c_1 = \lambda\, \sigma_1 ,$$

$$c_2 = \lambda\, \sigma_2 ,$$

$$\sigma_1 - \sigma_2 = \sigma ,$$

and

$$\beta = (2\, k_1 + 1)\, \sigma_1 + (2\, h_2 - 1)\, \sigma_2 ,$$

where σ_1 and σ_2 are of the zero order relative to ϵ, we deduce from Equation 170

$$\left.\begin{aligned} (\beta^2 - 2k^2 \mu_{13}\mu_{31})(K_1 + K_2) - \beta(\mu_{11} - \lambda k^2 \mu_{12})(K_1 - K_2) &= N_2 + N_1 , \\ \beta(\mu_{11} + \lambda k^2 \mu_{12})(K_1 + K_2) - \beta^2 (K_1 - K_2) &= N_2 - N_1 . \end{aligned}\right\} \tag{171}$$

The determinant of this system is

$$\Delta = \beta^2 (\mu_{11}^2 + 2k^2 \mu_{13}\mu_{31} - \lambda^2 k^4 \mu_{12}^2 - \beta^2), \tag{172}$$

and the determinants associated with $K_1 + K_2$ and $K_1 - K_2$ are

$$\Delta_1 = -\beta(\mu_{11} - \lambda k^2 \mu_{12} + \beta) N_1 + \beta(\mu_{11} - \lambda k^2 \mu_{12} - \beta) N_2 , \tag{173}$$

$$\Delta_2 = -\beta(\mu_{11} + \lambda k^2 \mu_{12} + \beta) N_1 - \beta(\mu_{11} + \lambda k^2 \mu_{12} - \beta) N_2 + 2k^2 \mu_{13}\mu_{31}(N_1 - N_2). \tag{174}$$

The solving of this system is a straightforward process and, generally speaking, the division by Δ does not introduce any small divisor.

For

$$i_1 - i_2 = -1 ,$$

the part independent from ϵ and k disappears in β. For small values of ϵ and k the value of $c_1 + c_2$ then will also become small. Thus, this case requires a special discussion, and we shall show that in the problems of integration, even for small values of ϵ and k, no small divisors will appear which would make the very coefficients of $\zeta^{j(c_1+c_2)}$ and $\zeta^{-j(c_1+c_2)}$ large.

A special consideration is also required for the case

$$i_1 = i_2 = 0,$$

because it is connected with the determination of the mean motion of the pericenter and with the finding of the first approximation to p_1.

In agreement with the theory of the long period effects by Poincaré (Reference 4), we have the following form of solution of $\zeta^{-c_2} p_1$, p_3 and of $\zeta^{-2} p_1$, p_3:

$$\sum N_{i_1, i_2}(\epsilon^2, k^2)\, \zeta^{(2i_1+1)c_1 + (2i_2-1)c_2} + \lambda e'^2 \sum M_{i_1, i_2}(\epsilon^2, k^2)\, \zeta^{(2j_1+1)c_1 + (2j_2-1)c_2} ;$$

and of ζ^{-c_2} w_1 and W_1:

$$\sum K_{i_1, i_2}(\epsilon^2, k^2)\, \zeta^{2i_1 c_1 + 2i_2 c_2} + \lambda e'^2 \sum L_{i_1, i_2}(\epsilon^2, k^2)\, \zeta^{2j_1 c_1 + 2j_2 c_2} ;$$

and of $w_3 - \chi_0$ and $\lambda e'^2 W_3$:

$$\lambda e'^2 \sum L_{i_1, i_2} (\epsilon^2, k^2) \zeta^{2j_1 c_1 + 2j_2 c_2}.$$

In addition

$$|i_1 + i_2| = 0 \ ,$$

$$|j_1 + j_2| = 1 ;$$

and for p_3,

$$N_{i_1, i_2} = - N_{-i_1-1, -i_2+1}$$

$$M_{i_1, i_2} = - M_{-i_1-1, -i_2+1};$$

but for P_3

$$N_{i_1, i_2} = N_{-i_1-1, -i_2+1}$$

$$M_{i_1, i_2} = M_{-i_1-1, -i_2+1}.$$

In our case we have to consider the terms

$$\zeta^{c_1+c_2} \quad \text{and} \quad \zeta^{-c_1-c_2} \ .$$

Retaining only these terms in P_3 and $\zeta^{-c_2} P_1$, we can set

$$P_3 = M_1 \ \zeta^{+c_1+c_2} + M_2 \ \zeta^{-c_1-c_2}$$

and

$$\zeta^{-c_2} P_1 = C_1 \ \zeta^{+c_1+c_2} + C_2 \ \zeta^{-c_1-c_2},$$

where M_1, M_2, C_1, and C_2 are the functions of ϵ^2 and k^2. If ϵ and k both are small, then we shall assume that they are of the same order of magnitude.

We have to set

$$N_1 = \mu_{13} k^2 M_1 + \beta C_1 \ ,$$

$$N_2 = \mu_{13} k^2 M_2 - \beta C_2 .$$

Then Equations 172 to 174 become

$$\Delta = \mu_{11}^2 \beta^2 + \cdots \ , \tag{175}$$

$$\Delta_1 = - \beta^2 \mu_{11} (C_1 + C_2) + k^2 \beta \mu_{11} \mu_{13} (M_2 - M_1) + \cdots \ , \tag{176}$$

$$\Delta_2 = -\beta^2 \mu_{11} (C_1 - C_2) + 2k^2 \beta \mu_{13} \mu_{31} (C_1 + C_2) \tag{177}$$

$$- 2k^4 \mu_{13}^2 \mu_{31} (M_1 - M_2) - k^2 \beta \mu_{13} (M_1 + M_2) + \cdots ,$$

where only the terms of the lowest order are written explicitly. We have

$$K_1 + K_2 = -\frac{1}{\mu_{11}} (C_1 + C_2) + \frac{k^2}{\beta} \frac{\mu_{13}}{\mu_{11}} (M_2 - M_1) + \cdots , \tag{178}$$

$$K_1 - K_2 = -\frac{1}{\mu_{11}} (C_1 + C_2) + \frac{k^2}{\beta} \frac{2\mu_{13}\mu_{31}}{\mu_{11}^2} (C_1 + C_2) \tag{179}$$

$$-\frac{k^4}{\beta^2} \cdot \frac{2\mu_{13}^2 \mu_{31}}{\mu_{11}^2} (M_1 - M_2) - \frac{k^2}{\beta} \frac{\mu_{13}}{\mu_{11}^2} (M_1 + M_2) + \cdots .$$

From differential Equations 157 to 160 we easily deduce as the first approximation for small values of k and ϵ:

$$c_1 = \frac{3}{4} \lambda \left(\chi_0 - \frac{3}{2} k^2 \right) + \cdots ,$$

$$c_2 = -\frac{3}{4} \lambda \left(\chi_0 + \frac{5}{2} \epsilon^2 \right) + \cdots ,$$

$$c_1 + c_2 = -\frac{3}{8} \lambda (3k^2 + 5\epsilon^2) + \cdots ,$$

$$\beta = -\frac{9}{8} k^2 - \frac{15}{8} \epsilon^2 + \cdots .$$

Consequently,

$$\left| \frac{k^2}{\beta} \right| < \frac{8}{9},$$

and from Equations 178 and 179 we can see that in the process of integration the order of the coefficients of long-period terms satisfying the condition $i_1 - i_2 = -1$ remains the same as that before the integration.

In the case $\alpha = c_1 - c_2$, we have

$$K_1 = 1,$$

$$\beta = \sigma_1 - \sigma_2 = \sigma$$

and Equations 170 become

$$(\sigma^2 - 2k^2 \mu_{13} \mu_{31}) (1 + K_2) - \sigma (\mu_{11} - \lambda k^2 \mu_{12}) (1 - K_2) = N_2 + N_1, \tag{180}$$

$$\sigma(\mu_{11} + k^2 \mu_{12}) (1 + K_2) - \sigma^2 (1 - K_2) = N_2 - N_1 . \tag{181}$$

In order to simplify the elimination we add the expression

$$(1 + K_2) + (1 - K_2) = 2 \quad . \tag{182}$$

The coefficients N_1 and N_2 are taken from the corresponding part in S_1.

Elimination of K_2 leads to

$$\begin{vmatrix} \sigma^2 - 2k^2 \mu_{12} \mu_{31} & -\sigma(\mu_{11} - k^2 \mu_{12}) & N_2 + N_1 \\ \sigma(\mu_{11} + k^2 \mu_{12}) & -\sigma^2 & N_2 - N_1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \tag{183}$$

for the determination of σ and, consequently, of c_1. Equation (183) can be written in a form suitable for the application of the method of successive approximations:

$$\sigma^2 = \mu_{11}^2 + 2k^2 \mu_{13} \mu_{31} - \lambda^2 k^4 \mu_{12}^2 + \tau, \tag{184}$$

where we put

$$\tau = \left(1 + \frac{\mu_{11} + \lambda k^2 \mu_{12}}{\sigma}\right)(N_2 + N_1)$$

$$+ (N_1 - N_2)\left(\mu_{11} - k^2 \mu_{12} + \sigma - \frac{2k^2}{\sigma} \mu_{13} \mu_{31}\right). \tag{185}$$

The terms contained in τ are of the higher orders. They contain either ϵ^2 or λ as a factor. The value of σ in Equation 185 can be taken from the preceding cycle of iteration.

To obtain the first approximation to c_1 and c_2 in the case of a large inclination, we neglect ϵ^2 and λ^2 and set

$$\sigma_2 = -\frac{3}{4}\chi_0 \, , \tag{186}$$

$$\chi_0 = \sqrt{1 - k^2} \, .$$

In accordance with Equations 162' to 164' and 184 we have

$$\sigma_1 - \sigma_2 = \frac{3}{2}\sqrt{1 - \frac{5}{2}k^2} \, . \tag{187}$$

Thus

$$c_1 = \frac{3}{2}\lambda\left(\sqrt{1 - \frac{5}{2}k^2} - \frac{1}{2}\sqrt{1 - k^2}\right), \tag{188}$$

$$c_2 = -\frac{3}{4}\lambda\sqrt{1-k^2}. \tag{189}$$

We see that for a small value of ϵ we have a resonance effect at $k^2 = 2/5$, $I \approx 39°$, or $I \approx 141°$. The critical argument is the argument of the perigee.

We have to determine the value of K_2. From Equation 181 we obtain

$$K_2 = \frac{\sigma - \mu_{11} - \lambda k^2 \mu_{12} + [(N_2 - N_1)/\sigma]}{\sigma + \mu_{11} + \lambda k^2 \mu_{12}}. \tag{190}$$

In the first approximation we have

$$K_2 = \frac{\sigma - \mu_{11}}{\sigma + \mu_{11}} = -\frac{3}{2}\,\frac{k^2}{\left(\sqrt{1-\frac{5}{2}k^2} + \sqrt{1-k^2}\right)^2}, \tag{191}$$

and

$$\left.\begin{aligned} p_1 &= \zeta^{+c_1} + K_2\,\zeta^{-c_1+2c_2}, \\ p_2 &= \zeta^{-c_1} + K_2\,\zeta^{+c_1-2c_2}, \\ w_1 &= \zeta^{+c_2}, \\ w_2 &= \zeta^{-c_2}, \end{aligned}\right\} \tag{192}$$

where K_2 is defined by means of Equation 191. Substituting these values in Equation 160', we have

$$D p_3 = \lambda\mu_{31}\,(1 + K_2)\,(\zeta^{-c_1+c_2} + \xi^{c_1-c_2}),$$

and taking Equation 187 into account we obtain the first approximations to p_3:

$$p_3 = \frac{2}{3}\,\frac{\mu_{31}}{\sqrt{1-\frac{5}{2}k^2}}\,(1 + K_2)\,(\zeta^{c_1-c_2} - \zeta^{-c_1+c_2}). \tag{193}$$

Using the solutions in Equations 192 and 193, we start the process of iteration and continue it till we reach the convergence. The integration of Equations 157' and 158' is a straightforward process and requires no additional explanation.

COMPARISON WITH DEVELOPMENTS OF DELAUNAY

The lunar theory of Delaunay gives the mean motions of the argument of the perigee g and of the longitude of the ascending node h in the form of power series in the mean values of n'/n, $\gamma = \sin I/2$, e, e' and a'/a. The information about the domain of convergence of these series cannot be obtained in an easy way. We shall attempt to obtain some information about the convergence using the closed-form expressions for $(c_1 - c_2)^2$ and c_2. Delaunay's development of dg/dt corresponds to $c_1 - c_2$ of our theory; dh/dt corresponds to c_2. We consider those parts of dg/dt, dh/dt, c_1, and c_2 which do not contain ϵ because previously we made the assumption that it is small. We mentioned also that if this assumption is made then the solution can be obtained in the form of trigonometric series with the usual arguments of the lunar theory.

From Equations 157' and 184 we have accurately up to λ^3,

$$(c_1 - c_2)^2 = \lambda^2 (\mu_{11}^2 + 2k^2 \mu_{13} \mu_{31}), \tag{194}$$

and, accurately up to λ^2,

$$c_2 = \lambda \nu_{11}. \tag{195}$$

If $\epsilon = 0$ then we have $k^2 = 1 - \chi_0^2$, and making use of Equations 161' to 164' we obtain with the same accuracy as before the closed-form expressions

$$(c_1 - c_2)^2 = \frac{9}{4}\lambda^2 \left[\frac{1}{2}(5\chi_0^2 - 3) + \frac{9}{16}\lambda \left(1 + \frac{2}{3} e'^2\right)(25\chi_0^2 - 9)\chi_0\right], \tag{196}$$

and

$$c_2 = -\frac{3}{4}\lambda\chi_0 + \frac{9}{64}\lambda^2 \left(1 + \frac{2}{3} e'^2\right)(3\chi_0^2 - 1). \tag{197}$$

The results 196 and 197 can be extended to include higher powers of λ and of the parallax. Of course, the parallactic factor must be properly modified. The solving of this problem will constitute the topic of future work. It will require keeping the higher orders of the "cross actions" of short-period terms and parallactic terms. In order to facilitate comparison of our results with those of Delaunay, we shall eliminate χ_0 from Equations 196 and 197 by means of

$$\chi_0 = 1 - 2\gamma^2.$$

and perform the expansion in powers of γ we obtain

$$(c_1 - c_2)^2 = \frac{9}{4}\lambda^2 \left\{(1 - 10\gamma^2 + 10\gamma^4) + \frac{9}{4}\lambda \left(1 + \frac{2}{3} e'^2\right)(4 - 33\gamma^2 + 75\gamma^4 - 50\gamma^6)\right\} \tag{198}$$

and

$$c_2 = -\frac{3}{4}\lambda(1 - 2\gamma^2) + \frac{9}{32}\lambda^2 \left(1 + \frac{2}{3} e'^2\right)(1 - 6\gamma^2 + 6\gamma^4). \tag{199}$$

Assuming that γ is small, we obtain from Equation 198:

$$c_1 - c_2 = \frac{3}{4}\lambda \left[\left(1 - 5\gamma^2 - \frac{15}{2}\gamma^4 + \ldots\right) + \lambda \left(1 + \frac{2}{3} e'^2\right)\left(\frac{9}{2} - \frac{117}{8}\gamma^2 + 45\gamma^4 + \ldots\right)\right], \tag{200}$$

keeping only the terms necessary to make the comparison with the results of Delaunay.

In the definition of λ the factor $(1 - e'^2)^{-3/2}$ is included. The presence of this factor in the expanded form in Delaunay's results is beyond any doubt: it can be seen directly if one looks carefully at the developments of dg/dt and dh/dt.

Thus, we have to compare the parts multiplied by λ and λ^2 with the corresponding γ-terms multiplied by n'/n and $(n')^2/n^2$ in the expansion of dg/dt and dh/dt, and we see that they are identical.

We can use Equations 196 and 197 for high inclinations without any reservation about the convergence relative to γ or χ_0, if I_0 is not in the neighborhood of the critical values $I \approx 39°$, or $I \approx 141°$. However, a few degrees of deviation of I from 39° or 141° is enough to warrant the development of the perturbations of the elements into trigonometric series.

In the close neighborhood of the critical inclination the theory of resonance must be applied. The critical inclinations evidently cannot be obtained easily from the developments of Delaunay.

It is of interest to mention that in the systems of Jupiter and Saturn the inclinations of the orbit planes of the satellites with respect to the orbit planes of the planets all lie between the limits given above. The inclination of the orbit plane of Jupiter VIII lies near the resonance case.

CONCLUSIONS

The Krylov-Bogolubov method for eliminating the short-period terms can be applied to solve the stellar three-body problem and, in particular, to solve the lunar problem. The perturbations in the elements $\vec{e}$, $\vec{c}$, and ℓ can be obtained in the form of Fourier series with the arguments linear in the independent variable.

The method of Krylov-Bogolubov can serve to amend the theory of Delaunay because it leads to series in a more compact form and their convergence can be investigated more easily. In addition, they are applicable to any orbital inclination up to the critical value.

Our next goal consists in the further application of the Krylov-Bogolubov method to establish the solution including the terms of higher order in λ and α. The work will be extended in two directions: to find the compact form of the analytical development and to arrange the computational scheme in order to use the method of successive approximation. The development will be based on the author's scheme (Reference 15) of computing the higher order effects in the Krylov-Bogolubov method.

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APPENDIX A

Symbols

ℓ - mean anomaly of B, the close companion in ternary system.

e - osculating eccentricity of orbit of B.

a - osculating semimajor axis of B.

π - osculating longitude of the perigee of B.

Ω - osculating longitude of the ascending node of B.

r - position vector of B relative to the central body A, $r = |r|$.

e - osculating Laplacian vector of B.

c - osculating areal velocity of B, c_1, c_2, c_3 - the notations for the components of c. At the end of the exposition the notations c_1 and c_2 designate the mean motions of the perigee and of the node.

c_1 - mean motion of longitude of the pericenter of B.

c_2 - mean motion of the longitude of the ascending node of B.

I - mutual orbital inclination of the close and of the distant companion C.

r′ - position vector of C relative to the center of masses of A and B.

f′ - true anomaly of C.

c′ - eccentricity of the orbit of the distant companion C.

R - disturbing function.

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