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ON SOME POSSIBLE MODIFICATIONS IN BROUWER'S THEORY OF THE GENERAL PERTURBATIONS IN RECTANGULAR COORDINATES

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ABSTRACT

The method of general perturbations in rectangular coordinates is the most direct of all methods of expansion of the perturbations into series, because it is intimately associated with the computation of ephemerides. In addition, unlike the method of variation of elliptic elements, the method of coordinates does not have the zero eccentricity as a singularity. Brouwer's theory of the general perturbations in the rectangular coordinates makes use of the variation of elements in the canonical form. However, if the perturbations are being developed into trigonometric series with purely numerical coefficients, the use of canonical elements is not of any advantage. This fact was recognized by Davis who re-wrote Brouwer's formulas in terms of the standard elliptic elements. Davis' formula contains two terms of order -1 in the eccentricity. The presence of these terms causes considerable numerical inconvenience in the case of nearly circular orbits. We suggest here use of the Eckert-Brouwer formula for the orbit correction as a foundation of a planetary theory. The application of this formula leads directly to a vectorial expression for perturbations which is free from the disadvantages mentioned above and is also convenient for the numerical computations.

The method of iteration is suggested in computing the effects of higher orders. The inclusion of the higher order terms is important not only in the planetary case, but also in the case of artificial celestial bodies moving in orbits in cislunar space, far away from the Earth. Such bodies in their motions resemble more planets or comets than satellites.

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INTRODUCTION

In our time, the planetary perturbations can be obtained with the same high accuracy by development into trigonometrical series as they can by using step-by-step numerical integration. This is the main cause for the revival of interest toward the general perturbations. In addition, an expansion of perturbations into series provides us with the possibility of a deeper insight into the nature of resonances in the motion of the artificial and of the natural celestial bodies, and this is a second reason for further pursuing the analytical or the semi-analytical methods of solution in celestial mechanics. The artificial celestial bodies moving in orbits in cislunar space, far away from the Earth, resemble a planet or a comet in their motions more than they resemble a satellite. In the earth-moon system, the accumulation of the long-period and of the resonance effect is greatly accelerated as compared to the speed of accumulation of similar effects in the planetary system. Thus the artificial bodies will provide us with an excellent check of our theories and will stimulate their further development. Of all methods of expansion of the perturbations into series, the method of general perturbations in rectangular coordinates seems to be the most direct; it is intimately associated with the computation of planetary ephermerides. Unlike the method of variation of elliptic osculating elements, this method does not have the zero eccentricity as a singularity. The first modern approach to the problem can be found in the work by Brown and Shook (Reference 1). A further extension of this idea is due to Brouwer (Reference 2). More recently, works using different approaches to the problem were published by Danby (Reference 3) and by Musen (References 4 and 5).

The central idea of Brouwer's theory is one form of variation of astronomical elements. This form requires the computation of Langrangian and Poissonian brackets, not for the osculating elements, but for the constant elements. Thus this method removes the influence of the variability of elements on the coefficients of the disturbing force components from the differential equations, and shifts this influence to the modified disturbing force. In the present paper, we express this modified disturbing force in terms of Faá De Bruno (Reference 6) differential operators, as we did in a planetary theory of a different form (References 4 and 5). The perturbation effects of higher order's are transferred from the elements to these operators. The application of the Faá De Bruno operators leads to the decomposition of the disturbing force in terms of multipoles with the momenta equal to the perturbations in the position vectors of the planets. In his work Brouwer used the system of canonical elements. The formula he deduced resembles the formula for the perturbations in the coordinates obtained by Brown and Shook (Reference 1). However, if the perturbations are being developed into trigonometric series with purely numerical coefficients, then the use of canonical elements is not of any advantage. Evidently, this fact was recognized by Davis (Reference 7), who transformed Brouwer's formula and re-wrote it in terms of the standard elliptic elements. Davis' formula contains, however, two terms of order -1 in the eccentricity. The presence of such terms causes considerable numerical inconvenience in the case of nearly circular orbits. We suggest here a modification of Brouwer's formula which is free from this disadvantage. In the classical planetary theories the relative positions of the orbital planes are taken into account by use of trigonometrical formulas. Such a system is definitely going out of fashion because it causes asymmetry in the development of higher order perturbations. Programming for digital computers very strongly favors the use of vectors and matrices. For all of these reasons, we discard the use of the canonical elements even as an intermediate step, and we suggest, instead, the use of the Eckert-Brouwer formula for the orbit correction as a foundation of a planetary theory (Reference 8). This formula in fact represents a solution of the variational equation of the two-body problem in terms of position and velocity vectors and also in terms of Gibbsian vectors and leads directly to a vectorial expression for perturbations which is both free from the disadvantages mentioned above and is convenient for numerical computations.

BASIC DIFFERENTIAL EQUATIONS

Here the case of two planets influencing each other is considered. Generalization to the case of a planetary system does not present any theoretical difficulty. The differential equation of the disturbed motion of the planet with mass m, as disturbed by the planet with the mass m', can be written in the form

$$\frac{d^2(\mathbf{r} + \delta\mathbf{r})}{dt^2} = \mu^2 \nabla \frac{1}{|\mathbf{r} + \delta\mathbf{r}|} + k^2 m' \nabla' \left(\frac{1}{|\mathbf{r}' + \delta\mathbf{r}'|} - \frac{1}{|\boldsymbol{\rho} + \delta\boldsymbol{\rho}|}\right), \tag{1}$$

where

$$\mu^2 = k^2 (1 + m).$$

By taking into account that

$$\frac{d^2\mathbf{r}}{dt^2} = \mu^2 \nabla \frac{1}{r} \tag{2}$$

and by making use of the operators

$$D = \nabla \exp \delta \mathbf{r} \cdot \nabla, \tag{3}$$

$$D' = \nabla' \exp \delta \mathbf{r}' \cdot \nabla', \text{ and} \tag{4}$$

$$D'' = \nabla' \exp \delta \boldsymbol{\rho} \cdot \nabla', \tag{5}$$

we can rewrite Equation 1 as

$$\frac{d^2 \delta \mathbf{r}}{dt^2} = \mu^2 (D - \nabla) \frac{1}{r} + k^2 m' \left(D' \frac{1}{r} - D'' \frac{1}{\rho}\right). \tag{6}$$

By taking the identity

$$\delta \mathbf{r} \cdot \nabla \nabla \frac{1}{r} = - \frac{1}{r^3} \left(I - \frac{3\,\mathbf{r}\,\mathbf{r}}{r^2}\right) \cdot \delta \mathbf{r}$$

into account we deduce that

$$\frac{d^2 \delta \mathbf{r}}{dt^2} + \frac{\mu^2}{r^3} \left(I - \frac{3\,\mathbf{r}\,\mathbf{r}}{r^2}\right) \cdot \delta \mathbf{r} = \mathbf{F}\ , \tag{7}$$

where

$$\mathbf{F} = \mu^2 (D - \nabla - \delta \mathbf{r} \cdot \nabla \nabla) \frac{1}{r} + k^2 m' \left(D' \frac{1}{r'} - D'' \frac{1}{\rho}\right)\ . \tag{8}$$

We make use of Brouwer's idea and apply the method of variation of constants to solve Equation 7, but avoid the use of canonical variables. The solution of Equation 2 for the undisturbed motion has the form

$$\mathbf{r} = \mathbf{r}(t;\ c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6)\ ,$$

where c_i are the constant elements. The general solution of the variational equation

$$\frac{d^2 \delta \mathbf{r}}{dt^2} + \frac{\mu^2}{r^3} \left(I - \frac{3\,\mathbf{r}\,\mathbf{r}}{r^2}\right) \cdot \delta \mathbf{r} = 0 \tag{7a}$$

has the form

$$\delta \mathbf{r} = \sum_{j=1}^{6} \delta c_j \frac{\partial \mathbf{r}}{\partial c_j}\ , \tag{9}$$

where δc_j are the arbitrary constants of integration. The method of variation of constants leads to the equations

$$\delta \mathbf{v} = \sum_{j=1}^{6} \delta c_j \frac{\partial \mathbf{v}}{\partial c_j}, \tag{10}$$

$$\sum_{j=1}^{6} \frac{d\delta c_j}{dt} \frac{\partial \mathbf{r}}{\partial c_j} = 0 \quad , \text{ and} \tag{11}$$

$$\sum_{j=1}^{6} \frac{d\delta c_j}{dt} \frac{\partial \mathbf{v}}{\partial c_j} = \mathbf{F} \cdot \tag{12}$$

It follows from Equations 11 and 12 that

$$\sum_{j=1}^{6} [c_i, c_j] \frac{d\delta c_j}{dt} = \frac{\partial \mathbf{r}}{\partial c_i} \cdot \mathbf{F}, \tag{13}$$

where $[c_i , c_j]$ are the Lagrangian brackets

$$[c_i, c_j] = \frac{\partial \mathbf{r}}{\partial c_i} \cdot \frac{\partial \mathbf{v}}{\partial c_j} - \frac{\partial \mathbf{r}}{\partial c_j} \cdot \frac{\partial \mathbf{v}}{\partial c_i} . \tag{14}$$

By solving Equation 13 with respect to $d\delta c_j / dt$ we obtain

$$\frac{d\delta c_i}{dt} = \sum_{j=1}^{6} (c_i, c_j) \frac{\partial \mathbf{r}}{\partial c_j} \cdot \mathbf{F}. \tag{15}$$

At first glance, Equations 13 and 15 have the same form as for osculating elements. However, the substantial difference between the standard method of variation of the osculating elements and Brouwer's idea is that the Lagrangian brackets in Equation 13 and the Poissonian brackets in Equation 15 are formed with the constant elements and not with the osculating elements. The sums $c_i + \delta c_i$ are not the osculating elements either. This means that the influence of the perturbations is removed from the elements and is transferred to the operators (Equations 3 through 5). This transfer produces considerable simplification in the computational procedure, as compared to the procedure for determining the general perturbations in the osculating elements.

From Equations 9 and 15 we obtain

$$\delta \mathbf{r} = \sum_{i=1}^{6} \frac{\partial \mathbf{r}}{\partial c_i} (\delta c_i) + \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{\partial \mathbf{r}}{\partial c_i} \int (c_i, c_j) \frac{\partial \mathbf{r}}{\partial c_j} \cdot \mathbf{F}\, dt \ , \tag{16}$$

where (δc_i) are the constants of integration. In order to shorten the number of trigonometric series we use the Hansen and Hill device. We distinguish between the time t under the integrals in Equation 16 and between the time contained in the undisturbed position vector $\mathbf{r}(t)$ and in its derivatives which stand outside the integrals. We designate temporarily by τ the time associated with the undisturbed motion and write $\bar{\mathbf{r}}$ for $\mathbf{r}(\tau)$. The mean anomaly $n\tau + \ell_0$ associated with $\bar{\mathbf{r}}$ is designated by $\bar{\ell}$. We consider τ and $\bar{\mathbf{r}}$ as constants until the integration is performed. After the integration, we replace τ by t again. Then taking

$$(c_i, c_j) = -(c_j, c_i)$$

into account we can write Equation 16 in a symmetrical form

$$\delta \mathbf{r} = \sum_{i=1}^{6} \frac{\partial \mathbf{r}}{\partial c_i} (\delta c_i) + \int \Gamma(t, \tau) \cdot \mathbf{F}\, dt \ , \tag{17}$$

where Γ is a dyadic defined by

$$\Gamma(t, \tau) = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} (c_i, c_j) \left(\frac{\partial \bar{\mathbf{r}}}{\partial c_i} \frac{\partial \mathbf{r}}{\partial c_j} - \frac{\partial \bar{\mathbf{r}}}{\partial c_j} \frac{\partial \mathbf{r}}{\partial c_i} \right) \ . \tag{18}$$

In a similar manner we obtain the perturbations in the velocity by,

$$\delta \mathbf{v} = \sum_{i=1}^{6} \frac{\partial \mathbf{v}}{\partial c_i} (\delta c_i) + \int \frac{\partial \Gamma(t, \tau)}{\partial \tau} \cdot \mathbf{F}\, dt \ . \tag{17'}$$

On the basis of the principle expressed by Equation 15 each differential equation for the perturbations of an osculating element can be transformed into the equation of our theory simply by replacing $\partial \Omega / \partial c_i$, the derivative of the disturbing function Ω, by $\partial \mathbf{r} / \partial c_i \cdot \mathbf{F}$. The "disturbing force" $\mathbf{F}$ is defined by Equation 8. If the element is a vector $\mathbf{c}$, then $\nabla_c \Omega$ should be replaced by $\nabla_c \mathbf{r} \cdot \mathbf{F}$. In his previous work (Reference 9) the author has obtained the following equations for the osculating

$\mathbf{P}, \mathbf{Q}$ and e:

$$\frac{d\mathbf{P}}{dt} = -\frac{1}{na^2\sqrt{1-e^2}}\ \mathbf{RR}\cdot\nabla_{\mathbf{Q}}\Omega + \frac{\sqrt{1-e^2}}{na^2 e}\ \mathbf{Q}\frac{\partial\Omega}{\partial e}, \tag{19}$$

$$\frac{d\mathbf{Q}}{dt} = +\frac{1}{na^2\sqrt{1-e^2}}\ \mathbf{RR}\cdot\nabla_{\mathbf{P}}\Omega - \frac{\sqrt{1-e^2}}{na^2 e}\ \mathbf{P}\frac{\partial\Omega}{\partial e}, \tag{20}$$

$$\frac{de}{dt} = +\frac{1-e^2}{na^2 e}\ \frac{\partial\Omega}{d\ell} + \frac{\sqrt{1-e^2}}{na^2 e}\ (\mathbf{P}\cdot\nabla_{\mathbf{Q}}\Omega - \mathbf{Q}\cdot\nabla_{\mathbf{P}}\Omega). \tag{21}$$

From

$$\mathbf{r} = \xi\mathbf{P} + \eta\mathbf{Q}, \quad \xi = c_1 = a(\cos u - e), \text{ and } \eta = s_1 = a\sqrt{1-e^2}\sin u$$

we have

$$\nabla_{\mathbf{P}}\mathbf{r} = \mathbf{I}c_1, \qquad \nabla_{\mathbf{Q}}\mathbf{r} = \mathbf{I}s_1,$$

$$\nabla_{\mathbf{P}}\mathbf{r}\cdot\mathbf{F} = c_1\mathbf{F}, \text{ and } \nabla_{\mathbf{Q}}\mathbf{r}\cdot\mathbf{F} = s_1\mathbf{F},$$

and the equations of our theory analogous to Equations 19 through 21 take the form:

$$\frac{d\delta\mathbf{P}}{dt} = +\frac{\sqrt{1-e^2}}{na^2 e}\ \mathbf{Q}\frac{\partial\mathbf{r}}{\partial e}\cdot\mathbf{F} - \frac{1}{na^2\sqrt{1-e^2}}\ s_1\ \mathbf{RR}\cdot\mathbf{F}, \tag{22}$$

$$\frac{d\delta\mathbf{Q}}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\ \mathbf{P}\frac{\partial\mathbf{r}}{\partial e}\cdot\mathbf{F} + \frac{1}{na^2\sqrt{1-e^2}}\ c_1\ \mathbf{RR}\cdot\mathbf{F}, \text{ and} \tag{23}$$

$$\frac{d\delta e}{dt} = +\frac{\sqrt{1-e^2}}{na^2 e}\left(\mathbf{r}\times\mathbf{R} + \sqrt{1-e^2}\ \frac{\partial\mathbf{r}}{\partial\ell}\right)\cdot\mathbf{F}. \tag{24}$$

The coefficients in these equations are formed by using the constant elements. Equations 22 and 23 can be replaced by the system

$$\delta\mathbf{P} = \delta\mathbf{\Psi}\times\mathbf{P}, \tag{25}$$

$$\delta\mathbf{Q} = \delta\mathbf{\Psi}\times\mathbf{Q}, \tag{26}$$

and

$$\frac{d\,\delta\boldsymbol{\Psi}}{dt} = + \frac{\sqrt{1-e^2}}{na^2\,e}\,\mathbf{R}\,\frac{\partial \mathbf{r}}{\partial e}\cdot\mathbf{F} + \frac{1}{na^2\sqrt{1-e^2}}\,\mathbf{r}\,\mathbf{R}\cdot\mathbf{F}\,. \tag{27}$$

A system of approximate equations resembling Equations 25 through 27 appears in Strömgren's theory of special perturbations (Reference 10). In our theory Equations 22 through 24 were exact. The theory presented in this paper can also be considered as an extension and improvement of Strömgren's results.

The classical equations

$$\frac{da}{dt} = \frac{2}{na}\,\frac{\partial\Omega}{\partial \ell}\,,\ \text{and}$$

$$\frac{d\ell}{dt} = -\frac{1-e^2}{na^2\,e}\,\frac{\partial\Omega}{\partial e} - \frac{2}{na}\,\frac{\partial\Omega}{\partial a}$$

have as their analogues in our theory the equations:

$$\frac{d}{dt}\,\frac{\delta a}{a} = + \frac{2}{na^2}\,\frac{\partial \mathbf{r}}{\partial \ell}\cdot\mathbf{F}\,,\ \text{and} \tag{28}$$

$$\frac{d\,\delta\ell_0}{dt} = -\left(\frac{1-e^2}{na^2\,e}\,\frac{\partial \mathbf{r}}{\partial e} + \frac{2}{na^2}\cdot a\,\frac{\partial \mathbf{r}}{\partial a}\right)\cdot\mathbf{F}\,. \tag{29}$$

Each formula for the differential correction of the orbital elements is, in fact, a solution of the variational equation (7a), and each such solution can serve as a foundation of a perturbation theory. However, the orbit correction formula by Eckert and Brouwer (Reference 8),

$$\delta\mathbf{r} = \delta\boldsymbol{\Psi}\times\mathbf{r} + \frac{\partial \mathbf{r}}{\partial \ell}\,\delta\ell_0 + a\,\frac{\partial \mathbf{r}}{\partial a}\cdot\frac{\delta a}{a} + \frac{\partial \mathbf{r}}{\partial e}\,\delta e\,, \tag{30}$$

where

$$a\,\frac{\partial \mathbf{r}}{\partial a} = \mathbf{r} - \frac{3}{2}\,n(t-t_0)\,\frac{\partial \mathbf{r}}{\partial \ell}\,, \tag{31}$$

$$\frac{\partial \mathbf{r}}{\partial e} = H\,\mathbf{r} + K\,\frac{\partial \mathbf{r}}{\partial \ell}\,, \tag{32}$$

$$H = -\frac{\cos u + e}{1-e^2}\,, \tag{33}$$

and

$$K = \left(\frac{2}{1 - e^2} + eH\right) \sin u, \tag{34}$$

is associated in a most intimate way with the determination of the general perturbations in $\mathbf{r}$ and $\mathbf{v}$. Direct application of this formula, by-passing the use of the canonical elements, leads immediately to a form of Γ which is in agreement with the general spirit of this theory.

Substituting Equations 24 and 27 through 29 into Equation 30 we obtain:

$$\Gamma = \frac{\sqrt{1 - e^2}}{na^2 e} \left[\frac{\partial \bar{\mathbf{r}}}{\partial e} \left(\mathbf{r} \times \mathbf{R} + \sqrt{1 - e^2}\, \frac{\partial \mathbf{r}}{\partial \ell}\right) - \left(\bar{\mathbf{r}} \times \mathbf{R} + \sqrt{1 - e^2}\, \frac{\partial \bar{\mathbf{r}}}{\partial \bar{\ell}}\right) \frac{\partial \mathbf{r}}{\partial e}\right]$$

$$+ \frac{2}{na^2} \left[\left(a \frac{\partial \bar{\mathbf{r}}}{\partial a}\right) \frac{\partial \mathbf{r}}{\partial \ell} - \frac{\partial \bar{\mathbf{r}}}{\partial \bar{\ell}} \left(a \frac{\partial \mathbf{r}}{\partial a}\right)\right] + \frac{1}{na^2 \sqrt{1 - e^2}}\, \mathbf{r} \times \bar{\mathbf{r}}\, \mathbf{R}. \tag{35}$$

The expressions

$$\mathbf{r} \times \mathbf{R} + \sqrt{1 - e^2}\, \frac{\partial \mathbf{r}}{\partial \ell} \quad \text{and} \quad \bar{\mathbf{r}} \times \mathbf{R} + \sqrt{1 - e^2}\, \frac{\partial \bar{\mathbf{r}}}{\partial \bar{\ell}}$$

contain the eccentricity as a factor which will be canceled with the eccentricity in the denominator. Our form of Γ makes it evident that $e = 0$ is not a singularity. The use of the standard elliptic elements and of Gibbsian vectors leads to a numerical theory which is valid also for nearly circular orbits. If the numerical theory of the general perturbations were to be developed for the osculating elliptic elements directly, then the eccentricity would appear in the denominator in several places, thus causing considerable difficulty in the case of nearly circular orbits.

Substituting

$$\mathbf{r} = c_1 \mathbf{P} + s_1 \mathbf{Q}, \quad c_1 = a(\cos u - e), \quad s_1 = a\sqrt{1 - e^2}\, \sin u,$$

$$\frac{\partial \mathbf{r}}{\partial \ell} = c_2 \mathbf{P} + s_2 \mathbf{Q}, \quad c_2 = -\frac{a^2}{r} \sin u, \quad \text{and} \quad s_2 = +\frac{a^2}{r} \sqrt{1 - e^2}\, \cos u \tag{36}$$

into Equation 35 we obtain the following decomposition of Γ:

$$\Gamma = \Gamma_{11}\, \mathbf{P}\mathbf{P} + \Gamma_{12}\, \mathbf{P}\mathbf{Q} + \Gamma_{21}\, \mathbf{Q}\mathbf{P} + \Gamma_{22}\, \mathbf{Q}\mathbf{Q} + \Gamma_{33}\, \mathbf{R}\mathbf{R}, \tag{37}$$

where

$$\Gamma_{11} = + \frac{\sqrt{1-e^2}}{na^2} (\bar{c}_4 c_3 - \bar{c}_3 c_4) + \frac{2}{na^2} (\bar{c}_5 c_2 - \bar{c}_2 c_5),$$

$$\Gamma_{12} = + \frac{\sqrt{1-e^2}}{na^2} (\bar{c}_4 s_3 - s_4 \bar{c}_3) + \frac{2}{na^2} (\bar{c}_5 s_2 - \bar{c}_2 s_5),$$

$$\Gamma_{21} = + \frac{\sqrt{1-e^2}}{na^2} (\bar{s}_4 c_3 - \bar{s}_3 c_4) + \frac{2}{na^2} (\bar{s}_5 c_2 - \bar{s}_2 c_5),$$

$$\Gamma_{22} = + \frac{\sqrt{1-e^2}}{na^2} (\bar{s}_4 s_3 - \bar{s}_3 s_4) + \frac{2}{na^2} (\bar{s}_5 s_2 - \bar{s}_2 s_5),$$

$$\Gamma_{33} = \frac{1}{na^2 \sqrt{1-e^2}} (c_1 \bar{s}_1 - s_1 \bar{c}_1),$$

$$c_3 = -\frac{1}{2} \frac{a^2}{r} \sqrt{1-e^2} \sin 2u, \qquad s_3 = a + \frac{a^2}{r} (\cos u - e) \cos u$$

$$c_4 = -a\left(1 + \frac{a}{r} \sin^2 u\right), \qquad s_4 = \frac{a}{\sqrt{1-e^2}} \cdot \frac{a}{r} (\cos u - e) \sin u,$$

$$c_5 = a \left[\cos u - e + \frac{3}{2} n (t - t_0) \frac{a}{r} \sin u\right], \text{ and}$$

$$s_5 = a \sqrt{1-e^2} \left[\sin u - \frac{3}{2} n (t - t_0) \frac{a}{r} \cos u\right].$$

The developments of c_i, s_i into the trigonometric series in ℓ with numerical coefficients can be obtained by using either the analytical methods or numerical Fourier analysis. The series for $\bar{c}_i$, $\bar{s}_i$ are obtained from the corresponding series for c_i, s_i by replacing ℓ by $\bar{\ell}$. After we obtained the developments of c_i, s_i, $\bar{c}_i$, $\bar{s}_i$, then the developments of Γ_{ij} into a double Fourier series with the arguments ℓ and $\bar{\ell}$ can be performed on an electronic computer without any great difficulty.

Setting

$$\mathbf{P} \cdot \mathbf{F} = F_1, \quad \mathbf{Q} \cdot \mathbf{F} = F_2, \text{ and } \mathbf{R} \cdot \mathbf{F} = F_3,$$

$$\frac{dW_1}{dt} = \Gamma_{11} F_1 + \Gamma_{12} F_2 , \tag{38}$$

$$\frac{dW_2}{dt} = \Gamma_{21} F_1 + \Gamma_{22} F_2 , \text{ and} \tag{39}$$

$$\frac{dW_3}{dt} = \Gamma_{33} F_3 , \tag{40}$$

we deduce

$$\begin{aligned} \delta \mathbf{r} &= (W_1 \mathbf{P} + W_2 \mathbf{Q} + W_3 \mathbf{R})_{\tau=t} + (\delta r) \\ &= \mathbf{P}\, \delta\xi + \mathbf{Q}\, \delta\eta + \mathbf{R}\, \delta\zeta , \end{aligned} \tag{41}$$

where

$$(\delta r) = \delta\Psi \times \mathbf{r} + \frac{\partial \mathbf{r}}{\partial \ell} \delta\ell_0 + a \frac{\partial \mathbf{r}}{\partial a} \frac{\delta a}{a} + \frac{\partial \mathbf{r}}{\partial e} \delta e ,$$

and the symbols

$$\delta\Psi,\ \delta\ell_0,\ \frac{\delta a}{a},\ \text{and } \delta e$$

now designate the constants of integration. The expressions for the partial derivatives are given by Equations 31, 32, and 36.

DECOMPOSITION OF THE DISTURBING FORCE

The system Equations 38 through 41 can be solved either by developing $\delta\mathbf{r}$ into a power series in the masses m and m′ or by the method of iteration. If we choose the first way we set:

$$\delta \mathbf{r} = \delta_1 \mathbf{r} + \delta_2 \mathbf{r} + \delta_3 \mathbf{r} + \cdots ,$$

$$\delta \mathbf{r}' = \delta_1 \mathbf{r}' + \delta_2 \mathbf{r}' + \delta_3 \mathbf{r}' + \cdots ,$$

$$\delta \rho = \delta_1 \rho + \delta_2 \rho + \delta_3 \rho + \cdots , \text{ and}$$

$$\delta_k \rho = \delta_k \mathbf{r}' - \delta_k \mathbf{r} ,$$

where $\delta_k \mathbf{r}$, $\delta_k \mathbf{r}'$ and $\delta_k \boldsymbol{\rho}$ are the kth order in m and m'. We have

$$\exp(\delta \mathbf{r} \cdot \nabla) = T_0 + T_1 + T_2 + T_3 + \cdots,$$

$$\exp(\delta \mathbf{r}' \cdot \nabla') = T_0' + T_1' + T_2' + T_3' + \cdots, \text{ and}$$

$$\exp(\delta \boldsymbol{\rho} \cdot \nabla') = T_0'' + T_1'' + T_2'' + T_3'' + \cdots,$$

where T_k, T_k', and T_k'' (k = 0, 1, 2, . . .) are Faá De Bruno differential operators. By setting

$$\delta_k = \delta_k \mathbf{r} \cdot \nabla ,$$

we have

$$T_0 = 1, \quad T_1 = \delta_1, \quad T_2 = \delta_2 + \frac{1}{2}\,\delta_1^2, \quad \text{and} \quad T_3 = \delta_3 + \delta_1\,\delta_2 + \frac{1}{6}\,\delta_1^3 \cdots \tag{42}$$

By replacing δ_k in Equation 42 by

$$\delta_k' = \delta_k \mathbf{r}' \cdot \nabla',$$

or by

$$\delta_k'' = \delta_k\, \delta\boldsymbol{\rho} \cdot \nabla',$$

we obtain the operators T_k' and T_k'', respectively.

In order to obtain the expansions of the operators

$$T_k,\ T_k',\ T_k'',\ D,\ D', \text{ and } D''$$

in powers of perturbations one can use formulas for the potentials of multipoles. By setting

$$\boldsymbol{\rho} = \mathbf{r} - \mathbf{a}$$

and designating the multipoles moments by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, . . . , and the del-operator with respect to $\mathbf{r}$ by ∇, we have:

$$\mathbf{a}_1 \cdot \nabla \frac{1}{\rho} = -\frac{\mathbf{a}_1 \cdot \boldsymbol{\rho}}{\rho^3},$$

$$\mathbf{a}_1 \cdot \nabla\, \mathbf{a}_2 \cdot \nabla \frac{1}{\rho} = + \frac{3}{\rho^5}\, \boldsymbol{\rho} \cdot \mathbf{a}_1\, \boldsymbol{\rho} \cdot \mathbf{a}_2 - \frac{1}{\rho^3}\, \mathbf{a}_1 \cdot \mathbf{a}_2 ,$$

$$\mathbf{a}_1 \cdot \nabla\, \mathbf{a}_2 \cdot \nabla\, \mathbf{a}_3 \cdot \nabla \frac{1}{\rho} =$$

$$- \frac{15}{\rho^7}\, \boldsymbol{\rho} \cdot \mathbf{a}_1\, \boldsymbol{\rho} \cdot \mathbf{a}_2\, \boldsymbol{\rho} \cdot \mathbf{a}_3 + \frac{3}{\rho^5} (\boldsymbol{\rho} \cdot \mathbf{a}_1\, \mathbf{a}_2 \cdot \mathbf{a}_3 + \boldsymbol{\rho} \cdot \mathbf{a}_2\, \mathbf{a}_3 \cdot \mathbf{a}_1 + \boldsymbol{\rho} \cdot \mathbf{a}_3\, \mathbf{a}_1 \cdot \mathbf{a}_2), \tag{43}$$

$$\mathbf{a}_1 \cdot \nabla\, \mathbf{a}_2 \cdot \nabla\, \mathbf{a}_3 \cdot \nabla\, \mathbf{a}_4 \cdot \nabla \frac{1}{\rho} =$$

$$+ \frac{105}{\rho^9}\, \boldsymbol{\rho} \cdot \mathbf{a}_1\, \boldsymbol{\rho} \cdot \mathbf{a}_2\, \boldsymbol{\rho} \cdot \mathbf{a}_3\, \boldsymbol{\rho} \cdot \mathbf{a}_4 - \frac{15}{\rho^7} (\boldsymbol{\rho} \cdot \mathbf{a}_2\, \boldsymbol{\rho} \cdot \mathbf{a}_3\, \mathbf{a}_1 \cdot \mathbf{a}_4 + \boldsymbol{\rho} \cdot \mathbf{a}_3\, \boldsymbol{\rho} \cdot \mathbf{a}_1\, \mathbf{a}_2 \cdot \mathbf{a}_4$$

$$+ \boldsymbol{\rho} \cdot \mathbf{a}_1\, \boldsymbol{\rho} \cdot \mathbf{a}_2\, \mathbf{a}_3 \cdot \mathbf{a}_4 + \boldsymbol{\rho} \cdot \mathbf{a}_1\, \boldsymbol{\rho} \cdot \mathbf{a}_4\, \mathbf{a}_2 \cdot \mathbf{a}_3 + \boldsymbol{\rho} \cdot \mathbf{a}_2\, \boldsymbol{\rho} \cdot \mathbf{a}_4\, \mathbf{a}_1 \cdot \mathbf{a}_3 + \boldsymbol{\rho} \cdot \mathbf{a}_3\, \boldsymbol{\rho} \cdot \mathbf{a}_4\, \mathbf{a}_1 \cdot \mathbf{a}_2)$$

$$+ \frac{3}{\rho^5} (\mathbf{a}_1 \cdot \mathbf{a}_4\, \mathbf{a}_2 \cdot \mathbf{a}_3 + \mathbf{a}_2 \cdot \mathbf{a}_4\, \mathbf{a}_3 \cdot \mathbf{a}_1 + \mathbf{a}_3 \cdot \mathbf{a}_4\, \mathbf{a}_1 \cdot \mathbf{a}_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

They remain valid if one of the moments is replaced by a matrix. Equations 38 through 41 require a knowledge of the components of **F** in the directions of **P**, **Q** and **R**. In order to obtain these components we have to replace one of the moments by **P**, **Q** or **R** respectively.

In a planetary theory we have to replace

$$\mathbf{a}_k \text{ by } \delta_k\, \mathbf{r}, \quad \delta_k\, \mathbf{r}', \text{ and } \delta_k\, \boldsymbol{\rho}$$

and

$$\nabla \text{ can mean either } \nabla \text{ or } \nabla' .$$

The application of the relations 43 to the expansion of Equation 8 was fully discussed in the author's previous work and therefore the details can be omitted in the present article. On occasion it might be more convenient to use the process of iteration instead of expanding $\delta\mathbf{r}$ and $\delta\mathbf{r}$ in powers of masses. Replacing one of the moments in Equation 43 by the idemfactor and setting the remaining moments equal to $\delta\mathbf{r}$, we have

$$\delta\mathbf{r} \cdot \nabla \nabla \frac{1}{r} = + \frac{3}{r^5}\, \mathbf{r}\, \mathbf{r} \cdot \delta\mathbf{r} - \frac{1}{r^3}\, \delta\mathbf{r} ,$$

$$\frac{1}{2}(\delta\mathbf{r}\cdot\nabla)^2\,\nabla\frac{1}{r} = -\frac{15}{2r^7}(\mathbf{r}\cdot\delta\mathbf{r})^2\,\mathbf{r} + \frac{3}{r^5}\left(\mathbf{r}\cdot\delta\mathbf{r}\;\delta\mathbf{r} + \frac{1}{2}\,\mathbf{r}\;\delta\mathbf{r}^2\right),$$

$$\frac{1}{6}(\delta\mathbf{r}\cdot\nabla)^3\,\nabla\frac{1}{r} = +\frac{35}{2r^9}(\mathbf{r}\cdot\delta\mathbf{r})^3\,\mathbf{r}$$

$$-\frac{15}{2r^7}\left[\mathbf{r}\;\mathbf{r}\cdot\delta\mathbf{r}\;\delta\mathbf{r}^2 + (\mathbf{r}\cdot\delta\mathbf{r})^2\;\delta\mathbf{r}\right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

From the last equations and from Equation 3 we deduce that

$$(\mathrm{D} - \nabla - \delta\mathbf{r}\cdot\nabla\nabla)\,\frac{1}{r} = \mathrm{A}\;\mathbf{r} + \mathrm{B}\,\delta\,\mathbf{r}, \tag{44}$$

where

$$\mathrm{A} = -\frac{15}{2r^7}\,\alpha^2 + \frac{3}{2r^5}\,\beta^2 + \frac{35}{2r^9}\,\alpha^3 - \frac{15}{2r^7}\,\alpha'\beta'^2 + \cdots, \tag{45}$$

$$\mathrm{B} = +\frac{3}{r^5}\,\alpha - \frac{15}{2r^7}\,\alpha^2 + \cdots, \tag{46}$$

$$\alpha = \mathbf{r}\cdot\delta\mathbf{r},\ \text{and}\ \beta^2 = \delta\mathbf{r}^2.$$

In an analogous way we deduce that

$$\mathrm{D}'\,\frac{1}{r'} = \mathrm{A}'\;\mathbf{r}' + \mathrm{B}'\;\delta\mathbf{r}', \tag{47}$$

$$\mathrm{A}' = -\frac{1}{r'^3} + \frac{3}{r'^5}\,\alpha' - \frac{15}{2r'^7}\,\alpha'^2 + \frac{3}{2r'^5}\,\beta'^2 + \frac{35}{2r'^9}\,\alpha'^3 - \frac{15}{2r'^7}\,\alpha\beta'^2 + \cdots, \tag{48}$$

$$\mathrm{B}' = -\frac{1}{r'^3} + \frac{3}{r'^5}\,\alpha' - \frac{15}{2r'^7}\,\alpha'^2 + \cdots, \tag{49}$$

$$\alpha' = \mathbf{r}'.\,\delta\mathbf{r}',$$

$$\beta'^2 = \delta\mathbf{r}'^2,$$

$$\mathrm{D}''\,\frac{1}{\rho} = \mathrm{A}''\;\boldsymbol{\rho} + \mathrm{B}''\;\delta\boldsymbol{\rho}, \tag{50}$$

$$A'' = -\frac{1}{\rho^3} + \frac{3}{\rho^5}\alpha'' - \frac{15}{2\rho^7}\alpha''^2 + \frac{3}{2\rho^5}\beta''^2 + \frac{35}{2\rho^9}\alpha''^3 - \frac{15}{2\rho^7}\alpha''\beta''^2 + \ldots\ldots, \tag{51}$$

$$B'' = -\frac{1}{\rho^3} + \frac{3}{\rho^5}\alpha'' - \frac{15}{2\rho^7}\alpha''^2 + \ldots\ldots,$$

$$\alpha'' = \boldsymbol{\rho}\cdot\delta\boldsymbol{\rho}, \text{ and}$$

$$\beta''^2 = \delta\boldsymbol{\rho}^2. \tag{52}$$

The expansions of A and B into double series in α and β^2 converge very fast. The negative powers of r, r′ and p appearing in the coefficients can be expanded into double Fourier series, with the arguments ℓ and ℓ', by means of harmonic analysis. By making use of Equations 44, 47 and 50 we can write Equation 8 in a compact form,

$$\mathbf{F} = (A\,\mathbf{r} + B\,\delta\,\mathbf{r}) + k^2\,m'\,[(A'\,\mathbf{r}' + \mathbf{B}'\,\delta\,\mathbf{r}') + (A''\,\boldsymbol{\rho} + B''\,\delta\boldsymbol{\rho})], \tag{53}$$

which is convenient for computing the perturbations by iteration. Decomposing Equation 53 along **P**, **Q** and **R** we obtain the expressions for F_1, F_2, F_3 to be used in association with Equations 38 through 41. In the author's earlier theories (References 4 and 5) $\delta\mathbf{r}$ was decomposed along $\mathbf{r}$, $\mathbf{v}$ and **R** or along $\mathbf{r}$, $\mathbf{R}\times\mathbf{r}$, **R**. The same form of **F** (Equation 53) can also be used conveniently in these earlier theories if the method of iteration is preferred.

CONCLUSION

The results given in this article represent a modification and extension of the results by Brouwer, Davis and Strömgren. Equation 8, the basic integral equation, which determines the perturbation $\delta\mathbf{r}$, can be solved by means of iteration with the help of Equations 35 and 44 through 53. The solution by expanding the perturbations into power series with respect to the masses can also be achieved using the system of operators given in the author's previous work. We suggest the application of the double harmonic analysis to expand the negative powers of the mutual distance.

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Appendix

Notations

k - the Gaussian constant

m - the mass of the disturbed planet; the mass of the Sun is one

$\mu^2 = k^2 (1 + m)$

m' - the mass of the disturbing planet

$\mathbf{r}$ - the undisturbed position vector of the planet m

$r = |\mathbf{r}|$

u - the undisturbed eccentric anomaly of m

$\mathbf{v}$ - the undisturbed velocity of m

$\mathbf{P}, \mathbf{Q}, \mathbf{R}$ - the Gibbsian vectors of m

a - the undisturbed semi-major axis of m

e - the undisturbed eccentricity of m

$n = \mu a^{-3/2}$ - the undisturbed anomalistic mean motion of m

ℓ_0 - the mean anomaly at the epoch of m

$\ell = nt + \ell_0$ - the undisturbed mean anomaly of m

$\mathbf{r}'$ - the undisturbed position vector of m'

$r' = |\mathbf{r}'|$

$\mathbf{r} + \delta\mathbf{r}$ - the disturbed position vector of m

$\delta_k\mathbf{r}$ - the perturbations of the kth order in the position vector of m

$\delta\mathbf{r}$ - the total perturbations in the position vector of m

$\delta\mathbf{v} = d\delta\mathbf{r}/dt$ - the perturbations in the velocity of m

$\mathbf{r}' + \delta\mathbf{r}'$ – the disturbed (actual) position vector of m'

$\delta_k\mathbf{r}'$ - the perturbations of the kth order in the position vector of m'

$\delta\mathbf{r}'$ - the total perturbations in the position vector of m'

$\boldsymbol{\rho} = \mathbf{r}' - \mathbf{r}$

$\rho = |\boldsymbol{\rho}|$

$\delta\boldsymbol{\rho} = \delta\mathbf{r}' - \delta\mathbf{r}$

$\mathbf{I}$ - the iden factor

∇ - the del-operator with respect to $\mathbf{r}$

∇' - the del-operator with respect to $\mathbf{r}'$

$\mathbf{D}'' = \nabla' \exp \delta\boldsymbol{\rho} \cdot \nabla'$

$\mathbf{D}' = \nabla' \exp \delta\mathbf{r}' \cdot \nabla'$

$\mathbf{D} = \nabla \exp \delta\mathbf{r} \cdot \nabla$

$\nabla_{\mathbf{P}}$ - the del-operator with respect to $\mathbf{P}$

$\nabla_{\mathbf{Q}}$ - the del-operator with respect to $\mathbf{Q}$

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