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Preprint No. EFINS 66-95

October 1966

*This research was supported by the National Aeronautics and Space Administration under Grant NASA NsG 96-60.

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Abstract

The stability of a simple equilibrium periodic system for the interstellar gas - magnetic field described in an earlier paper (Lerche, 1966) is discussed. It is demonstrated that the system is unstable with a growth time of the order of the free-fall time. The instability is of such a nature that the material tends to break up into clumps. In view of the fact that the periodic system is formed from a uniform atmosphere in a time of the same order, this calculation demonstrates that, at best, the periodic system can represent a transition phase of the interstellar gas - magnetic field system. It also shows that the formation of clumps of gas from a uniform atmosphere progresses in a time less than, or comparable to, the free-fall time.

I. Introduction

It has been demonstrated (Parker, 1966) that the interstellar gas, whose weight holds down the large scale galactic magnetic field threaded through the gas, is unstable to a Rayleigh-Taylor type instability of such a nature that the gas tends to accumulate in 'pockets' in the low regions of magnetic field. The gravity is provided largely by the galactic star system which takes little or no part in the motion. In fact, including self-gravity of the interstellar gas would only increase the rate of gas accumulation. The instability proceeds with a growth time of the order of the free-fall time provided that the intergalactic medium exerts pressures which are negligible compared to the cosmic ray gas pressure and the interstellar magnetic field pressure in the galaxy (about 10^{-12} dynes/cm^2).

While we have been unable to follow in detail the break-up process of the interstellar gas it has been demonstrated elsewhere (Lerche, 1966) that in at least one particular case it is energetically more favorable for the interstellar gas - magnetic field system to break up into a periodic array of current sheets than to remain as a uniform atmosphere. In particular we showed that by so breaking up there would be a reduction of some 6 percent in the total energy of the system. In the equilibrium current sheet system we envisage an infinite array of sheets as depicted in Figure 1. The current distribution is the same on all sheets and, in equilibrium, the currents point in the Z-direction, but are independent of z. Thus the equilibrium state is rigorously two dimensional in the x-y plane. Further, the material density on each sheet is held against the galactic gravity by the $\underset{\sim}{j} \times \underset{\sim}{B}$ force on each sheet and, since the galactic gravitational acceleration changes sign across the galactic plane ($y = 0$), we choose $j(y) = -j(-y)$. For a particular variation of $j(y)$ it was demonstrated that it was possible to set up such an equilibrium state. It was also possible to transform the periodic system back into a uniform atmosphere and to demonstrate that the uniform state had a greater total energy than the periodic array. Thus the infinite array of discrete sheets of gas is favored, as far as energy is concerned, over the uniform atmosphere.

However in the earlier paper (Lerche, 1966), hereinafter referred to as A, no account of the stability of the periodic current sheet system was given. It is the purpose of the present paper to demonstrate that the sheet system is an unstable equilibrium with an e-folding time of the order of the free-fall time. Since the original atmosphere is also unstable with an e-folding time of the same order, the current sheet system is, at best, a transition phase of the interstellar gas - magnetic field system.

We do not intend to consider a general perturbation in this paper but will restrict ourselves to two particular types of perturbation. The first we call a 'bending' mode since it affects both the position and direction of the currents on each sheet. The second we call a 'displacement' mode since it affects only the position of the currents on each sheet but not their direction.

Further we assume that current sheet - magnetic field system is embedded in a tenuous, conducting plasma in order that the hydromagnetic equations are applicable to this problem. The plasma exerts pressures and produces magnetic fields which are negligible in comparison to the stresses, fields, pressure and weight of the material current sheets. We also assume that any Alfvén waves which are generated can travel at a much higher velocity than the response speed of the material on the current sheets. Accordingly we can use the Biot - Savart law to calculate the magnetic field.

Finally we state that the calculation is performed for cold interstellar gas so that the current sheets are infinitely thin. This restriction is imposed in order that the algebra remain tractable.

II. The Bending Mode

With gravity in the y-direction we consider a perturbation of infinitesimal amplitude ξ in the x-z plane of such a nature that the nth current sheet takes up the position

$$x = 2na + (-1)^n \xi \sin kz \, e^{\gamma(y)t}, \quad n = 0, \pm 1, \pm 2, \dots \; ; \tag{1}$$

where $\gamma(y)$ is a function to be determined and ξ is a small constant displacement.

Further, since we are neglecting displacement currents, it follows that

$$\nabla \times \underset{\sim}{H}(\underset{\sim}{x}, t) = 4\pi c^{-1} \underset{\sim}{J}(\underset{\sim}{x}, t), \tag{2}$$

where $\underset{\sim}{J}$, the current density, is given by

$$\underset{\sim}{J}(\underset{\sim}{x}, t) = j(y) \sum_{n=-\infty}^{\infty} \hat{S}_n \, \delta\left[x - 2na - (-1)^n \xi \sin kz \, e^{\gamma(y)t}\right], \tag{3}$$

and the unit vector $\hat{S}_n$ is given by

$$\hat{S}_n = \frac{\hat{z} + \hat{x}\,(-1)^n k \xi \cos kz\, e^{\gamma(y)t}}{\sqrt{[1+\xi^2 k^2 \cos^2 kz\, e^{2\gamma(y)t}]}} \quad . \tag{4}$$

The solution to equation (2) can then be written

$$c\,\underset{\sim}{H}(\underset{\sim}{x},t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \; \frac{j(y')\,\delta[x'-2na-(-1)^n \xi \sin kz'\, e^{\gamma(y')t}]}{\sqrt{[1+\xi^2 k^2 \cos^2 kz'\, e^{2\gamma(y')t}]}} \times$$

$$[(x-x')^2+(y-y')^2+(z-z')^2]^{-3/2} \times \{-\hat{x}(y-y') + \hat{y}\,[x-x'-(-1)^n \xi\, k(z-z')\cos kz'\, e^{\gamma(y')t}]$$

$$+ \hat{z}\, k \xi (-1)^n (y-y') \cos kz'\, e^{\gamma(y')t}\} \quad . \tag{5}$$

Now the material density on, say, the rth current sheet is

$$\rho_r(x,y,z,t) = \sigma(y)\,\delta[x-2ra-(-1)^r \xi \sin kz\, e^{\gamma(y)t}] \quad . \tag{6}$$

The acceleration in terms of the unknown $\gamma(y)$ is readily computed from equation (1), so that from Newton's equation of motion the associated force must be

$$\underset{\sim}{F}_r = \hat{x}\,\sigma(y)(-1)^r \xi\, \gamma(y)^2 \sin kz\, e^{\gamma(y)t}\, \delta[x-2ra-\xi(-1)^r \sin kz\, e^{\gamma(y)t}]$$

$$+ O(\xi^2) \quad . \tag{7}$$

However, the force follows from the magnetic field and gravity as

$$\underset{\sim}{F}_r = \left[-\sigma(y)g(y)\hat{y} + j(y)c^{-1}\hat{S}_r \times \underset{\sim}{H}(\underset{\sim}{x},t)\right] \times \delta\left[x - 2ra - (-1)^r \xi \sin kz\, e^{\gamma(y)t}\right], \tag{8}$$

so that, equating the two expressions for $\underset{\sim}{F}_r$, the momentum equation is

$$\sigma(y)\,\xi\,(-1)^r \gamma(y)^2 \sin kz\, e^{\gamma(y)t}\hat{x} + O(\xi^2) = -\sigma(y)g(y)\hat{y} + j(y)c^{-1}\hat{S}_r \times \underset{\sim}{H}(\underset{\sim}{x},t), \tag{9}$$

where $x = 2ra + (-1)^r \xi \sin kz\, e^{\gamma(y)t}$.

For simplicity, and without loss of generality, we choose $r = 0$ when we have

$$\xi\sigma(y)\gamma(y)^2 \sin kz\, e^{\gamma(y)t} + O(\xi^2) = -\sigma(y)g(y)\hat{y} + j(y)c^{-1}\left[\hat{z} + \hat{x}\,k\cos kz\, e^{\gamma(y)t}\right] \times \underset{\sim}{H}(\underset{\sim}{x},t)\Big|_{x=\xi\sin kz\, e^{\gamma(y)t}}. \tag{10}$$

Now return to equation (5). With $x = \xi \sin kz\, e^{\gamma(y)t}$ we see that

$$c\,\underset{\sim}{H}(\underset{\sim}{x},t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} \frac{dz' \quad j(y')}{\left\{\left[2na - \xi\left(\sin kz\, e^{\gamma(y)t} - (-1)^n \sin kz'\, e^{\gamma(y')t}\right)\right]^2 + (y-y')^2 + (z-z')^2\right\}^{3/2}}$$
$$\times \Big[\!\Big[-\hat{x}(y-y') + \hat{y}\left\{-2na + \xi\left[\sin kz\, e^{\gamma(y)t} - (-1)^n e^{\gamma(y')t}\left(\sin kz' + k(z-z')\cos kz'\right)\right]\right\} + \hat{z}\,k\,\xi\,(-1)^n (y-y')\cos kz'\, e^{\gamma(y')t} \Big]\!\Big] + O(\xi^2). \tag{11}$$

Upon expanding equation (11) in an ascending series in the infinitesimal amplitude ξ and retaining terms only up to order ξ we obtain

$$
c\underset{\sim}{H} = -\hat{x}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}\frac{dz' \quad j(y')(y-y')}{[4n^2a^2+(y-y')^2+(z-z')^2]^{3/2}}
$$

$$
+\xi\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}\frac{dz' \quad j(y')}{[4n^2a^2+(y-y')^2+(z-z')^2]^{3/2}} \times
$$

$$
\Big\{\hat{y}\left[\sin kz\, e^{\gamma(y)t}-(-1)^n e^{\gamma(y')t}(\sin kz'+k(z-z')\cos kz')\right]
$$

$$
+\hat{z}(-1)^n k(y-y')\cos kz'\, e^{\gamma(y')t} - \hat{y}\,\frac{12n^2a^2\left[e^{\gamma(y)t}\sin kz-(-1)^n e^{\gamma(y')t}\sin kz'\right]}{[4n^2a^2+(y-y')^2+(z-z')^2]}\Big\}. \quad (12)
$$

Use of equation (12) in equation (10) and retaining only terms of order ξ or less leads to

$$
\hat{x}\,\sigma(y)\gamma(y)^2\xi\sin kz\, e^{\gamma(y)t} = -\sigma(y)g(y)\hat{y}
$$

$$
-\hat{y}\,\frac{2j(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{j(y')(y-y')\,dy'}{[4n^2a^2+(y-y')^2]}
$$

$$
-\hat{x}\,\frac{j(y)\xi}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}\frac{d\mu \quad j(y')\sin kz}{[4n^2a^2+(y-y')^2+\mu^2]^{3/2}} \times
$$

$$
\Big\{e^{\gamma(y)t}-(-1)^n e^{\gamma(y')t}(\cos k\mu+k\mu\sin k\mu) - 12n^2a^2\,\frac{\left[e^{\gamma(y)t}-(-1)^n\cos k\mu\, e^{\gamma(y')t}\right]}{[4n^2a^2+(y-y')^2+\mu^2]}\Big\}, \quad (13)
$$

where $\mu = z'-z$.

Now we have already chosen $\sigma(y)$ and $j(y)$ (but see A) in order that

$$\frac{2j(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{j(y')(y-y')\,dy'}{[4n^2a^2+(y-y')^2]} = -\sigma(y)g(y), \tag{14}$$

so that equation (13) reduces to

$$\sigma(y)\gamma(y)^2 e^{\gamma(y)t} = -\frac{j(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}\frac{d\mu \quad j(y')}{[4n^2a^2+(y-y')^2+\mu^2]^{3/2}} \times$$

$$\left\{e^{\gamma(y)t} - (-1)^n e^{\gamma(y')t}(\cos k\mu + k\mu\sin k\mu) - 12n^2a^2\frac{[e^{\gamma(y)t} - (-1)^n e^{\gamma(y')t}\cos k\mu]}{[4n^2a^2+(y-y')^2+\mu^2]}\right\}. \tag{15}$$

For times, τ , such that $\gamma\tau << 1$ we see that equation (15) becomes

$$\sigma(y)\gamma(y)^2 \simeq -\frac{j(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{\infty}\frac{d\mu \quad j(y')}{[4n^2a^2+(y-y')^2+\mu^2]^{3/2}} \times$$

$$\left\{1-(-1)^n(\cos k\mu + k\mu\sin k\mu) - 12n^2a^2\frac{[1-(-1)^n\cos k\mu]}{[4n^2a^2+(y-y')^2+\mu^2]}\right\}. \tag{16}$$

It can be shown (Gradshteyn and Ryzhik, 1965) that

$$\int_{-\infty}^{\infty}\frac{\cos k\mu \, d\mu}{(\mu^2+\lambda^2)^{3/2}} = \frac{2k}{|\lambda|}K_1(k|\lambda|), \tag{17a}$$

$$\int_{-\infty}^{\infty} \frac{k\mu \sin k\mu \, d\mu}{(\mu^2+\lambda^2)^{3/2}} = 2k^2 K_0(k|\lambda|) \, ,$$

(17b)

and

$$\int_{-\infty}^{\infty} \frac{\cos k\mu \, d\mu}{(\mu^2+\lambda^2)^{5/2}} = \frac{2k^2}{3\lambda^2}\left[K_0(k|\lambda|) + \frac{2K_1(k|\lambda|)}{k|\lambda|}\right] \, . \qquad \text{17c)}$$

Use of equations (17) in equation (16) leads to

$$\sigma(y)\gamma^2 = -\frac{2j'(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dy' \, j'(y') \Big[\frac{1}{\lambda^2} - \frac{8n^2a^2}{\lambda^4}$$

$$- (-1)^n k K_1(k\lambda)\lambda^{-1} - (-1)^n k^2 K_0(k\lambda) + (-1)^n 4n^2a^2k^2\lambda^{-2}K_0(k\lambda)$$

$$+ (-1)^n 8n^2a^2 k \lambda^{-3} K_1(k\lambda) \Big] \, , \qquad (18)$$

where

$$\lambda^2 = 4n^2a^2 + (y-y')^2 \quad \text{and } \lambda > 0.$$

In view of the fact that

$$\sum_{n=-\infty}^{\infty} \frac{1}{[4n^2a^2 + (y-y')^2]} = \frac{\pi}{2a(y-y')}\coth\left[\frac{\pi(y-y')}{2a}\right] \, , \qquad (19a)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{[4n^2a^2+(y-y')^2]^2} = \frac{\pi}{4a(y-y')^3} \times \left\{ \coth\left[\frac{\pi(y-y')}{2a}\right] + \frac{\pi(y-y')}{2a} \operatorname{cosech}^2\left[\frac{\pi(y-y')}{2a}\right] \right\}, \tag{19b}$$

we see that equation (18) can be written

$$\sigma(y)\gamma^2 = -\frac{2j(y)}{c^2}\int_{-\infty}^{\infty} dy'\, j(y') \left\{ \frac{\pi^2}{4a^2}\operatorname{cosech}^2\left[\frac{\pi(y-y')}{2a}\right] - k^2 K_0(k|y-y'|) - k\frac{K_1(k|y-y'|)}{|y-y'|} - k^2(y-y')\sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{K_0(k\lambda)}{\lambda^2} + k\sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{K_1(k\lambda)}{\lambda}\left[1 - \frac{2(y-y')^2}{\lambda^2}\right] \right\}, \tag{20}$$

where the primes on the sums over n denote the fact that the term $n = 0$ is to be omitted.

Use of A enables us to write

$$j(y) = \frac{j_0 \; y}{2^{1/4}\; |y| \sqrt{(\sinh|\pi y/2a|)}}, \tag{21}$$

and

$$\sigma(y) = \frac{\pi\sqrt{2}\; j_0^2}{g(|y|)\, c^2 \sinh|\pi y/2a|}. \tag{22}$$

Thus we have

$$\gamma^2 = -\frac{g(|Y|)}{2a}\frac{Y\sqrt{(\sinh|Y|)}}{|Y|}\int_{-\infty}^{\infty}\frac{dy'\;Y'}{|Y'|\sqrt{(\sinh|Y'|)}}\Big\{\operatorname{cosech}^2(Y-Y')$$

$$-\kappa^2 K_0(\kappa|Y-Y'|) - \kappa\frac{K_1(\kappa|Y-Y'|)}{|Y-Y'|}$$

$$-\kappa^2(Y-Y')^2\sum_{n=-\infty}^{\infty}{}' (-1)^n\frac{K_0[\kappa\sqrt{(n^2+\overline{Y-Y'}^2)}]}{[n^2+(Y-Y')^2]}$$

$$+\kappa\sum_{n=-\infty}^{\infty}{}' (-1)^n\frac{K_1[\kappa\sqrt{(n^2+\overline{Y-Y'}^2)}]}{\sqrt{[n^2+(Y-Y')^2]}}\times\left[1-\frac{2(Y-Y')^2}{(n^2+\overline{Y-Y'}^2)}\right]\Big\}, \tag{23}$$

where

$$2\pi\kappa = ak,\quad 2aY=\pi y \text{ and } 2aY'=\pi y'.$$

The author has been unable to perform the sums in equation (23) exactly. However some progress can be made in one case.

We suppose that the wave number is chosen so that $\kappa \gg 1$.

In this case

$$K_0(\kappa|Y-Y'|) : (Y-Y')^2\sum_{n=-\infty}^{\infty}{}'(-1)^n\frac{K_0\{\kappa\sqrt{[n^2+(Y-Y')^2]}\}}{[n^2+(Y-Y')^2]}$$

$$\gtrsim 1 : O(\kappa^{-1}e^{-2\kappa}),$$

and

$$K_1(\kappa|Y-Y'|) : |Y-Y'|\sum_{n=-\infty}^{\infty}{}'(-1)^n\frac{K_1\{\kappa\sqrt{[n^2+(Y-Y')^2]}\}}{\sqrt{[n^2+(Y-Y')^2]}}\times\left\{1-\frac{2(Y-Y')^2}{[n^2+(Y-Y')^2]}\right\}$$

$$\gtrsim 1 : O(\kappa^{-3/4}e^{-2\kappa}).$$

Thus to a good approximation we see that equation (23) can be written

$$\gamma^2 \simeq \frac{-gY}{2a|Y|}\sqrt{(\sinh|Y|)}\int_{-\infty}^{\infty}\frac{dY'\quad Y'}{\sqrt{(\sinh|Y'|)}\;|Y'|}\quad\times$$

$$\left[\operatorname{cosech}^2|Y-Y'| - \kappa^2 K_0\left(\kappa|Y-Y'|\right) - \kappa\frac{K_1\left(\kappa|Y-Y'|\right)}{\kappa|Y-Y'|}\right]. \qquad (24)$$

If we now set $Y' = Y + p$ we see that equation (24) becomes

$$\gamma^2 = \frac{-gY}{2a|Y|}\sqrt{(\sinh|Y|)}\int_0^{\infty} dp\left[\frac{Y+p}{|Y+p|\sqrt{(\sinh|Y+p|)}} + \frac{Y-p}{|Y-p|\sqrt{(\sinh|Y-p|)}}\right]\times$$

$$\left[\operatorname{cosech}^2 p - \kappa^2 K_0(\kappa p) - \kappa p^{-1}K_1(\kappa p)\right].$$

(25)

For $|Y| \gg 1$ it is clear that only the range $p \lesssim |Y|$ contributes a significant amount to the integral since when $p \gtrsim |Y|$ and $|Y| \gg 1$ the factor $\kappa p \gg p$, and then the factor $\operatorname{cosech}^2 p - \kappa^2 K_0(\kappa p) - \kappa p^{-1} K_1(\kappa p)$ reduces the integral to an insignificant amount.

Thus for $|Y| \gg 1$ we have

$$\gamma^2 \simeq -\frac{g}{a}\int_0^{\infty}\left[\frac{1}{\sinh^2 p} - \kappa^2 K_0(\kappa p) - \kappa\frac{K_1(\kappa p)}{p}\right]dp\,, \qquad (26)$$

and

$$\int_0^\infty \left[\frac{1}{\sinh^2 p} - \kappa^2 K_0(\kappa p) - \kappa \frac{K_1(\kappa p)}{p} \right] dp$$

$$= \left[-\coth p + \kappa K_1(\kappa p) \right]_{p=0}^{p=\infty} \equiv -1 \quad .$$

Thus

$$\gamma^2 \simeq g/a \quad , \tag{27}$$

provided $ak \gg 2\pi$ and $|y|\pi \gg 2a$.

For $|y|\pi \ll 2a$ but $ak \gg 2\pi$ it may appear at first sight that the dominant contribution to the integral in equation (25) is from the range $p \lesssim |Y|$

However in this range

$$\frac{1}{\sinh^2 p} - \kappa^2 K_0(\kappa p) - \kappa \frac{K_1(\kappa p)}{p}$$

may or may not be small. Thus to a good approximation we can write

$$\gamma^2 \simeq \left(g/2a \right) \times \left[\coth|Y| - \kappa K_1(\kappa|Y|) \right] \quad . \tag{28}$$

It is a simple matter to show that $\coth|Y| - \kappa K_1(\kappa|Y|)$ is a positive definite function for all values of $|Y|$. For $\kappa|Y| \ll 1$ but $\kappa \gg 1$ and $|Y| \ll 1$ equation (28) can be written

$$\gamma^2 \approx \frac{g y k^2}{16\pi} \ln\left(\frac{8}{ky}\right) \quad . \tag{29}$$

For $\kappa|Y| \gg 1$ but $|Y| \ll 1$ and $\kappa \gg 1$ we can write

$$\gamma^2 \simeq \frac{g}{\pi|y|} \tag{30}$$

We now have the set of results presented in table I. We see that the e-folding time for the instability is of the order of the free-fall time just as the uniform atmosphere was unstable with a similar e-folding time. However there is one fundamental difference. In the uniform atmosphere case the instability tries to move the material into the low regions of field whereas the bending mode tries to split the current sheets in a direction normal to gravity. This situation is depicted schematically in Figure 2. The hatched regions are where we expect the material from the sheets to accumulate due solely to the action of the bending mode.

We do not contend that the short wavelength $(ak \gg 2\pi)$ bending mode is the most unstable mode. However it demonstrates that even if no other wavelength exists for which the sheets are unstable this mode alone is sufficient to guarantee instability with a period of the order of the free fall time. Since this is also the period of instability of the uniform atmosphere (Parker, 1966) it raises the question: Can sheets be formed from a uniform atmosphere?

Before considering this question we shall first demonstrate that the current sheet system is also unstable to the displacement mode which tends to split the y pointing 'columns' of material into a discrete series of clumps on each columnar track.

III. The Displacement Mode

We consider a perturbation of constant infinitesimal amplitude ξ in the x-y plane of such a nature that the nth current sheet takes up the position

$$x = 2na + (-1)^n \xi \Phi(y) e^{\gamma t}, \quad n = 0, \pm 1, \pm 2, \ldots ; \tag{31}$$

where $\Phi(y)$ is a function which has yet to be specified but is taken to be independent of n. Further, since we are assuming that the Alfven waves travel at a much faster speed than the response speed of the material sheets, it follows that

$$c\underset{\sim}{H}(\underset{\sim}{x},t) = \int \frac{\underset{\sim}{J}(\underset{\sim}{x}',t) \times (\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} d^3\underset{\sim}{x}' , \tag{32}$$

where the current density, $\underset{\sim}{J}(\underset{\sim}{x},t)$, is given by

$$\underset{\sim}{J}(\underset{\sim}{x},t) = \hat{z}\, j(y) \sum_{n=-\infty}^{\infty} \delta\left[x - 2na - (-1)^n \xi \Phi(y) e^{\gamma t}\right] . \tag{33}$$

In this case we displace the currents on each sheet but leave their direction unaltered.

Computing the force exerted on the rth current sheet due to the displacement in an analagous manner to that of § II, and expanding to order ξ, leads to

$$\sigma(y)\Phi(y)\gamma^2 = \frac{j(y)}{c^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{j(y')\left[\Phi(y) - (-1)^n \Phi(y')\right]}{\left[4n^2a^2 + (y-y')^2 + z'^2\right]^{3/2}} \times \left\{ -1 + \frac{12 n^2 a^2}{\left[4n^2a^2 + (y-y')^2 + z'^2\right]} \right\} , \tag{34}$$

where we have chosen to consider the forces on the $r = 0$ sheet for simplicity.

Upon performing the integrals over z' we see that equation (34) becomes

$$\sigma(y)\,\Phi(y)\,\gamma^2 = \frac{2j(y)}{c^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} dy'\, j(y')\,[\Phi(y) - (-1)^n\Phi(y')] \times$$

$$\left\{ \frac{1}{[4n^2a^2 + (y-y')^2]} - \frac{2(y-y')^2}{[4n^2a^2 + (y-y')^2]^2} \right\} . \tag{35}$$

Now it is well known that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+\lambda^2)} = \frac{\pi}{\lambda}\coth(\pi\lambda) , \tag{36a}$$

$$\sum_{n=-\infty}^{\infty} \frac{2\lambda^2}{(n^2+\lambda^2)^2} = \frac{\pi}{\lambda}\left[\coth(\pi\lambda) + \pi\lambda\,\mathrm{cosech}^2(\pi\lambda)\right] , \tag{36b}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n^2+\lambda^2)} = \frac{\pi}{\lambda}\,\mathrm{cosech}(\pi\lambda) , \tag{36c}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{2\lambda^2(-1)^n}{(n^2+\lambda^2)^2} = \frac{\pi}{\lambda}\,\mathrm{cosech}(\pi\lambda)\,[1 + \pi\lambda\coth(\pi\lambda)] . \tag{36d}$$

Use of equations (36) enables equation (35) to be written

$$\sigma(y)\,\Phi(y)\,\gamma^2 = \frac{\pi^2 j(y)}{2a^2c^2}\int_{-\infty}^{\infty} dy'\, j(y')\,\mathrm{cosech}^2\left[\frac{\pi(y-y')}{2a}\right] \times$$

$$\left\{ \Phi(y')\cosh\left[\frac{\pi(y-y')}{2a}\right] - \Phi(y) \right\} . \tag{37}$$

In the present situation we have yet to specify $\Phi(y)$. As is usual in a normal mode analysis we should now set $\gamma =$ constant and look for the solutions of the integral equation (37) for $\Phi(y)$ and see what conditions this imposes on the values of γ i.e. we should solve an integral equation eigenvalue problem for $\Phi(y)$ and γ.

Further it is clear by inspection of equation (37) that, in general, we have a singular point as y' passes through y unless we treat the integral as a principal value integral. In view of A we see that equation (37) should indeed be interpreted as a principal value integral. Making use of equations (21) and (22) in equation (37) we obtain

$$\gamma^2 \Phi(y) = \frac{\pi g\, y}{4a^2|y|} \sqrt{\left(\sinh\left|\frac{\pi y}{2a}\right|\right)}\; P\int_{-\infty}^{\infty} \frac{d\mu\ (y+\mu)\, \operatorname{cosech}^2(\pi\mu/2a)}{|y+\mu|\ \sqrt{[\sinh|\pi(y+\mu)/2a|]}} \times$$

$$\left[\cosh\left(\frac{\pi\mu}{2a}\right)\Phi(y+\mu) - \Phi(y)\right]. \tag{38}$$

Changing variables in equation (38) through $2aY = \pi y$ and $2ap = \pi\mu$

we obtain

$$\gamma^2 \Phi(y) = \frac{g\, Y}{2a|Y|} \sqrt{(\sinh|Y|)}\; P\int_{-\infty}^{\infty} dp\ \frac{(p+Y)}{|p+Y|\ \sqrt{(\sinh|p+Y|)}} \times$$

$$\operatorname{cosech}^2 p\left[\cosh p\ \Phi(Y+p) - \Phi(Y)\right]. \tag{39}$$

The author has so far been unable to solve equation (39) exactly. However some progress can be made in two extreme cases.

a) $|Y| \gg 1$.

In such a case, provided $\Phi(y)$ is reasonably well behaved, we have that

$$\gamma^2 \simeq \frac{g}{2a}\int_{-\infty}^{\infty} dp \; \frac{(\cosh p - 1)}{\sinh^2 p} \,,$$

$$\equiv g/a \,. \tag{40}$$

Note that in this approximation <u>any</u> non-zero, reasonably well behaved $\Phi(y)$ leads to an unstable situation. However we obtain no knowledge of the way in which the current sheets move except that the material moves off its equilibrium position.

b) $|Y| \ll 1$.

In this case we get a significant contribution to the integral only if $p \lesssim |Y|$ as can be seen by inspection of equation (39) and in this regime we have

$$\gamma^2 \Phi(Y) \simeq \frac{g\,\Phi(Y)}{2a}\, P\int_{-|Y|}^{+|Y|} \operatorname{cosech}^2 p \,(\cosh p - 1)\, dp \,.$$

Thus

$$\gamma^2 \simeq \frac{g|Y|}{4a} \equiv \frac{\pi g |y|}{8a^2} \,. \tag{41}$$

Note that in this case γ is not constant but is a function of y. Despite this fact we feel that the results of equations (40) and (41) suggest very strongly that the current sheet system is unstable to the displacement mode with an e-folding time of the order of the free-fall time.

The motion of the material off the current sheets due solely to the displacement mode is depicted in Figure 8.

IV. Conclusion

We have shown that the equilibrium system of paper A is an unstable situation with an e-folding time of the order of the free-fall time. Since the uniform atmosphere forms material sheets in a time of the same order (Parker, 1966) it is doubtful that the gas passes through the quasi-equilibrium current sheet state. Even though the sequence "uniform atmosphere $\longrightarrow$ current sheets $\longrightarrow$ clumps" may not be the most rapid way for the material to break up into clumps, it demonstrates that the formation of clumps of gas from a uniform atmosphere occurs in a time less than, or of the order of, the free-fall time.

While the arguments presented here and in paper A have dealt with a particular model of the interstellar gas - magnetic field system we believe (but so far have been unable to prove) that their physical content is valid for a much wider class of situation than the simple system we have used for illustrative purposes.

We have done this stability calculation to demonstrate that the interstellar gas, which holds down the magnetic field threading through it, cannot give rise to an equilibrium state as depicted in paper A which can be considered stable over a period of the order of $10^6 - 10^7$ years (the galactic free-fall time) and further to show that discrete clumps of gas will form from a uniform atmosphere in a period of the same order.

Acknowledgments

The author is grateful to Professor E. N. Parker for many stimulating discussions concerning the behavior of the interstellar gas.

This research was supported by the National Aeronautics and Space Administration under grant NASA-NsG-96-60.

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TABLE I

	$\pi\lvert y\rvert \ll 2a$	$\pi\lvert y\rvert \gg 2a$
$\lvert y\rvert \gg 4k$	$\dfrac{g}{\pi\lvert y\rvert}$	$\dfrac{g}{a}$
$\lvert y\rvert \ll 4k$	$\dfrac{g\lvert y\rvert k^2}{16\pi} \ln\left(\dfrac{8}{k\lvert y\rvert}\right)$	$\dfrac{g}{a}$

Caption:

Values of γ^2 for $ak \gg 2\pi$

Captions:

Fig. 1. The equilibrium array of current sheets.

Fig. 2. Motion of material off the current sheets and into the hatched regions (columns in the y direction) due to the unstable bending mode. We expect $L \ll a$.

Fig. 3. Motion of material off the current sheets and into the hatched regions due solely to the displacement mode. We expect $\ell \gtrsim a$.

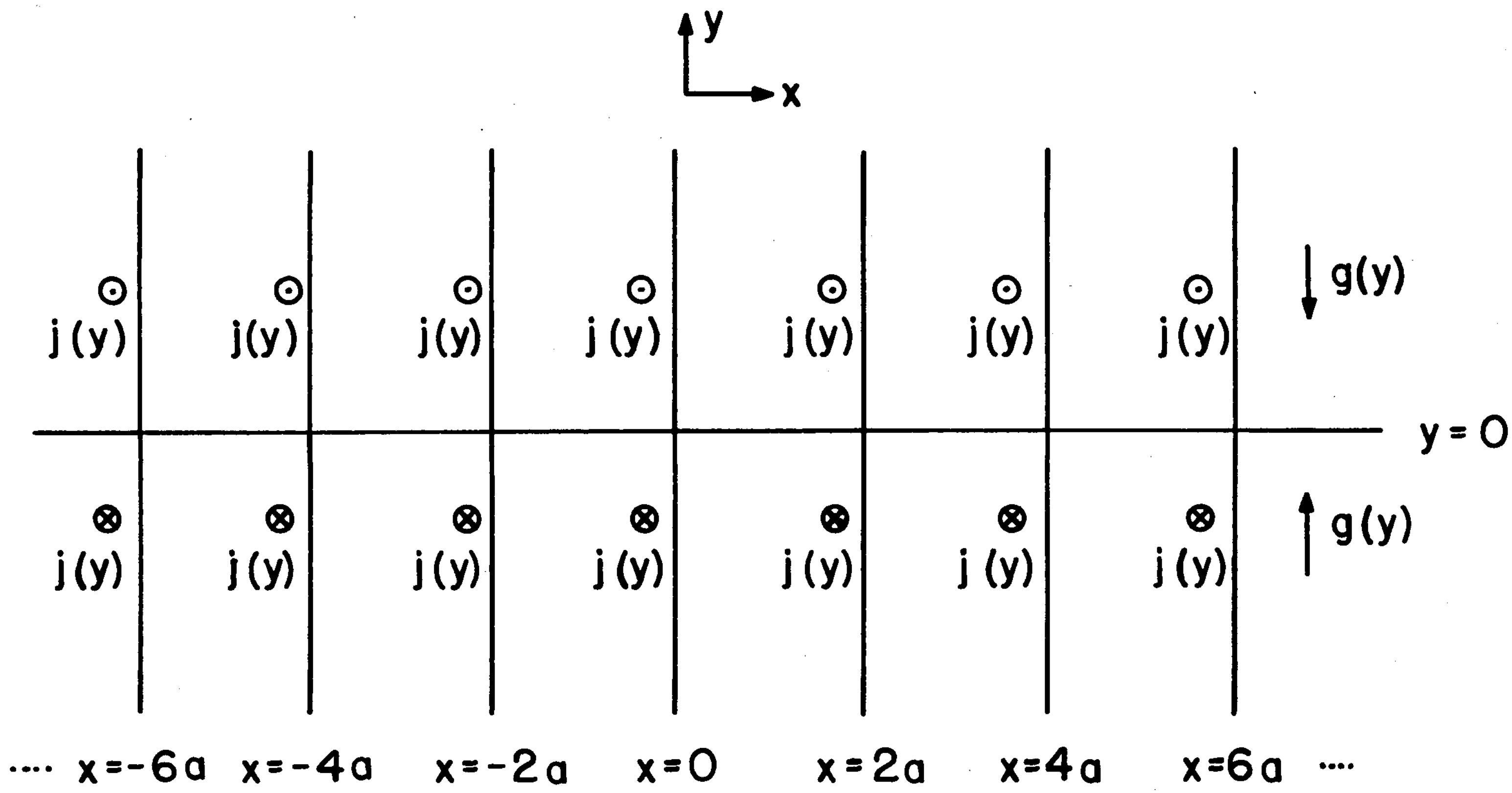

Fig.1

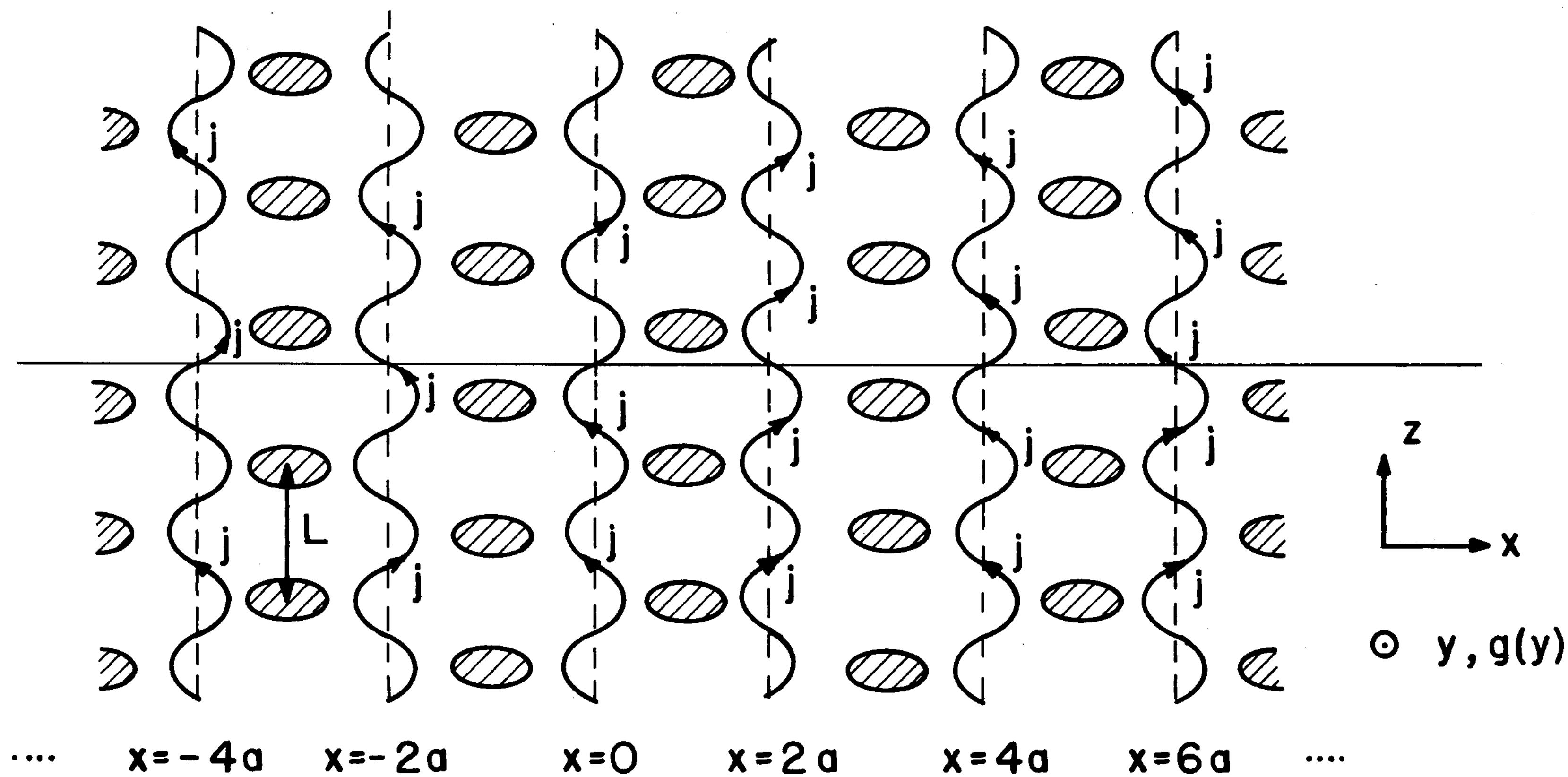

Fig. 2

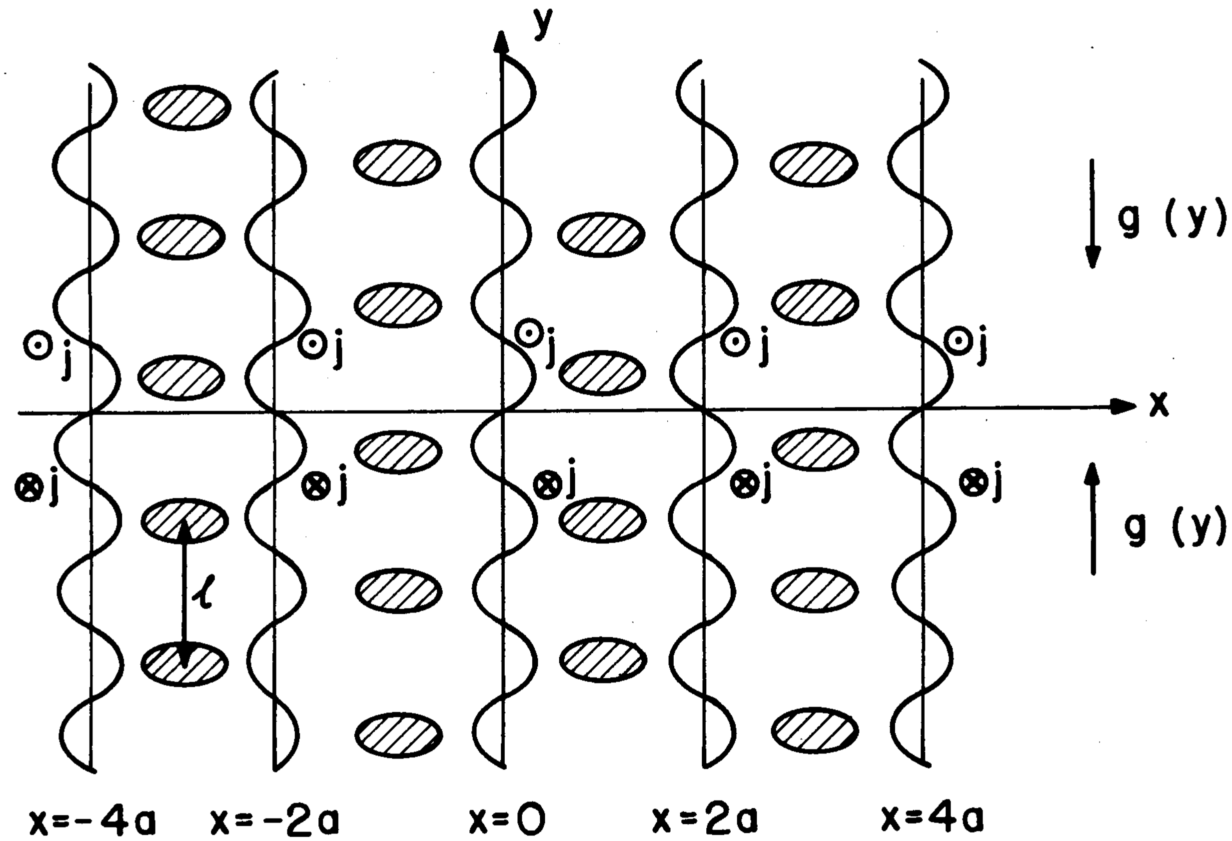

Fig. 3