Engineering Research Division

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STABILITY OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS

For the Period

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Prepared by:

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I. Personnel

The technical personnel currently being partially supported by the research grant are as follows:

1. William G. Vogt, Principal Investigator - 1/2 time

2. Charles G. Krueger, Scientific Investigator - 1/4 time

3. Gabe R. Buis, Graduate Student Assistant - full time

In addition several graduate students presently studying under Drs. Krueger or Vogt have expressed an interest in the general area but are not supported by the grant.

The main problem with staffing the research grant was that the announcement of NASA support for the research came at a time when most graduate students had already committed themselves to study both at the University of Pittsburgh and at institutions across the country. This, along with the fact that a background in functional analysis is required for this research, made the enlistment of well qualified graduate students quite difficult. As support for this research is continued, this problem will be taken care of. Since Mr. Gabe R. Buis, the one graduate student we did succeed in obtaining, has an excellent background and a consuming interest in the research, we expect some major contributions from him in this area.

II. Research Accomplished

The main research effort has been in examining the technical literature to determine the present status of results in three areas:

A. General Properties of Solutions to Partial Differential Equations;

B. Results on Stability of Solutions to Partial Differential Equations;

C. Possible Techniques for Studying Stability of Solutions to Partial Differential Equations.

A complete report on the bibliographical survey will be prepared when this search is completed. Preliminary results and conclusions are listed below.

A. General Properties of Solutions to Partial Differential Equations.

The general problem of finding conditions which assure the existence, uniqueness, and continuity of solutions to partial differential equations has been recognized by physicists and mathematicians ever since partial differential equations have been studied. If solutions exist, are unique, and are continuous both with respect to the initial conditions and time, they are said to be "well set" or "well posed" problems.

The classical approach to this problem often consisted of finding a solution to a particular problem for a sufficiently small interval of time and examining the properties of this solution to determine if the problem is well posed. While this approach yielded useful local results, little could be determined about global properties.

The functional analysis approach, virtually a revolution in thinking about these problems, has only recently yielded results on the global properties of solutions [1]. Further extensions to these earlier results have been made [2-4].

In stability studies, these global results are of paramount importance for any rigouous analysis of such problems because of the fact that solutions must be characterized over the entire interval of time $o \leq t < \infty$ and not just for some sufficiently small interval of time $o \leq t \leq T$. The investigators now think it possible to obtain sufficient conditions to assure global behavior of solutions using the concept of Lyapunov Functional, and are presently working on this problem.

B. Results on Stability of Solutions to Partial Differential Equations.

The literature search so far has revealed that, except for a very few isolated instances, stability studies of solutions to partial differential equations have consisted of eigenfunction expansions of solutions for particular problems assuming a priori that such solutions exist, are unique, and have the required continuity properties for all time $o \leq t < \infty$. With the same a priori assumption the investigators, for several particular problems, have succeeded in obtaining the same conditions for stability, using the Lyapunov functional approach. Usually these conditions are obtained by very simple formal operations not requiring knowledge of the solutions themselves.

The investigation will continue as the literature search progresses. However, because the validity of these formal operations on Lyapunov functionals is intrinsically dependent on the global properties of solutions mentioned above, mathematically rigouous applications to physical problems must await some of these expected results.

C. Possible Techniques for Studying Stability

Some of the possible techniques for studying stability are:

(1) Examination of solutions or approximate solutions;

(2) Lyapunov Functionals;

(3) Monotone Mappings;

(4) Fixed Point Theorems in Banach Spaces.

As mentioned previously, (1) above has been often used in the past, but rigorous justification for this approach is somewhat lacking and often the approach is very difficult if possible at all.

The Lyapunov functional approach (2) appears at present to have the most promise but is severely handicapped by the necessity for solutions having smoothness properties beyond those usually required on solutions in the ordinary sense. Additionally, compactness of certain function spaces is generally required for a successful application of this method. However, the method's usefullness is enhanced by the simplicity of the formal application of the method to physical problems. At present it would appear that the Lyapunov Functional Approach will be most useful even though it has some serious drawbacks.

The concepts of Lyapunov functional (2) and monotone mappings in Banach spaces (3) as used by Browder [4-5] appear to be intimately related. Browder has characterized a class of mappings in Banach spaces which in many respects is similar to gradients of Lyapunov functionals. From a mathematical point of view, these mappings appear to be more natural to stability studies in function spaces. Research on this interrelationship will be continued, especially as applied to the class of partial differential equations considered by Lions and Strauss [2].

The motion of fixed point theorems on Banach Spaces (4) can be used to characterize a motion $F(x,t)$ defined in a Banach Space which as $t \to \infty$ approximates a periodic motion. An initial success along this line was achieved by Jones [6], who considered the existence of periodic solutions of functional differential equations whose coefficients are operators defined on finite dimensional Euclidean spaces. It is possible to extend these ideas to equations of the type studied by Lions and Strauss [2] and also considering the case where a solution considered as a motion in Banach Space is asymptotic to an almost periodic motion.

III. Future Direction of Research

The bibliographical search will continue as before. Its completion is being delayed by the inaccessibility of some foreign sources. A report on this search will be prepared in the coming months.

The literature survey has shown that the main obstacle in extending Lyapunov's Second Method to partial differential equations is that for most problems in partial differential equations the question of being well-posed is in doubt. The outstanding problem in this research is thus to obtain conditions, which assure the existence, uniqueness and continuability of solutions to partial differential equations in the time interval $o \leq t < \infty$. A partial solution to this problem appears possible and research will be continued in this direction. In this research special emphasis will be placed on applying Lyapunov functionals to achieve the stated results.

Research on the application of Lyapunov functionals for obtaining conditions to assure the stability of the solutions of specific partial differential equations will be continued. This, of course, includes <u>nonlinear</u> partial differential equations. However the validity of these applications will to some extent depend on the research above.

The relation between Browder's monotone operators and Lyapunov functionals will be further explored with the hope that some of the stringent conditions on solutions to partial differential equations imposed by the Lyapunov functional approach will be awakened.

Jones' work [6] will be extended to include certain classes of partial differential equations.

Additional approaches to the stability problems will be explored as they occur.

IV. Problems

The main problem in the research is the difficulty of the research area itself, as is indicated in the original research proposal. Not only are general results characterizing solutions to partial differential equations sparse, but even these results are based on sophisticated mathematical concepts not usually studied by engineers or physicists. If such concepts as generalized solutions are employed in the development of Lyapunov functionals to obtain stability conditions, many questions must be raised and answered concerning the applicability of these conditions to ordinary solutions. To assure a general acceptance among engineers, a general theory must be developed along the lines of that for ordinary differential equations, with which most engineers are acquainted.

V. References

1 K. Jörgens, "Dan Anfangswertproblem im Grossen fur eine Klasse nichtlinearer Wellergleichungen," Math Zeitschr., vol. 77, 1961, pp. 295-308.

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3. J. Sather, "The Existence of a Global Classical Solution of the Initial-Boundary Value Problem for $\Box\mu + \mu^3 = f$", Archive for Rational Mechanics and Analysis, vol. 22, No. 4, 1966, pp. 292-307.

4. F. E. Browder, "Multivalued Monotone Nonlinear Mappings and Duality Mappings in Banach Spaces, Trans. Amer. Math. Soc., vol. 118, (1965), pp. 338-357.

5. F. E. Browder, "Nonlinear Monotone Operators and Convex Sets in Banach Space," Bull. Amer. Math. Soc., vol. 71 (1965), pp. 780-785.

6. G. S. Jones, "Periodic Motions in Banach Space and Applications to Functional-Differential Equations", Contributions to Differential Equations, vol III., No. 1, pp. 75-106.