ESTIMATION OF PARAMETERS IN COMPOUND WEIBULL DISTRIBUTIONS

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TECHNICAL MEMORANDUM X-53422

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ABSTRACT

The two-parameter Weibull distribution has been recognized as a use-ful model for survival populations associated with reliability studies and life testing experiments. In the analysis of atmospheric data, the distributions encountered are often a result of combining two or more component distributions. These compound distribucions are consequently of interest to aerospace scientists. Presented is a method for estima-tion of the parameters of a compound Weibull distribution with two shape parameters, two scale parameters and a proportionality factor. The most general case of estimation will be considered in addition to a number of special cases that may be of practical value.

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SUMMARY

The two-parameter Weibull distribution has been recognized as a useful model for survival populations associated with reliability studies and life testing experiments. In the analysis of atmospheric data, the distributions encountered are often a result of combining two or more component distributions. These compound distributions are consequently of interest to aerospace scientists. This paper presents a method for estimation of the parameters of a compound Weibull distribution with density function

$$f(x) = \alpha f_1(x) + (1 - \alpha)\, f_2(x), \qquad 0 < \alpha < 1$$

where

$$f_1(x) = \gamma_1\, \theta_1^{-1}\, x^{\gamma_1 - 1} \exp\left[-x^{\gamma_1}/\theta_1\right], \quad x \geq 0,\; \theta_1 > 0,\; \gamma_1 > 0$$

and

$$f_2(x) = \gamma_2\, \theta_2^{-1}\, x^{\gamma_2 - 1} \exp\left[-x^{\gamma_2}/\theta_2\right], \quad x \geq 0,\; \theta_2 > 0,\; \gamma_2 > 0.$$

The parameters required are α, the proportionality factor, γ_1, γ_2, θ_1 and θ_2. The most general case of estimation will be considered in addition to a number of special cases that may be of practical value.

I. INTRODUCTION

The Weibull distribution, derived in 1939 by W. Weibull, has been recognized as an appropriate model in reliability studies and life testing. Numerous methods for obtaining efficient estimates of the two parameters of this distribution have been outlined in recent years [4, 10,11].

In actual physical applications, however, a mixture of two Weibull distributions often seems to be a more desirable model. Distributions resulting from mixing two or more component distributions are designated as "mixed" or "compound." This situation is quite common in the analysis of atmospheric data and consequently is of interest to aerospace scientists. Compound normal, Poisson and exponential distributions have been studied by A. C. Cohen, Jr. [1,2,3]. A method for estimating parameters of mixed distributions using sample moments has been outlined by Paul R. Rider [7] who considered compound Poisson, binomial, and a special case of the compound Weibull distribution. A graphical procedure for estimation of mixed Weibull parameters in life-testing of electron tubes is given by John H. K. Kao [6]. Although graphical methods have value for locating outliers, deriving initial estimates, and for determining whether the distribution is as hypothesized, for estimation purposes the analytic approach is probably superior.

This paper represents an attempt at estimating, by the method of sample moments, the five parameters of the compound Weibull distribution with density function

$$f(x) = \alpha f_1(x) + (1 - \alpha) f_2(x), \quad 0 < \alpha < 1 \tag{1}$$

where

$$\left. \begin{array}{l} f_1(x) = \gamma_1 \theta_1^{-1} x^{\gamma_1 - 1} \exp[-x^{\gamma_1}/\theta_1], \quad x \geq 0, \ \theta_1 > 0, \ \gamma_1 > 0 \\ \\ f_2(x) = \gamma_2 \theta_2^{-1} x^{\gamma_2 - 1} \exp[-x^{\gamma_2}/\theta_2], \quad x \geq 0, \ \theta_2 > 0, \ \gamma_2 > 0 \end{array} \right\} \tag{2}$$

The parameters involved are two scale parameters θ_1 and θ_2, two shape parameters γ_1 and γ_2, and the proportionality parameter α which expresses the probability that a given observation x_i comes from the population f_1.

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The compound cumulative distribution function is defined

$$F(x) = \alpha F_1(x) + (1 - \alpha) F_2(x) = 1 - \alpha \exp [-x^{\gamma_1}/\theta_1] - (1 - \alpha) \exp [-x^{\gamma_2}/\theta_2].$$

$$(3)$$

Figures 1 and 2 illustrate a generalized mixed Weibull probability density function and its corresponding distribution function.

The most general case of estimation will be considered in which all five parameters must be estimated from the data. Also, a number of special cases will be investigated in which certain parameters are known in advance of sampling or are restricted in some manner. Included will be the special case where $\gamma_1 = 1$, i.e.,

$$f_1(x) = \theta_1^{-1} \exp [-x/\theta_1], \qquad (4)$$

which is the well-known exponential distribution. Estimating procedures are greatly simplified in these special cases as there are fewer sample moments involved in the estimating equations. Also, sampling errors are reduced because of the elimination of the need for higher order moments.

II. ESTIMATION IN THE GENERAL CASE

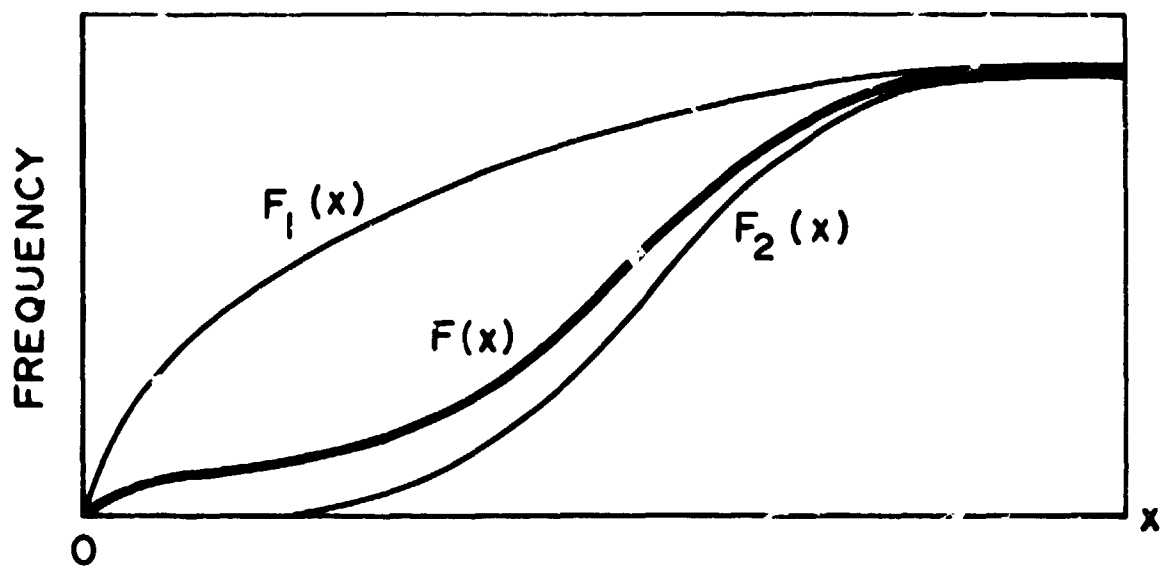The rth theoretical moment about the origin of f(x) is given by

$$\mu_r' = \alpha \int_0^\infty x^r f_1(x) \, dx + (1 - \alpha) \int_0^\infty x^r f_2(x) \, dx, \qquad (5)$$

where $f_1(x)$ and $f_2(x)$ are defined as in equations (2). The first five theoretical moments about the origin of (1) follow as
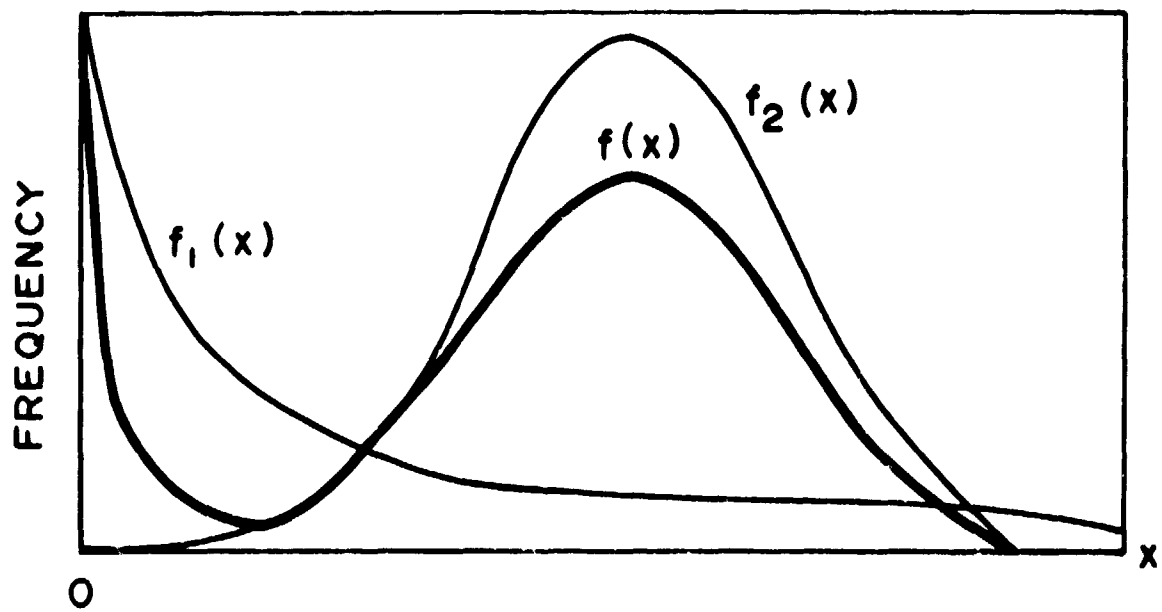
MIXED WEIBULL PROBABILITY DENSITY FUNCTION

FIGURE 1

MIXED WEIBULL CUMULATIVE DISTRIBUTION FUNCTION

FIGURE 2

$$\mu_1' = \alpha\, \theta_1^{1/\gamma_1} \Gamma\left(\frac{1}{\gamma_1} + 1\right) + (1 - \alpha)\theta_2^{1/\gamma_2} \Gamma\left(\frac{1}{\gamma_2} + 1\right)$$

$$\mu_2' = \alpha\, \theta_1^{2/\gamma_1} \Gamma\left(\frac{2}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{2/\gamma_2} \Gamma\left(\frac{2}{\gamma_2} + 1\right)$$

$$\mu_3' = \alpha\, \theta_1^{3/\gamma_1} \Gamma\left(\frac{3}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{3/\gamma_2} \Gamma\left(\frac{3}{\gamma_2} + 1\right) \qquad (6)$$

$$\mu_4' = \alpha\, \theta_1^{4/\gamma_1} \Gamma\left(\frac{4}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{4/\gamma_2} \Gamma\left(\frac{4}{\gamma_2} + 1\right)$$

$$\mu_5' = \alpha\, \theta_1^{5/\gamma_1} \Gamma\left(\frac{5}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{5/\gamma_2} \Gamma\left(\frac{5}{\gamma_2} + 1\right)$$

where Γ is the gamma function, i.e.,

$$\Gamma(k) = \int_0^\infty y^{k-1}\, e^{-y}\, dy.$$

Employing the technique of equating population moments to corresponding sample moments, the set of equations (6) becomes

$$m_1' = \alpha\, \theta_1^{1/\gamma_1} \Gamma\left(\frac{1}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{1/\gamma_2} \Gamma\left(\frac{1}{\gamma_2} + 1\right)$$

$$m_2' = \alpha\, \theta_1^{2/\gamma_1} \Gamma\left(\frac{2}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{2/\gamma_2} \Gamma\left(\frac{2}{\gamma_2} + 1\right)$$

$$m_3' = \alpha\, \theta_1^{3/\gamma_1} \Gamma\left(\frac{3}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{3/\gamma_2} \Gamma\left(\frac{3}{\gamma_2} + 1\right) \qquad (7)$$

$$m_4' = \alpha\, \theta_1^{4/\gamma_1} \Gamma\left(\frac{4}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{4/\gamma_2} \Gamma\left(\frac{4}{\gamma_2} + 1\right)$$

$$m_5' = \alpha\, \theta_1^{5/\gamma_1} \Gamma\left(\frac{5}{\gamma_1} + 1\right) + (1 - \alpha)\, \theta_2^{5/\gamma_2} \Gamma\left(\frac{5}{\gamma_2} + 1\right)$$

where m_i' $(i = 1, 2, \ldots, 5)$ is the ith noncentral moment of the sample.

The set of equations (7) is a system of five equations which must be solved simultaneously for estimates of the five parameters α, θ_1, θ_2, γ_1 and γ_2. For convenience in handling these equations, we will make the following transformations where necessary in this paper. (This notation will be used unless stated otherwise.)

Let

$$v = \theta_1^{1/\gamma_1} \qquad \beta_1 = \Gamma\left(\frac{1}{\gamma_1} + 1\right) \qquad \psi_1 = \Gamma\left(\frac{1}{\gamma_2} + 1\right)$$

$$n = \theta_2^{1/\gamma_2} \qquad \beta_2 = \Gamma\left(\frac{2}{\gamma_1} + 1\right) \qquad \psi_2 = \Gamma\left(\frac{2}{\gamma_2} + 1\right) \qquad (8)$$

$$\beta_3 = \Gamma\left(\frac{3}{\gamma_1} + 1\right) \qquad \psi_3 = \Gamma\left(\frac{3}{\gamma_2} + 1\right).$$

Thus, the first three equations of (7) become

$$m_1' = \alpha v \beta_1 + (1 - \alpha)\, n \psi_1 \qquad (9)$$

$$m_2' = \alpha v^2 \beta_2 + (1 - \alpha)\, n^2 \psi_2 \qquad (10)$$

$$m_3' = \alpha v^3 \beta_3 + (1 - \alpha)\, n^3 \psi_3. \qquad (11)$$

Solving (9) for v, substituting this expression into (10), and then solving for n yields

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}. \qquad (12)$$

Substituting the expression for n from (12) back into (9) and solving for v gives

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1}\left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}\right] \quad (13)$$

Upon substituting the expression for n from (12) and the expression for v from (13) into equation (11), we have one equation in the three unknowns α, γ_1 and γ_2. At this point, it is obvious that explicit expressions for the unknown parameters α, γ_1 and γ_2 cannot be obtained. Therefore, it is suggested that the following procedure be used to obtain a graphical estimate of α, the proportionality parameter. This method is essentially that of Kao [6] and is based upon the fact that a simple Weibull cumulative distribution becomes a straight line in ln versus 'n-ln coordinates. This method of estimating α will produce a relatively small error in the estimating procedure since α is limited to the range $0 < \alpha < 1$.

(1) Plot the sample cumulative distribution function for the mixed data on special ln versus ln-ln paper and visually fit a curve (called Weibull plot) among these points.*

(2) Starting at each end of the Weibull plot, draw two tangent lines and denote them by $\widehat{\alpha F_1}$ and $\overline{(1 - \alpha)F_2}$ which are estimates of $\alpha F_1(x)$ and $(1 - \alpha)F_2(x)$, respectively.

(3) At the intersection of $\overline{(1 - \alpha)F_2}$ with the upper borderline drop a vertical line whose intersection with $\widehat{\alpha F_1}$ as read from the percent scale gives our estimate of α.

See Figure 4 under Section IV for an illustration of this method.

Once an estimate of α has been determined graphically, solve equation (11) for γ_1 and γ_2 by the following iterative procedure. This procedure is taken from Cohen [1] and is a modified Newton-Raphson method.

Assume a value for γ_1 and solve equation (11) for a first approximation to γ_2. These first approximations can be substituted into (12) and (13) to obtain first approximations to θ_1 and θ_2. The first set of approximations are then introduced into the fourth equation of (6) to approximate the fourth non-central theoretical moment, μ_4'.

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*Special Weibull graph paper is available from Cornell University, Ithaca, New York.

Let $\gamma_{1(i)}$ denote the ith approximation to γ_1 and let $\mu'_{4(i)}$ denote the ith approximation (corresponding to $\gamma_{1(i)}$) to μ'_4. It should be relatively easy to find approximations $\gamma_{1(i)}$ and $\gamma_{1(i+1)}$ such that the sample moment m'_4 is in the interval $[\mu'_{4(i)}, \mu'_{4(i+1)}]$. Once the interval between $\gamma_{1(i)}$ and $\gamma_{1(i+1)}$ has been narrowed sufficiently, the required estimate γ_1 can be obtained by a simple linear interpolation as indicated below.

γ_1	μ'_4
$\gamma_{1(i)}$	$\mu'_{4(i)}$
γ_1	m'_4
$\gamma_{1(i+1)}$	$\mu'_{4(i+1)}$

The required estimate of γ_2 can subsequently be obtained from equation (11). Once γ_1 and γ_2 have been determined by equation (11), estimates for θ_1 and θ_2 are obtained from equations (13) and (12), respectively.

Unfortunately, the quadratic solutions in equations (12) and (13) result in more than one set of estimates. The problem of non-unique sets of estimates was considered by Karl Pearson [15] and A. C. Cohen, Jr. [1] in connection with mixtures of two normal distributions. Pearson suggested choosing the set of estimates which gives closest agreement between the sixth sample moment and the sixth theoretical moment after equating the first five sample moments to the corresponding theoretical moments. This procedure is followed for all acceptable sets of estimates.

Cohen [1] suggests, as an alternate procedure for resolving the problem of multiple sets of estimates, that we might choose the set of estimates which produces the smallest Chi-Square Index of Dispersion when observed frequencies are compared with expected frequencies.

In the general case of estimation considered here, we are concerned with a mixture of two Weibull distributions where the proportionality factor α is estimated graphically from the cumulative frequencies and the four remaining parameters are estimated by equating the first four sample moments to corresponding theoretical moments. When confronted with more than one set of acceptable estimates, we adopt Pearson's suggested procedure and choose the set which produces the closest agreement between

the fifth noncentral moment of the sample m_5' and the theoretical moment μ_5' given by the final equation of (6).

The calculations described above may be carried out iteratively with relative ease using the computer program included in this paper as an Appendix. First approximations to initiate the iterative process may be obtained using a graphical method such as that of Kao [6].

III. ESTIMATION IN SPECIAL CASES

A number of special cases that may be of practical value in which certain parameters are known or are restricted in some manner are considered.

1. θ_1 known. With θ_1 known, we must estimate the parameters α, θ_2, γ_1 and γ_2 only. If we let $v = \theta_1^{1/\gamma_1}$, equations (9), (10) and (11) become

$$m_1' = \alpha\, \theta_1^{1/\gamma_1}\, \beta_1 + (1 - \alpha)n\psi_1, \tag{14}$$

$$m_2' = \alpha\, \theta_1^{2/\gamma_1}\, \beta_2 + (1 - \alpha)n^2\psi_2, \tag{15}$$

$$m_3' = \alpha\, \theta_1^{3/\gamma_1}\, \beta_3 + (1 - \alpha)n^3\psi_3. \tag{16}$$

Solving (14) for α gives

$$\alpha = \frac{m_1' - n\psi_1}{\theta_1^{1/\gamma_1}\, \beta_1 - n\psi_1}. \tag{17}$$

Inserting this expression for α into equation (15) and solving for n, we obtain after considerable algebraic manipulation

$$n = \frac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2} \tag{18}$$

Substituting this expression for n back into (17) gives

$$\alpha = \cfrac{m_1' - \psi_1 \left[\dfrac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2}\right]}{\theta_1^{1/\gamma_1}\beta_1 - \psi_1 \left[\dfrac{v^2\psi_1\beta_2 - m_2'\psi_1 \pm \sqrt{(m_2')^2\psi_1^2 - 2m_2'v^2\psi_1^2\beta_2 + v^4\psi_1^2\beta_2^2 - 4m_1'v^3\beta_1\beta_2\psi_2 + 4(m_1')^2v^2\beta_2\psi_2 + 4m_2'v^2\beta_1^2\psi_2 - 4m_1'm_2'v\beta_1\psi_2}}{2v\beta_1\psi_2 - 2m_1'\psi_2}\right]} \quad (19)$$

Now, upon substituting the expression for n from (18) and the expression for α from (19) into equation (16), we have an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative procedure described in the general case. Once γ_1 and γ_2 have been determined, we obtain our estimates for θ_2 and α from equations (18) and (19), respectively. As in the general case, the positive and negative roots resulting from the quadratic solution in (18) unfortunately results in more than one set of estimates. As before, it is suggested that the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve be used.

2. θ_2 known. With θ_2 known we need only estimate α, θ_1, γ_1 and γ_2. If we let $n = \theta_2^{1/\gamma_2}$, equations (9), (10) and (11) become

$$m_1' = \alpha v\beta_1 + (1 - \alpha)\, \theta_2^{1/\gamma_2}\, \psi_1, \quad (20)$$

$$m_2' = \alpha v^2\beta_2 + (1 - \alpha)\, \theta_2^{2/\gamma_2}\, \psi_2, \quad (21)$$

$$m_3' = \alpha v^3\beta_3 + (1 - \alpha)\, \theta_2^{3/\gamma_2}\, \psi_3. \quad (22)$$

Solving (20) for α gives

$$\alpha = \frac{m_1' - \theta_2^{1/\gamma_2}\, \psi_1}{v\beta_1 - \theta_2^{1/\gamma_2}\, \psi_1}. \quad (23)$$

Inserting this expression for α into equation (21) and solving for v, we obtain after considerable algebraic manipulation

$$v = \frac{m_2'\beta_1 - n^2\beta_1\psi_2 \pm \sqrt{n^4\beta_1^2\psi_2^2 - 2n^2m_2'\beta_1^2\psi_2 + (m_2')^2\beta_1^2 - 4m_1'm_2'n\psi_1\beta_2 + 4m_2'n^2\psi_1^2\beta_2 + 4(m_1')^2n^2\beta_2\psi_2 - 4m_1'n^3\psi_1\psi_2\beta_2}}{2m_1'\beta_2 - 2n\psi_1\beta_2} \quad (24)$$

Substituting this expression for v back into (23) gives

$$\alpha = \frac{m_1' - \theta_2^{1/\gamma_2}\psi_1}{\beta_1\left[\frac{m_2'\beta_1 - n^2\beta_1\psi_2 \pm \sqrt{n^4\beta_1^2\psi_2^2 - 2n^2m_2'\beta_1^2\psi_2 + (m_2')^2\beta_1^2 - 4m_1'm_2'n\psi_1\beta_2 + 4m_2'n^2\psi_1^2\beta_2 + 4(m_1')^2n^2\beta_2\psi_2 - 4m_1'n^3\psi_1\psi_2\beta_2}}{2m_1'\beta_2 - 2n\psi_1\beta_2} - \theta_2^{1/\gamma_2}\psi_1\right]} \quad (25)$$

Now, upon substituting the expression for v from (24) and the expression for α from (25) into equation (22), we obtain an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative process described previously. Once γ_1 and γ_2 are determined, we obtain our estimates for θ_1 and α from equations (24) and (25). As in case 1, θ_1 known, the positive and negative roots resulting from the quadratic solution in (24) gives more than one set of estimates. Again, it is suggested that we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

3. γ_1 known. If γ_1 is known we must estimate α, θ_1, θ_2 and γ_2 only. Solving equation (9) for v gives

$$v = \frac{m_1' - (1 - \alpha)\, n\psi_1}{\alpha\beta_1} . \quad (26)$$

Inserting this expression for v into equation (10) and solving for n, we obtain

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} . \quad (27)$$

Substituting this expression for n back into (26) gives

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1}\left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}\right] \quad (28)$$

Upon substituting the expression for n from (27) and the expression for v from (28) into equation (11), we obtain an equation in the two unknowns α and γ_2 which may be solved by the iterative procedure described in the general case. With α and γ_2 determined we may solve equations (27) and (28) for θ_2 and θ_1, respectively. As before, the positive and negative roots which result from the quadratic solution in (27) give more than one set of estimates. Again, we choose the set of estimates which gives the closest agreement to the fifth noncentral moment of the sample.

An alternate method for estimation in this case would be to estimate α graphically as in the general case and then solve equation (11) for γ_2 after the substitution of the expression for n from (27) and the expression for v from (28) into equation (11). As before, θ_2 and θ_1 would then be obtained from equations (27) and (28).

4. γ_2 known. With γ_2 known, we must estimate α, θ_1, θ_2 and γ_1 only. As in case 3, γ_1 known, solving equation (9) for v gives

$$v = \frac{m_1' - (1 - \alpha)\, n\psi_1}{\alpha\beta_1}. \quad (29)$$

Inserting this expression for v into equation (10) and solving for n, we obtain

$$n = \frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2}.$$

$$(30)$$

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When we substitute this expression for n back into (29), we get

$$v = \frac{m_1' - (1-\alpha)\psi_1}{\alpha\beta_1} \left[\frac{m_1'(1-\alpha)\psi_1\beta_2 \pm \sqrt{-(m_1')^2\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_2 + m_2'\alpha(1-\alpha)^2\beta_2\beta_1^2\psi_1^2 + m_2'\alpha^2(1-\alpha)\beta_1^4\psi_2}}{(1-\alpha)^2\psi_1^2\beta_2 + \alpha(1-\alpha)\beta_1^2\psi_2} \right] \quad (31)$$

Upon inserting the expression for n from (30) and the expression for v from (31) into equation (11), we obtain an equation in the two unknowns α and γ_1 which may be solved by the iterative procedure described pre-viously. With α and γ_1 determined in this manner, we now solve equa-tions (30) and (31) for θ_2 and θ_1, respectively. As before, we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

As in case 3, γ_1 known, an alternate method for estimation would be to estimate α, the proportionality parameter graphically, and then solve equation (11) directly for γ_1 after the substitution of the expression for n from (30) and the expression for v from (31) into equation (11).

5. $\gamma_1 = 1$. This is a special case of case 3, γ_1 known. Thus, case 3 is reduced to mixing an exponential distribution with a Weibull distribution where $f_1(x)$ in equations (2) is an exponential distribution and $f_2(x)$ is a Weibull distribution. We need only estimate α, γ_2, θ_1 and θ_2.

With $\gamma_1 = 1$, equations (9), (10) and (11) become

$$m_1' = \alpha v + (1 - \alpha) n\psi_1 \quad (32)$$

$$m_2' = 2\alpha v^2 + (1 - \alpha) n^2\psi_2 \quad (33)$$

$$m_3' = 6\alpha v^3 + (1 - \alpha) n^3\psi_3, \quad (34)$$

where the only change in notation from previous cases is $v = \theta_1$.

Solving equation (32) for v gives

$$v = \frac{m_1' - (1 - \alpha) \, n\psi_1}{\alpha} \, .$$

(35)

Substituting this expression for v into equation (33) and solving for n gives

$$n = \frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2(1-\alpha)^2\psi_1^2 + \alpha(1-\alpha)\psi_2} \, .$$

(36)

Inserting this expression for n back into (35) gives

$$v = \frac{m_1'}{\alpha} - \psi_1 \left[\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2\alpha(1-\alpha)\psi_1^2 + \alpha^2\psi_2} \right] \, .$$

(37)

Now, upon substituting the expression for n from (36) and the expression for v from (37) into equation (34), we obtain an equation in the two unknowns α and γ_2 which we may solve using the iterative procedure described in previous cases. With minor simplifications, equation (34) becomes

$$m_3' = 6\alpha \left[\frac{m_1'}{\alpha} - \psi_1 \left(\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2\alpha(1-\alpha)\psi_1^2 + \alpha^2\psi_2} \right) \right]^3 +$$

(38)

$$(1-\alpha)\psi_3 \left[\frac{2m_1'(1-\alpha)\psi_1 \pm \sqrt{2m_2'\alpha(1-\alpha)^2\psi_1^2 - 2(m_1')^2\alpha(1-\alpha)\psi_2 + m_2'\alpha^2(1-\alpha)\psi_2}}{2(1-\alpha)^2\psi_1^2 + \alpha(1-\alpha)\psi_2} \right]^3$$

Once α and γ_2 have been determined from equation (38) we may obtain our estimates for θ_2 and θ_1 from equations (36) and (37), respectively.

As in the other cases, we choose the set of estimates which gives the closest agreement between the fifth noncentral moment of the sample and the corresponding "fitted" compound curve.

An alternate method of estimation would be to estimate α graphically as in the general case and then solve equation (38) directly for γ_2. As above, estimates for θ_2 and θ_1 would then be obtained from equations (36) and (37).

6. $\gamma_1 = \gamma_2 =$ unknown. Changing our notation slightly, we will let $\gamma_1 = \gamma_2 = \gamma$. Thus, we must estimate α, γ, θ_1 and θ_2. Now, let

$$v = \theta_1^{1/\gamma} \qquad \beta_1 = \Gamma\left(\frac{1}{\gamma} + 1\right) \qquad \beta_3 = \Gamma\left(\frac{3}{\gamma} + 1\right)$$

$$n = \theta_2^{1/\gamma} \qquad \beta_2 = \Gamma\left(\frac{2}{\gamma} + 1\right).$$

With this notation, equations (9), (10) and (11) become

$$m_1' = \alpha v \beta_1 + (1 - \alpha)\, n\beta_1 \tag{39}$$

$$m_2' = \alpha v^2 \beta_2 + (1 - \alpha)\, n^2\beta_2 \tag{40}$$

$$m_3' = \alpha v^3 \beta_3 + (1 - \alpha)\, n^3\beta_3. \tag{41}$$

As before, solving equation (39) for v gives

$$v = \frac{m_1' - (1 - \alpha)\, n\beta_1}{\alpha\beta_1}. \tag{42}$$

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Inserting this expression for v into equation (40) and subsequently solving for n, we have

$$n = \frac{m_1'(1-\alpha)\beta_2 \pm \sqrt{(1-\alpha)\beta_2\,\alpha[m_2'\beta_1^2 - (m_1')^2\beta_2]}}{(1-\alpha)\beta_1\beta_2}.$$ (43)

Substituting this expression for n back into (42) gives

$$v = \frac{m_1'}{\alpha\beta_1} - \left[\frac{m_1'(1-\alpha)\beta_2 \pm \sqrt{\alpha(1-\alpha)\alpha[m_2'\beta_1^2 - (m_1')^2\beta_2]}}{\alpha\beta_1\beta_2}\right].$$ (44)

Upon substituting the expression for n from (43) and the expression for v from (44) into equation (41), we obtain an equation in the two unknowns α and γ which may be solved by the iterative process described in the general case. We may now solve equations (43) and (44) for θ_2 and θ_1. As in the other cases, the quadratic solution in equation (43) results in more than one set of estimates. Again, we use the set of estimates which gives the closest agreement to the fifth noncentral moment of the sample.

As before, an alternate solution would be to estimate α graphically and then solve the resulting equation (41) directly for γ.

In the event that α is known in advance of sampling we may solve equation (41) directly for γ and subsequently obtain θ_2 and θ_1 from (43) and (44).

7. $\gamma_1 = \gamma_2 =$ known. If we let $\gamma_1 = \gamma_2 = \gamma$, the first three equations of (7) become

$$m_1' = \alpha\theta_1^{1/\gamma}\,\Gamma\left(\frac{1}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{1/\gamma}\,\Gamma\left(\frac{1}{\gamma}+1\right)$$ (45)

$$m_2' = \alpha\theta_1^{2/\gamma}\,\Gamma\left(\frac{2}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{2/\gamma}\,\Gamma\left(\frac{2}{\gamma}+1\right)$$ (46)

$$m_3' = \alpha\theta_1^{3/\gamma}\,\Gamma\left(\frac{3}{\gamma}+1\right) + (1-\alpha)\,\theta_2^{3/\gamma}\,\Gamma\left(\frac{3}{\gamma}+1\right).$$ (47)

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Thus, we must estimate α, θ_1 and θ_2 only. For simplification, we will let

$$c_j = \frac{m_j'}{\Gamma\left(\frac{j}{\gamma} + 1\right)}.$$

Now, equations (45), (46) and (47) become

$$\alpha\theta_1^{1/\gamma} + (1 - \alpha)\,\theta_2^{1/\gamma} = c_1 \tag{48}$$

$$\alpha\theta_1^{2/\gamma} + (1 - \alpha)\,\theta_2^{2/\gamma} = c_2 \tag{49}$$

$$\alpha\theta_1^{3/\gamma} + (1 - \alpha)\,\theta_2^{3/\gamma} = c_3. \tag{50}$$

Solving equation (48) for α we have

$$\alpha = \frac{c_1 - \theta_2^{1/\gamma}}{\theta_1^{1/\gamma} - \theta_2^{1/\gamma}}. \tag{51}$$

Substituting the expression for α from (51) into (49) and (50), we have after considerable algebraic manipulation the equations

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma} = c_1\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) - c_2 \tag{52}$$

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) = c_1\left(\theta_1^{2/\gamma} + \theta_1^{1/\gamma}\,\theta_2^{1/\gamma} + \theta_2^{2/\gamma}\right) - c_3. \tag{53}$$

Inserting the expression for $\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}$ from (52) into the left side of equation (53), we obtain after simplifying

$$\theta_1^{1/\gamma}\,\theta_2^{1/\gamma}\, c_1 - c_2\left(\theta_1^{1/\gamma} + \theta_2^{1/\gamma}\right) + c_3 = 0. \tag{54}$$

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Solving (54) for $\theta_1^{1/\gamma}$ gives

$$\theta_1^{1/\gamma} = \frac{\theta_2^{1/\gamma} c_2 - c_3}{\theta_2^{1/\gamma} c_1 - c_2} \; . \tag{55}$$

Substituting this expression for $\theta_1^{1/\gamma}$ back into equation (52), we have after simplification the quadratic equation in $\theta_i^{1/\gamma}$

$$(c_2 - c_1^2) \, [\theta_i^{1/\gamma}]^2 - (c_3 - c_1 c_2) \, \theta_i^{1/\gamma} + (c_1 c_3 - c_2^2) = 0, \tag{56}$$

whose solution for θ_i is

$$\theta_i = \left[2(c_1^2 - c_2)^{-1} \left[(-c_3 + c_1 c_2) \pm (c_3^2 - 6 c_1 c_2 c_3 - 3 c_1^2 c_2^2 + 4 c_1^3 c_3 + 4 c_2^3)^{1/2} \right] \right]^{\gamma} .$$

$$\tag{57}$$

Without loss of generality, we may impose the restriction that $\theta_1 < \theta_2$. Thus, we obtain θ_1 and θ_2 from equation (57) using the negative and positive roots, respectively. Once we have determined θ_1 and θ_2, we obtain our estimate for α from equation (51).

8. $\theta_1 = \theta_2 = $ known. If we let $\theta_1 = \theta_2 = \theta$, the first three equations of (7) become

$$m_1' = \alpha \theta^{1/\gamma_1} \beta_1 + (1 - \alpha) \, \theta^{1/\gamma_2} \psi_1 \tag{58}$$

$$m_2' = \alpha \theta^{2/\gamma_1} \beta_2 + (1 - \alpha) \, \theta^{2/\gamma_2} \psi_2 \tag{59}$$

$$m_3' = \alpha \theta^{3/\gamma_1} \beta_3 + (1 - \alpha) \, \theta^{3/\gamma_2} \psi_3 \; . \tag{60}$$

Thus, we must estimate the parameters α, γ_1 and γ_2 only. As previously, solving (58) for α gives us

$$\alpha = \frac{m_1' - \theta^{1/\gamma_2}\psi_1}{\theta^{1/\gamma_1}\beta_1 - \theta^{1/\gamma_2}\psi_1}. \tag{61}$$

Substituting this expression for α into equation (59), we have after simplifying

$$m_1'\left(\theta^{2/\gamma_1}\beta_2 - \theta^{2/\gamma_2}\psi_2\right) + m_2'\left(\theta^{1/\gamma_2}\psi_1 - \theta^{1/\gamma_1}\beta_1\right)$$

$$- \theta^{2/\gamma_1}\beta_2 \, \theta^{1/\gamma_2}\psi_1 + \theta^{2/\gamma_2}\psi_2 \, \theta^{1/\gamma_1}\beta_1 = 0. \tag{62}$$

Equation (62) is an equation in the two unknowns γ_1 and γ_2 which may be solved by the iterative procedure described for the general case of estimation. Once γ_1 and γ_2 are determined, we obtain our estimate for α from equation (61).

9. $\theta_1 = \theta_2 =$ unknown. For this special case, it is suggested that we solve for α graphically and subsequently follow the procedure outlined for the general case of estimation.

IV. AN ILLUSTRATIVE EXAMPLE

To illustrate the estimation procedure outlined in this paper for the general case, we will consider a sample of 2000 observations selected from a mixed population constructed by combining two Weibull distributions with $\gamma_1 = 2.0000$, $\theta_1 = 10.0000$, $\gamma_2 = 0.8000$, $\theta_2 = 1.0000$ and $\alpha = 0.8000$. The sample is summarized in Table I. For the sample selected, $m_1' = 2.4708$, $m_2' = 8.6270$, $m_3' = 36.3408$, $m_4' = 174.9190$ and $m_5' = 935.3733$.

TABLE I

A SAMPLE OF 2000 OBSERVATIONS FROM A MIXED WEIBULL POPULATION

CLASSES	CLASS MARKS	f_1	f_2	f	CUMULATIVE FREQUENCY IN PER CENT
0 - 0.5	0.25	40	175	215	10.75
0.5 - 1.0	0.75	113	78	191	20.30
1.0 - 1.5	1.25	170	47	217	31.15
1.5 - 2.0	1.75	205	30	235	42.90
2.0 - 2.5	2.25	216	20	236	54.70
2.5 - 3.0	2.75	206	14	220	65.70
3.0 - 3.5	3.25	180	10	190	75.20
3.5 - 4.0	3.75	147	7	154	82.90
4.0 - 4.5	4.25	112	5	117	88.75
4.5 - 5.0	4.75	80	4	84	92.95
5.0 - 5.5	5.25	54	3	57	95.80
5.5 - 6.0	5.75	34	2	36	97.60
6.0 - 6.5	6.25	20	1	21	98.65
6.5 - 7.0	6.75	11	1	12	99.25
7.0 - 7.5	7.25	6	1	7	99.60
7.5 - 8.0	7.75	3	1	4	99.80
8.0 - 8.5	8.25	1	1	2	99.90
8.5 - 9.0	8.75	1		1	99.95
9.0 - 9.5	9.25	1		1	100.00
		1600	400	2000	

In the above table, f_1 = class frequencies from $f_1(x)$, f_2 = class frequencies from $f_2(x)$, and f = class frequencies from the resulting mixed distribution.

Figure 3 is a graph of the compound density function and its component distributions. Notice at this point that $\gamma \leq 1$ produces a J-shaped function while $\gamma > 1$ produces a bell-shaped curve.

Employing the graphical technique described in Section I provides an estimate of α, the proportionality parameter, equal to 0.80 as shown on Figure 4.

Once our estimate of α has been determined graphically, we solve equation (11) iteratively for first approximations to γ_1 and γ_2. Corresponding first approximations for θ_1 and θ_2 are obtained from equations (13) and (12), respectively. Each set of first approximations is introduced into the fourth equation of (6) to approximate the fourth noncentral theoretical moment, μ_4'. Each set of first approximations is also substituted into the final equation of (6) to approximate the fifth noncentral theoretical moment μ_5', as suggested by Pearson [15]. The set of estimates which gives the closest agreement between the fifth noncentral moment of the sample m_5' and the corresponding "fitted" compound curve given by the final equation of (6) is the required set of estimates. Utilizing the computer program given in the Appendix, we find that our estimate of γ_1 lies between 1.90 and 2.10. The corresponding estimates for the remaining parameters, μ_4' and μ_5', are as follows:

γ_1	γ_2	θ_1	θ_2	μ_4'	μ_5'
1.90	2.30	11.52	1.29	343.8875	1926.6399
2.10	0.80	6.74	0.82	64.6006	287.5141

Our next approximation to γ_1 is obtained by simple linear interpolation as indicated below.

γ_1	μ_4'
1.90	343.8875
2.0210	174.9190
2.10	64.6006

Substituting $\gamma_1 = 2.0210$ into equation (11), solving for γ_2 and subsequently solving equations (13) and (12) for θ_1 and θ_2, we obtain $\gamma_2 = 0.8339$, $\theta_1 = 10.1496$ and $\theta_2 = 0.9929$. Introducing this set of approximations into the final equation of (6) gives $\mu_5' = 935.3646$. This value of μ_5' is in such close agreement with the corresponding sample moment $m_5' = 935.3733$ that we are justified in accepting this set of approximations as our final estimates. However, if further preciseness is desired, this iterative process may be continued to any desired degree of accuracy.
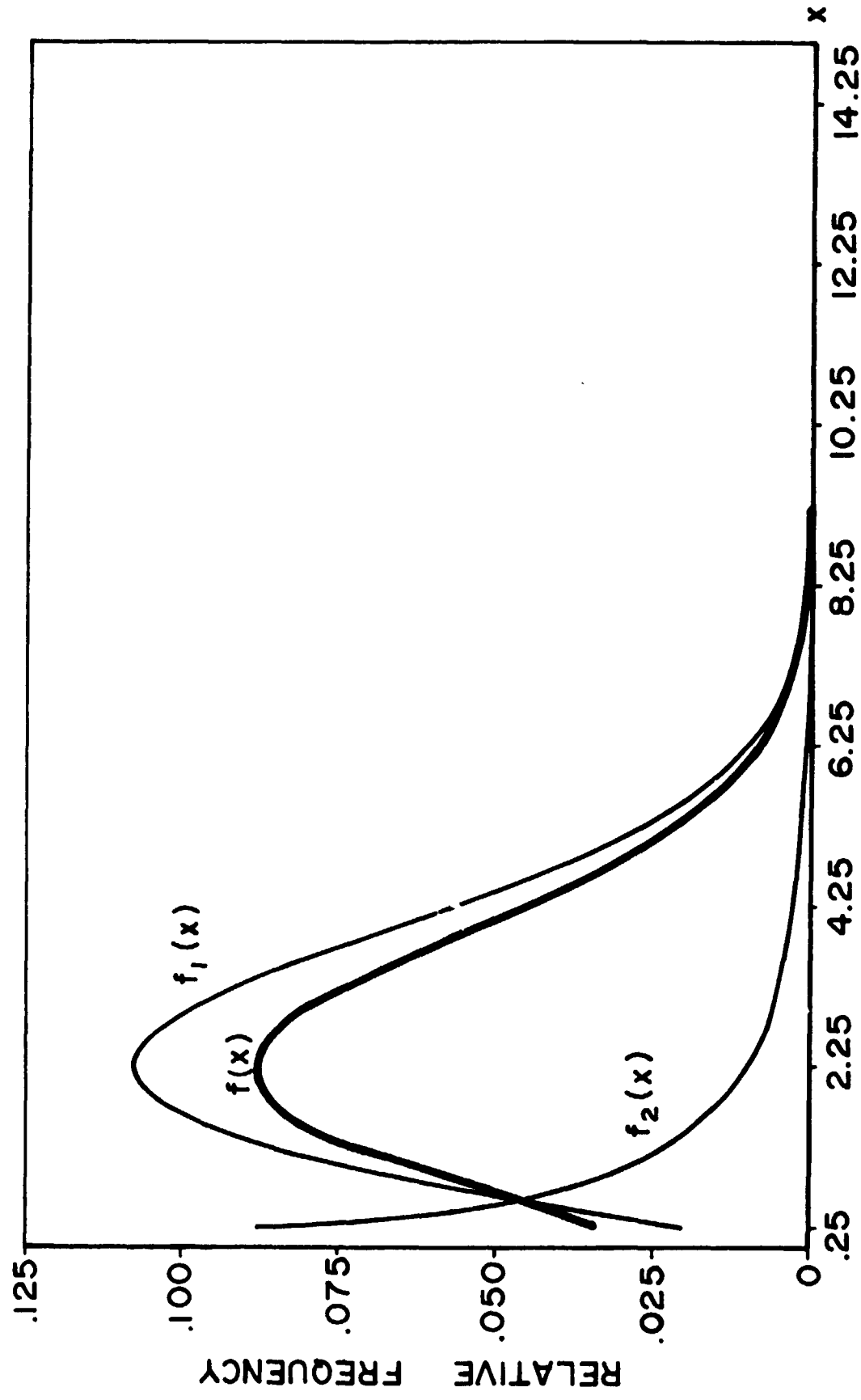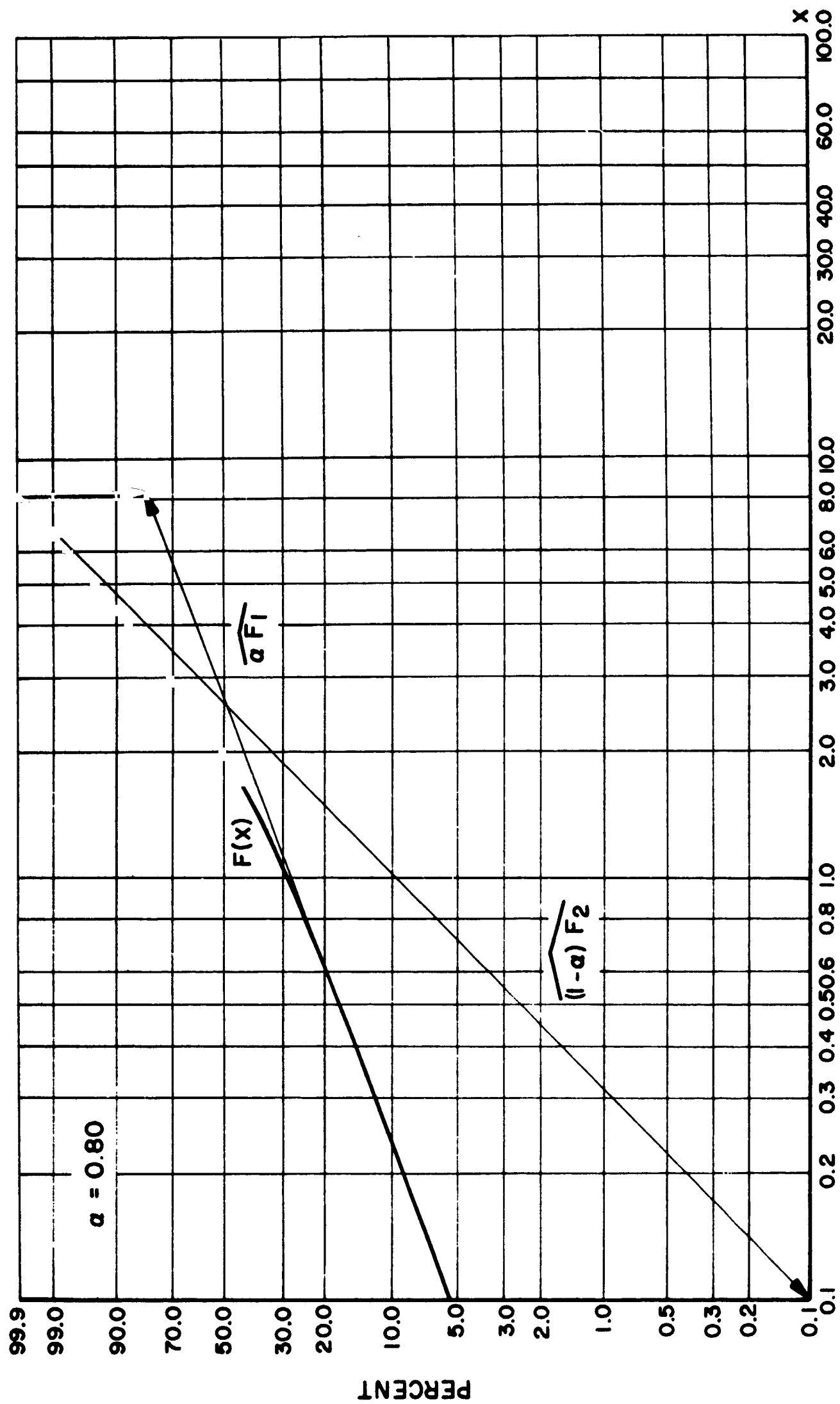
ILLUSTRATIVE EXAMPLE OF MIXED WEIBULL DENSITY FUNCTION

FIGURE 3

WEIBULL PLOT
FIGURE 4

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The computer program outlined in the Appendix gives all possible solutions to the estimating equations. Specifically, equations (12) and (13) produce four solutions resulting from the four combinations of the positive and negative signs prefixing the radicals; i.e., the combinations are (-,-), (-,+), (+,-) and (+,+). It was discovered that the computer program printout gave closest agreement between the theoretical moment μ_5' and the sample moment m_5' when the combination (-,-) was used.

Comparisons of expected with observed frequencies, along with a comparison of observed and expected distribution functions, are presented in Table II. It also seems appropriate to compare the observed frequencies for the mixed sample with these same frequencies assuming that the sample fits a simple Weibull distribution. This will prove or disprove that our mixed sample could be treated as a simple Weibull distribution. Cohen's maximum likelihood estimation procedure [4], was used to derive estimates for the parameters, and the resulting expected frequencies were obtained. Notations used in Table II are as follows:

f_o = observed frequencies for mixed data.

F_o = observed distribution function for mixed data.

f_e = expected frequencies using estimates derived in this report.

F_e = expected distribution function using estimates derived in this report.

f_{es} = expected frequencies assuming data fits a simple Weibull distribution.

F_{es} = expected distribution function assuming data fits a simple Weibull distribution.

The agreement between observed frequencies for the sample and expected frequencies using the derived estimates is very good as shown in Table II. The corresponding observed and expected distribution functions are in very close agreement with the maximum absolute difference of 0.0050 occurring at the class 3.5 - 4.0. Comparing this value with the Kolmogorov-Smirnov statistic, we see that

$$D_\alpha^N = D_{.01}^{2000} = 0.0364$$

gives an excellent "goodness-of-fit" at the 99 percent level of confidence.

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TABLE II

OBSERVED AND EXPECTED FREQUENCIES FOR 2000 OBSERVATIONS
FROM A MIXED WEIBULL DISTRIBUTION

CLASSES	f_o	f_e	f_{es}	F_o	F_e	F_{es}
0 - 0.5	215	211.0	143.3	.1075	.1055	.0717
0.5 - 1.0	191	192.9	243.2	.2030	.2020	.1932
1.0 - 1.5	217	219.1	271.9	.3115	.3116	.3292
1.5 - 2.0	235	237.8	265.8	.4290	.4305	.4621
2.0 - 2.5	236	238.9	239.9	.5470	.5499	.5821
2.5 - 3.0	220	222.1	204.6	.6570	.6610	.6843
3.0 - 3.5	190	191.7	166.8	.7520	.7569	.7677
3.5 - 4.0	154	154.3	130.9	.8290	.8340	.8332
4.0 - 4.5	117	116.2	99.4	.8875	.8921	.8828
4.5 - 5.0	84	82.1	73.3	.9295	.9332	.9195
5.0 - 5.5	57	54.6	52.6	.9580	.9605	.9458
5.5 - 6.0	36	34.3	36.8	.9760	.9776	.9642
6.0 - 6.5	21	20.4	25.2	.9865	.9878	.9768
6.5 - 7.0	12	11.5	16.9	.9925	.9935	.9852
7.0 - 7.5	7	6.2	11.1	.9960	.9966	.9908
7.5 - 8.0	4	3.2	7.1	.9980	.9982	.9943
8.0 - 8.5	2	1.6	4.5	.9990	.9990	.9966
8.5 - 9.0	1	.8	2.8	.9995	.9994	.9980
9.0 - 9.5	1	.4	1.7	1.0000	.9997	.9988

Comparing F_o with F_{es} shows a maximum absolute difference in the distribution functions of 0.0358 occurring at the class 0 - 0.5. Since the Kolmogorov-Smirnov statistic

$$D_\alpha^N = D_{.05}^{2000} = 0.0304,$$

this value of 0.0358 is sufficient to reject the hypothesis that our mixed distribution could be considered a simple Weibull distribution.

As an alternate goodness-of-fit test for agreement between observed frequencies and expected frequencies using the derived estimates, the X^2 index was calculated and the results are as follows:

$$\chi^2 = 1.2669 \qquad \text{d.f.} = 10 \qquad P(\chi^2 > 1.2669 = 0.995).$$

Thus, in consideration of the low χ^2 index of dispersion, we may conclude that we have an excellent fit for the chosen sample.

V. CONCLUSIONS

It is an accepted fact that the method of moments is not (except for distributions such as the normal, binomial, and Poisson) the most efficient procedure for estimating the parameters of a frequency distribution. Methods having maximum efficiency, such as the method of maximum likelihood are more desirable. However, in the case of the mixed Weibull distribution with its five parameters, the maximum likelihood estimating equations are almost intractable.

The central, noncentral, and factorial moments of this distribution were invest gated, and it was discovered that the noncentral moments possessed optimum characteristics for the development of estimating equations. A comparison of noncentral sample moments with the theoretical moments for the sample selected showed an error of 0.09 percent for the first moment, 0.44 percent for the second moment, 1.64 percent for the third moment, 4.94 percent for the fourth moment, and 12.73 percent for the fifth moment. This progressive increasing percentage of nonagreement between sample moments and theoretical moments illustrates the large sampling errors involved in the use of higher order moments. Using only the first three sample moments with their relatively low percentage of error in the estimating equations produced very good agreement between final estimates and the population parameters γ_1, γ_2, θ_1 and θ_2. Sheppard's corrections for grouped data were applied to the sample moments in order to increase this agreement, but produced no significant change in the results; therefore, the corrections were not used in the estimating equations.

This paper presents an estimating procedure that produced very good results for the sample chosen. The use of electronic digital computers makes the somewhat involved method practical and applicable to experiments in which the mixed Weibull distribution is the appropriate statistical model.

In the author's opinion, the estimating procedures outlined in this report warrant further investigation for increase in efficiency, and improvement and simplification of form for the estimating equations involved.

APPENDIX

COMPUTER PROGRAM FOR ESTIMATING THE PARAMETERS
OF A MIXED WEIBULL DISTRIBUTION

ESTIMATION IN MIXTURES OF TWO WEIBULL DISTRIBUTIONS

```fortran
      DIMENSION XT(50),FT(50)
      READ(60,1)DG1,DG2,FG1,FG2
      READ(60,1)RGAM1,RGAM2,FN,ALP
    1 FORMAT(4E10.5)
      READ(60,2)NV1,K
    2 FORMAT(I2/I2)
      READ(60,3)(XT(I),I=1,NV1)
      READ(60,3)(FT(I),I=1,NV1)
    3 FORMAT(8F10.5)
      WRITE(61,4)
      DM1=0.0
      DM2=0.0
      DM3=0.0
      DM4=0.0
      DM5=0.0
      DO 5 I=1,NV1
      FD1=XT(I)
      FD2=FD1*XT(I)
      FD3=FD2*XT(I)
      FD4=FD3*XT(I)
      FD5=FD4*XT(I)
      DM1=DM1+FD1*FT(I)
      DM2=DM2+FD2*FT(I)
      DM3=DM3+FD3*FT(I)
      DM4=DM4+FD4*FT(I)
      DM5=DM5+FD5*FT(I)
    5 CONTINUE
      DM1=DM1/FN
      DM2=DM2/FN
      DM3=DM3/FN
      DM4=DM4/FN
      DM5=DM5/FN
      WRITE(61,70)DM1,DM2,DM3,DM4,DM5
      GAM1=RGAM1
      GAM2=RGAM2
   25 TRM1=1.0/GAM1+1.0
      R1=GAMMA(TRM)
      TRM2=2.0/GAM1+1.0
      R2=GAMMA(TRM)
      TRM3=3.0/GAM1+1.0
      R3=GAMMA(TRM)
      WRITE(61,92)
   90 TRM=1.0/GAM2+1.0
      PS1=GAMMA(TRM)
      TRM2=2.0/GAM2+1.0
      PS12=GAMMA(TRM)
      PS13=3.0/GAM2+1.0
      PS13=GAMMA(TRM)
      FP=ALP*(1.0-ALP)*R1*PS12*PS12
      FP1=(1.0-ALP)**2*PS11**2*R2
      DFN=2.0*(FP+FP1)
      FP2=-4.0*DM1**2*FR*R2
      FR3=4.0*DM2*ALP*FR1*R1**2
      FRA=2.0*DM2*ALP**2*(1.0-ALP)*FR1**4*PS12
      TRM=FR2+FR3+FR4

   91 IF(TRM1)92,91,91
   91 TRM1=SQRT(TRM)
      FR5=2.0*DM1*(1.0-ALP)*PS11*R2
      FNP=(FR5+TRM1)/DFN
      FNM=(FR5-TRM1)/DFN
      FR6=(DM1-(1.0-ALP)*PS11)/(ALP*R1)
      VP=FR6*FNP
      VM=FR6*FNM
      GO TO (13,14,15,16)K
   13 V=VP
      FN=FNP
      GO TO 10
   14 V=VP
      FN=FNM
      GO TO 10
   15 V=VM
      FN=FNP
      GO TO 10
   16 V=VM
      FN=FNM
      GO TO 10
   17 IF(FG2-GAM2-.00001)28,29,27
   27 GAM2=GAM2+DG2
      GO TO 90
   28 IF(FG1-GAM1-.00001)100,100,29
   29 GAM1=GAM1+DG1
      GAM2=RGAM2
      GO TO 25
   10 TH1=FXP(GAM1*ALOG(V))
      TH2=FXP(GAM2*ALOG(FN))
      FR=5.0/GAM1+1.0
      TRV=GAMMA(FR)
      FR=5.0/GAM2+1.0
      TRM1=GAMMA(FP)
      FP1=ALP*(TH1**(5.0/GAM1))
      FP2=(1.0-ALP)*(TH2**(5.0/GAM2))
      UP5=FR1*TRV+FR2*TRM1
      WRITE(61,26)GAM1,GAM2,TH1,TH2,UP5
      GO TO 12
  100 STOP
    4 FORMAT(1H1)
   70 FORMAT(3X,6HDM1   =F14.7,3X,6HDM2   =F14.7,3X,6HDM3   =F14.7,
     13X,6HDM4   =F14.7,3X,6HDM5   =F14.7/)
   92 FORMAT(//)
   26 FORMAT(3X,6HGAM1  =F15.8,3X,6HGAM2  =F15.8,3X,
     16HTH1   =F15.8,3X,6HTH2   =E15.8,3X,6HU5    =E15.8)
      END
```

```fortran
      FUNCTION GAMMA(X)
75    FORMAT(53H GAMMA FUNCTION OF A NEGATIVE INTEGER IS NOT DEFINED.)
5     IF(X) 10,74,15
10    N=-X
      FN=-N-1
      V=X-FN
      IF(V-1.)20, 80, 20
15    N=X
      FN=N
      V=X-FN
20    GAMMA=1.+V*(.42278434 +V*(.41184025 +V*(.08157821 +V*
     1(.07424708   +V*(-.00021090747 +V*(.01097369  +V*(-.00246674R
     2+V*(.00152307681  -V*(.00246623420  -V*.0000677105  )))))))
      IF(FN-2.) 37,25,30
24    GAMMA=1.
25    RETURN
30    N=N-1
      DO 35 I=2,N
      FI=I
35    GAMMA=GAMMA*(FI+V)
      RETURN
37    N=2.-FN
      DO 40 I=1,N
      FI=2-I
40    GAMMA=GAMMA/(FI+V)
      RETURN
80    WRITE(61,75)
      STOP
      END
      FINIS
```
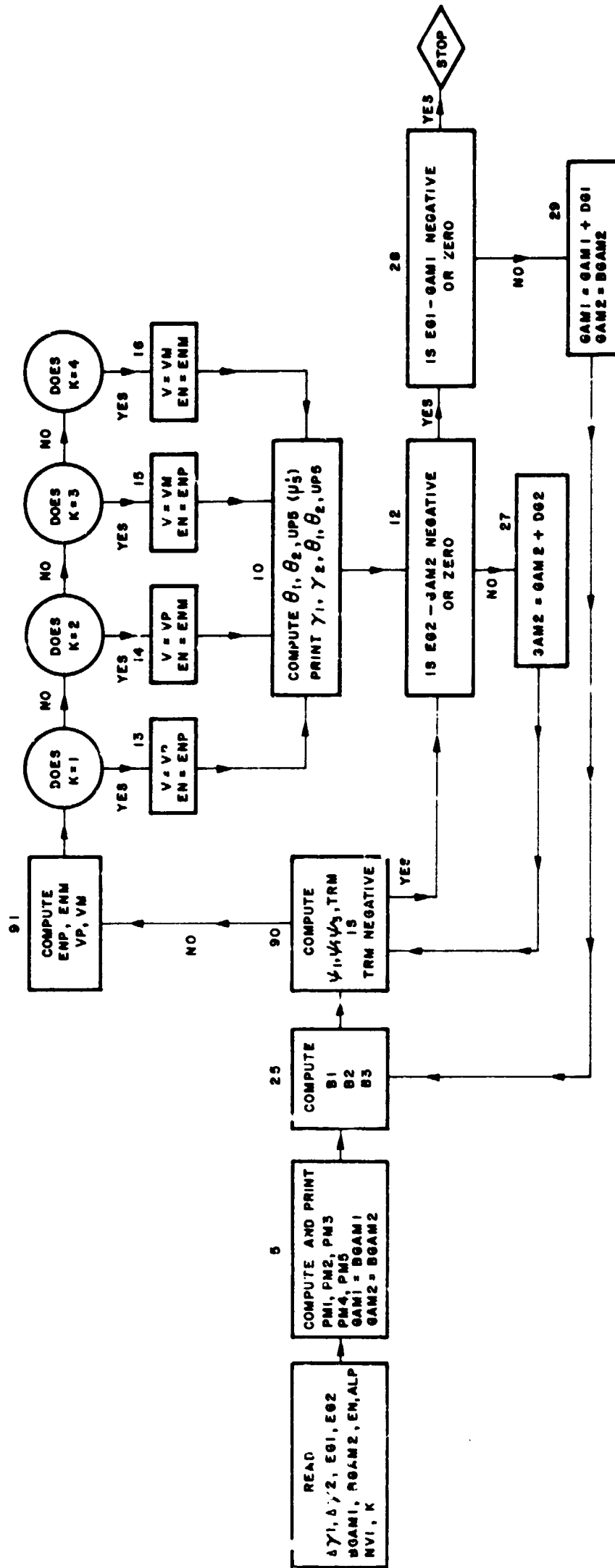
GAM00001
GAM00002
GAM00003
GAM00004
GAM00005
71M00006
GAM00007
GAM00008
GAM00009
GAM00010
GAM00011
GAM00012
GAM00013
GAM00014
GAM00015
GAM00016
GAM00017
GAM00018
GAM00019
GAM00020
GAM00021
GAM00022
GAM00023
GA100024
GAM00025
GAM00026
C5107570
GAM00030

NOTE: TRM IS THE QUANTITY UNDER THE RADICAL IN EQUATIONS 12 AND 13.
VP IS THE VALUE OF EQUATION 13 WHEN THE POSITIVE VALUE IS USED.
VM IS THE VALUE OF EQUATION 13 WHEN THE NEGATIVE VALUE IS USED.

ENP IS THE VALUE OF EQUATION 12 WHEN THE POSITIVE VALUE IS USED.
ENM IS THE VALUE OF EQUATION 12 WHEN THE NEGATIVE VALUE IS USED.

COMPUTER PROGRAM FLOW CHART FOR ESTIMATING THE PARAMETERS OF A MIXED WEIBULL DISTRIBUTION

READ
$\Delta\gamma_1, \Delta\gamma_2$, EG1, EG2
BGAM1, BGAM2, EN, ALP
NVI, K

5 COMPUTE AND PRINT
PM1, PM2, PM3
PM4, PM5
GAM1 = BGAM1
GAM2 = BGAM2

25 COMPUTE
B1
B2
B3

90 COMPUTE
ψ_1, ψ_2, ψ_3, TRM
IS
TRM NEGATIVE

91 COMPUTE
ENP, ENM
VP, VM

13 DOES K=1 — YES → V = VP EN = ENP

14 DOES K=2 — YES → V = VP EN = ENM

15 DOES K=3 — YES → V = VM EN = ENP

16 DOES K=4 — YES → V = VM EN = ENM

10 COMPUTE θ_1, θ_2, UP5 (μ_5')
PRINT $\gamma_1, \gamma_2, \theta_1, \theta_2$, UP5

12 IS EG2 − JAM2 NEGATIVE OR ZERO

27 JAM2 = GAM2 + DG2

28 IS EG1 − GAM1 NEGATIVE OR ZERO — YES → STOP

29 GAM1 = GAM1 + DG1
GAM2 = BGAM2

LOAD DATA

FIRST, TWO TABLES ARE READ IN STARTING IN XI AND FI

DG1 - $\Delta\gamma_1$

DG2 - $\Delta\gamma_2$

EG1 - TERMINAL γ_1

EG2 - TERMINAL γ_2

BGAM1 - BEGINNING γ_1

BGAM2 - BEGINNING γ_2

EN - SAMPLE SIZE   2000 USED FOR THIS PROGRAM

ALP - α .8 USED

NVI - NUMBER OF VALUES IN THE TABLES

K - AN INTEGER $1 \rightarrow 4$

THE COMPUTED μ_5' SHOULD APPROACH m_5'

THIS CAN BE ACCOMPLISHED BY A SERIES OF RUNS VARYING $\Delta\gamma_1$, $\Delta\gamma_2$, γ_1, AND γ_2

m_1' - PM1 $\qquad$ θ_1 - TH1

m_2' - PM2 $\qquad$ θ_2 - TH2

m_3' - PM3

m_4' - PM4

m_5' - PM5

β_1 - B1

β_2 - B2

β_3 - B3

ψ_1 - PSI1

ψ_2 - PSI2

ψ_3 - PSI3

μ_5' - UP5

N - EN $\qquad$ SAMPLE SIZE

n - EN $\qquad$ ONCE THE ABOVE N IS USED THE LOCATION EN IS NO LONGER NEEDED.
$\qquad\qquad\qquad$ IT IS USED TO STORE n.

v - V

ENP - IN EQUATION 12, IF THE POSITIVE VALUE OF THE RADICAL IS USED.

ENM - IF THE NEGATIVE VALUE IS USED.

VP - IN EQUATION 13, IF THE POSITIVE VALUE OF THE RADICAL IS USED.

VM - IF THE NEGATIVE VALUE IS USED

THE FOLLOWING FOUR COMBINATIONS ARE POSSIBLE, DEPENDING UPON WHAT NUMBER
IS READ INTO LOCATION K:

v	n	k	
+	+	1	REGARDLESS OF WHICH COMBINATION IS USED, BEFORE THE
+	-	2	PROGRAM IS CONTINUED, THE VALUE OF EQUATION 13 IS
-	+	3	STORED IN V AND THE VALUE OF EQUATION 12 IS STORED
-	-	4	IN EN.

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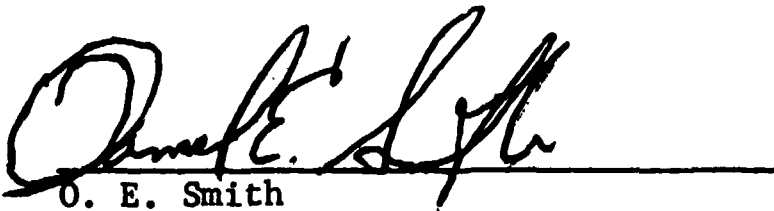   London, Vol. 185, 1894 pp. 71-110.

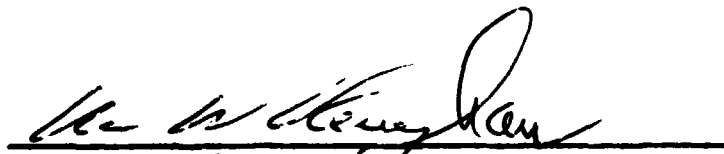# ESTIMATION OF PARAMETERS IN COMPOUND WEIBULL DISTRIBUTIONS

by Lee W. Falls

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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