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TRANSFER MATRICES AND STIFFNESS MATRICES FOR UNIFORM BEAMS

by

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SUMMARY

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Transfer matrices and stiffness matrices are developed for uniform beams in a steady state vibration at an arbitrary frequency. Assumptions of small bending deflection theory are used. Effects of shear deformation, rotary inertia, and cross-sectional warping are neglected.

In this report, the following matrices are presented:

Beam transfer and stiffness matrices without intermediate loads;
Beam transfer and stiffness matrices for intermediate concentrated force and couple;
Beam transfer matrix for several intermediate concentrated forces and moments; and
Beam transfer matrix for continuous applied force and moment distribution

Author

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LIST OF SYMBOLS

A, B, C, D	deflection coefficients of beam in bending mode; to be determined from boundary conditions
d	deflection
E	Young's modulus of elasticity
E_o, E_1, . . E_L	elements of a stiffness matrix defined in equations (81) and (82)
f	continuous force per unit length of beam
f	subscript for force
F_7, F_8, F_9, F_{10}	defined in equation (15)
h_1, h_2, h_5	defined in equation (35)
I	area moment of inertia
I_1	A 4 x 4 matrix defined in equation (41)
K, $\bar{K}$	Stiffness matrix
L	length of uniform beam
L	L end of beam
m	continuous moment per unit length of beam
m	subscript for moment
M	moment
O	O end of beam
O	subscript for maximum amplitude
P	load
t	time function
T, $\bar{T}$	transfer matrix
V	shear
w, W	beam bending deflection
x	beamwise coordinate

LIST OF SYMBOLS con't.

α	frequency parameter
β	$= 1 - \zeta$
γ, ζ	$= \alpha x/L$
θ	slope
μ	beam mass per unit length
ω	angular response frequency

1.0 INTRODUCTION

In the following analysis, transfer matrices and stiffness matrices are developed for uniform beams in a state of steady vibration at an arbitrary frequency ω. The usual assumption of small bending deflection theory is used and all effects of shear deformation, rotary inertia and cross-section warping are neglected.

The first transfer matrices to be developed relate the oscillatory deflection, slope, bending moment and shear at the one end of the beam to the oscillatory deflection, slope, bending moment and shear at the other end of the beam. From these transfer matrices, it is possible to solve for the dynamic stiffness matrix of the beam which relates the bending moment and shear loads at the two ends of the beam to the deflections and slopes at the two ends.

Transfer matrices are then developed for a uniform beam with an oscillatory concentrated force and couple acting at an arbitrary point along the span. A dynamic stiffness matrix is also obtained for this case which relates the bending moments and shears at the two ends of the beam, and relates the applied force and couple, to the deflections and slopes at the two ends of the beam and at the point of application of the force and couple. It is then shown how these transfer and stiffness matrices can be generalized to account for an arbitrary number of forces and couples applied at arbitrary discrete points along the span.

Finally, transfer matrices are developed for the case in which a continuous distribution of forces and couples are applied along the span of the beam. This case is an obvious generalization of the case of a number of concentrated forces and couples applied at discrete spanwise locations.

1.1 Beam Transfer Matrix Without Intermediate Applied Loads

A free-body diagram of a uniform beam is shown in Figure 1 below along with the positive directions chosen for the end deflections, slopes, bending moments and shears. The symbols

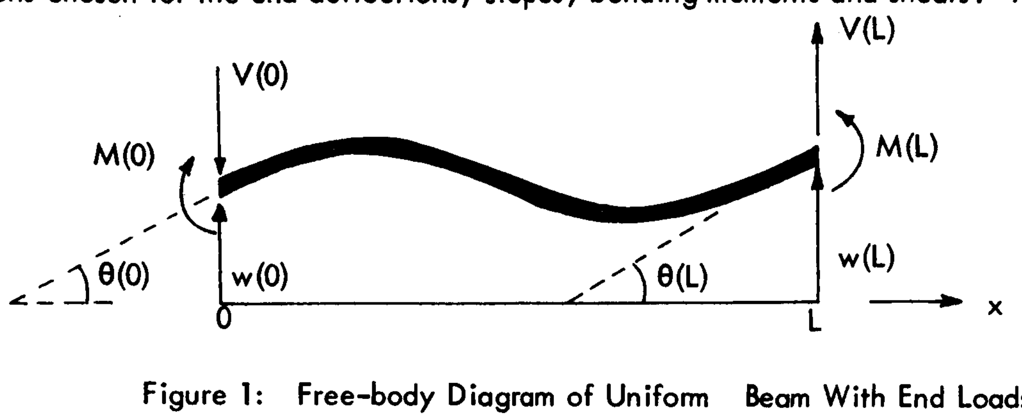


Figure 1: Free-body Diagram of Uniform Beam With End Loads.

chosen for the end deflections and loads are

$w(0),\ w(L)$ = end deflections

$\theta(0),\ \theta(L)$ = end slopes

$M(0),\ M(L)$ = end bending moments

$V(0),\ V(L)$ = end shears

The equation of motion for a uniform beam is

$$\frac{\partial^4 W(x,t)}{\partial x^4} + \frac{\mu}{EI} \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{1}$$

where

$W(x,t)$ = beam bending deflection

μ = beam mass per unit length

EI = beam bending stiffness

For harmonic motion at frequency ω, the bending deflection can be expressed in the complex form

$$W(x,t) = w(x)\ e^{i\omega t} \tag{2}$$

Upon substituting (2) into (1), the following ordinary differential equation for the spanwise deflection amplitude, $w(x)$, is obtained

$$\frac{d^4 w(x)}{d x^4} = \left(\frac{a}{L}\right)^4 w(x) \tag{3}$$

where the nondimensional frequency parameter a and the response frequency ω are related by the equation

$$\omega = \left(\frac{a}{L}\right)^2 \sqrt{\frac{EI}{\mu}} \tag{4}$$

The general solution of (3) can be set down immediately, and has the form

$$w(x) = A \cosh \zeta + B \sinh \zeta + C \cos \zeta + D \sin \zeta \tag{5}$$

where

$$\zeta = \alpha x/L$$

A, B, C, D = deflection coefficients to be determined from the boundary conditions on the beam.

In terms of the deflection $w(x)$, the slope $\theta(x)$, bending moment $M(x)$, and shear $V(x)$ are defined as

$$\begin{aligned} \theta(x) &= d\, w(x)/dx \\ M(x) &= E I\, d^2 w(x)/dx^2 \\ V(x) &= E I\, d^3 w(x)/dx^3 \end{aligned} \tag{6}$$

In terms of the deflection coefficients A, B, C, D and the non-dimensional frequency parameter α, the deflection, slope, bending moment and shear at the end $x = 0$ are

$$\begin{aligned} w(0) &= A + C \\ L\theta(0) &= \alpha (B + D) \\ (L^2/E I)\, M(0) &= \alpha^2 (A - C) \\ -(L^3/E I)\, V(0) &= \alpha^3 (B - D) \end{aligned} \tag{7}$$

The simultaneous solution of (7) for A, B, C, D gives

$$A = \frac{1}{2}\left[w(0) + \frac{1}{\alpha^2} \frac{L^2 M(0)}{E I} \right]$$

$$B = \frac{1}{2}\left[\frac{1}{\alpha} L\theta(0) - \frac{1}{\alpha^3} \frac{L^3 V(0)}{E I} \right]$$

$$C = \frac{1}{2}\left[w(0) - \frac{1}{\alpha^2} \frac{L^2 M(0)}{EI} \right]$$

$$D = \frac{1}{2}\left[\frac{1}{\alpha} L\theta(0) + \frac{1}{\alpha^3} \frac{L^3 V(0)}{EI} \right]$$

which in matrix form becomes

$$\begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} = T_1 \begin{vmatrix} w(0) \\ L\theta(0) \\ L^2M(0)/EI \\ -L^3V(0)/EI \end{vmatrix} \tag{8}$$

where the 4×4 matrix T_1 is

$$T_1 = \frac{1}{2} \begin{vmatrix} 1 & 0 & \frac{1}{\alpha^2} & 0 \\ 0 & \frac{1}{\alpha} & 0 & -\frac{1}{\alpha^3} \\ 1 & 0 & -\frac{1}{\alpha^2} & 0 \\ 0 & \frac{1}{\alpha} & 0 & \frac{1}{\alpha^3} \end{vmatrix} \tag{9}$$

The deflection, slope, bending moment and shear at the end $x = L$, where $\zeta = \alpha$, can be obtained from (5) and (6); and the results can be written in matrix form as follows:

$$\begin{vmatrix} w(L) \\ L\theta(L) \\ L^2M(L)/EI \\ -L^3V(L)/EI \end{vmatrix} = T_2 \begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} \tag{10}$$

where the 4×4 matrix T_2 is

$$T_2 = \begin{vmatrix} \cosh \alpha & \sinh \alpha & \cos \alpha & \sin \alpha \\ \alpha \sinh \alpha & \alpha \cosh \alpha & -\alpha \sin \alpha & \alpha \cos \alpha \\ \alpha^2 \cosh \alpha & \alpha^2 \sinh \alpha & -\alpha^2 \cos \alpha & -\alpha^2 \sin \alpha \\ \alpha^3 \sinh \alpha & \alpha^3 \cosh \alpha & \alpha^3 \sin \alpha & -\alpha^3 \cos \alpha \end{vmatrix} \tag{11}$$

The beam transfer matrix between the deflections and loads at $x = 0$ and the deflections and loads at $x = L$ can be obtained by combining (8) and (10) to give

$$\begin{vmatrix} W(L) \\ L\theta(L) \\ L^2M(L)/EI \\ -L^3V(L)/EI \end{vmatrix} = \overline{T}(0,L) \begin{vmatrix} W(0) \\ L\theta(0) \\ L^2M(0)/EI \\ -L^3V(0)/EI \end{vmatrix} \tag{12}$$

where the transfer matrix $\overline{T}(0,L)$ is given by the equation

$$\overline{T}(0,L) = T_2 \cdot T_1 \tag{13}$$

Multiplying, according to (13), the matrices T_1 and T_2 as given by (9) and (11) results in the following matrix for $T(0,L)$:

$$\overline{T}(0,L) = \frac{1}{2} \begin{vmatrix} F_9 & F_7/\alpha & F_{10}/\alpha^2 & F_8/\alpha^3 \\ \alpha F_8 & F_9 & F_7/\alpha & F_{10}/\alpha^2 \\ \alpha^2 F_{10} & \alpha F_8 & F_9 & F_7/\alpha \\ \alpha^3 F_7 & \alpha^2 F_{10} & \alpha F_8 & F_9 \end{vmatrix} \tag{14}$$

where the functions F_7 , F_8 , F_9 , F_{10} are

$$
\begin{aligned}
F_7 &= \sinh \alpha + \sin \alpha \\
F_8 &= \sinh \alpha - \sin \alpha \\
F_9 &= \cosh \alpha + \cos \alpha \\
F_{10} &= \cosh \alpha - \cos \alpha
\end{aligned}
\tag{15}
$$

The particular notation used in (14) for the functions of α given in (15) is the same as that used by Bishop and Johnson in Reference 1, except for the signs of F_8 and F_{10}. Tabulated numerical values of these functions can be found in Reference 1.

The elements of the transfer matrix, $\overline{T}(0,L)$, are nondimensional. It is sometimes more convenient to use the dimensional form of this matrix, denoted by $T(0,L)$, which is defined by the equations

$$
\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix}
\tag{16}
$$

where $T(0,L)$ is

$$
T(0,L) = \frac{1}{2} \left| \begin{array}{c:c:c:c}
F_9 & \frac{L}{\alpha} F_7 & \frac{1}{EI} (\frac{L}{\alpha})^2 F_{10} & -\frac{1}{EI} (\frac{L}{\alpha})^3 F_8 \\
\frac{\alpha}{L} F_8 & F_9 & \frac{1}{EI} (\frac{L}{\alpha}) F_7 & -\frac{1}{EI} (\frac{L}{\alpha})^2 F_{10} \\
EI (\frac{\alpha}{L})^2 F_{10} & EI (\frac{\alpha}{L}) F_8 & F_9 & -\frac{L}{\alpha} F_7 \\
-EI (\frac{\alpha}{L})^3 F_7 & -EI (\frac{\alpha}{L})^2 F_{10} & -\frac{\alpha}{L} F_8 & F_9
\end{array} \right|
\tag{17}
$$

It is often convenient to reverse the order of the transfer matrix equation (16) and write

$$\begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} = T(L,0) \begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} \tag{18}$$

where the transfer matrix $T(L,0)$ is the inverse of $T(0,L)$, and is

$$T(L,0) = \frac{1}{2} \left| \begin{array}{c:c:c:c} F_9 & -\frac{L}{\alpha} F_7 & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10} & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 F_8 \\ -\frac{\alpha}{L} F_8 & F_9 & -\frac{1}{EI}\left(\frac{L}{\alpha}\right) F_7 & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10} \\ EI\left(\frac{\alpha}{L}\right)^2 F_{10} & -EI\left(\frac{\alpha}{L}\right) F_8 & F_9 & \frac{L}{\alpha} F_7 \\ EI\left(\frac{\alpha}{L}\right)^3 F_7 & -EI\left(\frac{\alpha}{L}\right)^2 F_{10} & \frac{\alpha}{L} F_8 & F_9 \end{array} \right| \tag{19}$$

The nondimensional form of $T(L0)$ is defined by the equations

$$\begin{vmatrix} w(0) \\ L\theta(0) \\ L^2 M(0)/EI \\ -L^3 V(0)/EI \end{vmatrix} = \overline{T}(L,0) \begin{vmatrix} w(L) \\ L\theta(L) \\ L^2 M(L)/EI \\ -L^3 V(L)/EI \end{vmatrix} \tag{20}$$

where $\overline{T}(L,0)$ is

$$\overline{T}(L,0) = \frac{1}{2} \begin{vmatrix} F_9 & -F_7/\alpha & F_{10}/\alpha^2 & -F_8/\alpha^3 \\ -\alpha F_8 & F_9 & -F_7/\alpha & F_{10}/\alpha^2 \\ \alpha^2 F_{10} & -\alpha F_8 & F_9 & -F_7/\alpha \\ -\alpha^3 F_7 & \alpha^2 F_{10} & -\alpha F_8 & F_9 \end{vmatrix} \tag{21}$$

1.2 Beam Stiffness Matrix Without Intermediate Applied Loads

The dynamic stiffness matrix for the uniform beam without intermediate loads is a matrix which when postmultiplied by the column matrix of the deflections and slopes at the two ends of the beam will be equal to the column matrix of the bending moments and shears at the two ends of the beam. Denoting the dynamic stiffness matrix by K, it is clear that one possible definition of K is given by the equation

$$\begin{vmatrix} M(0) \\ V(0) \\ M(L) \\ V(L) \end{vmatrix} = K \begin{vmatrix} w(0) \\ \theta(0) \\ w(L) \\ \theta(L) \end{vmatrix} \tag{22}$$

which is often expressed alternatively as follows:

$$\begin{vmatrix} L^2 M(0)/EI \\ -L^3 V(0)/EI \\ L^2 M(L)/EI \\ -L^3 V(L)/EI \end{vmatrix} = \overline{K} \begin{vmatrix} w(0) \\ L\theta(0) \\ w(L) \\ L\theta(L) \end{vmatrix} \tag{23}$$

In order to use the above matrix equations to determine the form of K it is convenient to define the following matrices:

$$d(0) = \begin{vmatrix} w(0) \\ \theta(0) \end{vmatrix} \qquad d(L) = \begin{vmatrix} w(L) \\ \theta(L) \end{vmatrix}$$

$$P(0) = \begin{vmatrix} M(0) \\ V(0) \end{vmatrix} \qquad P(L) = \begin{vmatrix} M(L) \\ V(L) \end{vmatrix} \tag{24}$$

$$T(L,0) = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix}$$

The 2 x 2 matrices T_{11}, T_{12}, T_{21}, T_{22} can be obtained from (19) and are

$$T_{11} = \frac{1}{2}\left|\begin{array}{c|c} F_9 & -\frac{L}{\alpha} F_7 \\ \hline -\frac{\alpha}{L} F_8 & F_9 \end{array}\right| , \qquad T_{12} = \frac{1}{2}\left|\begin{array}{c|c} \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10} & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 F_8 \\ \hline -\frac{1}{EI}\left(\frac{L}{\alpha}\right) F_7 & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10} \end{array}\right|$$

$$T_{21} = \frac{1}{2}\left|\begin{array}{c|c} EI\left(\frac{\alpha}{L}\right)^2 F_{10} & -EI\left(\frac{\alpha}{L}\right) F_8 \\ \hline EI\left(\frac{\alpha}{L}\right)^3 F_7 & -EI\left(\frac{\alpha}{L}\right)^2 F_{10} \end{array}\right| , \quad T_{22} = \frac{1}{2}\left|\begin{array}{c|c} F_9 & \frac{L}{\alpha} F_7 \\ \hline \frac{\alpha}{L} F_8 & F_9 \end{array}\right| \tag{25}$$

From (18) and from the matrix definitions in (24), it follows that

$$\begin{vmatrix} d(0) \\ P(0) \end{vmatrix} = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \cdot \begin{vmatrix} d(L) \\ P(L) \end{vmatrix} \tag{26}$$

The two matrix equations represented by (26) are

$$d(0) = T_{11}\, d(L) + T_{12}\, P(L) \tag{27}$$

$$P(0) = T_{21}\, d(L) + T_{22}\, P(L) \tag{28}$$

Solving (27) for P(L) gives

$$P(L) = T_{12}^{-1}\, d\,(0) - T_{12}^{-1}\, T_{11}\, d\,(L) \tag{29}$$

The load matrix P(0) can then be determined in terms of the deflections by substituting (29) in (28) giving

$$P(0) = \left[T_{21} - T_{22}\, T_{12}^{-1}\, T_{11} \right] d(L) + T_{22}\, T_{12}^{-1}\, d(0) \tag{30}$$

Equations (29) and (30) can be combined into the single matrix equation

$$\begin{vmatrix} P(0) \\ P(L) \end{vmatrix} = \left|\begin{array}{c|c} T_{22}\, T_{12}^{-1} & T_{21} - T_{22}\, T_{12}^{-1}\, T_{11} \\ \hline T_{12}^{-1} & -\, T_{12}^{-1}\, T_{11} \end{array}\right| \cdot \begin{vmatrix} d(0) \\ d(L) \end{vmatrix} \tag{31}$$

The inverse matrix T_{12}^{-1} can be shown to be equal to

$$T_{12}^{-1} = \frac{1}{h_0} \left|\begin{array}{c|c} -\,EI\,(\frac{\alpha}{L})^2\, F_{10} & -\,EI\,(\frac{\alpha}{L})\, F_8 \\ \hline EI\,(\frac{\alpha}{L})^3\, F_7 & EI\,(\frac{\alpha}{L})^2\, F_{10} \end{array}\right| \tag{32}$$

where

$$h_0 = \cosh \alpha \, \cos \alpha - 1 \tag{33}$$

Forming the product of T_{22} and T_{12}^{-1} gives

$$T_{22}\, T_{12}^{-1} = \frac{1}{2h_0} \left|\begin{array}{c|c} EI\,(\frac{\alpha}{L})^2\, h_5 & 2\,EI\,(\frac{\alpha}{L})\, h_1 \\ \hline 2\,EI\,(\frac{\alpha}{L})^3\, h_2 & EI\,(\frac{\alpha}{L})^2\, h_5 \end{array}\right| \tag{34}$$

where

$$h_1 = \cosh\alpha \ \sin\alpha - \sinh\alpha \ \cos\alpha$$

$$h_2 = \cosh\alpha \ \sin\alpha + \sinh\alpha \ \cos\alpha \qquad (35)$$

$$h_5 = 2 \ \sinh\alpha \ \sin\alpha$$

Forming the product of $-T_{12}^{-1}$ and T_{11} gives

$$-T_{12}^{-1} T_{11} = \frac{1}{2h_0} \left| \begin{array}{c|c} EI\left(\frac{\alpha}{L}\right)^2 h_5 & -2EI\left(\frac{\alpha}{L}\right) h_1 \\ \hline -2EI\left(\frac{\alpha}{L}\right)^3 h_2 & EI\left(\frac{\alpha}{L}\right)^2 h_5 \end{array} \right| \qquad (36)$$

Forming the product $T_{22} T_{12}^{-1} T_{11}$ and determining the quantity $(T_{21} - T_{22} T_{12}^{-1} T_{11})$ gives

$$T_{21} - T_{22} T_{12}^{-1} T_{11} = \frac{1}{h_0} \left| \begin{array}{c|c} -EI\left(\frac{\alpha}{L}\right)^2 F_{10} & EI\left(\frac{\alpha}{L}\right) F_8 \\ \hline -EI\left(\frac{\alpha}{L}\right)^3 F_7 & EI\left(\frac{\alpha}{L}\right)^2 F_{10} \end{array} \right| \qquad (37)$$

Now substituting (24), (32), (34), (36) and (37) into (31) and using the definition for K in (22) gives the following dynamic stiffness matrix for a uniform beam without intermediate applied loads:

$$K = \frac{EI}{2h_0} \left| \begin{array}{c|c|c|c} \left(\frac{\alpha}{L}\right)^2 h_5 & 2\left(\frac{\alpha}{L}\right) h_1 & -2\left(\frac{\alpha}{L}\right)^2 F_{10} & 2\left(\frac{\alpha}{L}\right) F_8 \\ 2\left(\frac{\alpha}{L}\right)^3 h_2 & \left(\frac{\alpha}{L}\right)^2 h_5 & -2\left(\frac{\alpha}{L}\right)^3 F_7 & 2\left(\frac{\alpha}{L}\right)^2 F_{10} \\ -2\left(\frac{\alpha}{L}\right)^2 F_{10} & -2\left(\frac{\alpha}{L}\right) F_8 & \left(\frac{\alpha}{L}\right)^2 h_5 & -2\left(\frac{\alpha}{L}\right) h_1 \\ 2\left(\frac{\alpha}{L}\right)^3 F_7 & 2\left(\frac{\alpha}{L}\right)^2 F_{10} & -2\left(\frac{\alpha}{L}\right)^3 h_2 & \left(\frac{\alpha}{L}\right)^2 h_5 \end{array} \right| \qquad (38)$$

1.3 Beam Transfer Matrix for Intermediate Concentrated Force and Couple

A diagram of a uniform beam which has a concentrated force F and couple, or moment, M applied at point $x = x_1$, and bending moments and shears applied at the ends, is shown in Figure 2 below.

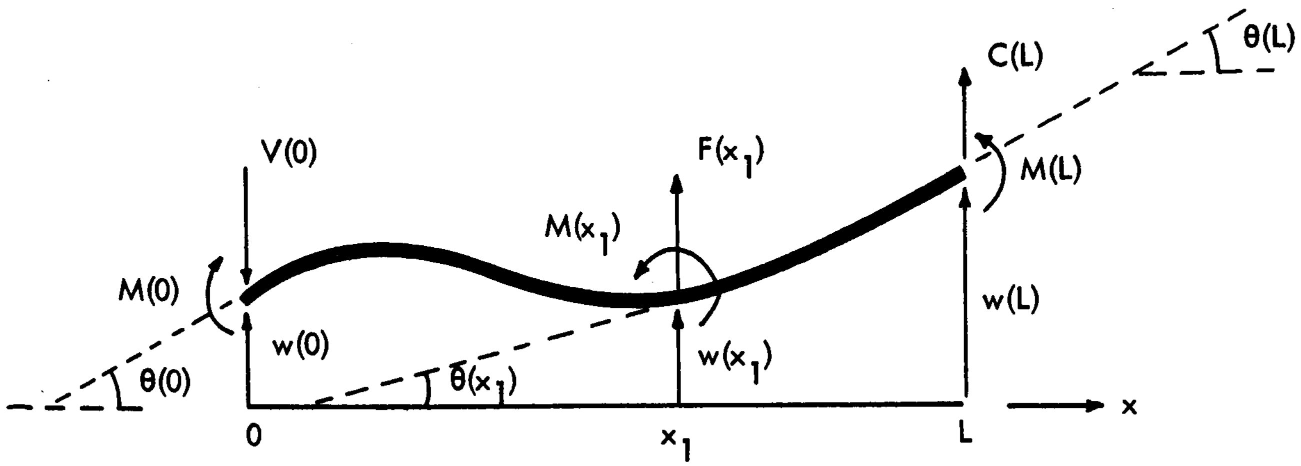


Figure 2: Diagram of Uniform Beam with Intermediate Applied Concentrated Force and Couple.

In order to obtain the transfer matrix from $x = 0$ to $x = L$ it is necessary to develop transfer matrices from $x = 0$ to $x = x_1 - 0$, from $x = x_1 - 0$ to $x = x_1 + 0$, and from $x = x_1 + 0$ to $x = L$. For this purpose a free-body diagram of the beam is presented in Figure 3 which shows the end loads, internal loads and applied loads.

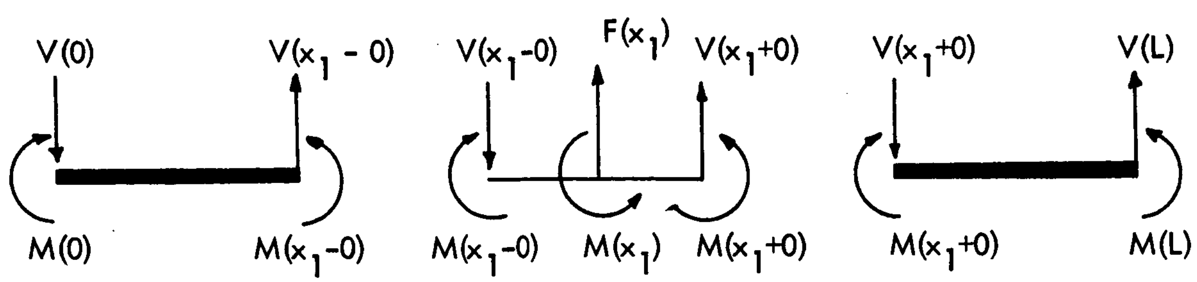


Figure 3: Free-body Diagram of Beam Showing Internal and Applied Loads.

By comparison with (16), the transfer matrix equation from $x = 0$ to $x = x_1 - 0$ is

$$\begin{vmatrix} w(x_1 - 0) \\ \theta(x_1 - 0) \\ M(x_1 - 0) \\ V(x_1 - 0) \end{vmatrix} = T(0, x_1 - 0) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} \tag{39}$$

where

$T(0, x_1 - 0)$ = transfer matrix obtained from (17) by replacing L by x_1 and replacing a by a_1.

The deflections and slopes at $x_1 - 0$ and $x_1 + 0$ are equal. However, from equilibrium of the central span between $x_1 - 0$ and $x_1 + 0$, the loads at $x = x_1 + 0$ are

$$\begin{aligned} M(x_1 + 0) &= M(x_1 - 0) - M(x_1) \\ V(x_1 + 0) &= V(x_1 - 0) - F(x_1) \end{aligned} \tag{39.1}$$

Thus the transfer matrix equation from $x_1 - 0$ to $x_1 + 0$ is

$$\begin{vmatrix} w(x_1 + 0) \\ \theta(x_1 + 0) \\ M(x_1 + 0) \\ V(x_1 + 0) \end{vmatrix} = I \begin{vmatrix} w(x_1 - 0) \\ \theta(x_1 - 0) \\ M(x_1 - 0) \\ V(x_1 - 0) \end{vmatrix} - I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{40}$$

where I is a 4 x 4 unit matrix and I_1 is

$$I_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{41}$$

By comparison with (16), the transfer matrix equation from $x = x_1 + 0$ to $x = L$ is

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(x_1 + 0,\ L) \begin{vmatrix} w(x_1 + 0) \\ \theta(x_1 + 0) \\ M(x_1 + 0) \\ V(x_1 + 0) \end{vmatrix} \tag{42}$$

$T(x_1 + 0,\ L)$ = transfer matrix obtained from (17) by replacing L by $(L - x_1)$ and by replacing a by a_2.

Substitution of (39) into (40) and substituting the resulting equation into (42) gives the transfer matrix equation from $x = 0$ to $x = L$, namely

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(x_1 + 0,\ L) \cdot T(0,\ x_1 - 0) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} - T(x_1 + 0, L)\, I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{43}$$

Clearly the matrix product $T(x_1 + 0,\ L) \cdot T(0,\ x_1 - 0)$ must be equal to $T(0, L)$ as defined by (17) so that (43) can be written as

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} - T(x_1 + 0,\ L)\ I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{44}$$

It is to be noted now that the tranfer matrix $T(0,L)$ contains the quantities a and L whereas the matrix $T(x_1 + 0,\ L)$ contains the quantities a_2 and $(L - x_1)$. However, since the frequency of vibration ω, the stiffness EI, and the mass per unit length are the same for the span $x_1 + 0$ to L as for the entire beam, then from the frequency equation (4) it follows that

$$\left[\sqrt{\frac{\mu}{EI}}\ \omega^2\right]^{1/2} = \frac{a_2}{L - x_1} = \frac{a}{L}$$

or

$$a_2 = \left[1 - \frac{x_1}{L}\right] a \tag{45}$$

Thus in $T(x_1 + 0,\ L)$, the ratio $a_2/(L - x_1)$ can be replaced by a/L, but a_2 must be computed from (45). Thus the matrix product $T(x_1 + 0, L)\ I_1$

$$T(x_1 + 0, L)\ I_1 = \frac{1}{2} \left|\begin{array}{c:c:c:c} 0 & 0 & \frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a_2) & -\frac{1}{EI}\left(\frac{L}{a}\right)^3 F_8(a_2) \\ 0 & 0 & \frac{1}{EI}\left(\frac{L}{a}\right) F_7(a_2) & -\frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a_2) \\ 0 & 0 & F_9(a_2) & -\frac{L}{a} F_7(a_2) \\ 0 & 0 & -\frac{a}{L} F_8(a_2) & F_9(a_2) \end{array}\right| \tag{46}$$

The transfer matrix equation can be left in the form shown in (44) or it can be written in the following alternate form

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T_1(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{47}$$

where the rectangular transfer matrix $T_1(0,L)$ is defined by (48)

(See next page for Equation 48)

$$T_1(0,L) = \frac{1}{2} \left[\begin{array}{c:c:c:c:c:c} F_9(\alpha) & (\frac{L}{\alpha}) F_7(\alpha) & \frac{1}{EI}(\frac{L}{\alpha})^2 F_{10}(\alpha) & -\frac{1}{EI}(\frac{L}{\alpha})^3 F_8(\alpha) & \frac{-1}{EI}(\frac{L}{\alpha})^2 F_{10}(\alpha_2) & \frac{1}{EI}(\frac{L}{\alpha})^3 F_8(\alpha_2) \\ (\frac{\alpha}{L}) F_8(\alpha) & F_9(\alpha) & \frac{1}{EI}(\frac{L}{\alpha}) F_7(\alpha) & -\frac{1}{EI}(\frac{L}{\alpha})^2 F_{10}(\alpha) & \frac{-1}{EI}(\frac{L}{\alpha}) F_7(\alpha_2) & \frac{1}{EI}(\frac{L}{\alpha})^2 F_{10}(\alpha_2) \\ EI(\frac{\alpha}{L})^2 F_{10}(\alpha) & EI(\frac{\alpha}{L}) F_8(\alpha) & F_9(\alpha) & -(\frac{L}{\alpha}) F_7(\alpha) & -F_9(\alpha_2) & (\frac{L}{\alpha}) F_7(\alpha_2) \\ -EI(\frac{\alpha}{L})^3 F_7(\alpha) & -EI(\frac{\alpha}{L})^2 F_{10}(\alpha) & -(\frac{\alpha}{L}) F_8(\alpha) & F_9(\alpha) & (\frac{\alpha}{L}) F_8(\alpha_2) & -F_9(\alpha_2) \end{array} \right] \quad (48)$$

The transfer matrix from $x = L$ to $x = 0$ can be obtained by the following steps:

$$\begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} = T(x_1 - 0,0) \begin{vmatrix} w(x_1 - 0) \\ \theta(x_1 - 0) \\ M(x_1 - 0) \\ V(x_1 - 0) \end{vmatrix}$$

$$= T(x_1 - 0,0) \begin{vmatrix} w(x_1 + 0) \\ \theta(x_1 + 0) \\ M(x_1 + 0) \\ V(x_1 + 0) \end{vmatrix} + T(x_1 - 0,0)\, I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix}$$

$$= T(x_1 - 0,0) \cdot T(x_1+0,L) \begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} + T(x_1 - 0,0)\, I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix}$$

$$= T(L,0) \begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} + T(x_1 - 0,0)\, I_1 \begin{vmatrix} 0 \\ 0 \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{49}$$

where $T(L,0)$ is defined by (19) and I_1 is defined by (41). The transfer matrix $T(x_1 - 0,0)$ can be obtained from (19) by replacing L by x_1 and replacing a by a_1. The quantity a_1 is defined by the equation

$$\left[\sqrt{\frac{\mu}{EI}}\,\omega^2\right]^{1/2} = \frac{\alpha_1}{x_1} = \frac{\alpha}{L} \tag{50}$$

Thus the matrix product $T(x_1 - 0,0)\ I_1$ is equal to

$$\begin{vmatrix} 0 & 0 & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha_1) & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 F_8(\alpha_1) \\ 0 & 0 & -\frac{1}{EI}\left(\frac{L}{\alpha}\right) F_7(\alpha_1) & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha_1) \\ 0 & 0 & F_9(\alpha_1) & \frac{L}{\alpha} F_7(\alpha_1) \\ 0 & 0 & \frac{\alpha}{L} F_8(\alpha_1) & F_9(\alpha_1) \end{vmatrix} \tag{51}$$

It is possible to write (49) in the following form

$$\begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} = T_2(L,0) \begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \\ M(x_1) \\ F(x_1) \end{vmatrix} \tag{52}$$

where the rectangular transfer matrix in (52) is equal to the following:

(See next page for equation 53)

$$T_2(L,0) = \frac{1}{2} \left[\begin{array}{c:c:c:c:c:c} F_9(\alpha) & -\frac{L}{\alpha} F_7(\alpha) & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha) & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 F_8(\alpha) & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha_1) & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 F_8(\alpha_1) \\ -\frac{\alpha}{L} F_8(\alpha) & F_9(\alpha) & -\frac{1}{EI}\left(\frac{L}{\alpha}\right) F_7(\alpha) & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha) & -\frac{1}{EI}\left(\frac{L}{\alpha}\right) F_7(\alpha_1) & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 F_{10}(\alpha_1) \\ EI\left(\frac{\alpha}{L}\right)^2 F_{10}(\alpha) & -EI\left(\frac{\alpha}{L}\right) F_8(\alpha) & F_9(\alpha) & \frac{L}{\alpha} F_7(\alpha) & F_9(\alpha_1) & \frac{L}{\alpha} F_7(\alpha_1) \\ EI\left(\frac{\alpha}{L}\right)^3 F_7(\alpha) & -EI\left(\frac{\alpha}{L}\right)^2 F_{10}(\alpha) & \frac{\alpha}{L} F_8(\alpha) & F_9(\alpha) & \frac{\alpha}{L} F_8(\alpha_1) & F_9(\alpha_1) \end{array} \right] \quad (53)$$

1.4 Beam Stiffness Matrix For Intermediate Concentrated Force and Couple

The dynamic stiffness matrix for the beam shown in Figure 2 is a matrix which relates the three pairs of loads acting on the beam to the three pairs of deflections at $x = 0$, $x = x_1$ and $x = L$. In keeping with the definition of the stiffness matrix given in (22), the stiffness matrix for the beam segment between $x = 0$ and $x = x_1 - 0$ is defined by the equation,

$$\begin{vmatrix} M(0) \\ V(0) \\ M(x_1 - 0) \\ V(x_1 - 0) \end{vmatrix} = K(0, x_1 - 0) \begin{vmatrix} w(0) \\ \theta(0) \\ w(x_1) \\ \theta(x_1) \end{vmatrix} \tag{54}$$

where

$K(0, x_1 - 0)$ = stiffness matrix obtained from (38) by replacing L by x_1 and replacing a by a_1 where $a_1 = x_1 a / L$

Similarly, the stiffness matrix for the beam segment between $x = x_1 + 0$ and $x = L$ is defined by the equation

$$\begin{vmatrix} M(x_1 + 0) \\ V(x_1 + 0) \\ M(L) \\ V(L) \end{vmatrix} = K(x_1 + 0, L) \begin{vmatrix} w(x_1) \\ \theta(x_1) \\ w(L) \\ \theta(L) \end{vmatrix} \tag{55}$$

$K(x_1 + 0, L)$ = stiffness matrix obtained from (38) by replacing L by $L - x_1$ and replacing a by a_2 where $a_2 = a\,(1 - x_1/L)$.

It is convenient now to write the stiffness matrices $K(0, x_1 - 0)$ and $K(x_1 + 0, L)$ in the forms

$$K(0,x_1 - 0) = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \tag{56}$$

$$K(x_1 + 0,L) = \begin{vmatrix} K'_{11} & K'_{12} \\ K'_{21} & K'_{22} \end{vmatrix} \tag{57}$$

Using a notation similar to that defined in (24), the equations (54) and (55) can be written in the form of simultaneous equations,

$$P(0) = K_{11}\, d(0) + K_{12}\, d(x_1) \tag{58}$$

$$P(x_1 - 0) = K_{21}\, d(0) + K_{22}\, d(x_1) \tag{59}$$

$$P(x_1 + 0) = K'_{11}\, d(x_1) + K'_{12}\, d(L) \tag{60}$$

$$P(L) = K'_{21}\, d(x_1) + K'_{22}\, d(L) \tag{61}$$

However, (39.1) can be written in a similar matrix form,

$$P(x_1) = P(x_1 - 0) - P(x_1 + 0) \tag{62}$$

Substituting (59) and (60) into (62) gives

$$P(x_1) = K_{21}\, d(0) + (K_{22} - K'_{11})\, d(x_1) - K'_{12}\, d(L) \tag{63}$$

The stiffness matrix can now be developed immediately from (58), (61) and (63), and is defined by the equation

$$\begin{vmatrix} P(0) \\ P(x_1) \\ P(L) \end{vmatrix} = \left|\begin{array}{c|c|c} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} - K'_{11} & - K'_{12} \\ 0 & K'_{21} & K'_{22} \end{array}\right| \cdot \begin{vmatrix} d(0) \\ d(x_1) \\ d(L) \end{vmatrix} \tag{64}$$

In expanded form, (64) can be expressed as

$$
\begin{vmatrix} M(0) \\ V(0) \\ M(x_1) \\ F(x_1) \\ M(L) \\ V(L) \end{vmatrix} = K(0,x_1,L) \begin{vmatrix} w(0) \\ \theta(0) \\ w(x_1) \\ \theta(x_1) \\ w(L) \\ \theta(L) \end{vmatrix} \tag{65}
$$

where the stiffness matrix $K(0,x_1,L)$ is given by Equation 66

(See next page for Equation 66)

$$K(0,x_1,L)=\frac{EI}{2}\begin{bmatrix}
\left(\frac{\alpha}{L}\right)^2\frac{h_5(\alpha_1)}{h_0(\alpha_1)} & 2\left(\frac{\alpha}{L}\right)\frac{h_1(\alpha_1)}{h_0(\alpha_1)} & -2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_1)}{h_0(\alpha_1)} & 2\left(\frac{\alpha}{L}\right)\frac{F_8(\alpha_1)}{h_0(\alpha_1)} & 0 & 0 \\
2\left(\frac{\alpha}{L}\right)^3\frac{h_2(\alpha_1)}{h_0(\alpha_1)} & \left(\frac{\alpha}{L}\right)^2\frac{h_5(\alpha_1)}{h_0(\alpha_1)} & -2\left(\frac{\alpha}{L}\right)^3\frac{F_7(\alpha_1)}{h_0(\alpha_1)} & 2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_1)}{h_0(\alpha_1)} & 0 & 0 \\
-2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_1)}{h_0(\alpha_1)} & -2\left(\frac{\alpha}{L}\right)\frac{F_8(\alpha_1)}{h_0(\alpha_1)} & \left(\frac{\alpha}{L}\right)^2\left[\frac{h_5(\alpha_1)}{h_0(\alpha_1)}-\frac{h_5(\alpha_2)}{h_0(\alpha_2)}\right] & -2\left(\frac{\alpha}{L}\right)\left[\frac{h_1(\alpha_1)}{h_0(\alpha_1)}-\frac{h_1(\alpha_2)}{h_0(\alpha_2)}\right] & 2\left(\frac{\alpha}{L}\right)\frac{F_{10}(\alpha_2)}{h_0(\alpha_2)} & -2\left(\frac{\alpha}{L}\right)\frac{F_8(\alpha_2)}{h_0(\alpha_2)} \\
2\left(\frac{\alpha}{L}\right)^3\frac{F_7(\alpha_1)}{h_0(\alpha_1)} & 2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_1)}{h_0(\alpha_1)} & -2\left(\frac{\alpha}{L}\right)^3\left[\frac{h_2(\alpha_1)}{h_0(\alpha_1)}-\frac{h_2(\alpha_2)}{h_0(\alpha_2)}\right] & \left(\frac{\alpha}{L}\right)^2\left[\frac{h_5(\alpha_1)}{h_0(\alpha_1)}-\frac{h_5(\alpha_2)}{h_0(\alpha_2)}\right] & 2\left(\frac{\alpha}{L}\right)^3\frac{F_7(\alpha_2)}{h_0(\alpha_2)} & -2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_2)}{h_0(\alpha_2)} \\
0 & 0 & -2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_2)}{h_0(\alpha_2)} & -2\left(\frac{\alpha}{L}\right)\frac{F_8(\alpha_2)}{h_0(\alpha_2)} & \left(\frac{\alpha}{L}\right)^2\frac{h_5(\alpha_2)}{h_0(\alpha_2)} & -2\left(\frac{\alpha}{L}\right)\frac{h_1(\alpha_2)}{h_0(\alpha_2)} \\
0 & 0 & 2\left(\frac{\alpha}{L}\right)^3\frac{F_7(\alpha_2)}{h_0(\alpha_2)} & 2\left(\frac{\alpha}{L}\right)^2\frac{F_{10}(\alpha_2)}{h_0(\alpha_2)} & -2\left(\frac{\alpha}{L}\right)^3\frac{h_2(\alpha_2)}{h_0(\alpha_2)} & \left(\frac{\alpha}{L}\right)^2\frac{h_5(\alpha_2)}{h_0(\alpha_2)}
\end{bmatrix} \tag{66}$$

1.5 Beam Transfer Matrix for Several Intermediate Concentrated Forces and Moments

A diagram of a uniform beam which has a number of concentrated forces $F(x_n)$ and couples $M(x_n)$, and bending moments and shears at the ends, is shown in Figure 4 below. Equation (44)

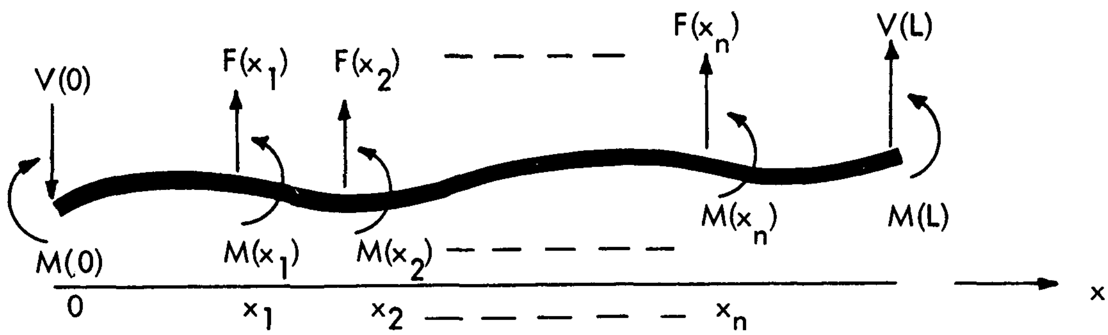


Figure 4: Diagram of Uniform Beam with Intermediate Applied Forces, Couples, and End Loads.

gives the transfer matrix equation for a single applied force and couple at x_1. This equation is easily generalized to the following for n intermediate forces and couples shown in Figure 4,

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} - \sum_{r=1}^{n} T(x_r + 0, L)\, I_1 \cdot \begin{vmatrix} 0 \\ 0 \\ M(x_r) \\ V(x_r) \end{vmatrix} \qquad (67)$$

where the transfer matrices $T(x_r + 0, L)\, I_1$ are defined as

$$T(x_r+0,L) = \frac{1}{2}\begin{vmatrix} 0 & 0 & \frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a_r) & -\frac{1}{EI}\left(\frac{L}{a}\right)^3 F_8(a_r) \\ 0 & 0 & \frac{1}{EI}\left(\frac{L}{a}\right) F_7(a_r) & -\frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a_r) \\ 0 & 0 & F_9(a_r) & -\frac{L}{a} F_7(a_r) \\ 0 & 0 & -\left(\frac{a}{L}\right) F_8(a_r) & F_9(a_r) \end{vmatrix} \tag{68}$$

and where a_r is equal to

$$a_r = \left[1 - \frac{x_r}{L}\, a\right] \tag{69}$$

Similarly a generalization of (49) gives the transfer matrix equation

$$\begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} = T(L,0) \begin{vmatrix} w(L) \\ \theta(L) \\ m(L) \\ V(L) \end{vmatrix} + \sum_{r=1}^{n} T(x_r - 0,0)\, I_1 \begin{vmatrix} 0 \\ 0 \\ m(x_r) \\ F(x_r) \end{vmatrix} \tag{70}$$

where the transfer matrix $T(x_r - 0,0) \cdot I_1$ is

$$T(x_r - 0,0)\cdot I_1 = \frac{1}{2}\begin{vmatrix} 0 & 0 & \frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a'_r) & \frac{1}{EI}\left(\frac{L}{a}\right)^3 F_8(a'_r) \\ 0 & 0 & -\frac{1}{EI}\left(\frac{L}{a}\right) F_7(a'_r) & -\frac{1}{EI}\left(\frac{L}{a}\right)^2 F_{10}(a'_r) \\ 0 & 0 & F_9(a'_r) & \frac{L}{a} F_7(a'_r) \\ 0 & 0 & \frac{a}{L} F_8(a'_r) & F_9(a'_r) \end{vmatrix} \tag{71}$$

where

$$a'_r = x_r\, a/L \tag{72}$$

1.6 Beam Stiffness Matrix for Several Intermediate Concentrated Forces and Couples

Using the notation of (24), the stiffness matrix for the beam segment between $x = 0$ and $x = x_1 - 0$ can be obtained from (54) and (56) and is defined by the equation

$$\begin{vmatrix} P(0) \\ P(x_1 - 0) \end{vmatrix} = \left|\begin{array}{c|c} K_{11}(\alpha_1'') & K_{12}(\alpha_1'') \\ K_{21}(\alpha_1'') & K_{22}(\alpha_1'') \end{array}\right| \cdot \begin{vmatrix} d(0) \\ d(x_1) \end{vmatrix} \tag{73}$$

The notation $K_{ij}(\alpha_1'')$ denotes that the matrices K_{ij}, as defined by (54) and (56), are functions of L/α and α_1'' where

$$\alpha_1'' = x_1 \alpha / L \tag{74}$$

The load transfer equation across the point x_1 can be written in a manner similar to (62), namely

$$P(x_1) = P(x_1 - 0) - P(x_1 + 0) \tag{62}$$

The stiffness matrix equation for the beam segment between $x = x_1 + 0$ and $x = x_2 - 0$ is similar to that defined by (60), namely,

$$\begin{vmatrix} P(x_1 + 0) \\ P(x_2 - 0) \end{vmatrix} = \left|\begin{array}{c|c} K_{11}(\alpha_2'') & K_{12}(\alpha_2'') \\ K_{21}(\alpha_2'') & \end{array}\right| \cdot \begin{vmatrix} d(x_1) \\ d(x_2) \end{vmatrix} \tag{75}$$

where

$$\frac{\alpha_2''}{x_2 - x_1} = \frac{\alpha}{L} \tag{76}$$

From (73) and (75), the load matrix $P(x_1)$ in (62) can be expressed in the form

$$P(x_1) = \left|\begin{array}{c|c|c} K_{21}(\alpha_1'') & K_{22}(\alpha_1'') - K_{11}(\alpha_2'') & -K_{12}(\alpha_2'') \end{array}\right| \begin{vmatrix} d(0) \\ d(x_1) \\ d(x_2) \end{vmatrix} \tag{77}$$

Equation (77) is easily generalized to give the stiffness matrix equation for the loads at x_r , $(1 \le r \le n)$

$$P(x_r) = \left| K_{21}(\alpha''_r) \;\vdots\; K_{22}(\alpha''_r) - K_{11}(\alpha''_{r+1}) \;\vdots\; - K_{12}(\alpha''_{r+1}) \right| \left| \begin{matrix} d(x_r - 1) \\ d(x_r) \\ d(x_{r+1}) \end{matrix} \right| \tag{78}$$

where $x_0 = 0$ and $x_n + 1 = L$, and where

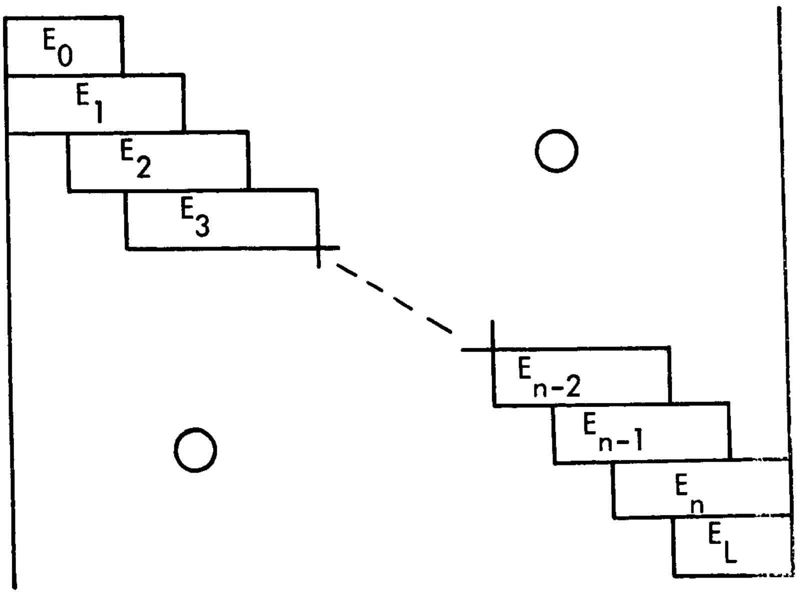
$$\frac{\alpha''_r}{x_r - x_{r-1}} = \frac{\alpha}{L} \tag{79}$$

the general stiffness matrix can now be defined by the equation

$$\left| \begin{matrix} P(0) \\ P(x_1) \\ \vdots \\ P(x_n) \\ P(L) \end{matrix} \right| = K(0, x_1, \ldots, x_n, L) \left| \begin{matrix} d(0) \\ d(x_1) \\ \vdots \\ d(x_n) \\ d(L) \end{matrix} \right| \tag{80}$$

where the stiffness matrix $K(0, x_1, \ldots, x_n, L)$ is

$K(0, x_1, \ldots, x_n, L) =$



(81)

and where the submatrices E_i are

$$E_0 = \left| K_{11}(\alpha''_1) \;\vdots\; K_{12}(\alpha''_1) \right|$$

$$E_r = \left| K_{21}(\alpha''_r) \;\vdots\; K_{22}(\alpha''_r) - K_{11}(\alpha''_{r=1}) \;\vdots\; - K_{12}(\alpha''_{r+1}) \right| , \quad 1 \le r \le n \tag{82}$$

$$E_L = \left| K_{21}(\alpha''_{n+1}) \;\vdots\; K_{22}(\alpha''_{n+1}) \right|$$

$$\alpha''_1 = x_1 \alpha / L$$

$$\alpha''_r = (x_r - x_r - 1)\, \alpha / L \tag{83}$$

$$\alpha''_L = (L - x_n)\, \alpha / L$$

$$K_{11}(\alpha'') = \frac{EI}{2h_0(\alpha'')} \left| \begin{array}{c|c} (\frac{\alpha}{L})\, h_5(\alpha'') & 2(\frac{\alpha}{L})\, h_1(\alpha'') \\ \hline 2(\frac{\alpha}{L})^3\, h_2(\alpha'') & (\frac{\alpha}{L})^2\, h_5(\alpha'') \end{array} \right|$$

$$K_{12}(\alpha'') = \frac{EI}{2h_0(\alpha'')} \left| \begin{array}{c|c} -2(\frac{\alpha}{L})\, F_{10}(\alpha'') & 2(\frac{\alpha}{L})\, F_8(\alpha'') \\ \hline -2(\frac{\alpha}{L})^3\, F_7(\alpha'') & 2(\frac{\alpha}{L})^2\, F_{10}(\alpha'') \end{array} \right|$$

$$K_{21}(\alpha'') = \frac{EI}{2h_0(\alpha'')} \left| \begin{array}{c|c} -2(\frac{\alpha}{L})^2 F_{10}(\alpha'') & -2(\frac{\alpha}{L})\, F_8(\alpha'') \\ \hline 2(\frac{\alpha}{L})^3\, F_7(\alpha'') & 2(\frac{\alpha}{L})^2\, F_{10}(\alpha'') \end{array} \right| \tag{84}$$

$$K_{22}(\alpha'') = \frac{EI}{2h_0(\alpha'')} \left| \begin{array}{c|c} (\frac{\alpha}{L})^2\, h_5(\alpha'') & -2(\frac{\alpha}{L})\, h_1(\alpha'') \\ \hline -2(\frac{\alpha}{L})^3\, h_2(\alpha'') & (\frac{\alpha}{L})^2\, h_5(\alpha'') \end{array} \right|$$

1.7 Beam Transfer Matrix for Continuous Applied Force and Moment Distribution

A diagram of a uniform beam which has a continuous distribution of applied forces and moments along the span, and bending moments and shears at the beams ends, is shown in Figure 5 below. The beam transfer matrix for this

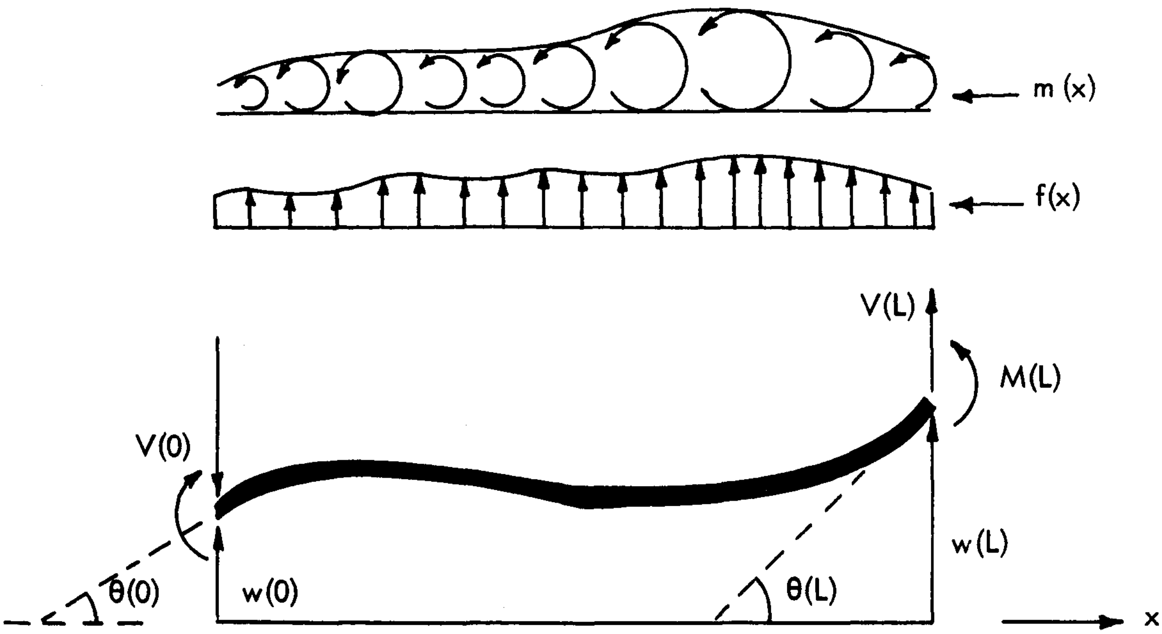


Figure 5: Diagram of Uniform Beam with End Loads and Continuous Distributions of Applied Forces and Moments

case can be developed by allowing the number of discrete applied forces and couples shown in Figure 4 to increase without bound, and passing to the limit in (67) and (70).

Let the continuous force per unit length f(x), and the continuous moment per unit length m(x), be expressed in the form

$$f(x) = f_0 \psi_f(x)$$

$$m(x) = m_0 \psi_m(x) \tag{85}$$

where $\psi_f(x)$ and $\psi_m(x)$ are distribution functions which have a maximum value of unity. The quantities f_0 and m_0 thus represent the maximum amplitude along the span of the applied force and moment distribution respectively.

The incremental force dF(x) and the incremental moment dM(x) acting over an incremental length dx at x are

$$dF(x) = f_0 \, \psi_f(x) \; dx$$
$$dM(x) = m_0 \, \psi_m(x) \; dx \qquad (86)$$

Substituting dF(x) and DM(x) for $F(x_r)$ and $M(x_r)$ in (67) and passing to the limit as dx approaches zero gives

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} - \int_0^L T(x,L) \, I_1 \cdot \begin{vmatrix} 0 \\ 0 \\ f_0 \, \psi_f(x) \\ m_0 \, \psi_m(x) \end{vmatrix} dx \qquad (87)$$

The transfer matrix $T(x_r + 0,\ L) \ \ I_1$ in (67) contains the parameter a_r which is defined by (69). In the limiting case, a_r will be replaced by β where

$$\beta = \left[1 - \frac{x}{L} \, a \right] \qquad (88)$$

Thus β is a continuous function of x. From a knowledge of the transfer matrix defined in (68), the transfer matrix equation (87) can be written in the form

$$\begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} = T(0,L) \begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} - T_c(L) \begin{vmatrix} 0 \\ 0 \\ m_0 \\ f_0 \end{vmatrix} \qquad (89)$$

where the "continuous" transfer matrix T_c is equal to

$$T_c(L) = \frac{1}{2} \begin{vmatrix} 0 & 0 & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 \int_0^L \psi_f(x)\, F_{10}(\beta)\, dx & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 \int_0^L \psi_m(x)\, F_8(\beta)\, dx \\ 0 & 0 & \frac{1}{EI}\left(\frac{L}{\alpha}\right) \int_0^L \psi_f(x)\, F_7(\beta)\, dx & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 \int_0^L \psi_m(x)\, F_{10}(\beta)\, dx \\ 0 & 0 & \int_0^L \psi_f(x)\, F_9(\beta)\, dx & -\frac{L}{\alpha}\int_0^L \psi_m(x)\, F_7(\beta)\, dx \\ 0 & 0 & -\frac{\alpha}{L}\int_0^L \psi_f(x)\, F_8(\beta)\, dx & \int_0^L \psi_m(x)\, F_9(\beta)\, dx \end{vmatrix} \tag{90}$$

Similarly, in the limit (70) becomes

$$\begin{vmatrix} w(0) \\ \theta(0) \\ M(0) \\ V(0) \end{vmatrix} = T(L,0) \begin{vmatrix} w(L) \\ \theta(L) \\ M(L) \\ V(L) \end{vmatrix} + T_c(0) \begin{vmatrix} 0 \\ 0 \\ m_0 \\ f_0 \end{vmatrix} \tag{91}$$

where the transfer matrix $T_c(0)$ can be obtained from (71):

$$T_c(0) = \frac{1}{2} \begin{vmatrix} 0 & 0 & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 \int_0^L \psi_f(x)\, F_{10}(\gamma)\, dx & \frac{1}{EI}\left(\frac{L}{\alpha}\right)^3 \int_0^L \psi_m(x) F_8(\gamma)\, dx \\ 0 & 0 & -\frac{1}{EI}\left(\frac{L}{\alpha}\right) \int_0^L \psi_f(x) F_7(\gamma)\, dx & -\frac{1}{EI}\left(\frac{L}{\alpha}\right)^2 \int_0^L \psi_m(x) F_{10}(\gamma)\, dx \\ 0 & 0 & \int_0^L \psi_f(x)\, F_9(\gamma)\, dx & \frac{L}{\alpha}\int_0^L \psi_m(x)\, F_7(\gamma)\, dx \\ 0 & 0 & \frac{\alpha}{L}\int_0^L \psi_f(x)\, F_8(\gamma)\, dx & \int_0^L \psi_m(x)\, F_9(\gamma)\, dx \end{vmatrix} \tag{92}$$

where from (72)

$$\gamma = \alpha x/L . \tag{93}$$