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by

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SUMMARY

The use of Langmuir probes in ionized gas flows is investigated. First the basic theory of electrostatic probes is investigated and the regimes in which the classical theory is applicable are noted; then a survey is made of literature which modifies the basic theory when the assumptions of classical theory are not met.

A survey is then made of literature dealing with the application of basic probe theory to a flowing ionized gas. The modified equations are given. The possible sources of error are enumerated. Finally recommendations are made as to when the classical Langmuir probe can be used in an ionized gas flow, and the equations to use it are given where applicable.

It is determined that in most cases the classical theory can be used, with the basic equations unmodified by the flow. In some cases a modification is necessary and the necessary equations are supplied.

Author

INTRODUCTION

The use of an electrostatic probe technique to measure the charged particle density and electron temperature in an ionized gas was first developed by Langmuir and Mott-Smith in 1924. Since then the theory of the so-called "Langmuir Probe" has been reexamined and developed by a number of investigators. The "classical" Langmuir theory has not been modified very much as a result of these investigations, but theories have been developed for using the probe techniques in circumstances which are outside the limits of the restrictions of classical theory. This paper is concerned with applying the classical theory to measure the parameters of an ionized gas flow, on the assumption that it meets the conditions imposed by the classical theory.

The basic assumptions made in classical probe theory are

1. The sheath thickness d is much less than any mean free paths λ involved in the gas (i.e. neutral-neutral, ion-ion, electron-electron, ion-electron, ion-neutral, electron-neutral).

2. The probe diameter a is much less than the mean free paths.
3. The sheath thickness d must be much less than the probe dimension a.

The reason for assumption 1 is that we want no collisions in the sheath, so that the current can be found by assuming that the collected particles are in free-fall to the probe under action of the electric field. Assumptions 1, 2, and 3 insure that the probe does not disturb the plasma and so truly acts like a probe.

In this paper we will consider the theory of a plane probe. This is usually a metallic surface (such as the end of a wire) which is inserted into the plasma and collects charged particles from it. By measuring the current collected by the probe as a function of the potential applied to it, one can determine the electron temperature, the electron and ion density, the floating potential, and the plasma potential, the last two potentials with respect to an arbitrary reference and with respect to each other.

SYMBOLS

a	probe diameter
A	probe area
$d_{e,i}$	thickness of electron and ion sheaths, respectively
e	electronic charge
f	Maxwellian distribution function
h	Debye length
$i_{e,i}$	collected electron and ion currents, respectively
j	total current density
$j_{e,i}$	electron and ion current density, repectively
$j_{e,i_{sat.}}$	saturation electron and ion current density, respectively
k	Boltzmann constant
$m_{e,i}$	electron and ion masses, repectively
M_1	free stream Mach number
$n_{e,i}$	electron and ion number density, repectively
n_o	equlibrium electron and ion density in plasma
S	speed ratio
$T_{e,i}$	electron and ion temperatures, repectively
$v_{e,i}$	electron and ion velocity in sheath, respectively

v_{io}	ion velocity at sheath edge in case where $T_i << T_e$
γ	ratio of specific heats, $\frac{c_p}{c_v}$
ϵ_o	permitivity of free space
θ	angle between flow direction and normal to probe surface
λ	mean free path
φ_f	floating potential of probe with respect to an arbitrary reference point
φ_g	potential of probe with respect to an arbitrary reference point
φ_p	plasma potential with respect to arbitrary reference point
φ	potential of probe with respect to plasma potential
φ_o	kinetic energy of ions at sheath edge divided by electronic charge
ω_{pi}	ion plasma frequency $\frac{n_o e^2}{m_i \epsilon_o}$

QUALITATIVE DESCRIPTION OF PROBE CHARACTERISTIC

Let φ_g be the potential of the probe with respect to some fixed reference point, which is arbitrary. At the point φ_p, which is called the plasma potential, the probe is at the same potential as the plasma. Since at this potential there is no potential gradient between the probe and the plasma, no field exists, and the charged particles flow to the probe because of their thermal velocities. Since the ions are much more massive than the electrons, they move much slower than the electrons, and the probe collects predominantly electron current.

If the potential is increased so that $\varphi_g > \varphi_p$, the electrons are accelerated toward the probe and the ions are repelled so that the ion current vanishes. Therefore an excess of negative charge builds up near the probe surface until the total charge is equal to the positive charge on the probe. This layer of charge is called a sheath. It acts to shield the probe field so that outside of it there is very little electric field and the plasma is undisturbed. This sheath is very thin usually, and it is this fact which makes the probe a true probe in the sense that its effect is not felt outside the sheath. The electron current entering the sheath is that due to random thermal motions. As long as the sheath area remains constant, the current entering it will be constant. Since the sheath area does not change very much as a function of voltage, this condition is satisfied and the current levels off to an almost flat curve, region I of Figure 1. This is electron saturation current.

As soon as φ_g is made less than φ_p, the probe begins to repel electrons and accelerate ions. Chen (reference 1) calls this region the transition region of

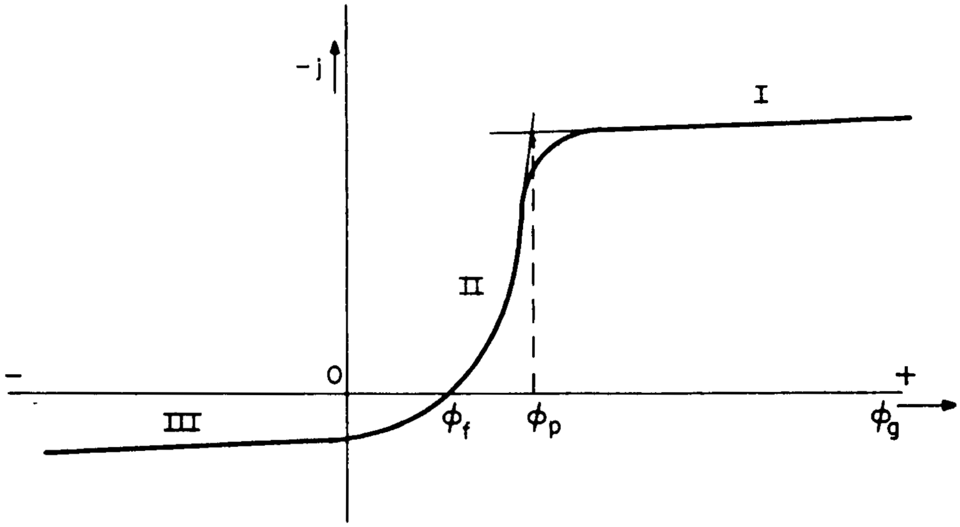

Figure 1. Typical Probe Characteristic

the characteristic. The electron current decreases as φ_g decreases in this region which corresponds to region II of Figure 1. If the electron distribution were Maxwellian, the curve in region II, neglecting the ion current, would be exponential. At the point φ_f, which is called the floating potential, the retarding field is just strong enough to repel all electrons except for a flux equal to the flux of ions, and therefore the net current is zero. If an insulated electrode were inserted into the plasma it would assume this potential.

If φ_g is decreased to a low enough value, we pass into region III in which almost all the electrons are repelled and an ion sheath is formed and a corresponding ion saturation current is collected. At first glance it would seem that this region is very similar to region I, with the exception of the mass difference between the ions and electrons. It would seem that the current collected is the current due to random thermal motions of the ions. The next section will demonstrate that sheath formation is different when the colder species is collected, and the equation for the saturation current (if the ions are colder than the electrons) is not as simple as we would think.

Basic Theory

In the theory that follows in Sections A, B, C, and D, we will assume that $T_e \gg T_i$. In Section E we will discuss the modification to the theory if the reverse is true, i. e. if $T_e << T_i$.

A. <u>Electron Saturation Current ($\varphi_g > \varphi_p$).</u> - The saturation electron current, which is the current density due to random thermal motions entering the sheath, is given by Dickson (reference 2) as (see Appendix A)

$$j_{e_{sat.}} = - en_o \sqrt{\frac{kT_e}{2\pi m_e}} \tag{1}$$

where this result is a standard derivation of kinetic theory. The measured current i_e is given by

$$i_e = j_{e_{sat.}} A \tag{2}$$

where A is the area of the sheath. We will assume that it is equal to the area of the probe, since this is very nearly always true, although for very large potentials the sheath might become thick enough so that the area does change slightly. This current is space-charge limited, and the classical Child-Langmuir law for space-charge limited current flow may be used to relate the sheath thickness to the potential φ, where φ is defined as $\varphi = \varphi_g - \varphi_p$. The expression, as given by Chen is

(see Appendix B)

$$j_{e_{sat.}} = -\left(\frac{4\varepsilon_o}{9d_e^2}\right)\left(\frac{2e}{m_e}\right)^{\frac{1}{2}} \varphi^{\frac{3}{2}} \qquad (3)$$

where d_e is the electron sheath thickness. Since j_e saturation is constant, we note that d_e must vary as $\varphi^{\frac{3}{4}}$, and in fact the expression for d_e is given by French (reference 3) as

$$d_e = \frac{\varphi^{\frac{3}{4}}}{\left(\frac{9}{8}\frac{j_{e_{sat.}}}{\varepsilon_o}\sqrt{\frac{2m_e}{e}}\right)^{\frac{1}{2}}} \qquad (4)$$

Since the sheath thickness is the range over which the major effect of the probe is felt, a rough approximation for the sheath thickness is that it is of the order of magnitude of the classical Debye length h of a plasma, where this is given by French to be

$$h = \sqrt{\frac{k\varepsilon_o T_e}{e^2 n_o}} = 69.0\sqrt{\frac{T_e}{n_o}} \text{ meters} \qquad (5)$$

B. Ion Saturation Current ($\varphi_g << \varphi_p$). - One can write an expression analogous to the Eq. (1) for the current due to the random thermal motions of the ions

$$j_{i_{sat.}} = en_o\sqrt{\frac{kT_i}{2\pi m_i}} \qquad (6)$$

where n_o is the density of charged particles in the plasma, and where, since $n_i = n_e$ in the plasma, we use n_o for both. We can also write the Child-Langmuir law for the space-charge limited ion current density as

$$j_{i_{sat.}} = \left(\frac{4\epsilon_o}{9d_i^2}\right)\left(\frac{2e}{m_i}\right)^{\frac{1}{2}} |\varphi|^{\frac{3}{2}} \qquad (7)$$

where d_i is the thickness of the ion sheath. Again d_i is found to be

$$d_i = \frac{|\varphi|^{\frac{3}{4}}}{\left(\frac{9}{8}\frac{j_i sat}{\epsilon_o}\sqrt{\frac{2m_i}{e}}\right)^{\frac{1}{2}}}, \tag{8}$$

It has been found that if the correct value of n_o were substituted in (6) and we solved for T_i we would obtain values of T_i which are many orders of magnitude larger than the actual value as determined by other means. Thus, Eq. (6) is in error. It has been found by several authors (first by Bohm in 1949) that when the ions are at a lower temperature than the electrons, they must enter the sheath with an energy which depends on the electron temperature. The ions must have an energy greater than or equal to $\frac{1}{2}kT_e$ upon entering the sheath for a stable sheath to exist at all. Thus in general the ion saturation current cannot be used in the form (6) in order to find the number density of ions n_o. An approximate expression for the saturation ion current was found by Bohm, Burhop and Massey (1949) and is given by

$$j_{i_{sat.}} = \frac{e}{2} n_o \left(\frac{kT_e}{m_i}\right)^{\frac{1}{2}} \tag{9}$$

This expression gives an order of magnitude check on the plasma density. In reality, the ion current must depend on the probe voltage and there are some theories for it, but all suffer from the fact that the electric field from the probe accelerates ions from large distances and thus collisions and external electric fields could influence the probe current.

Chen has found that if condition (2) of the basic assumptions is not satisfied, i.e. if $\frac{\lambda}{a} \to 0$, then the saturation currents can still be found by multiplying Eqs. (1) and (9) by a factor $\frac{3\lambda}{4a}$. Thus, (1) becomes

$$j_{e_{sat.}} = -en_o \frac{3\lambda}{4a}\sqrt{\frac{kT_e}{2\pi m_e}} \tag{1a}$$

and (9) becomes

$$j_{i_{sat.}} = \frac{en_o}{2}\frac{3\lambda}{4a}\sqrt{\frac{kT_e}{m_i}} \tag{9a}$$

Chen does not state which mean free path λ he means, but physically it would appear to be the ion-neutral and electron-neutral mean free paths.

C. Transition Region ($\varphi_g \leq \varphi_p$). – This region is denoted by region II on Figure 1. In this region all the ions entering the sheath are collected, and in addition, those electrons having energies in excess of $|\varphi|$ e are collected. For a Maxwellian distribution of electrons the current density is the same regardless of the sheath and probe sizes and even of the shape of the probe, according to Chen. Let us assume such a Maxwellian distribution. Then, if the electron distribution is in thermal equilibrium under the action of the potential gradient we know that the density follows the Boltzmann law

$$n_e = n_o e^{-\frac{e|\varphi|}{kT_e}} \tag{10}$$

and the distribution is still Maxwellian everywhere; only the density is changed by the potential.

The random thermal current density striking the probe is then given by

$$j = j_i + j_e = j_{i_{sat.}} + j_{e_{sat.}} e^{-\frac{e|\varphi|}{kT_e}} \tag{11}$$

Taking the logarithm of j_e we find that

$$\ell n j_e = \ell n j_{e_{sat.}} + \ell n e^{-\frac{e|\varphi|}{kT_e}} = \ell n j_{e_{sat.}} - \frac{e}{kT_e} |\varphi| \tag{12}$$

If we differentiate both sides with respect to $|\varphi|$ we see that

$$\frac{d \ell n j_e}{d|\varphi|} = - \frac{e}{kT_e} = - \frac{11,600}{T_e} \tag{13}$$

Thus, if we plot $\ell n j_e$ against $|\varphi|$ we should obtain a straight line of slope $-\frac{e}{kT_e}$, so that by measuring this slope we are able to determine T_e. If we do indeed obtain this straight line, we can assume that our original assumption of Maxwellian distribution is correct, as Dickson points out. If the plot does not yield a straight line, on the other hand, we cannot say for sure that the distribution is not Maxwellian, since other effects may affect the curve. If the distribution is not Maxwellian, then the preceding theory does not apply, and the probe readings will not yield any information concerning the electron temperature.

In order to determine n_o, we substitute the value of T_e which we have just obtained into the saturation current equation and solve for n_o. This can be Eqs. (1), (9), (1a) or (9a), depending upon which current was measured and upon the appropriate conditions.

D. <u>Floating Potential ($\varphi_g=\varphi_f$)</u> - The floating potential is the potential at which the flux of electrons and ions to the probe is equal, i.e. the current is zero. We can solve for φ at that point.

$$j = j_{i_{sat.}} + j_{e_{sat.}} e^{-\frac{e|\varphi|}{kT_e}} j = \frac{e}{2} n_o \left(\frac{kT_e}{m_i}\right)^{\frac{1}{2}} - e n_o \left(\frac{kT_e}{2\pi m_e}\right)^{\frac{1}{2}} e^{-\frac{e|\varphi|}{kT_e}} = 0 \tag{14}$$

where we use Eqs. (1) and (9) for $j_{e_{sat.}}$ and $j_{i_{sat.}}$ respectively

$$e^{\frac{e|\varphi|}{kTe}} = \sqrt{\frac{2}{\pi} \frac{m_i}{m_e}} \tag{15}$$

Let us call the value of φ at floating potential φ_f'. In terms of φ_f and φ_p as defined in Figure 1, φ_f' is given by

$$\varphi_f' = \varphi_f - \varphi_p \tag{16}$$

and we note that it is negative. Solving (15) for φ_f' we find that it is given by

$$\varphi_f' = -\frac{kT_e}{2e} \ln \frac{2m_i}{\pi m_e} \tag{17}$$

We can rewrite (16) in the form

$$\varphi_p = \varphi_f - \varphi_f' \tag{16a}$$

Thus, if we place an insulated probe into the plasma and we observe that it reaches the potential φ_f with respect to an arbitrary reference point, we can calculate the potential of the plasma with respect to the same reference point by applying (17) in (16a).

Another standard method of determining φ_p is to say that it is at the intersection of the extrapolations of the semi-logarithmic plot of regions I and II.

E. Modification if $T_i >> T_e$ - French has investigated the case where the electrons are the colder species. He found that in this situation the procedure used to find T_e does not change at all. However, the arguments used in regions I and III are now reversed. The electrons are constrained to enter the sheath with an energy greater than or equal to $\frac{1}{2}kT_i$. Thus, the electrons and ions reverse roles. Eq. (1) can be applied to region III by replaceing T_e and m_e by T_i and m_i respectively. Eq. (9) can be applied to region I by replacing T_e with T_i and m_i with m_e.

Let us look at the ratio of electron current to ion current in both temperature regimes. They are

$$\frac{j_{e_{sat.}}}{j_{i_{sat.}}} = \frac{\frac{en_o}{2}\left(\frac{2kT_e}{\pi m_e}\right)^{\frac{1}{2}}}{\frac{en_o}{2}\left(\frac{kT_e}{m_i}\right)^{\frac{1}{2}}} = \left(\frac{2m_i}{\pi m_e}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\sqrt{\frac{m_i}{m_e}} \qquad T_e >> T_i$$

$$\frac{j_{e_{sat.}}}{j_{i_{sat.}}} = \frac{\frac{en_o}{2}\left(\frac{kT_i}{m_e}\right)^{\frac{1}{2}}}{\frac{en_o}{2}\left(\frac{2kT_i}{\pi m_i}\right)^{\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{m_i}{m_e}} \qquad T_i \gg T_e \tag{19}$$

(18)

Thus we see that by observing the ratio of the saturation currents we can determine which species is at the higher temperature, the ions or the electrons. When the ions are the warmer species, the saturation current ratio will be larger than in the cold ion case by a factor $\frac{\pi}{2}$. It has been pointed out in private correspondence with M. Abele that in order for the ions to be considered the higher temperature species, the actual condition on T_i should be that $T_i > \frac{m_i}{m_e} T_e$.

APPLICATION TO MEASUREMENTS IN A FLOWING IONIZED GAS

Very little work has been done in the application of classical Langmuir probe theory in measuring the parameters of an ionized gas flow. French has written a paper on probes in a very low density flowing plasma, where the probe is very small compared to the neutral-neutral mean free paths but can be either in continuum or free-molecule flow with respect to charged-particle collisions. Talbot (reference 4) has attempted to analyze the problem of a probe placed at the stagnation point of a blunt body. It appears that he has mistakenly used classical probe theory where it is not directly applicable, but it seems that his equations can be modified so as to make his method work.

French points out that the essential change in probe theory due to having mass motion is that the energy of the ions will be non-isotropic. Let us define a quantity S, which French calls the speed ratio. He does not define what it represents, but it appears that it is the ratio of the flow velocity to the thermal velocity. Because of the mass difference between the ions and electrons, the electron speed ratio is negligible compared to the ion and atom speed ratios. Thus we expect the usual retarding field method to find T_e to be unaffected by mass motion. This can be checked by testing two identical probes, one parallel to the flow and one transverse. If the probe is parallel to the flow, the probe theory should not change at all. If the probe is transverse and $T_i \gg T_e$, then the part of the characteristic biased to collections is the same as before, but the part biased to collect electrons is uncertain because the reflected ions cause an increased density in front of the probe and because the electron collection depends on an effective ion temperature which can't be defined. If S is the ion speed ratio, $j_{i_{sat.}}$ is given by

$$j_{i_{sat.}} = en_o\sqrt{\frac{kT_i}{2\pi m_i}}\left[e^{-S^2\sin^2\theta} + S\sin\theta\sqrt{\pi}\,(1+\mathrm{erf}\sin\theta)\right] \tag{20}$$

as derived by French, where θ is the angle between the normal to the probe surface and the flow direction. In the case where $T_e \gg T_i$, ion collection is uncertain and the electron collection is the same as before, since S is so small for the electrons. He says that the density increase caused by the reflected ions will integrate out to to zero around the probe.

Since the electrons have a low speed ratio they do not form a shock wave but they follow the ion concentration profile in order to maintain plasma neutrality. The electron temperature is not expected to change very much in going through the shock front. When the probe is biased to collect ions, no shock will form, since the probe is a sink for ions. An area parallel to the flow will cause no shock even when reflecting ions, and the electron current can be used to determine n_o. The electron temperatures, when found for transverse and parallel probes, can be compared to determine to what extent the electron compression through the shock is isothermal. The transverse probe should measure a higher plasma concentration when reflecting ions. The free stream density can be found approximately by using the normal shock relation

$$\frac{n_{o_1}}{n_{o_2}} = \frac{1}{\gamma+1}\left[(\gamma-1) + \frac{2}{m_1^2}\right]$$

French found that charged-particle collisions in the sheath did not influence his current readings to any appreciable extent. He was mainly concerned with placing a small probe in a flow to determine the free stream parameters without disturbing the flow, but the method should work if the probe is placed on a body and the flow over the body is examined, except for the possible modifications needed if the mean free paths are too small, as discussed previously.

In Talbot's stagnation point probe, he attempts to apply classical probe theory. He has a probe buried in the stagnation point of a large blunt body (large compared with the probe dimensions). Talbot doesn't specifically relate the probe diameter to the mean free paths, but it appears that it is much larger, so that the classical theory would have to be used in the modified form [as in (1a) and (9a)] in order to find the ion density. The determination of the electron temperature is the same as before. The sheath is assumed to be much thinner than any of the mean free paths involved, and the electron temperature is assumed to be much higher than the ion temperature. The density and electron temperature are found from the current versus voltage characteristic in the usual way. This gives the values of n_o and T_e at the outer edge of the sheath, a height d above the surface. He then takes the already known solution for stagnation point boundary layer flow and matches it to the sheath solution, since he knows the ion concentration and electron temperature at $y = d$, and these are the boundary conditions. From the standard flow analysis and shock relations he deduces the free stream parameters.

The analysis by Talbot seems to be inconsistent in two respects. In the first place, as previously mentioned, he applies the classical probe equations even though his probe diameter is much larger than the mean free paths. As explained above, this can be corrected by using the modified saturation current relations. Secondly, he insists upon using the saturation ion current to determine the density, but without the correct modifications as derived by Bohm and numerous others. His reasoning is that the ion thermal current should be limited by the free stream ion current and not by the electron temperature. This may be the case, but he applies the equation for ion thermal current involving T_i, which is known to be incorrect. Talbot says that he does not want to use electron current because it might be too much of a drain on the plasma. The large majority of investigators use the electron current measurement and obtain reasonably good results, and it is felt that Talbot's method would work with that modification.

The effect of mass motion does not affect the theory in the case of Talbot's probe because the flow velocities (and thus the speed ratios) of all the species are very low that far down in the boundary layer.

Some papers have been published concerning the use of the stagnation point probe in the case of a high density flow, where the mean free paths are much smaller than the sheath thickness. The most notable is the work done by Chung at Aerospace Corporation, and by Pollin (reference 5) at the Harry Diamond Laboratories. In this case the current flow through the sheath is collision dominated and the classical free-fall relations cannot be applied. We are only concerned in this paper with situations in which the free-fall relations do apply.

SOURCES OF ERROR

Wehner and Medicus (reference 6) have written an excellent paper enumerating several of the possible sources of error in making and interpreting probe measurements. Some of these sources, as enumerated by Loeb, (reference 7) are:

a) If the probe is larger than the sheath thickness it disturbs the plasma, and the modification given is only a rough approximation.

b) The strongly negative probe disturbs the plasma at the sheath edge, the measurement is not of undisturbed plasma.

c) It is assumed that all carriers entering the sheath surface will reach the probe and register as current, and that only such carriers as represent the original plasma outside the sheath will reach the probe and yield probe current. This condition is not fulfilled and the failure is greater, the more negative the probe and the thicker the sheath. The possible causes are

1) collisions of electrons with molecules in the sheath which direct some electrons from the probe

2) reflection of electrons and ions from the probe surface (low accommodation coefficient)

3) current caused by ionization in the sheath due to electron impact and high energy photons

4) emission of electrons by the probe surface due to photons from the sheath and secondary electrons liberated from the metal surface by positive ion and meta-stable atom impact.

5) hot probes emitting thermionically also can cause this.

The report by Wiener and Medicus discusses this problem in great detail, and shows how to recognize and compensate for some of the effects.

d) contact potential differences between reference electrode and probe can falsify some readings.

e) The theory given assumes only electrons and positive ions to be carriers. If there were negative ions it would not hold.

f) Plasma oscillations can throw the probe curve off.

g) A non-maxwellian distribution can make the results invalid.

CONCLUSIONS

It is concluded that classical Langmuir probe theory can be applied to ionized gas flows as long as the restrictions imposed by classical theory are satisfied, i.e. the relations between probe size and sheath thickness to mean free path. It will apply exactly as derived for a static plasma if the probe surface is parallel to the flow direction. If it is not parallel, the modified formula derived by French may be applied (for the case of higher ion temperatures) or else the usual formula for $T_e \ll T_i$. Since the lower temperature species saturation current is uncertain, it is always wise to use the higher temperature species saturation current to determine the density when possible. The method described can be applied to determine which species is hotter. In the case of $T_e = T_i$, no Bohm condition is necessary.

The advantage of electrostatic probes over all other plasma diagnostic techniques is that local measurements can be made at a point. Almost all other methods, such as spectroscopy or microwave propagation, supply information averaged over a large volume of plasma. The plot of current versus voltage may be obtained continuously in a steady state plasma flow, or point by point in a pulsed flow, the probe bias being changed from pulse to pulse, or the entire curve may be obtained in a few microseconds in a pulsed flow by the use of a fastsweeping voltage source. Chen has estimated the time response of the probe to be of the order of $\frac{1}{\omega_{pi}}$ where ω_{pi} is the ion plasma frequency, and this is so low that for almost any conceivable sweep frequency the change in the sheath thickness can be assumed to take place instantaneously.

We expect, according to Loeb, that probes will yield fairly reliable electron temperatures (certainly order of magnitude), and ion and electron densities of perhaps within half an order of magnitude.

APPENDIX A. DERIVATION OF THERMAL CURRENT DENSITY

Let us derive the expression for the current density entering the sheath due to the random thermal motions of the charged particles. We need only look at the electron current, since the ion current is exactly analogous.

If the sheath surface is taken to lie in the y-z plane, then the current flows in the +x direction. This current is given by $-en_o\bar{v}_x$, where $-en_o$ is the electron charge density, and $\bar{v}_x$ is the average of the x-component of the random thermal velocity. Elementary kinetic theory says that $\bar{v}_x$ is given by integrating v_x with the distribution function over the range of x velocities from 0 to ∞. We assume a Maxwellian distribution, and the distribution function f is thus given by

$$f = \left(\frac{m_e}{2\pi kT_e}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT_e}} \tag{A-1}$$

where v^2 is given by

$$v^2 = v_x^2 + v_y^2 + v_z^2 \tag{A-2}$$

The current density j_e is then given by

$$j_{e_{sat.}} = -en_o \int_{v_x=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x f dv_x dv_y dv_z$$

$$= -en_o\left(\frac{m_e}{2\pi kT_e}\right)^{\frac{3}{2}} \int_0^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} v_x e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT_e}} dv_z \tag{A-3}$$

These integrals are in standard forms that can be found in any tables of definite integrals, and when evaluated yield the result that

$$j_{e_{sat.}} = -en_o\sqrt{\frac{kT_e}{2\pi m_e}} \tag{A-4}$$

APPENDIX B DERIVATION OF CHILD-LANGMUIR LAW

The following derivation of the Child-Langmuir law for space-charge limited current is as given by Chen: Suppose we have two infinite plane-parallel plates, one of which emits particles and is at zero potential, and the other of which is perfectly absorbing and is at a potential φ, as shown in Figure B-1.

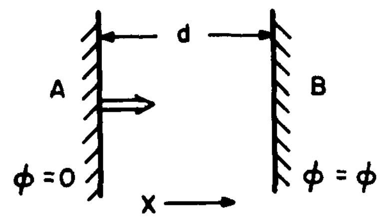

Figure B-1

Then, if a species of particle with charge -e and mass m_e is emitted at zero velocity at plane A, its velocity at a position where the potential is φ is given by

$$v_x = \left(\frac{2e\varphi}{m_e}\right)^{\frac{1}{2}} \qquad \text{(B-1)}$$

where the assumption is made that the particle falls freely through the potential drop φ, i.e., suffers no collisions or other hinderances of a mechanical character in its fall. If the emitted current density is j_e, the particle density at x will be

$$n(x) = \frac{j_{e_{sat.}}}{-ev_x(x)} = \frac{j_{e_{sat}}}{-e}\left[\frac{2e\varphi(x)}{m_e}\right]^{-\frac{1}{2}} \qquad \text{(B-2)}$$

Poisson's equation becomes

$$\frac{d^2\varphi}{dx^2} = \frac{j_{e_{sat}}}{\epsilon_o}\left(\frac{2e\varphi}{m_e}\right)^{-\frac{1}{2}} \qquad \text{(B-3)}$$

Multiplying by $\frac{d\varphi}{dx}$ and integrating from $x=0$ we have

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 = \frac{j_{e_{sat.}}}{\epsilon_o} \int_o^{\varphi} \left(\frac{2e\varphi}{m_e}\right)^{-\frac{1}{2}} d\varphi = \frac{j_{e_{sat.}}}{\epsilon_o} \left(\frac{2m_e}{e}\right)^{\frac{1}{2}} \varphi^{\frac{1}{2}} + \left(\frac{d\varphi}{dx}\right)_o \tag{B-4}$$

In the case of space-charge limited current flow $\left(\frac{d\varphi}{dx}\right)_0$ vanishes. Then we have

$$\varphi^{-\frac{1}{4}} \, d\varphi = \left(\frac{2j_{e_{sat.}}}{\epsilon_o}\right)^{\frac{1}{2}} \left(\frac{2m_e}{e}\right)^{\frac{1}{4}} dx \tag{B-5}$$

Integrating from $x=0$ to $x=d$ we have

$$\frac{4}{3}\varphi^{\frac{3}{4}} = \left(\frac{2j_{e_{sat.}}}{\epsilon_o}\right)^{\frac{1}{2}} \left(\frac{2m_e}{e}\right)^{\frac{1}{4}} d \tag{B-6}$$

or j_e is given by

$$j_{e_{sat.}} = \left(\frac{4\epsilon_o}{9d^2}\right)\left(\frac{2e}{m_e}\right)^{\frac{1}{2}} \varphi^{\frac{3}{2}} \tag{B-7}$$

which is the Child-Langmuir 3/2 power law for space-charge limited current flow between two planes separated by a distance d with a potential φ between them. Obviously plane A corresponds to the plasma and plane B corresponds to the probe.

Physically what is happening is as follows: Particles are emitted at plane A and are accelerated toward plane B. Thus a stream of particles flows from A to B. Since all the particles are of the same sign, they repel each other. The stream of particles repels other particles trying to flow to the plane B, and a cloud of charged particles develops near plane A. This cloud is the space charge and it is what limits the current.

APPENDIX C DERIVATION OF THE BOHM CONDITION

Let us examine a gas of electrons. Suppose it is being acted upon by an electric field $\underline{E}$ and a pressure gradient ∇P_e. In order for the electrons to reach an equilibrium configuration, the force on a volume of electron gas must be zero. If n_e is the density of the electron gas at a point where the potential is φ, and n_o is the density where $\varphi=\varphi_o$, the force balance equation is

$$-n_e e\underline{E} - \nabla P_e = 0, \tag{C-1}$$

but we know that

$$\underline{E} = -\nabla\varphi \quad \text{and} \quad P_e = n_e kT_e \tag{C-2}$$

so that (C-1) becomes

$$n_e e\nabla\varphi - kT_e \nabla n_e = 0 \tag{C-3}$$

$$\nabla\varphi = \frac{kT_e}{e}\,\frac{\nabla n_e}{n_e} \tag{C-4}$$

If we integrate (C-4) for the one-dimensional problem we obtain

$$\varphi = \frac{kT_e}{e}\,\ell n\, n_e + c_1 , \tag{C-5}$$

but since $n_e = n_o$ at $\varphi = \varphi_o$ this becomes

$$\varphi - \varphi_o = \frac{kT_e}{e}\,\ell n\,\frac{n_e}{n_o} \tag{C-6}$$

and thus n_e is given by

$$n_e = n_o e^{\frac{e(\varphi - \varphi_o)}{kT_e}} \tag{C-7}$$

Now let us consider the ion current. Let n_o be the ion density at the sheath edge and let v_{io} be the velocity of the ions at that point. Then the current density of ions entering the sheath is given by $en_o v_{io}$. Since the ion current is continuous through the sheath, $n_o v_{io}$ must be equal to $n_i v_i$ where n_i is the density at the point in the sheath where the ion velocity is v_i.

$$n_i v_i = n_o v_{io} \tag{C-8}$$

The kinetic energy of the ions at the sheath edge is given by

$$\frac{1}{2} m_i v_{io}^2 = e\varphi_o \tag{C-9}$$

where φ_o is a constant chosen so that it is equal to $\frac{m_i v_{io}^2}{2e}$, and the kinetic energy at any point in the sheath is given by

$$\frac{1}{2} m_i v_i^2 = q\varphi \tag{C-10}$$

Thus v_{io} and v_i are given by

$$v_{io} = \sqrt{\frac{2\varphi_o}{m_i}}, \quad v_i = \sqrt{\frac{2\varphi}{m_i}} \tag{C-11}$$

and n_i is given by

$$n_i = n_o \frac{v_{io}}{v_i} = n_o \sqrt{\frac{\varphi_o}{\varphi}} \tag{C-12}$$

Poisson's equation is then given by

$$\frac{\partial^2 (\varphi - \varphi_o)}{\partial x^2} = - \frac{e}{\epsilon_o} (n_i - n_e) \tag{C-13}$$

If we assume that $\frac{q(\varphi - \varphi_o)}{kT_e} << 1$, n_e is given by

$$n_e = n_o e^{\frac{e(\varphi - \varphi_o)}{kT_e}} \approx n_o \left(1 + \frac{e(\varphi - \varphi_o)}{kT_e}\right) \qquad \text{(C-14)}$$

Let us examine $\varphi^{-\frac{1}{2}}$.

$$\varphi^{-\frac{1}{2}} = \frac{1}{\sqrt{\varphi}} = \frac{1}{\sqrt{\frac{e\varphi}{ekT_e} kT_e}} = \frac{1}{\sqrt{\frac{kT_e}{e}}\sqrt{\frac{e\varphi}{kT_e}}} = \frac{1}{\sqrt{\frac{kT_e}{e}}\sqrt{\frac{e\varphi}{kT_e} - \frac{e\varphi_o}{kT_e} + \frac{e\varphi_o}{kT_e}}}$$

$$= \frac{1}{\sqrt{\frac{kT_e}{e}}\sqrt{\frac{e(\varphi - \varphi_o)}{kT_e} + \frac{e\varphi_o}{kT_e}}} = \frac{1}{\sqrt{\frac{kT_e}{e}}\sqrt{\frac{\frac{e(\varphi-\varphi_o)}{kT_e}}{\frac{e\varphi_o}{kT_e}}+1}\sqrt{\frac{e\varphi_o}{kT_e}}} = \frac{1}{\sqrt{\varphi_o}}\left[\frac{\frac{e(\varphi-\varphi_o)}{kT_e}}{\frac{e\varphi_o}{kT_e}} + 1\right]^{-\frac{1}{2}}$$

$$\varphi^{-\frac{1}{2}} \approx \frac{1}{\sqrt{\varphi_o}}\left[1 - \frac{\frac{e(\varphi-\varphi_o)}{2kT_e}}{\frac{e\varphi_o}{kT_e}}\right] = \frac{1}{\sqrt{\varphi_o}}\left[1 - \frac{(\varphi - \varphi_o)}{2\varphi_o}\right] \qquad \text{(C-15)}$$

Substituting into Poisson's equation we obtain

$$\frac{\partial^2(\varphi - \varphi_o)}{\partial x^2} = -\frac{e}{\epsilon_o}\left[n_o\sqrt{\varphi_o}\ \frac{1}{\sqrt{\varphi_o}}\left[1 - \frac{(\varphi-\varphi_o)}{2\varphi_o}\right] - n_o\left[1 + \frac{e(\varphi-\varphi_o)}{kT_e}\right]\right] = \frac{en_o}{\epsilon_o}\left[\frac{1}{2\varphi_o} + \frac{e}{kT_e}\right](\varphi-\varphi_o)$$

(C-16)

This can be rewritten as

$$\frac{\partial^2(\varphi - \varphi_o)}{\partial x^2} - \frac{en_o}{\epsilon_o}\left[\frac{1}{2\varphi_o} + \frac{e}{kT_e}\right](\varphi - \varphi_o) = 0 \qquad \text{(C-17)}$$

If $\left[\frac{1}{2\varphi_o} + \frac{e}{kT_e}\right] < 0$, the solution of (C-17) for $(\varphi - \varphi_o)$ will oscillate, and this would mean that the sheath is unstable. Thus, we have a criterion for a stable sheath, namely

$$\frac{1}{2\varphi_o} + \frac{e}{kT_e} \geq 0 \text{ for stability} \tag{C-18}$$

This can be written as

$$|\varphi_o| \geq \left| \frac{kT_e}{2e} \right| \tag{C-19}$$

for a stable sheath. Since $e\varphi_o = \frac{m_i v_{io}^2}{2}$, this is the same as

$$\frac{m_i v_{io}^2}{2} \geq \frac{kT_e}{2} \tag{C-20}$$

or the kinetic energy of the ions at the sheath edge must be greater than $\frac{1}{2}kT_e$ for a stable sheath to exist.

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