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ASSESSMENT OF SLOSH COUPLING WITH SPACE VEHICLE

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FOREWORD

This report was prepared by the Convair Division of General Dynamics for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of the Resources Management Office, Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center with Mr. Robert S. Ryan acting as Project Manager.

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ABSTRACT

Results of an analytical study to determine the role of the action of the liquid in attitude stability equations of liquid propellant space vehicles are presented.

The planar perturbation equations of motion for a liquid propellant launch vehicle have been formulated.

The vehicle motion was treated as a summation of perturbations from a known reference motion and motion in which vehicle body axes remain coincident with reference axes. The effects of the liquid in the perturbation equations were isolated and identified.

On replacing the liquid motion by a simple mechanical system, the planar perturbation equations of motion for the vehicle were again derived.

The role of the liquid motions and mechanical motions were then compared; it was shown that the mechanical system did exactly duplicate the action of the liquid. The analysis was then extended to tanks of arbitrary shape having rotational symmetry. Again, the mechanical system was shown to duplicate the action of the liquid. However, the equation expressing the constancy of pressure at the free surface was found to contain an additional term proportional to the angular displacement of the body axes with respect to the reference axes. This anomaly was ascribed to the fact that in many previous analyses, the equations were not perturbed but arbitrarily linearized with respect to body rates.

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1/ INTRODUCTION

INTRODUCTION

Recent Saturn I and Atlas/Centaur flight data have disclosed certain liquid propellant responses which seem to be at variance with predicted theoretical analyses. Informal technical discussions with MSFC and GD/C personnel indicate that these anomalies do not result from non-linearities of the propellant motions and/or the elasticity of the tank walls which are normally neglected in conventional treatments.

The usual method of obtaining the forces exerted on a launch vehicle by its oscillating propellants is to compute the liquid reaction forces under the assumptions that the motion of the propellant tanks do not depend on the dynamic state of the vehicle, and then to consider that these reaction forces may be treated as generalized forces acting on the vehicle in an arbitrary state of motion. Such a procedure has been questioned by various analysts.

Moreover, a perusal of past efforts discloses a lack of agreement as to the exact analytic statement of the problem. In many instances, the analysis of the liquid motion is rather confusing and assumptions used are difficult to justify. To obtain resulting ordinary linear differential equations with constant coefficients, products of vehicle body rates, both linear and angular, and liquid motion are discarded. There are many other inconsistencies too lengthy to enumerate.

In this study we shall re-examine the manner in which the action of the liquid is represented in the attitude stability equations of liquid propellant space vehicles, the object being one of uncovering errors which might give rise to apparent discrepancies between theory and flight data.

To this end, we choose to conduct the investigation in the following way.

(1) Review briefly, and mostly without proofs, certain fundamental results from theoretical hydrodynamics necessary to describe the motion of a heavy liquid enclosed in a rigid vessel which is itself in motion.

Attitude control studies of space vehicles necessitate the use of reference systems whose origins remain in coincidence with the origin of vehicle body axes or tank-fixed axes, and are thus axes moving with respect to an inertial reference. It is convenient to express all quantities, both absolute and relative, in terms of the moving coordinates. Therefore, the pertinent equations, boundary conditions, etc., will be written exclusively in terms of moving coordinates.

An energy formulation of the system (vessel plus liquid) will be written for six degrees of freedom.

(2) These concepts will then be extended to the case of the planar motion of a liquid propellant vehicle having a single tank and engine. For simplicity the tank will be taken to be a prismatic cylinder. The results will apply trivially to more than one tank and engine.

(3) The planar equations of motion will be used to obtain the perturbation equations of motion. To this end, we shall treat the vehicle motion as a summation of perturbations from a known reference motion and motion in which vehicle body axes remain coincident with reference axes. The role of the liquid motions in the perturbation equations will then be isolated and identified.

(4) The liquid in the propellant tank will be replaced by a simple mechanical system (system of pendula plus discrete mass and moment of inertia), and planar perturbation equations for the entire vehicle will be derived. The effects of the mechanical system motions in the perturbation equations will be isolated and identified.

(5) The role of the liquid motions and mechanical system motions will be compared to see if the mechanical system can duplicate the action of the liquid.

(6) The analysis will be extended to tanks of arbitrary shape having rotational symmetry, such as commonly occurs in most vehicles. The role of the liquid motions and mechanical system motions will again be compared.

2/ FUNDAMENTAL RESULTS FROM THEORETICAL HYDRODYNAMICS

BASIC EQUATIONS FOR MOTION OF A HEAVY LIQUID ENCLOSED IN A PARTLY FILLED VESSEL WHICH IS ITSELF IN MOTION

Consider the motion of a frictionless liquid enclosed in a partly filled vessel which is itself in motion, Fig. 1. To describe the motion of the system, take a cartesian frame of reference fixed relatively to the container, say an origin o and three axes ox, oy, oz. Reference oxyz is orientated in such a manner that oz is measured positively along the outward directed normal to the undisturbed free surface. Thus the free surface, denoted by S(t), coincides with plane xoy (the plane $z = o$) when the container and liquid are at rest.

Let

$$z = \zeta (x, y, t) \tag{2.1}$$

be the equation of S(t) when it is displaced. Denote by $\Sigma(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by S(t) and $\Sigma(t)$. Let Σ, τ, and S represent the corresponding values of $\Sigma(t)$, $\tau(t)$ and S(t) in the undisturbed position. All surfaces are assumed to be piece wise smooth.

Suppose that at time t the vessel is coincident with inertial space and that it is moving relatively to inertial space with motion described by an observer in inertial space as a velocity $\bar{u}$ of o and an angular velocity $\bar{\omega}$. Then the position vector $\bar{r}$ of a particular liquid particle $P \epsilon \tau(t)$ at time t is the same for an observer moving with the vessel as it is for an observer in inertial space.

The point P, if rigidly attached to the moving frame of reference oxyz, has the velocity

$$\bar{V} = \bar{u} + \bar{\omega} \times \bar{r} . \tag{2.2}$$

Thus, if P is fixed in inertial space instead of in oxyz, it will appear to an observer in oxyz to move with velocity $-\bar{V}$.

Denote by $\bar{q}(P, t)$, $\bar{v}(P, t)$ the velocities of the liquid particle at point $P(x, y, z) \epsilon \tau(t)$ at time t as estimated by observers in inertial space and

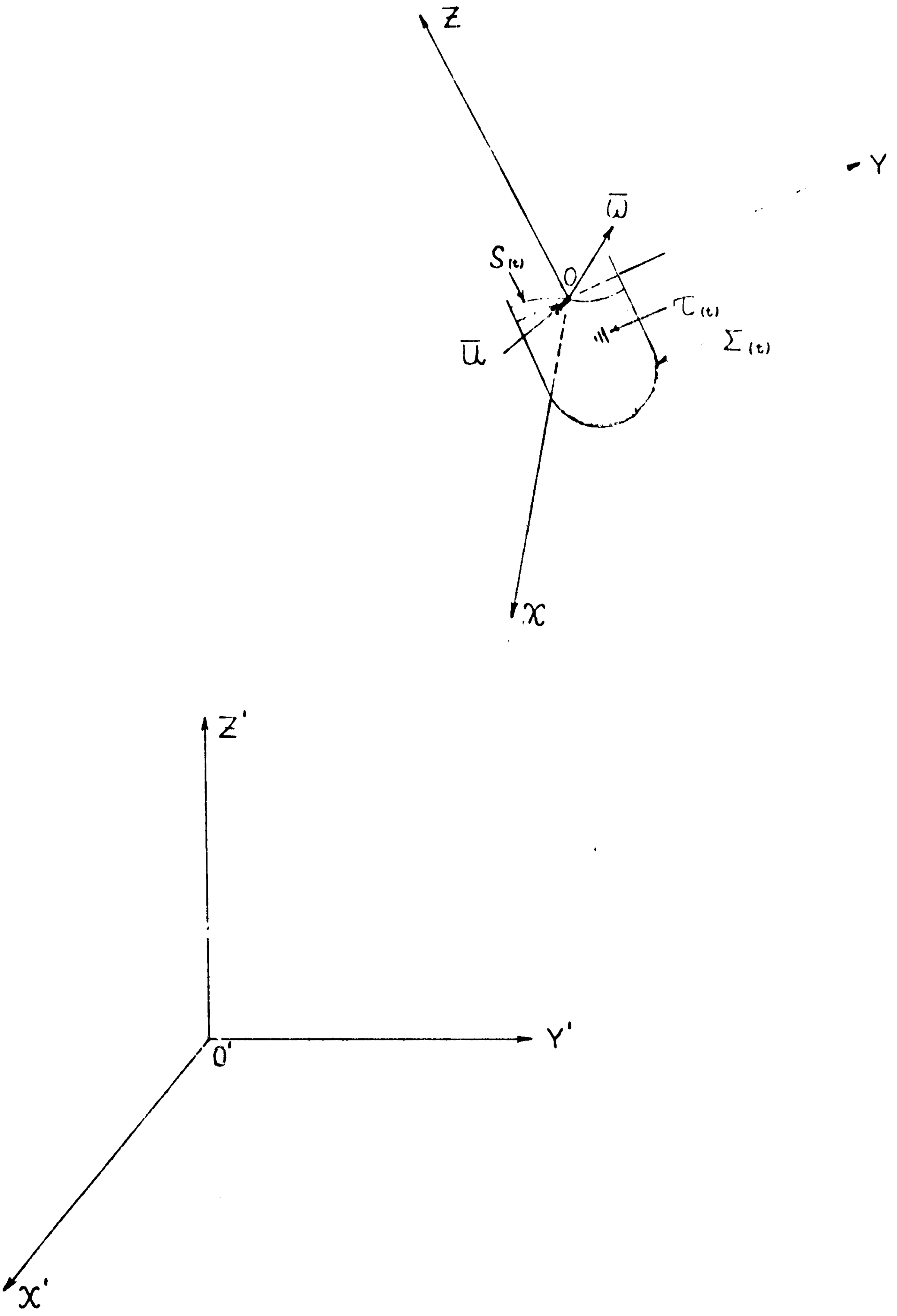

Figure 1. Partially filled vessel in motion.

oxyz respectively. Then

$$\bar{q} = \bar{V} + \bar{v}, \ \bar{v} = \frac{d\bar{r}}{dt} , \tag{2.3}$$

in which the position vector $\bar{r}$ is referred to the moving reference oxyz.

Assume the liquid to be homogeneous and incompressible throughout the motion. Moreover, neglect interfacial tension forces and capillary contact effects between liquid and boundary. Then, the motion of the liquid in $\tau(t)$, when referred to the moving frame of reference oxyz, is completely described by the following formulae:

Equation of motion

$$\frac{d\bar{q}}{dt} + \bar{\omega} \times \bar{q} = \frac{d\bar{v}}{dt} + 2\bar{\omega} \times \bar{v} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \bar{a} = \bar{f} - \frac{1}{\rho}\nabla p , \ P \epsilon \tau(t) ,$$

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + [(\bar{q} - \bar{V}) \nabla] \bar{q}, \ \frac{dv}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \nabla) \bar{v}, \ \bar{a} = \frac{d\bar{u}}{dt} + \bar{\omega} \times \bar{u}, \ P \epsilon \tau(t) \tag{2.4}$$

Equation of continuity

$$\nabla \bar{q} = \nabla \bar{v} = 0 , \ P \epsilon \tau(t) \tag{2.5}$$

Boundary conditions (kinematical)

$$q_n - V_n = v_n = 0 , \ P \epsilon \Sigma(t) \tag{2.6}$$

$$q_n - V_n = \zeta_t \cos (n, z), \ P \epsilon S(t)$$

Boundary conditions (physical)

$$p \perp \Sigma(t), \tag{2.7}$$

$$p(x, y, \zeta, t) = \text{const.}, \ P \epsilon S(t)$$

Forces and moments

$$-\bar{F}_1 = \rho \iiint_{\tau(t)} \frac{\partial \bar{q}}{\partial t} d\tau + \rho \iiint_{\tau(t)} (\bar{\omega} \times \bar{q}) d\tau - \rho \iiint_{\tau(t)} \bar{f} d\tau + \rho \iint_{S(t)} \bar{q}\zeta_t \cos(n, z) ds \tag{2.8}$$

$$= \rho \iiint_{\tau(t)} \frac{\partial \bar{v}}{\partial t}\, d\tau + 2\rho \iiint_{\tau(t)} (\bar{\omega} \times \bar{v}) d\tau + \rho \iiint_{\tau(t)} (\dot{\bar{\omega}} \times \bar{r}) d\tau + \rho \iiint_{\tau(t)} [\bar{\omega} \times (\bar{\omega} \times \bar{r})] d\tau$$

$$- \rho \iiint_{\tau(t)} (\bar{f} - \bar{a}) d\tau + \rho \iint_{(t)} \bar{v}\, \zeta_t \cos(n, z) ds,$$

$$-L_1 = \rho \iiint_{\tau(t)} (\bar{r} \times \frac{\partial \bar{q}}{\partial t}) d\tau + \rho \iiint_{\tau(t)} [\bar{\omega}(\bar{r} \cdot \bar{q}) - \bar{q}(\bar{\omega} \bar{r})] d\tau - \rho \iiint_{\tau(t)} (\bar{r} \times \bar{f}) d\tau$$

$$+ \rho \iint_{S(t)} (\bar{r} \times \bar{q}) \zeta_t \cos(n, z) ds$$

$$= \rho \iiint_{\tau(t)} (\bar{r} \times \frac{\partial \bar{v}}{\partial t}) d\tau + 2\rho \iiint_{\tau(t)} [\bar{\omega}(\bar{r}\bar{v}) - \bar{v}(\bar{\omega}\bar{r})] d\tau + \rho \iiint_{\tau(t)} (\bar{\omega}\bar{r})(\bar{r} \times \bar{\omega}) d\tau$$

$$+ \rho \iiint_{\tau(t)} [\dot{\bar{\omega}}(\bar{r}\bar{r}) - \bar{r}(\dot{\bar{\omega}}\bar{r})] d\tau - \rho \iiint_{\tau(t)} [\bar{r} \times (\bar{f} - \bar{a})] d\tau$$

$$+ \rho \iint_{S(t)} (\bar{r} \times \bar{v}) \zeta_t \cos(n, z) ds$$

in which $\bar{f}$ is the vector of body forces (such as gravity) per unit mass; ρ is the mass density; p is the pressure intensity at point P(x, y, z) (independent of direction); $\bar{a}$ is the absolute acceleration of 0 as measured by an observer in inertial space.

Formula (2.4) is the second law of motion applied to a liquid particle of infinitesimal volume, and can also be obtained immediately from Euler's equation on application of the classical expression for rates of chance of a vector viewed from inertial and moving space.[1] Note that the equation of motion is expressed in either of two forms; one of which is in terms of the absolute velocity $\bar{q}$, and the other in terms of the relative velocity $\bar{v}$.

Formula (2.5) states that the net flow rate of liquid into any small volume must be zero. It likewise is expressed in either of two forms.

Formula (2.6) is the kinematic condition which must be satisfied at the solid boundary $\Sigma(t)$, namely, that the component of velocity normal to the boundary must be equal to the velocity component of the boundary normal to itself. Note that the component of relative velocity normal to $\Sigma(t)$ is zero, $v_n = 0$;

[1]See notes.

the component of the absolute velocity normal to $\Sigma(t)$ is equal to V_n, the velocity of the boundary normal to itself. This follows from the relationship (2.3).

Formula (2.6₂) is the kinematic condition which must be satisfied at the free surface, Lord Kelvin's condition, and is a corollary of (2.5).[2] $\zeta_t \left(= \frac{\partial \zeta}{\partial t}\right)$ is the apparent velocity of movement of the free surface S(t) in the direction oz; i.e., ζ_t is the velocity of movement along the straight line x = const., y = const., of the point of intersection of the surface $z = \zeta$ with this straight line.

Formula (2.7₁) states that for a frictionless liquid in contact with a rigid boundary, the liquid thrust shall be normal to the boundary.

Formula (2.7₂) expresses the constancy of pressure at the free surface S(t).

Formula (2.8₁), expressed in either of two forms, is the computation of the force resulting from the action of the liquid motion on the vessel surface $\Sigma(t)$. It is obtained from an integration of formula (2.4) throughout $\tau(t)$ and application of the divergence theorem and formulae (2.5, 6, 7).

Formula (2.8₂) is the computation for the moments about the origin of the forces exerted by the liquid on the vessel surface $\Sigma(t)$.

There is an important corollary to formula (2.4) concerning the vorticity of liquid elements. Let the extraneous body forces be conservative,

$$\bar{f} = -\nabla\Omega \tag{2.9}$$

then, on introducing formula (2.9) into (2.4) and operating with $\nabla \times$ on the resulting equation of motion, we get

$$\frac{d}{dt}(\nabla \times \bar{q}) + \bar{\omega} \times (\nabla \times \bar{q}) = [(\nabla \times \bar{q})\nabla]\bar{q}\ , \tag{2.10}$$

$$\frac{d}{dt}(\nabla \times \bar{q}) = \frac{\partial}{\partial t}(\nabla \times \bar{q}) + [(\bar{q} - \bar{V})\nabla](\nabla \times \bar{q})\ ,$$

and (2.11)

$$\frac{d}{dt}(\nabla \times \bar{v}) + \bar{\omega} \times (\nabla \times \bar{v}) + 2\dot{\bar{\omega}} = 2\nabla(\bar{\omega}\,\bar{v}) - \bar{\omega} \times (\nabla \times \bar{v}) + [(\nabla \times \bar{v})\nabla]\bar{v}\ ,$$

---

[2]See notes.

$$\frac{d}{dt}(\nabla \times \bar{v}) = \frac{\partial}{\partial t}(\nabla \times \bar{v}) + (\bar{v}\nabla)(\nabla \times \bar{v}) ,$$

in which $(\nabla \times \bar{q})$ is the absolute vorticity of liquid elements and $(\nabla \times \bar{v})$ the relative vorticity of liquid elements, that is, the vorticity as measured by an observer in inertial and moving space respectively. $(\nabla \times \bar{q})$ and $(\nabla \times \bar{v})$ are not independent. Indeed, operating on (2.3) with $\nabla \times$ we see

$$(\nabla \times \bar{q}) = 2\bar{\omega} + (\nabla \times \bar{v}) \tag{2.12}$$

Consider a liquid particle which has no vorticity $(\nabla \times \bar{q}) = 0$; from (2.10), it follows that

$$\frac{d(\nabla \times \bar{q})}{dt} = 0 ,$$

and therefore the particle never acquires vorticity. This implies that if

$$(\nabla \times \bar{q}) = 0 \tag{2.13}$$

At some time $t = t_o$, then it is zero for all time, and

$$(\nabla \times \bar{v}) = -2\bar{\omega} \tag{2.14}$$

using (2.12).

There is a significant corollary to formula (2.5) concerning the flux of the liquid. From the divergence theorem

$$\iint_{\Sigma(t) + S(t)} v_n \, ds = \iiint_{\tau(t)} \nabla \bar{v} \, d\tau ,$$

which, in light of formulae (2.5, 6), gives

$$\iint_{S(t)} \zeta_t \cos(n, z) \, ds = 0 . \tag{2.15}$$

It is known that the most general solution of formula (2.5) can be expressed in either of the two forms [1]

$$\bar{q} = \bar{q}_o + \bar{q}_1 , \; \bar{q}_o = \nabla \phi , \; \nabla \bar{q}_1 = 0 , \tag{2.16}$$

$$\bar{v} = \bar{v}_o + \bar{v}_1 , \; \bar{v}_o = \nabla \psi , \; \nabla \bar{v}_1 = 0 .$$

Vectors $\bar{q}_0$ and $\bar{v}_0$ are the irrotational components of the absolute and relative velocity vectors respectively. $\bar{q}_1$, $\bar{v}_1$ are the corresponding vortical components.

Substituting (2.16) into formula (2.5), we obtain

$$\nabla \phi = \nabla \psi = 0 \, , \tag{2.17}$$

which states that the irrotational components of the absolute and relative velocity vectors must satisfy Laplace's equation.

When there is no vorticity

$$\bar{q}_1 = 0 \, , \tag{2.18}$$

$$(\nabla \times \bar{v}_1) = -2 \, \bar{\omega} \, ,$$

formulae (2, 10, 11) are satisfied identically and the absolute velocity is completely determined to within an arbitrary additive function of time by the velocity potential ϕ.

Formula (2.18_2) is satisfied identically if we take

$$\bar{v}_1 = -\bar{\omega} \times \bar{r} \tag{2.19}$$

Thus the relative velocity becomes

$$\bar{v} = \nabla \psi - \bar{\omega} \times \bar{r} \tag{2.20}$$

Introducing formulae (2.16_1) with $\bar{q}_1 = 0$ and (2.20) into (2.4) and integrating the resulting equation of motion, we get

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \Omega + \frac{1}{2} v^2 - \frac{1}{2} V^2 = D(t) \tag{2.21}$$

$$\frac{p}{\rho} + \frac{\partial \psi}{\partial t} + \Omega + \bar{a}\bar{r} + \frac{1}{2} v^2 - \frac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t)$$

where

$$v^2 = \nabla \psi - |\bar{\omega} \times \bar{r}|^2 = |\nabla \phi - \bar{V}|^2 \, , \; V^2 = |\bar{V}|^2 \, , \; \bar{f} = -\nabla \Omega \, ,$$

and D(t), C(t) are instantaneous constants, that is, functions of t only. Therefore at a given instant the constants have the same value throughout the liquid. Expression (2.21) is the unsteady pressure equation expressed in either of two forms.

Substituting formulae (2.16_1) with $\bar{q}_1 = 0$ and (2.20) into the boundary conditions (2.6), we obtain

$$\frac{\partial \phi}{\partial n} = \bar{u}\bar{n} + \omega_x [y \cos(n, z) - z \cos(n, y)] + \omega_y [z \cos(n, x) - x \cos(n, z)]$$
$$+ \omega_z [x \cos(n, y) - y \cos(n, x)], \quad P \in \Sigma(t), \tag{2.22}$$

$$\frac{\partial \phi}{\partial n} = \bar{u}\bar{n} + \omega_x [y \cos(n, z) - z \cos(n, y)] + \omega_y [z \cos(n, x) - x \cos(n, z)]$$
$$+ \omega_z [x \cos(n, y) - y \cos(n, x)] + \zeta_t \cos(n, z), \quad P \in S(t),$$

$$\frac{\partial \psi}{\partial n} = \omega_x [y \cos(n, z) - z \cos(n, y)] + \omega_y [z \cos(n, x) - x \cos(n, z)]$$
$$+ \omega_z [x \cos(n, y) - y \cos(n, x)], \quad P \in \Sigma(t),$$

$$\frac{\partial \psi}{\partial n} = \omega_x [y \cos(n, z) - z \cos(n, y)] + \omega_y [z \cos(n, x) - x \cos(n, z)]$$
$$+ \omega_z [x \cos(n, y) - y \cos(n, x)] + \zeta_t \cos(n, z), \quad P \in S(t),$$

where cos (n, x), ... denote the cosines of the angles formed by the outward directed normal to the surface of contact at the point P.

Introduce a new potential φ such that

$$\phi = \bar{u}\bar{r} - \omega_x yz - \omega_z xy - \omega_y xz + \int_0^t \tfrac{1}{2} u^2 \, dt + \varphi, \tag{2.23}$$

$$\psi = -\omega_x yz - \omega_y xz - \omega_z xy + \varphi,$$

$$\phi = \psi + \bar{u}\bar{r} + \int_0^t \tfrac{1}{2} u^2 \, dt.$$

then it follows from (2.17, 21, 22) that

$$\Delta \varphi = 0, \quad P \in \tau(t), \tag{2.24}$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \dot{\omega}_x yz - \dot{\omega}_y xz - \dot{\omega}_z xy + \bar{a}\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t), \quad P \in \tau(t), \tag{2.25}$$

$$\frac{\partial \varphi}{\partial n} = 2\left[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)\right], \quad P \epsilon \Sigma(t) \qquad (2.26)$$

$$2\left[\omega_x y \cos(n, z) + \omega_y z \cos(n, x) + \omega_z x \cos(n, y)\right] + \zeta_t \cos(n, z),$$

$$P \epsilon \; S(t) \; .$$

Since the pressure must be independent of position at the free surface we require

$$\frac{\partial \varphi}{\partial t} - \dot{\omega}_x \, yz - \dot{\omega}_y \, xz - \dot{\omega}_z \, xy + \bar{a}\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = 0, \quad P \epsilon \; S(t), \; (z = \zeta), \qquad (2.27)$$

which is the condition (2.7a). The forces and moments (2.8) become

$$-F_{1_x} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial x} \, d\tau - \rho \iiint_{\tau(t)} f_x \, d\tau + \rho \, a_x \iiint_{\tau(t)} d\tau \qquad (2.28)$$

$$-\rho(\omega_y^2 + \omega_z^2) \iiint_{\tau(t)} x \, d\tau + \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y \, d\tau + \rho(\omega_z \omega_x - \dot{\omega}_y) \iiint_{\tau(t)} z \, d\tau$$

$$-F_{1_y} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y} \, d\tau - \rho \iiint_{\tau(t)} f_y \, d\tau + \rho \, a_y \iiint_{\tau(t)} d\tau$$

$$+ \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x \, d\tau - \rho(\omega_x^2 + \omega_z^2) \iiint_{\tau(t)} y \, d\tau + \rho(\omega_z \omega_y - \dot{\omega}_x) \iiint_{\tau(t)} z \, d\tau$$

$$-F_{1_z} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} \, d\tau - \rho \iiint_{\tau(t)} f_z \, d\tau + \rho \, a_z \iiint_{\tau(t)} d\tau$$

$$+ \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} x \, d\tau + \rho(\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y \, d\tau - \rho(\omega_x^2 + \omega_y^2) \iiint_{\tau(t)} z \, d\tau$$

$$-L_{1_x} = \rho \iiint_{\tau(t)} \left[y \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) - z \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial t}\right)\right] d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \left(y \frac{\partial v^2}{\partial z} - z \frac{\partial v^2}{\partial y}\right) d\tau$$

$$- \rho \iiint_{\tau(t)} (y f_z - z f_y) \, d\tau + \rho \, a_z \iiint_{\tau(t)} y \, d\tau - \rho \, a_y \iiint_{\tau(t)} z \, d\tau$$

$$+ \rho(\omega_x \omega_z - \dot{\omega}_z) \iiint_{\tau(t)} x y \, d\tau - \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x z \, d\tau$$

$$+ \rho(\omega_y\ \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} y^2\, d\tau + \rho(\omega_z^2 - \omega_y^2) \iiint\limits_{\tau(t)} y\, z\, d\tau$$

$$- \rho(\omega_y\ \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} z^2\, d\tau$$

$$-L_{1_y} = \rho \iiint\limits_{\tau(t)} [z \frac{\partial}{\partial x}(\frac{\partial \varphi}{\partial t}) - x \frac{\partial}{\partial z}(\frac{\partial \varphi}{\partial t})]\, d\tau + \frac{1}{2}\rho \iiint\limits_{\tau(t)} (z \frac{\partial v^2}{\partial x} - x \frac{\partial v^2}{\partial z})\, d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (z\, f_x - x\, f_z)\, d\tau - \rho a_z \iiint\limits_{\tau(t)} x\, d\tau + \rho a_x \iiint\limits_{\tau(t)} z\, d\tau - \rho(\omega_x\ \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} x^2\, d\tau$$

$$- \rho(\omega_y\, \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} x\, y\, d\tau + \rho(\omega_x^2 - \omega_z^2) \iiint\limits_{\tau(t)} x\, z\, d\tau + \rho(\omega_x\, \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} y\, z\, d\tau$$

$$+ \rho(\omega_x\ \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} z^2\, d\tau$$

$$-L_{1_z} = \rho \iiint\limits_{\tau(t)} [x \frac{\partial}{\partial y}(\frac{\partial \varphi}{\partial t}) - y \frac{\partial}{\partial x}(\frac{\partial \varphi}{\partial t})]\, d\tau + \frac{1}{2}\rho \iiint\limits_{\tau(t)} (x \frac{\partial v^2}{\partial y} - y \frac{\partial v^2}{\partial x})\, d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (x\, f_y - y\, f_x)\, d\tau + \rho a_y \iiint\limits_{\tau(t)} x\, d\tau - \rho\ a_x \iiint\limits_{\tau(t)} y\, d\tau + \rho(\omega_x\ \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} x^2\, d\tau$$

$$+ \rho(\omega_y^2 - \omega_x^2) \iiint\limits_{\tau(t)} x\, y\, d\tau + \rho(\omega_y\ \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} x\, z\, d\tau - \rho(\omega_x\, \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} y^2\, d\tau$$

$$- \rho(\omega_x\, \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} y\, z\, d\tau$$

Given $\bar{u}$, $\bar{\omega}$, certain initial conditions and the geometry of the vessel, we can in principle find φ. With φ known, the motion of the liquid and its important consequences are completely determined. It should be pointed out that without further knowledge of $\bar{\omega}$ and $\bar{u}$, except insofar as they are given functions of time, we cannot logically discard nonlinearities involving them.

The problem as stated is somewhat academic because in dynamic studies of aerospace vehicles $\bar{u}$ and $\bar{\omega}$ are not given functions of time which are independent of vehicle motion and orientation with respect to inertial space. In reality they depend on the body rates of the vehicle and consequently are not known until the total motion of the system is known.

SUMMARY OF FUNDAMENTAL RESULTS

The basic equations describing the motion of a liquid enclosed in a partly filled vessel which is itself in motion have been derived. Principal assumptions employed in the derivation were:

(1) frictionless liquid;
(2) homogeneous liquid;
(3) liquid incompressible throughout motion;
(4) interfacial tension forces and capillary contact effects between liquid and boundaries neglected;
(5) liquid motion irrotational as viewed by an observer in inertial space.

The pertinent formulae derived are:

(1) Velocities

$$\bar{q} = \bar{u} - \omega_x \nabla y z - \omega_y \nabla x z - \omega_z \nabla x y + \nabla\varphi \tag{2.29}$$

$$\bar{v} = - \omega_x \nabla y z - \omega_y \nabla x z - \omega_z \nabla x y + \nabla\varphi$$

(2) Equation of motion

$$\frac{p}{\rho} + \frac{\partial\varphi}{\partial t} - \dot{\omega}_x\, y z - \dot{\omega}_y\, x z - \dot{\omega}_z\, x y + \bar{a}\,\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = C(t) \tag{2.30}$$

(3) Equation of continuity

$$\Delta\varphi = 0$$

(4) Boundary conditions (kinematical)

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 2\,[\omega_x\, y \cos(n, z) + \omega_y\, z \cos(n, x) + \omega_z\, x \cos(n, y)], & P \in \Sigma(t), \\ 2\,[\omega_x\, y \cos(n, z) + \omega_y\, z \cos(n, x) + \omega_z\, x \cos(n, y)] + \zeta_t \cos(n, z), & P \in S(t), \end{cases} \tag{2.31}$$

$$\iiint_{S(t)} \zeta_t \cos(n, z)\, ds = 0$$

(5) **Constancy of pressure at free surface**

$$\frac{\partial\varphi}{\partial t} - \dot{\omega}_x y z - \dot{\omega}_y x z - \dot{\omega}_z x y + \bar{a}\,\bar{r} + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}(\bar{\omega} \times \bar{r})^2 = 0 , \quad z = \zeta \qquad (2.32)$$

(6) **Forces and moments**

$$-F_{1_x} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial x}\, d\tau - \rho \iiint_{\tau(t)} f_x\, d\tau + \rho a_x \iiint_{\tau(t)} d\tau \qquad (2.33)$$

$$-\rho(\omega_y^2 + \omega_z^2) \iiint_{\tau(t)} x\, d\tau + \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} y\, d\tau + \rho(\omega_z \omega_x - \dot{\omega}_y) \iiint_{\tau(t)} z\, d\tau$$

$$-F_{1_y} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\, d\tau - \rho \iiint_{\tau(t)} f_y\, d\tau + \rho a_y \iiint_{\tau(t)} d\tau$$

$$+\rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x\, d\tau - \rho(\omega_x^2 + \omega_z^2) \iiint_{\tau(t)} y\, d\tau + \rho(\omega_z \omega_y - \dot{\omega}_x) \iiint_{\tau(t)} z\, d\tau$$

$$-F_{1_z} = \rho \iiint_{\tau(t)} \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z}\, d\tau - \rho \iiint_{\tau(t)} f_z\, d\tau + \rho a_z \iiint_{\tau(t)} d\tau$$

$$+\rho(\omega_x \omega_z - \dot{\omega}_y) \iiint_{\tau(t)} x\, d\tau + \rho(\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y\, d\tau - \rho(\omega_x^2 + \omega_y^2) \iiint_{\tau(t)} z\, d\tau$$

$$-L_{1_x} = \rho \iiint_{\tau(t)} \left[y \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial t}\right) - z \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}\right)\right] d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \left(y \frac{\partial v^2}{\partial z} - z \frac{\partial v^2}{\partial y}\right) d\tau$$

$$-\rho \iiint_{\tau(t)} (y f_z - z f_y)\, d\tau + \rho a_z \iiint_{\tau(t)} y\, d\tau - \rho a_y \iiint_{\tau(t)} z\, d\tau$$

$$+\rho(\omega_x \omega_z - \dot{\omega}_z) \iiint_{\tau(t)} x y\, d\tau - \rho(\omega_x \omega_y - \dot{\omega}_z) \iiint_{\tau(t)} x z\, d\tau$$

$$+\rho(\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} y^2\, d\tau + \rho(\omega_z^2 - \omega_y^2) \iiint_{\tau(t)} y z\, d\tau$$

$$-\rho(\omega_y \omega_z - \dot{\omega}_x) \iiint_{\tau(t)} z^2\, d\tau$$

$$-L_{1_y} = \rho \iiint\limits_{\tau(t)} [z \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t}) - x \frac{\partial}{\partial z} (\frac{\partial \varphi}{\partial t})] d\tau + \frac{1}{2}\rho \iiint\limits_{\tau(t)} (z \frac{\partial v^2}{\partial x} - x \frac{\partial v^2}{\partial z}) d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (z f_x - x f_z) d\tau - \rho a_z \iiint\limits_{\tau(t)} x\, d\tau + \rho a_x \iiint\limits_{\tau(t)} z\, d\tau - \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} x^2 d\tau$$

$$- \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} x y\, d\tau + \rho(\omega_x^2 - \omega_z^2) \iiint\limits_{\tau(t)} x z\, d\tau + \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} y z\, d\tau$$

$$+ \rho(\omega_x \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} z^2 d\tau$$

$$-L_{1_z} = \rho \iiint\limits_{\tau(t)} [x \frac{\partial}{\partial y} (\frac{\partial \varphi}{\partial t}) - y \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t})] d\tau + \frac{1}{2} \rho \iiint\limits_{\tau(t)} (x \frac{\partial v^2}{\partial y} - y \frac{\partial v^2}{\partial x}) d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (x f_y - y f_x) d\tau + \rho a_y \iiint\limits_{\tau(t)} x\, d\tau - \rho a_x \iiint\limits_{\tau(t)} y\, d\tau + \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} x^2 d\tau$$

$$+ \rho (\omega_y^2 - \omega_x^2) \iiint\limits_{\tau(t)} x y\, d\tau + \rho (\omega_y \omega_z - \dot{\omega}_x) \iiint\limits_{\tau(t)} x z\, d\tau - \rho (\omega_x \omega_y - \dot{\omega}_z) \iiint\limits_{\tau(t)} y^2 d\tau$$

$$- \rho (\omega_x \omega_z - \dot{\omega}_y) \iiint\limits_{\tau(t)} y z\, d\tau$$

Thus, with $\bar{u}$, $\bar{\omega}$, certain initial conditions and the geometry of the vessel given, φ can presumably be determined. Consequently the motion of the liquid in $\tau(t)$ is known. It should be pointed out that the determination of φ involves difficulties of a fundamental nature. A number of algorithms pertaining to the non-linear oscillations of a liquid enclosed in partly filled prismatic cylinders have been published but all of them are still very clumsy and, most important, no one has thus far succeeded in proving their convergence. Moreover, the very question of the existence of periodic solutions of non-linear systems still remains open.

STOKES' PROBLEM

Consider the classical problem of the motion of a heavy liquid enclosed in a completely filled vessel which is itself in motion.

Denote by τ the volume of the liquid cavity (the plane S is no longer a free surface). Let the volume τ be set in motion in any manner. Moreover, assume that this motion is known to us, i.e., the instantaneous translational and angular velocities of the vessel are known. The velocity potential of the absolute motion of the liquid must satisfy

$$\Delta \phi = 0$$

The boundary condition is

$$\frac{\partial \phi}{\partial n} = \bar{u}\,\bar{n} + \omega_x\,[y \cos(n, z) - z \cos(n, y)] + \omega_y\,[z \cos(n, x) - x \cos(n, z)]$$

$$+ \omega_z\,[x \cos(n, y) - y \cos(n, x)], \quad P \epsilon\, \Sigma + S\,. \tag{2.34}$$

This condition may be satisfied by writing

$$\phi = u_x\,\phi_x^* + u_y\,\phi_y^* + u_z\,\phi_z^* + \omega_x\,\varphi_x^* + \omega_y\,\varphi_y^* + \omega_z\,\varphi_z^*\,,$$

where ϕ_x^* , . . . are harmonic and satisfy the boundary conditions

$$\frac{\partial \phi_x^*}{\partial n} = \cos(n, x), \quad \frac{\partial \phi_y^*}{\partial n} = \cos(n, y), \quad \frac{\partial \phi_z^*}{\partial n} = \cos(n, z), \quad P \epsilon\, \Sigma + S\,,$$

$$\frac{\partial \varphi_x^*}{\partial n} = y \cos(n, z) - z \cos(n, y), \quad P \epsilon\, \Sigma + S\,, \tag{2.35}$$

$$\frac{\partial \varphi_y^*}{\partial n} = z \cos(n, x) - x \cos(n, z), \quad P \epsilon\, \Sigma + S\,,$$

$$\frac{\partial \varphi_z^*}{\partial n} = x \cos(n, y) - y \cos(n, x), \quad P \epsilon\, \Sigma + S\,,$$

so that ϕ_x^* , . . . depend solely on the geometry of τ and not on its motion. Such is the classical problem of Stokes. ϕ_x^* , ... are sometimes called Stokes potentials.

If for ϕ_x^* , ϕ_y^* , ϕ_z^*, φ_x^* , φ_y^*, φ_z^* we substitute x, y, z, $-zy+\varphi_x^*$, $-xz+\varphi_y^*$, $-xy+\varphi_z^*$ we get

$$\varphi = \bar{u}\,\bar{r} - \omega_x (y z - \varphi_x^*) - \omega_y (x z - \varphi_y^*) - \omega_z (x y - \varphi_z^*) , \tag{2.36}$$

where φ_x^* , ... are harmonic and satisfy conditions

$$\frac{\partial \varphi_x^*}{\partial n} = 2 y \cos (n, z) , \; P \epsilon \Sigma + S , \tag{2.37}$$

$$\frac{\partial \varphi_y^*}{\partial n} = 2 z \cos (n, x) , \; P \epsilon \Sigma + S ,$$

$$\frac{\partial \varphi_z^*}{\partial n} = 2 x \cos (n, y) , \; P \epsilon \Sigma + S .$$

Note that the differential system

$$\Delta \varphi_x^* = 0 , \; P \epsilon \tau ,$$

$$\frac{\partial \varphi_x^*}{\partial n} = 2 y \cos (n, z), \; P \epsilon \Sigma + S$$

is equivalent to the variational problem for the functional

$$G_x^* = \frac{1}{2} \iiint_{\tau} (\nabla \varphi_x^*)^2 d\tau - 2 \iint_{S+\Sigma} \varphi_x^* \, y \cos (n, z) \, ds .$$

Similarly,

$$G_y^* = \frac{1}{2} \iiint_{\tau} (\nabla \varphi_y^*)^2 d\tau - 2 \iint_{S+\Sigma} \varphi_y^* \, z \cos (n, x) \, ds ,$$

$$G_z^* = \frac{1}{2} \iiint_{\tau} (\nabla \varphi_z^*)^2 d\tau - 2 \iint_{S+\Sigma} \varphi_z^* \, x \cos (n, y) \, ds .$$

These expressions may be combined into a single formula

$$G_i^* (\varphi^*) = \frac{1}{2} \iiint_{\tau} (\nabla \varphi^*)^2 d\tau - 2 \iint_{S+\Sigma} \varphi^* \, \eta_i \, ds \tag{2.38}$$

where

$$\eta_i = \begin{cases} y \cos(n, z) & \text{for } i = 1 , \\ z \cos(n, x) & \text{for } i = 2 , \\ x \cos(n, y) & \text{for } i = 3 . \end{cases}$$

Thus the problem of determining Stokes' potentials is equivalent to finding the extremum of functional (2.38). The method of Ritz may be used to solve this variational problem.

ENERGY FORMULATION

Consider the motion of the entire system (liquid plus solid). Denote by T the total kinetic energy of the system. Thus

$$T = T_l + T_s ,$$

$$T_l = \frac{1}{2}\iiint_{\tau(t)} \rho |q|^2 d\tau = \frac{1}{2}\iiint_{\tau(t)} \rho \, |\bar{V} + \bar{v}|^2 \, d\tau \, , \quad T_s = \frac{1}{2} \iiint_{\tau^*} \rho \, |\bar{V}|^2 \, d\tau^* \, ,$$

where T_l is the kinetic energy of the liquid in $\tau(t)$ and T_s is the kinetic energy of the solid, i.e., vessel proper. τ^*, ρ^* represent the volume and mass density of the vessel proper,respectively. Let Π_l be the potential energy of the liquid in τ (t),

$$\Pi_l = \iiint_{\tau(t)} \rho \; \Omega \; d\tau$$

$$\bar{f} = - \nabla \Omega$$

then, the equations of motion for the complete system may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T}{\partial \bar{u}} = \bar{F}_{ext} + \iiint_{\tau^*} \rho^* \bar{f} \, d\tau^* + \iiint_{\tau(t)} \rho \; \bar{f} \, d\tau \tag{2.39}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T}{\partial \bar{u}} = \bar{L}_{ext} + \iiint_{\tau^*} \rho^* \, (\bar{r} \times \bar{f}) \, d\tau^* + \iiint_{\tau(t)} \rho \, (\bar{r} \times \bar{f}) \, d\tau \, ,$$

$$\frac{d}{dt} \, (T_l + \Pi_l) = - \iint_{S(t) + \Sigma(t)} p \, (V_n + v_n) \, ds = - \iint_{S(t) + \Sigma(t)} p \, V_n \, ds \, ,$$

$$p \, (x, \, y, \, \zeta, \, t) = 0 \, , \quad P \epsilon S(t) \, ,$$

$$\nabla \bar{v} = 0 \, , \quad P \epsilon \tau(t)$$

$$v_n = \begin{cases} 0 \, , \; P \epsilon \Sigma(t) \, , \\ \zeta_t \cos(n, \, z) \, , \; P \epsilon \; S(t) \, , \end{cases}$$

in which $\iiint_{\tau(t)} \rho \bar{f} \, d\tau$, $\iiint_{\tau^*} \rho^* \bar{f} \, d\tau^*$ are the resultant of body forces per unit mass acting on the liquid in $\tau(t)$ and on the solid respectively, and $\bar{F}_{ext}$, $\bar{L}_{ext}$ are externally applied forces and moments.

Formulae (2.39_1) and (2.39_2) are Kirchhoff's equations in vector form (Lagrange's equations referred to moving coordinates).

Formula (2.39_3) states that the rate of change of total energy of any portion of the liquid (assumed incompressible) as it moves about is equal to the rate of working of the pressures on the boundary. In deriving this expression it was assumed that $\frac{\partial \Omega}{\partial t} = 0$. The unsteady pressure equation may be derived from the formula. Moreover, condition (2.39_4) is a corollary of (2.39_3) for the irrotational motion of a liquid enclosed in a fixed vessel.

Formulae (2.39_1) and (2.39_2) can be written as

$$\frac{d}{dt}\left(\frac{\partial T_s}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_s}{\partial \bar{u}} = \bar{F}_{ext} + \iiint_{\tau^*} \rho^* \bar{f} \, d\tau^* + \iiint_{\tau(t)} \rho \bar{f} \, d\tau - \frac{d}{dt}\left(\frac{\partial T_l}{\partial \bar{u}}\right) - \bar{\omega} \times \frac{\partial T_l}{\partial \bar{u}} ,$$

$$\frac{d}{dt}\left(\frac{\partial T_s}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_s}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_s}{\partial \bar{u}} = \bar{L}_{ext} + \iiint_{\tau^*} \rho^* (\bar{r} \times \bar{f}) \, d\tau^* + \iiint_{\tau(t)} \rho (\bar{r} \times \bar{f}) \, d\tau$$

$$- \frac{d}{dt}\left(\frac{\partial T_l}{\partial \bar{\omega}}\right) - \bar{\omega} \times \frac{\partial T_l}{\partial \bar{\omega}} - \bar{u} \times \frac{\partial T_l}{\partial \bar{u}} .$$

Now, if the liquid had been absent ($T_l = 0$), the right side of these equations would have contained only $\bar{F}_{ext} + \iiint_{\tau^*} \bar{f} \, d\tau^*$ and $\bar{L}_{ext} + \iiint_{\tau^*} \rho^* (\bar{r} \times \bar{f}) \, d\tau^*$. The action of the liquid pressures must therefore be represented by the remaining terms on the right. Thus the action of the liquid is represented by the force and moment

$$-F_l = \frac{d}{dt}\left(\frac{\partial T_l}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_l}{\partial \bar{u}} - \iiint_{\tau(t)} \rho \bar{f} \, d\tau ,$$

$$-\bar{L}_l = \frac{d}{dt}\left(\frac{\partial T_l}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_l}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_l}{\partial \bar{u}} - \iiint_{\tau(t)} \rho (\bar{r} \times \bar{f}) \, d\tau ,$$

which can be verified from formulae (2.8). To be sure

$$\frac{\partial T_l}{\partial \bar{u}} = \iiint_{\tau(t)} (\bar{u} + \bar{\omega} \times \bar{r} + \bar{v}) \rho \, d\tau ,$$

$$\frac{\partial T_1}{\partial \bar{\omega}} = \iiint\limits_{\tau(t)} (\bar{r} \times \bar{u} + \bar{r} \times (\bar{\omega} \times \bar{r}) + \bar{r} \times \bar{v})\, \rho\, d\tau \,,$$

$$\bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint\limits_{\tau(t)} (\bar{\omega} \times \bar{u} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \bar{\omega} \times \bar{v})\, \rho\, d\tau \,,$$

$$\bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} = \iiint\limits_{\tau(t)} [\bar{\omega} \times (\bar{r} \times \bar{u}) - (\bar{\omega}\ \bar{r})(\bar{\omega} \times \bar{r}) + \bar{\omega} \times (\bar{r} \times \bar{v})]\, \rho\, d\tau \,,$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{u}}\right) = \iiint\limits_{\tau(t)} [\dot{\bar{u}} + \dot{\bar{\omega}} \times r + \bar{\omega} \times \bar{v} + \frac{\partial \bar{v}}{\partial t} + (\bar{v}\,\nabla)\,\bar{v}]\, d\tau, \quad [\frac{d}{dt}(\rho\, d\tau) = 0]$$

$$= \iiint\limits_{\tau(t)} (\dot{\bar{u}} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \bar{v})\, \rho\, d\tau + \iiint\limits_{\tau(t)} \frac{\partial(\rho\,\bar{v})}{\partial t} d\tau + \iint\limits_{S(t)} \rho\, \bar{v}\, \zeta_t \cos(n, z)\, ds \,,$$

$$\rho\, [\frac{\partial \bar{v}}{\partial t} + (\bar{v}\,\nabla)\,\bar{v}] = \frac{\partial(\rho\bar{v})}{\partial t} + \nabla\,(\rho\bar{v} : \bar{v}) \,,$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\bar{v}) = 0 \,,$$

$$\iiint\limits_{\tau(t)} \nabla(\rho\bar{v} : \bar{v})\, d\tau = \iint\limits_{S(t)+\Sigma(t)} \rho\, \bar{v}\, v_n\, ds - \iint\limits_{S(t)} \rho\, \bar{v}\, \zeta_t \cos(n, z)\, ds \,,$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{\omega}}\right) = \iiint\limits_{\tau(t)} [\bar{v} \times \bar{u} + \bar{r} \times \dot{\bar{u}} + \bar{v} \times (\bar{\omega} \times \bar{r}) + \bar{r} \times (\dot{\bar{\omega}} \times \bar{r}) + \bar{r} \times (\bar{\omega} \times \bar{v})]\, \rho\, d\tau$$

$$+ \iiint\limits_{\tau(t)} [\bar{r} \times \frac{\partial(\rho\bar{v})}{\partial t}]\, d\tau + \iint\limits_{S(t)} \rho(\bar{r} \times \bar{v})\, \zeta_t \cos(n, z)\, ds \,,$$

and therefore

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{u}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint\limits_{\tau(t)} \frac{\partial(\rho\bar{v})}{\partial t}\, d\tau + 2 \iiint\limits_{\tau(t)} \rho\, (\bar{\omega} \times \bar{v})\, d\tau + \iiint\limits_{\tau(t)} \rho\, (\dot{\bar{\omega}} \times \bar{r})\, d\tau$$

$$+ \iiint\limits_{\tau(t)} \rho\, [\bar{\omega} \times (\bar{\omega} \times \bar{r})]\, d\tau + \iiint\limits_{\tau(t)} \rho\, \bar{a}\, d\tau + \iint\limits_{S(t)} \rho\bar{v}\, \zeta_t \cos(n, z)\, ds$$

$$= -\bar{F}_1 + \iiint\limits_{\tau(t)} \rho\, \bar{f}\, d\tau \,,$$

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \bar{\omega}}\right) + \bar{\omega} \times \frac{\partial T_1}{\partial \bar{\omega}} + \bar{u} \times \frac{\partial T_1}{\partial \bar{u}} = \iiint\limits_{\tau(t)} \rho\, [\bar{r} \times \frac{\partial(\rho\bar{v})}{\partial t}]\, d\tau + 2 \iiint\limits_{\tau(t)} \rho[\bar{\omega}(\bar{r}\ \bar{v}) - \bar{v}(\bar{\omega}\ \bar{r})]\, d\tau$$

$$+ \iiint_{\tau(t)} \rho (\bar{\omega}\ \bar{r})\ (\bar{r} \times \bar{\omega})\ d\tau + \iiint_{\tau(t)} \rho\, [\dot{\bar{\omega}} (\bar{r}\ \bar{r}) - \bar{r}\ (\dot{\bar{\omega}}\ \bar{r})]\ d\tau$$

$$+ \iiint_{\tau(t)} \rho (\bar{r} \times \bar{a})\ d\tau + \iint_{S(t)} \rho (\bar{r} \times \bar{v})\ \zeta_t \cos (n,\ z)\ ds$$

$$= -\bar{L}_1 + \iiint_{\tau(t)} \rho (\bar{r} \times \bar{f})\ d\tau$$

as adduced.

If the motion of the liquid is irrotational when measured in inertial space, then

$$\bar{v} = \nabla \psi - \bar{\omega} \times \bar{r}\ ,$$

$$= \nabla \varphi - \bar{\omega} \times \bar{r} - \omega_x \nabla y\, z - \omega_y \nabla x\, z - \omega_z \nabla x\, y\ ,$$

using (2.23) and formulae (2.39) become

$$a_x \{\rho \iiint_{\tau(t)} d\tau + \iiint_{\tau^*} \rho^* d\tau^*\} - (\omega_y^2 + \omega_z^2)\ \{\rho \iiint_{\tau(t)} x\ d\tau + \iiint_{\tau^*} \rho^* x\ d\tau^*\} \tag{2.40}$$

$$+ (\omega_y\ \omega_x - \dot{\omega}_z)\ \{\rho \iiint_{\tau(t)} y\ d\tau + \iiint_{\tau^*} \rho^*\ y\ d\tau^*\} + (\omega_x\ \omega_z + \dot{\omega}_y)\ \{\rho \iiint_{\tau(t)} z\ d\tau + \iiint_{\tau^*} \rho^*\ z\ d\tau^*\}$$

$$- 2\ \dot{\omega}_y\ \rho \iiint_{\tau(t)} z\ d\tau - \rho \iiint_{\tau(t)} f_x\ d\tau - \iiint_{\tau^*} \rho^* f_x\ d\tau^* + \rho \iiint_{\tau(t)} \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2}\ \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial x}\ d\tau = F_{xt_x}$$

$$a_y \{\rho \iiint_{\tau(t)} d\tau + \iiint_{\tau^*} \rho^*\ d\tau^*\} + (\omega_x\ \omega_y + \dot{\omega}_z)\ \{\rho \iiint_{\tau(t)} x\ d\tau + \iiint_{\tau^*} \rho^* x\ d\tau^*\}$$

$$- (\omega_x^2 + \omega_z^2)\ \{\rho \iiint_{\tau(t)} y\ d\tau + \iiint_{\tau^*} \rho^* y\ d\tau^*\} + (\omega_y\ \omega_z - \dot{\omega}_x)\ \{\rho \iiint_{\tau(t)} z\ d\tau + \iiint_{\tau^*} \rho^* z\ d\tau^*\}$$

$$- 2\ \dot{\omega}_z\ \rho \iiint_{\tau(t)} x\ d\tau - \rho \iiint_{\tau(t)} f_y\ d\tau - \iiint_{\tau^*} \rho^* f_y\ d\tau^* + \rho \iiint_{\tau(t)} \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2}\ \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\ dv = F_{ext_y}$$

$$a_z \{\rho \iiint\limits_{\tau(t)} d\tau + \iiint\limits_{\tau^*} \rho^* d\tau^*\} + (\omega_x \; \omega_z - \dot{\omega}_y) \{ \rho \iiint\limits_{\tau(t)} x \, d\tau + \iiint\limits_{\tau^*} \rho^* x \, d\tau^* \}$$

$$+ (\omega_y \, \omega_z + \dot{\omega}_x) \{ \rho \iiint\limits_{\tau(t)} y \, d\tau + \iiint\limits_{\tau^*} \rho^* y \, d\tau^*\} - (\omega_x^2 + \omega_y^2) \{\rho \iiint\limits_{\tau(t)} z \, d\tau + \iiint\limits_{\tau^*} \rho^* z \, d\tau^*\}$$

$$- 2 \, \dot{\omega}_x \, \rho \iiint\limits_{\tau(t)} y \, d\tau - \rho \iiint\limits_{\tau(t)} f_z \, d\tau - \iiint\limits_{\tau^*} \rho^* f_z \, d\tau^* + \rho \iiint\limits_{\tau(t)} \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) d\tau$$

$$+ \tfrac{1}{2} \, \rho \iiint\limits_{\tau(t)} \frac{\partial v^2}{\partial z} \, d\tau = F_{ext_z}$$

$$a_z \{\rho \iiint\limits_{\tau(t)} y \, d\tau + \iiint\limits_{\tau^*} \rho^* y \, d\tau^*\} - a_y \{ \rho \iiint\limits_{\tau(t)} z \, d\tau + \iiint\limits_{\tau^*} \rho^* z \, d\tau^*\}$$

$$+ (\omega_x \; \omega_z - \dot{\omega}_y) \{ \rho \iiint\limits_{\tau(t)} x \, y \, d\tau + \iiint\limits_{\tau^*} \rho^* x \, y \, d\tau^*\} - (\omega_x \, \omega_y + \dot{\omega}_z) \{\rho \iiint\limits_{\tau(t)} x \, z \, d\tau + \iiint\limits_{\tau^*} \rho^* x \, z \, d\tau^*\}$$

$$+ (\omega_y \, \omega_z + \dot{\omega}_x) \{\rho \iiint\limits_{\tau(t)} y^2 \, d\tau + \iiint\limits_{\tau^*} \rho^* y^2 \, d\tau^*\} + (\omega_z^2 - \omega_y^2) \{\rho \iiint\limits_{\tau(t)} y \, z \, d\tau + \iiint\limits_{\tau^*} \rho^* \, y \, z \, d\tau^*\}$$

$$- (\omega_y \, \omega_z - \dot{\omega}_x) \{\rho \iiint\limits_{\tau(t)} z^2 \, d\tau + \iiint\limits_{\tau^*} \rho^* z^2 \, d\tau^*\} + 2 \, \dot{\omega}_z \, \rho \iiint\limits_{\tau(t)} x \, z \, d\tau - 2 \, \dot{\omega}_x \iiint\limits_{\tau(t)} y^2 \, d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (y \, f_z - z \, f_y) \, d\tau - \iiint\limits_{\tau^*} \rho^* (y \, f_z - z \, f_y) \, d\tau^* + \rho \iiint\limits_{\tau(t)} [y \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) - z \, \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right)] \, d\tau$$

$$+ \tfrac{1}{2} \, \rho \iiint\limits_{\tau(t)} \left(y \, \frac{\partial v^2}{\partial z} - z \, \frac{\partial v^2}{\partial y}\right) d\tau = L_{ext_x}$$

$$- a_z \{ \rho \iiint\limits_{\tau(t)} x \, d\tau + \iiint\limits_{\tau^*} \rho^* \, x \, d\tau^*\} + a_x \{\rho \iiint\limits_{\tau(t)} z \, d\tau + \iiint\limits_{\tau^*} \rho^* z \, d\tau^*\}$$

$$- (\omega_x \, \omega_z - \dot{\omega}_y) \{\rho \iiint\limits_{\tau(t)} x^2 \, d\tau + \iiint\limits_{\tau^*} \rho^* \, x^2 \, d\tau^*\} - (\omega_y \, \omega_z + \dot{\omega}_x) \{\rho \iiint\limits_{\tau(t)} x \, y \, d\tau + \iiint\limits_{\tau^*} \rho^* x \, y \, d\tau^*\}$$

$$+ (\omega_x^2 - \omega_z^2) \{ \rho \iiint\limits_{\tau(t)} x \, z \, d\tau + \iiint\limits_{\tau^*} \rho^* x \, z \, d\tau^* \} + (\omega_x \, \omega_y - \dot{\omega}_z) \{\rho \iiint\limits_{\tau(t)} yz \, d\tau + \iiint\limits_{\tau^*} \rho^* \, y \, z \, d\tau^*\}$$

$$+ (\omega_x \, \omega_z + \dot{\omega}_y) \{ \rho \iiint\limits_{\tau(t)} z^2 \, d\tau + \iiint\limits_{\tau^*} \rho^* z^2 \, d\tau^*\} + 2 \, \dot{\omega}_x \, \rho \iiint\limits_{\tau(t)} x \, y \, d\tau - 2 \, \dot{\omega}_y \, \rho \iiint\limits_{\tau(t)} z^2 \, d\tau$$

$$- \rho \iiint\limits_{\tau(t)} (z \, f_x - x \, f_z) \, d\tau - \iiint\limits_{\tau^*} \rho^* (z \, f_x - x \, f_z) \, d\tau^* + \rho \iiint\limits_{\tau(t)} [z \, \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial t}\right) - x \, \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right)] \, d\tau$$

$$+ \frac{1}{2} \rho \iiint_{\tau(t)} (z \frac{\partial v^2}{\partial x} - x \frac{\partial v^2}{\partial z}) \, d\tau = L_{ext_y} ,$$

$$a_y \{\rho \iiint_{\tau(t)} x \, d\tau + \iiint_{\tau^*} \rho^* x \, d\tau^*\} - a_x \{\rho \iiint_{\tau(t)} y \, d\tau + \iiint_{\tau^*} \rho^* y \, d\tau^*\}$$

$$+ (\omega_x \omega_y + \dot{\omega}_z) \{\rho \iiint_{\tau(t)} x^2 \, d\tau + \iiint_{\tau^*} \rho^* x^2 \, d\tau^*\} + (\omega_y^2 - \omega_x^2) \{\rho \iiint_{\tau(t)} x \, y \, d\tau + \iiint_{\tau^*} \rho^* x \, y \, d\tau^*\}$$

$$+ (\omega_y \omega_z - \dot{\omega}_x) \{\rho \iiint_{\tau(t)} x \, z \, d\tau + \iiint_{\tau^*} \rho^* x \, z \, d\tau^*\} - (\omega_x \omega_y - \dot{\omega}_z) \{\rho \iiint_{\tau(t)} y^2 \, d\tau + \iiint_{\tau^*} \rho^* y^2 \, d\tau^*\}$$

$$- (\omega_x \omega_z + \dot{\omega}_y) \{\rho \iiint_{\tau(t)} y \, z \, d\tau + \iiint_{\tau^*} \rho^* y \, z \, d\tau^*\} - 2 \dot{\omega}_z \rho \iiint_{\tau(t)} x^2 \, d\tau + 2 \dot{\omega}_y \rho \iiint_{\tau(t)} y \, z \, d\tau$$

$$- \rho \iiint_{\tau(t)} (x f_y - y f_x) \, d\tau - \iiint_{\tau^*} \rho^* (x f_y - y f_x) \, d\tau^* + \rho \iiint_{\tau(t)} [x \frac{\partial}{\partial y} (\frac{\partial \varphi}{\partial t}) - y \frac{\partial}{\partial x} (\frac{\partial \varphi}{\partial t})] \, d\tau$$

$$\frac{1}{2} \rho \iiint_{\tau(t)} (x \frac{\partial v^2}{\partial y} - y \frac{\partial v^2}{\partial x}) \, d\tau = L_{ext_z} ,$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \dot{\omega}_x \, y \, z - \dot{\omega}_y \, x \, z - \dot{\omega}_z \, x \, y + \bar{a} \, \bar{r} + \Omega + \frac{1}{2} v^2 - \frac{1}{2} (\bar{\omega} \times \bar{r})^2 = C(t), \quad P \epsilon \, \tau(t),$$

$$\frac{\partial \varphi}{\partial t} - \dot{\omega}_x \, y \, z - \dot{\omega}_y \, x \, z - \dot{\omega}_z \, x \, y + \bar{a} \, \bar{r} + \Omega + \frac{1}{2} v^2 - \frac{1}{2} (\bar{\omega} \times \bar{r})^2 = 0, \quad P \epsilon \, S(t), \quad z = \zeta ,$$

$$\Delta \varphi = 0 \quad P \epsilon \, \tau(t)$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 2 \, [\omega_x \, y \cos (n, z) + \omega_y \, z \cos (n, x) + \omega_z \, x \cos (n, y)] , \quad P \epsilon \, \Sigma(t) , \\ 2 \, [\omega_x \, y \cos (n, z) + \omega_y \, z \cos (n, x) + \omega_z \, x \cos (n, y)] + \zeta_t \cos (n, z) , \\ \quad P \epsilon \, S(t) . \end{cases}$$

Thus, with the external forces and moments, certain initial conditions and the geometry of the vessel given, the motion of the system can, in principle, be determined from (2.40).

3/ PLANAR EQUATIONS FOR THE MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

PLANAR EQUATIONS FOR THE MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

Consider the planar motion of the launch vehicle illustrated in Figure 2. The vehicle consists of an engine, a main body which is assumed to be rigid and a rigid tank partly filled with a frictionless liquid. For simplicity, the tank is assumed to be a prismatic cylinder, Figure 3.

To describe the motion of the system take three cartesian frames of reference $o' x' y' z'$ fixed relatively to the vehicle at a distance l below the "capped" free surface, $o\,x\,y\,z$ fixed relatively to the container and $o_e\, x_e\, y_e\, z_e$ fixed relatively to the swivelling engine. Reference $o\,x\,y\,z$ is oriented in such a manner that $o\,z$ is measured positively along the outward directed normal to the undisturbed free surface. Thus the free surface, denoted by $S(t)$, coincides with plane $x\,o\,y$ (the plane $z = 0$) when the vehicle and liquid are at rest.

Let

$$z = \zeta\,(x,\ y,\ t)$$

be the equation of $S(t)$ when it is displaced. Denote by $\Sigma\,(t) = \Sigma_1(t) + \Sigma_2(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by $S(t)$ and $\Sigma(t)$. Let Σ, τ and S represent the values of $\Sigma(t)$, $\tau(t)$ and $S(t)$ in the undisturbed position. In addition let C be the boundary curve of $\Sigma_1(t)$. All surfaces are presumed to be piecewise smooth.

Coordinate systems $o\,x\,y\,z$ and $o_e\, x_e\, y_e\, z_e$ are related to $o' x' y' z'$ as follows:

$$z' = z + l\ , \qquad z' = -\,l_1 - (z_e \cos\beta + y_e \sin\beta)\ ,$$

$$y' = y\ , \qquad y' = y_e \cos\beta - z_e \sin\beta\ ,$$

$$x' = x\ , \qquad x' = x_e$$

Suppose that at time t the vehicle is coincident with inertial space and that it is moving relatively to inertial space with motion described by an observer in inertial space as a velocity $\bar{u}$ of O' and an angular velocity $\bar{\omega}$. Then the position

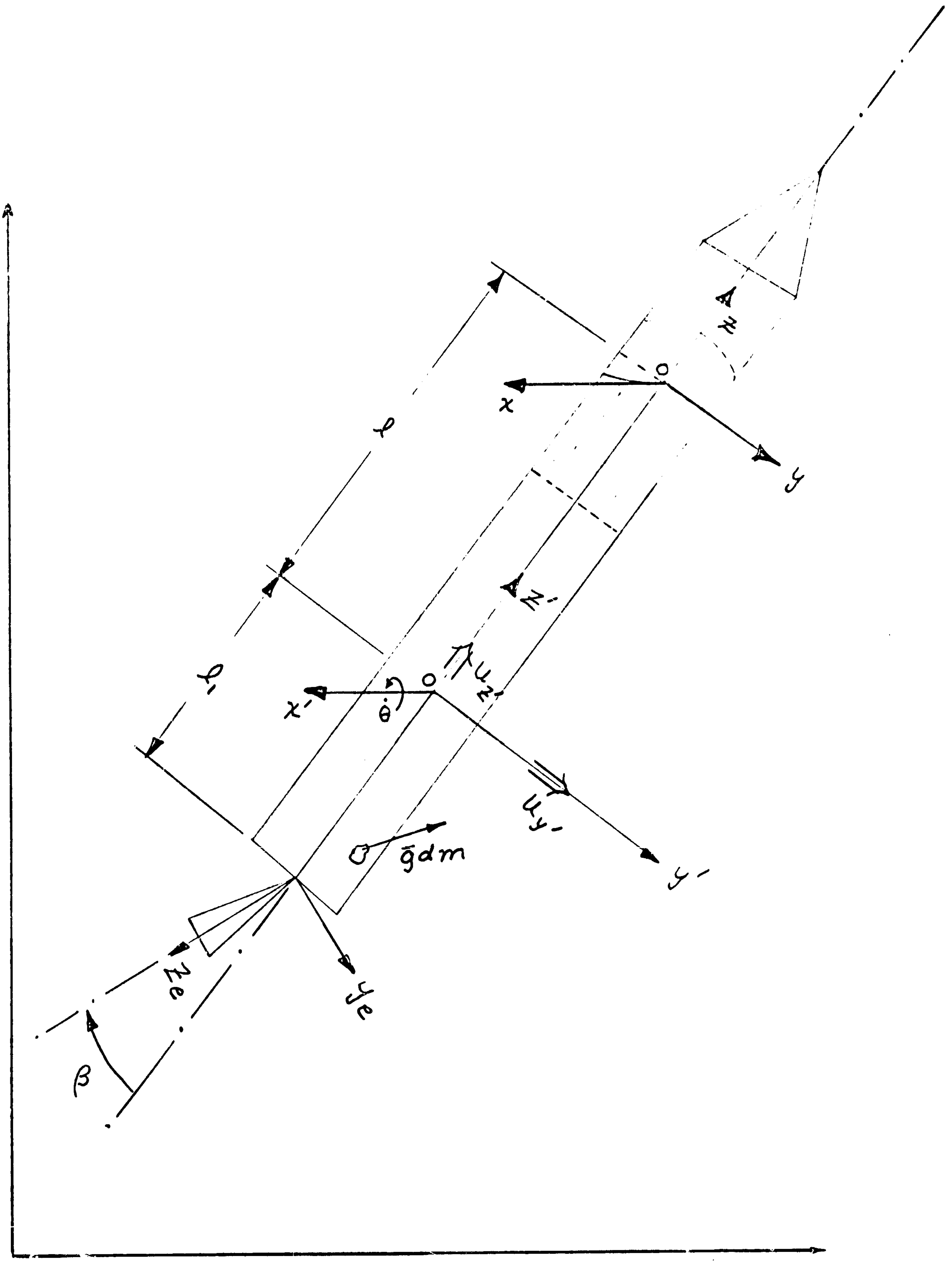

Figure 2. Launch vehicle experiencing planar motion.

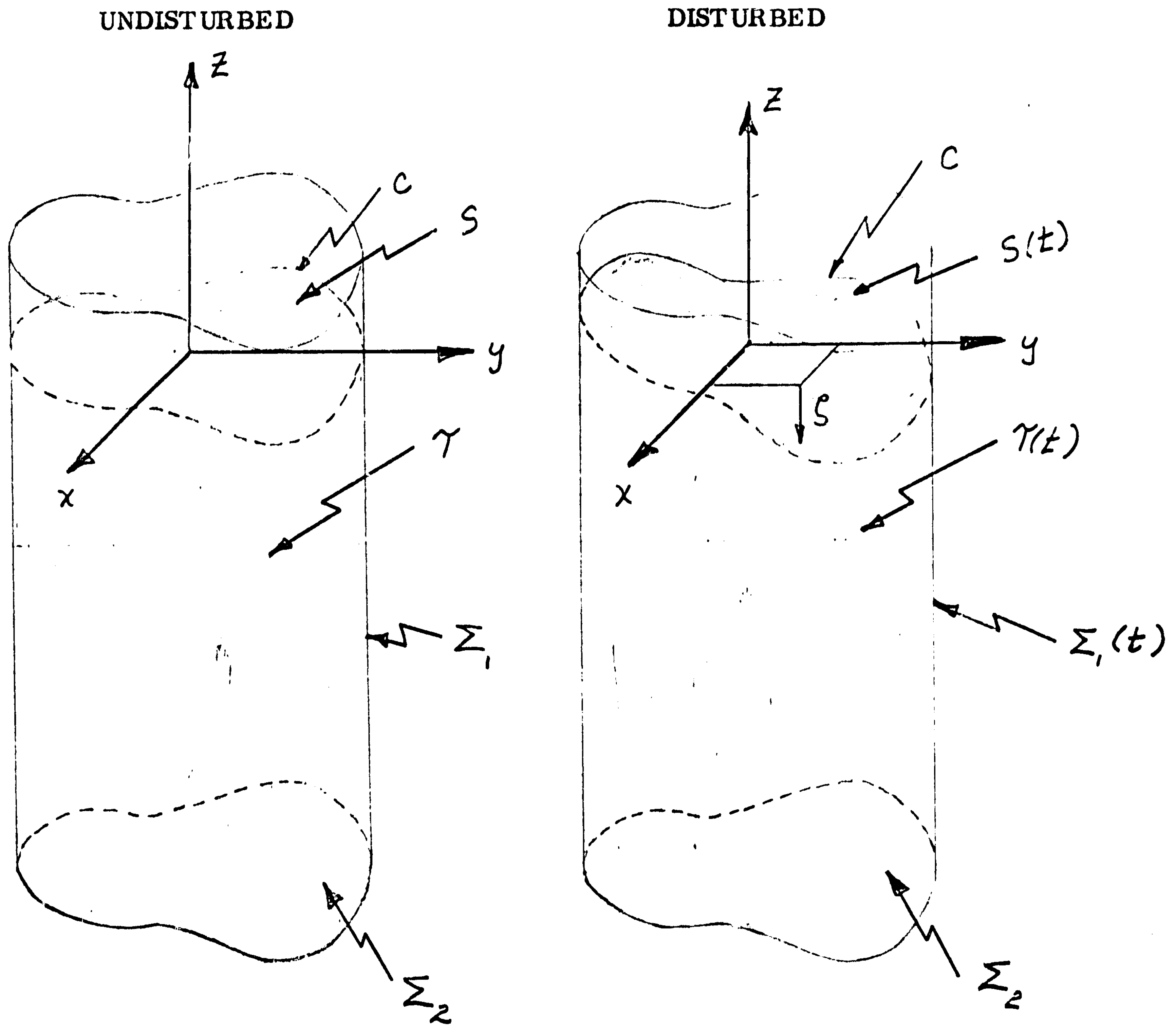

Figure 3. Prismatic cylindrical propellant tank.

vector $\bar{r}'$ of a particular point P on the vehicle at time t is the same for an observer moving with the vehicle as it is for an observer in inertial space.

The point P, if rigidly attached to the vehicle, has the velocity

$$\bar{V} = \{0, (u_y' - z'\dot{\theta}), (u_z' + y'\dot{\theta})\} .$$

Denote by $\bar{q}$ (P, t), $\bar{v}$ (P, t) the velocities of the liquid particle $P \epsilon \tau(t)$ at time t as estimated by observers in inertial space and o x y z respectively. Then

$$\bar{q} = \{0, (u_y' - (z + l)\dot{\theta}, (u_z' + y\dot{\theta})\} + \bar{v} ,$$

in which $\bar{q}$, $\bar{v}$ are referred to moving reference o x y z.

The velocity of a generic mass point on the engine is, referring to Figure 2,

$$\bar{V} = \{0, [u_y' + (\dot{\theta} - \dot{\beta})(z_e \cos\beta + y_e \sin\beta) + l_1 \dot{\theta}], [u_z' - (\dot{\theta} - \dot{\beta})(z_e \sin\beta - y_e \cos\beta)]\} .$$

The total kinetic energy of the vehicle is therefore

$$T = T_{\text{main body}} + T_{\text{engine}} + T_{\text{liquid}} ,$$

where

$$T_{\text{main body}} = \tfrac{1}{2}(u_y'^2 + u_z'^2)\iiint_{\tau^*} dm + \tfrac{1}{2}\dot{\theta}^2 \iiint_{\tau^*}(y'^2 + z'^2)\, dm + \dot{\theta}\{u_z' \iiint_{\tau^*} y'\, dm$$

$$- u_y' \iiint_{\tau^*} z'\, dm\} ,$$

$$\begin{aligned} T_{\text{engine}} &= \tfrac{1}{2}(u_y'^2 + u_z'^2)\iiint_{\tau_e} dm + \tfrac{1}{2}\dot{\theta}^2 \iiint_{\tau_e}[y_e^2 + (l_1 + z_e)^2]\, dm \qquad (3.1) \\ &+ \dot{\theta}\{u_z' \iiint_{\tau_e} y_e\, dm + u_y' \iiint_{\tau_e}(l_1 + z_e)\, dm\} + \tfrac{1}{2}\dot{\beta}^2 \iiint_{\tau_e}(y_e^2 + z_e^2)\, dm \\ &- \dot{\beta}\dot{\theta}\iiint_{\tau_e}(y_e^2 + z_e^2)\, dm - \dot{\beta}\{[(u_y' + l_1\dot{\theta})\cos\beta - u_z' \sin\beta]\iiint_{\tau_e} z_e\, dm \\ &+ [(u_y' + l_1\dot{\theta})\sin\beta + u_z' \cos\beta]\iiint_{\tau_e} y_e\, dm\} + \dot{\theta}\{[(u_y' + l_1\dot{\theta})(\cos\beta - 1) \\ &- u_z' \sin\beta]\iiint_{\tau_e} z_e\, dm + [(u_y' + l_1\dot{\theta})\sin\beta - u_z'(\cos\beta - 1)]\iiint_{\tau_e} y_e\, dm\} , \end{aligned}$$

$$T_{\text{liquid}} = \tfrac{1}{2}\,(u_y^2 + u_z^2{}')\iiint_{\tau(t)} \rho\, d\tau + \tfrac{1}{2}\,\dot{\theta}^2 \iiint_{\tau(t)} [y^2 + (z+1)^2]\,\rho\, d\tau + \tfrac{1}{2}\iiint_{\tau(t)} v^2\, \rho\, d\tau$$

$$+ \dot{\theta}\,\{u_z{}' \iiint_{\tau(t)} y\, \rho\, d\tau - u_y{}' \iiint_{\tau(t)} (1+z)\,\rho\, d\tau + u_y{}' \iiint_{\tau(t)} v_y\, \rho\, d\tau$$

$$+ u_z{}' \iiint_{\tau(t)} v_z\, \rho\, d\tau + \dot{\theta}\iiint_{\tau(t)} [y\, v_z - (z+1)\, v_y]\,\rho\, d\tau\ .$$

Assume the liquid to be homogeneous and incompressible throughout the motion. Neglect interfacial tension forces and capillary contact effects between liquid and tank walls. Moreover, assume that the liquid is not draining from the tank. Then, with the absolute velocity of liquid particles irrotational, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_y{}'}\right) - \dot{\theta}\,\frac{\partial T}{\partial u_z{}'} = F_{ext_y}{}' + \rho \iiint_{\tau(t)} g_y{}'\, d\tau + \iiint_{\tau^*} g_y{}'\, dm + \iiint_{\tau_e} g_y{}'\, dm\ ,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_z{}'}\right) + \dot{\theta}\,\frac{\partial T}{\partial u_y{}'} = F_{ext_z}{}' \quad \rho \iiint_{\tau(t)} g_z{}'\, d\tau + \iiint_{\tau^*} g_z{}'\, dm + \iiint_{\tau_e} g_z{}'\, dm\ , \qquad (3.2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + u_y{}'\,\frac{\partial T}{\partial u_z{}'} - u_z{}'\,\frac{\partial T}{\partial u_y{}'} = L_{ext_x}{}' + \rho \iiint_{\tau(t)} [y\, g_z{}' - (1+z)\, g_y{}']\, d\tau$$

$$+ \iiint_{\tau^*} (y'\, g_z{}' - z'\, g_y{}') + \iiint_{\tau_e} [y_e \cos\beta - z_e \sin\beta)\, g_z{}'$$

$$+ (l_1 + z_e \cos\beta + y_e \sin\beta)\, g_y{}'\,]\, dm\ ,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = \iiint_{\tau_e} (g_z{}' \sin\beta - g_y{}' \cos\beta)\, z_e\, dm - \iiint_{\tau_e} (g_z{}' \cos\beta + g_y{}' \sin\beta)\, y_e\, dm$$

$$+ Q_\beta\ ,$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \ddot{\theta}\, y\,(1+z) + a_y\, y + a_z (z+1) + \Omega + \tfrac{1}{2}\, v^2 - \tfrac{1}{2}\,\dot{\theta}^2\, [y^2 + (z+1)^2] = C(t),\quad P \in \tau(t)\ ,$$

$$\frac{\partial \varphi}{\partial t} - \ddot{\theta}\, y\,(1+z) + a_y\, y + a_z\,(z+1) + \Omega + \tfrac{1}{2}\, v^2 - \tfrac{1}{2}\,\dot{\theta}^2 [y^2 + (z+1)^2] = 0,\quad P \in S(t),\ z = \zeta,$$

$$\bar{v} = \nabla \varphi + (0,\ 0,\ -2\, y\ \dot{\theta})\ ,\quad P \in \tau(t)\ ,$$

$$\Delta\, \varphi = 0\ ,\quad P \in \tau(t)$$

$$\frac{\partial \varphi}{\partial n} = \begin{cases} 2\,\dot{\theta}\, y \cos(n, z)\,, & P \in \Sigma_1(t), \Sigma_2(t)\,, \\ 2\,\dot{\theta}\, y \cos(n, z) + \zeta_t \cos(n, z)\,, & P \in S(t)\,, \end{cases}$$

in which

$$\bar{g} = (0\,,\ g_{y'}\,,\ g_{z'})$$

is the vector of body forces per unit mass, and

$$\bar{g} = -\nabla\Omega\ .$$

Q_β is the generalized force which, in general, is associated with the spring moment per unit angle of the hinge when locked, with the damping moment of the hinge, with the engine actuator moment, and with the engine actuator position required by the autopilot.

Formulae $(3.2_{1,2,3})$ express the dynamic equilibrium of the complete system. Formula (3.2_4) is the Euler-Lagrange equation which describes the motion of the engine. Formula (3.2_5) is the unsteady pressure equation which is obtained from a quadrature of the Euler-Lagrange equation for motion of the liquid subject to the above assumptions. Formula (3.2_6) expresses the constancy of pressure at the free surface of the liquid. Formula (3.2_7) is a corollary of the assumption of absolute irrotational motion.

Using (3.1, 2), we get

$$a_{y'}\{\iiint_{\tau^*} dm + \iiint_{\tau_e} dm + \rho\iiint_{\tau(t)} d\tau\} - \dot{\theta}^2\{\iiint_{\tau^*} y'\, dm + \iiint_{\tau_e} y_e\, dm + \rho\iiint_{\tau(t)} y\, d\tau\} \quad (3.3)$$

$$-\ddot{\theta}\{\iiint_{\tau^*} z'\, dm + \iiint_{\tau_e}(z_e + l_1)\, dm + \rho\iiint_{\tau(t)}(z+1)\, d\tau\} - \iiint_{\tau^*} g_{y'}\, dm - \rho\iiint_{\tau(t)} g_{y'}\, dm$$

$$-\iiint_{\tau_e} g_{y'}\, dm + \rho\iiint_{\tau(t)} \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho\iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\, d\tau + [\ddot{\theta}(\cos\beta + 1) - 2\dot{\beta}\dot{\theta}\sin\beta$$

$$+(\dot{\theta}^2 + \dot{\beta}^2)\sin\beta - \ddot{\beta}\cos\beta]\iiint_{\tau_e} z_e\, dm + [\ddot{\theta}\sin\beta + 2\dot{\beta}\dot{\theta}\cos\beta - (\dot{\theta}^2 + \dot{\beta}^2)\cos\beta$$

$$+\dot{\theta}^2 - \ddot{\beta}\sin\beta]\iiint_{\tau_e} y_e\, dm + 2\ddot{\theta}\, l_1\iiint_{\tau_e} dm = F_{ext_{y'}}\ ,$$

$$a_{z'}\{\iiint_{\tau^*} dm + \iiint_{\tau_e} dm + \rho\iiint_{\tau(t)} d\tau\} - \dot{\theta}^2\{\iiint_{\tau^*} z'\, dm + \iiint_{\tau_e}(z_e + l_1)\, dm + \rho\iiint_{\tau(t)}(1 + z)\ \cdots\}$$

$$+ \ddot{\theta}\{ \iiint_{\tau^*} y'\,dm + \iiint_{\tau_e} y_e\,dm + \rho \iiint_{\tau(t)} y\,d\tau \} - \iiint_{\tau^*} g_z'\,dm - \rho \iiint_{\tau(t)} g_z'\,d\tau - \iiint_{\tau_e} g_z'\,dm$$

$$- 2\ddot{\theta}\rho \iiint_{\tau(t)} y\,d\tau + \rho \iiint_{\tau(t)} \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z}\,d\tau + [\ddot{\theta}\sin\beta + 2\dot{\beta}\dot{\theta}\cos\beta$$

$$- (\dot{\theta}^2 + \dot{\beta}^2)\cos\beta - \dot{\theta}^2 - \ddot{\beta}\sin\beta] \iiint_{\tau_e} z_e\,dm + [\ddot{\theta}(\cos\beta - 1) - 2\dot{\beta}\dot{\theta}\sin\beta$$

$$+ (\dot{\theta}^2 + \dot{\beta}^2)\sin\beta - \ddot{\beta}\cos\beta] \iiint_{\tau_e} y_e\,dm + 2\,l_1\,\dot{\theta}^2 \iiint_{\tau_e} dm = F_{ext_{z'}} ,$$

$$a_{z'}\{ \iiint_{\tau^*} y'\,dm + \iiint_{\tau_e} y_e\,dm + \rho \iiint_{\tau(t)} y\,d\tau \} - a_{y'}\{ \iiint_{\tau^*} z'\,dm + \iiint_{\tau_e} (l_1 + z_e)\,dm$$

$$+ \rho \iiint_{\tau(t)} (1 + z)\,d\tau \} + \ddot{\theta}\{ \iiint_{\tau^*} (y'^2 + z'^2)\,dm + \iiint_{\tau_e} [y_e^2 + (l_1 + z_e)^2]\,dm$$

$$+ \rho \iiint_{\tau(t)} [y^2 + (1 + z)^2]\,d\tau\} - \iiint_{\tau^*} (y' g_z' - z' g_y)\,d\tau - \rho \iiint_{\tau(t)} [y\,g_z' - (1 + z)\,g_y']\,d\tau$$

$$- \iiint_{\tau_e} [(y_e\cos\beta - z_e\sin\beta)\,g_z' + (l_1 + z_e\cos\beta + y_e\sin\beta)\,g_y']\,dm$$

$$- 2\rho\ddot{\theta} \iiint_{\tau(t)} y^2\,d\tau + \rho \iiint_{\tau(t)} [y \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial t}\right) - (z + 1)\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial t}\right)]\,d\tau + \tfrac{1}{2}\rho \iiint_{\tau(t)} [y\frac{\partial v^2}{\partial z}$$

$$- (z + 1)\frac{\partial v^2}{\partial y}]\,d\tau - \ddot{\beta} \iiint_{\tau_e} (y_e^2 + z_e^2)\,dm + \{a_{y'}(\cos\beta + 1) - a_{z'}\sin\beta$$

$$+ l_1[2\ddot{\theta}(\cos\beta - 1) - 2\dot{\beta}\dot{\theta}\sin\beta + \dot{\beta}^2\sin\beta - \ddot{\beta}\cos\beta]\} \iiint_{\tau_e} z_e\,dm$$

$$+ \{a_{y'}\sin\beta + a_{z'}(\cos\beta - 1) + l_1\,[2\ddot{\theta}\sin\beta + 2\dot{\beta}\dot{\theta}\cos\beta - \dot{\beta}^2\cos\beta$$

$$- \ddot{\beta}\sin\beta]\} \iiint_{\tau_e} y_e\,dm + 2\,a_y'\,l_1 \iiint_{\tau_e} dm = L_{ext_{x'}} ,$$

$$\ddot{\beta} \iiint_{\tau_e} (y_e^2 + z_e^2)\,dm - \ddot{\theta} \iiint_{\tau_e} (y_e^2 + z_e^2)\,dm + [a_{z'}\sin\beta - a_{y'}\cos\beta - l_1\ddot{\theta}\cos\beta$$

$$+ l_1\dot{\theta}^2\sin\beta] \iiint_{\tau_e} z_e\,dm - [a_{y'}\sin\beta + a_{z'}\cos\beta + l_1\ddot{\theta}\sin\beta + l_1\dot{\theta}^2\cos\beta] \iiint_{\tau_e} y_e\,dm$$

$$- \iiint_{\tau_e} (g_{z'} \sin\beta - g_{y'} \cos\beta)\, z_e \, dm + \iiint_{\tau_e} (g_{z'} \cos\beta + g_{y'} \sin\beta)\, y_e \, dm = Q_\beta \;,$$

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} - \ddot{\theta}\, y\, (1 + z) + a_{y'}\, y + a_{z'}\, (z + 1) + \tfrac{1}{2} v^2 + \Omega - \tfrac{1}{2} \dot{\theta}^2 \left[y^2 + (1 + z)^2 \right] = C(t), \;\; P \epsilon\, \tau(t),$$

$$\frac{\partial \varphi}{\partial t} - \ddot{\theta}\, y\, (1 + z) + a_{y'}\, y + a_{z'}\, (1 + z) + \tfrac{1}{2} v^2 + \Omega - \tfrac{1}{2} \dot{\theta}^2 \left[y^2 + (1 + z)^2 \right] = 0, \;\; P \epsilon\, S(t), \;\; z = \zeta \;,$$

$$\bar{v} = \nabla \varphi + (0,\, 0,\, -2\, y\, \dot{\theta}) \,, \;\; P \epsilon\, \tau(t) \;,$$

$$\Delta \varphi = 0, \;\; P \epsilon\, \tau(t) \;,$$

$$\frac{\partial \varphi}{\partial n} = 0, \;\; P \epsilon\, \Sigma_1(t) \;,$$

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial z} = 2\, \dot{\theta}\, y, \;\; P \epsilon\, \Sigma_2(t), \;\; z = -h \;,$$

$$\frac{\partial \varphi}{\partial n} = (2\, \dot{\theta}\, y + \zeta_t) \cos(n,\, z) \,, \;\; P \epsilon\, S(t)$$

$$\bar{g} = (0,\, g_{y'},\, g_{z'}) \;,$$

$$\bar{g} = - \nabla \Omega \;,$$

$$a_{y'} = \dot{u}_{y'} - \dot{\theta}\, u_{z'} \;, \;\; a_{z'} = \dot{u}_{z'} + \dot{\theta}\, u_{y'} \;.$$

We now make the following simplifications and identifications:

(1) The specific body force are independent of position,

$$\bar{g} = \text{const};$$

(2) Origin of reference system $o'x'y'z'$ is at center of mass of vehicle when engine is "locked-out" and liquid "frozen" solid,

$$\iiint_{\tau^*} y' \, dm + \iiint_{\tau_e} y_e \, dm + \rho \iiint_{\tau} y \, d\tau = 0 \;,$$

$$\iiint_{\tau^*} z' \, dm + \iiint_{\tau_e} (l_1 + z_e) \, dm + \rho \iiint_{\tau} (1 + z) \, d\tau = 0 \;;$$

(3) Engine is symmetrical about its line of centers,

$$\iiint y_e \, dm = 0 \;;$$

(4) Origin of coordinate system oxyz is such that

$$\iiint_{\tau} y\, d\tau = \iint_{S} dx\, dy \int_{-h}^{0} y\, dz = h \iint_{S} y\, dx\, dy = 0,$$

the tank is symmetrical about longitudinal axis of vehicle;

(5) Total moment of inertia of system about axis $o'x'$ is

$$I_{o'x'} = I_{o'x'}^{mb} + I_{o'x'}^{e} + I_{o'x'}^{l},$$

$$I_{o'x'}^{mb} = \iiint_{\tau^*} (y'^2 + z'^2)\, dm \quad ,$$

$$I_{o'x'}^{e} = \iiint_{\tau_e} [y_e^2 + (l_1 + z_e)^2]\, dm \quad ,$$

$$I_{o'x'}^{l} = \iiint_{\tau} [y^2 + (l + z)^2]\, d\tau \quad ;$$

(6) Total mass of system is

$$M = M_{mb} + M_e + M_l \quad ,$$

$$M_{mb} = \iiint_{\tau^*} dm \quad ,$$

$$M_e = \iiint_{\tau_e} dm \quad ,$$

$$L_l = \rho \iiint_{\tau} d\tau \quad ;$$

(7) Moment of inertia of engine about axis $o_e x_e$ is

$$I_{o_e y_e}^{e} = \iiint_{\tau_e} (y_e^2 + z_e^2)\, dm \quad ;$$

(8) First moment of mass of engine about axis $o_e y_e$ is

$$S_{o_e y_e}^{e} = \iiint_{\tau_e} z_e\, dm \quad ;$$

(9) Second moment of mass of liquid about axis oz is

$$I^{1}_{Oz} = \rho \iiint_{\tau} y^2 \, d\tau \;;$$

(10) Various volume integrals associated with liquid are evaluated as

$$\rho \iiint_{\tau(t)} d\tau = \rho \iiint_{\tau} d\tau + \rho \iint_{S} \zeta \, dx\, dy \;,$$

$$\rho \iiint_{\tau(t)} y \, d\tau = \rho \iiint_{\tau} y \, d\tau + \rho \iint_{S} y \, \zeta \, dx\, dy \;,$$

$$\rho \iiint_{\tau(t)} (1 + z) \, d\tau = \rho \iiint_{\tau} (1 + z) d\tau + \rho \iint_{S} (\zeta^2/2 + 1 \, \zeta) \, dx\, dy \;,$$

$$\rho \iiint_{\tau(t)} [y^2 + (1 + z)^2] \, d\tau = \rho \iiint_{\tau} [y^2 + (1 + z)^2] \, d\tau + \rho \iint_{S} [1^2 + y^2) \, \zeta + 1 \, \zeta^2 + \zeta^3/3] \, dx\, dy;$$

(11) Potential Ω is

$$\Omega = - g_y' \, y - g_z' \, (z + 1) \;.$$

With these simplifications and definitions, formulae (3.3) may be written as follows:

Component of force equation along axis $o'y'$

$$\begin{aligned} &M \, (a_y' - g_y') + \rho (a_y' - g_y') \iint_{S} \zeta \, dx\, dy - \rho \, \ddot{\theta} \iint_{S} (1 \, \zeta + \zeta^2/2) \, dx\, dy - \rho \, \dot{\theta}^2 \iint_{S} y \, \zeta \, dx\, dy \\ &+ \rho \iiint_{\tau(t)} \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial y} d\tau + [\, \ddot{\theta} (\cos \beta + 1) - 2 \, \dot{\beta} \, \dot{\theta} \sin \beta \\ &+ (\dot{\theta}^2 + \dot{\beta}^2) \sin \beta - \ddot{\beta} \cos \beta \,] \, S^{e}_{o_e y_e} + 2 \, l_1 \, \ddot{\theta} \, M_e = F_{ext_y}' \end{aligned} \tag{3.4}$$

Component of force equation along axis $o'z'$

$$\begin{aligned} &M \, (a_z' - g_z') + \rho (a_z' - g_z') \iint_{S} \zeta \, dx\, dy - \rho \, \ddot{\theta} \iint_{S} y \, \zeta \, dx\, dy - \rho \, \dot{\theta}^2 \iint_{S} (1 \, \zeta + \zeta^2/2) \, dx\, dy \\ &+ \rho \iiint_{\tau(t)} \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial t}\right) d\tau + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} d\tau - [\, \ddot{\theta} \sin \beta + 2 \, \dot{\beta} \, \dot{\theta} \cos \beta \end{aligned} \tag{3.5}$$

$$- (\dot{\theta}^2 + \dot{\beta}^2) \cos\beta - \dot{\theta}^2 - \ddot{\beta} \sin\beta \,] S_{o_e^\circ y_e} + 2\, l_1 M_e \dot{\theta}^2 = F_{ext_z'}$$

Moment equation about axis $o'x'$

$$(I_{o'x'} - 2 I^1_{oz}) \ddot{\theta} + \rho (a_{z'} - g_{z'}) \iint_S y \zeta \, dx\, dy - \rho (a_{y'} - g_{y'}) \iint_S (l \zeta + \zeta^2/2)\, dx\, dy$$

$$+ \rho \ddot{\theta} \iint_S [(l^2 - y^2) \zeta + \zeta^2 + \zeta^3/3]\, dx\, dy + \rho \iiint_{\tau(t)} [y \frac{\partial}{\partial z} (\frac{\partial\varphi}{\partial t}) - (l + z) \frac{\partial}{\partial y} (\frac{\partial\varphi}{\partial t})]\, d\tau$$

$$+ \tfrac{1}{2} \rho \iiint_{\tau(t)} [y \frac{\partial v^2}{\partial z} - (l + z) \frac{\partial v^2}{\partial y}]\, d\tau - I_{o_e^\circ x_e} \ddot{\beta} + \{(a_{y'} - g_{y'})(\cos\beta + 1) \tag{3.6}$$

$$- (a_{z'} - g_{z'}) \sin\beta + l_1 [2 \ddot{\theta} (\cos\beta - 1) - 2 \dot{\beta} \dot{\theta} \sin\beta + \dot{\beta}^2 \sin\beta$$

$$- \ddot{\beta} \cos\beta] \} S_{o_e^\circ y_e} + 2\, l_1 (a_{y'} - g_{y'}) M_e = L_{ext_x'}$$

Engine equation

$$I_{o_e^\circ x_e} (\ddot{\beta} - \ddot{\theta}) + [(a_{z'} - g_{z'}) \sin\beta - (a_{y'} - g_{y'}) \cos\beta - l_1 \ddot{\theta} \cos\beta \tag{3.7}$$

$$+ l_1 \dot{\theta}^2 \sin\beta] S_{o_e^\circ y_e} = Q_\beta$$

Constancy of pressure at free surface

$$\frac{\partial\varphi}{\partial t} - \ddot{\theta} y (l + \zeta) + (a_{y'} - g_{y'}) y + (a_{z'} - g_{z'}) (l + \zeta) + \tfrac{1}{2} v^2 \tag{3.8}$$

$$- \tfrac{1}{2} [y^2 + (l + \zeta)^2] \dot{\theta}^2 = 0, \quad z = \zeta$$

Kinematics of liquid

$$\bar{v} = \nabla\varphi + (0, 0, -2 \dot{\theta} y), \quad P \epsilon \tau(t) \tag{3.9}$$

$$\Delta\varphi = 0, \quad P \epsilon \tau(t)$$

$$\frac{\partial\varphi}{\partial n} = 0, \quad P \epsilon \Sigma_1(t)$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial z} = 2 \dot{\theta} y, \quad P \epsilon \zeta(t), \; z = -h,$$

$$\frac{\partial\varphi}{\partial n} = (2 \dot{\theta} y + \zeta_t) \cos(n, z), \quad P \epsilon S(t).$$

The solution to the Neumann problem (3.9), subject to the restriction $\iint_S \zeta_t \, dx\, dy = 0$, may be taken in the form of a series

$$\varphi(x, y, z, t) = \sum_1^\infty (\alpha_i(t) \cosh k_i (z + h) + \gamma_i(t) \sinh k_i z) \varphi_i (x, y) + \alpha_0(t) \tag{3.10}$$

This function is harmonic as indicated by (3.4_1), and satisfies (3.4_2) if the infinitely many values k_i^2 ($i = 1, 2, \ldots$) are the values of k^2 (eigenvalues) for which the two-dimensional scalar Helmholtz equation

$$\Delta \varphi + k^2 \varphi = 0, \ P \in S , \tag{3.11}$$

has a non-zero solution satisfying

$$\frac{\partial \varphi}{\partial n} = 0, \ P \in \Sigma_1 \text{ (or C)} . \tag{3.12}$$

Functions φ_i ($i = 1, 2, \ldots$) are the correspondin solutions (eigenfunctions) of (3.6).

We point out, without proof, several important properties of the system of functions φ_i and the numbers k_i^2:

(1) Of the infinite number of numbers k_i^2, all are real and positive;

(2) The set of functions φ_i is orthogonal

$$(\varphi_i, \varphi_i) \equiv \iint\limits_S \varphi_i \ \varphi_j \, dx \, dy = \begin{cases} 0, & i \neq j \\ \|\varphi_i\|^2, & i = j \end{cases}, \tag{3.13}$$

and can be normalized;

(3) Any function $\mu(x, y)$ which has continuous second-order derivatives in and on the boundary of S and which is orthogonal to a constant:

$$(\mu, 1) \equiv \iint \mu(x, y) \, dx \, dy = 0 ,$$

and which satisfies the boundary conditions, may be expressed as a uniformly convergent series of Eigen functions

$$\mu(x, y) = \sum_1^{\infty} c_i \ \varphi_i(x, y) ; \tag{3.14}$$

(4) If function μ which is continuously twice-differentiable satisfies the condition (3.7), then the series (3.9) not only converges uniformly to μ, but the series obtained from it by termwise differentiation also converges in the mean to the corresponding derivative of μ;

(5) In addition to the condition for orthogonality, written above, the following equations hold:

$$\iint \nabla \varphi_i \nabla \varphi_i \, dx\, dy = \begin{cases} 0, & i \neq j \\ k_i^2 \; \|\varphi_i\|^2, & i = j \end{cases} \qquad (3.15)$$

We are now in a position to expand $2 y \dot{\theta}$ and ζ (x, y, t) as series of the form

$$2 \dot{\theta} y = 2 \dot{\theta} \{ \sum_1^{\infty} \eta_i (t) \varphi_i (x, y) + \eta_0 (t) \} , \qquad (3.16)$$

$$\zeta (x, y, t) = \sum_1^{\infty} \xi_i (t) \varphi_i (x, y) + \xi_0 (t) .$$

Such expansions are possible, since for a fixed value of t functions $2 \dot{\theta} (y - \eta_0)$ and ζ (x, y, t) - ξ_0 (t) are developable in series of the form (3.14). The coefficients of these series depend, in general, on t. Choosing η_0 and ξ_0 so that y - η_0 and $\zeta - \xi_0$ are orthogonal to a constant (unity), we get

$$\eta_i (t) = \frac{(y, \varphi_i)}{\|\varphi_i\|^2} , \qquad (3.17)$$

$$\eta_0 (t) = \frac{(y, 1)}{A} = 0 , \quad (y, 1) = 0 ,$$

and

$$\xi_i (t) = \frac{(\zeta, \varphi_i)}{\|\varphi_i\|^2} , \qquad (3.18)$$

$$\xi_0 (t) = \frac{(1, \zeta)}{A} ,$$

respectively, using the condition for orthogonality (3.13). Here A is the area of the cross section of the cylinder. Now since

$$\frac{\partial \zeta}{\partial n} = 0 , \quad P \epsilon C ,$$

the series obtained from (3.16_2) by termwise differentiation converges in the mean to the corresponding derivative of ζ. Note that this implies a 90° contact angle. For contact angles other than this such an expansion as (3.16_2) is still possible with, of course, certain modifications.

For (3.5) to satisfy the boundary condition (3.9_4) it is necessary that

$$\gamma_i (t) = 2 \dot{\theta} \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} . \qquad (3.19)$$

To satisfy the boundary condition (3.9) it is necessary for

$$\sum_1^\infty \{k_i \varphi_i \sinh k_i (\zeta + h) - \nabla \varphi_i \nabla \zeta \cosh k_i (\zeta + h)\} \alpha_i (t)$$

$$+ \sum_1^\infty \{k_i \varphi_i \cosh k_i \zeta - \nabla \varphi_i \nabla \zeta \sinh k_i \zeta\} \gamma_i (t) = + 2 \dot{\theta} y + \zeta_t$$

$$= \sum_1^\infty \gamma_i (t) k_i \cosh k_i h \varphi_i + \zeta_t$$

however

$$\psi_i = - \frac{1}{k_i^2} \Delta \psi_i = - \frac{1}{k_i^2} \nabla (\nabla \psi_i) ,$$

so that this expression can be written, on application of a known vector identity, in the form

$$\sum_1^\infty \alpha_i (t) \nabla \{\frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i\} + \sum_1^\infty \gamma_i (t) \nabla \{\frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \nabla \varphi_i\} \qquad (3.20)$$

$$= - \zeta_t = - \sum_1^\infty \dot{\xi}_i (t) \varphi_i - \dot{\xi}_o (t) ,$$

using (3.18). Multiplying both sides of this equality by φ_j and integrating over S, we get

$$- \|\varphi_i\|^2 \dot{\xi}_i = \sum_1^\infty \alpha_j (t) \iint_S \varphi_i \nabla \{\frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\} d x d y$$

$$+ \sum_1 \gamma_j (t) \iint_S \varphi_i \nabla \{\frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\} d x d y$$

but

$$\varphi_i \nabla \{\frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\} = \nabla \{\varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j\}$$

$$- \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_i \nabla \varphi_j ,$$

$$\varphi_i \nabla \{\frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\} = \nabla \{\varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j\}$$

$$- \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_i \nabla \varphi_j ,$$

and moreover

$$\iint_S \nabla \{ \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \} \, dx\, dy = \int_C \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \frac{\partial \varphi_j}{\partial n} \, d L = 0 \ ,$$

$$\iint_S \nabla \{ \varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \} \, dx\, dy = 0 \ ,$$

in accordance with the divergence theorem and boundary condition (3.12). Therefore, it follows

$$\dot{\xi}_i = \sum_1^\infty C_{ij} \alpha_j + 2 \dot{\theta} \mu_i \tag{3.21}$$

where

$$C_{ij} = \frac{1}{\|\varphi_i\|^2} \iint_S \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \nabla \varphi_i \, dx\, dy \ , \tag{3.22}$$

$$\mu_i = \frac{1}{\|\varphi_i\|^2} \sum_1^\infty \frac{(y, \varphi_j)}{k_j \ \varphi_j \ ^2 \cosh k_j h} \iint_S \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \nabla \varphi_i \, dx\, dy \ .$$

If we integrate both sides of (3.20) over S we obtain the constraint expressing the constancy of volume. To be sure

$$\iint_S \nabla \{ \frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i \} \, dx\, dy = \int_C \frac{\sinh k_i (\zeta + h)}{k_i} \frac{\partial \varphi_i}{\partial n} \, d L = 0 \ ,$$

$$\iint_S \nabla \{ \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \nabla \varphi_i \} \, dx\, dy = \int_C \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \frac{\partial \varphi_i}{\partial n} \, d L = 0 \ ,$$

and

$$\sum_1^\infty \dot{\xi}_i \iint_S \varphi_i \, dx\, dy + \dot{\xi}_o \iint_S dx\, dy = A \dot{\xi}_o \ ,$$

so that

$$A \dot{\xi}_o = 0 \ .$$

But the volume is given by

$$\tau(t) = \iint\limits_S dx\,dy \int_{-h}^{\zeta} dz = h\iint\limits_S dx\,dy + \iint\limits_S \zeta\, dx\,dy = \tau_0 + A\,\xi_0\,,$$

and

$$\dot{\tau}(t) = 0 = A\,\dot{\xi}_0$$

as adduced.[3]

Substituting the expansions for φ and ζ in expression (3.8), multiplying the resulting equation by φ_i and integrating over S, we get, after considerable manipulation and rearranging indices,

$$(a_z' - g_z')\|\varphi_i\|^2\,\xi_i + \sum_1^\infty \dot{\alpha}_j \iint\limits_S \varphi_i\,\varphi_j \cosh k_j(\zeta + h)\,dx\,dy + (a_y' - g_y')(y, \varphi_i) \tag{3.23}$$

$$+ \tfrac{1}{2}\sum_1^\infty \sum_1^\infty \alpha_i\,\alpha_l \iint\limits_S \varphi_i\,[\nabla\varphi_j\,\nabla\varphi_l \cosh k_j(\zeta + h)\cosh k_l(\zeta + h) + k_j\,k_l\,\varphi_j\,\varphi_l \sinh k_j(\zeta + h)\,\cdot$$

$$\sinh k_l\,(\zeta + h)]\,dx\,dy + \{2\sum_1^\infty \frac{(y, \varphi_j)}{k_j\,\|\varphi_j\|^2\cosh k_j\,h} \iint\limits_S \varphi_i\,\varphi_j \sinh k_j\,\zeta\,dx\,dy$$

$$- \sum_1^\infty \xi_j \iint\limits_S y\,\varphi_i\,\varphi_j\,dx\,dy - \xi_0\,(y, \varphi_i)\}\,\ddot{\theta} + 2\,\{\tfrac{1}{2}\sum_1^\infty \sum_1^\infty \alpha_l \frac{(y, \varphi_j)}{k_j\,\|\varphi_j\|^2\cosh k_j\,h}\,\cdot$$

$$\iint\limits_S \varphi_i\,[\nabla\varphi_j\,\nabla\varphi_l \sinh k_j\,\zeta\cosh k_l(\zeta + h) + k_j\,k_l\,\varphi_j\,\varphi_l \cosh k_j\,\xi\,\sinh k_l(\zeta + h)]\,dx\,dy$$

$$+ \tfrac{1}{2}\sum_1^\infty \sum_1^\infty \alpha_j \frac{(y, \varphi_l)}{k_l\,\|\varphi_l\|^2\cosh k_l\,h} \iint\limits_S \varphi_i\,[\nabla\varphi_j\,\nabla\varphi_l \cosh k_j\,(\zeta + h)\sinh k_l\,\zeta$$

$$+ k_j\,k_l\,\varphi_j\,\varphi_l \sinh k_j\,(\zeta + h)\cosh k_l\,\zeta\,]\,dx\,dy - 2\sum_1^\infty \alpha_j\,k_j \iint\limits_S y\,\varphi_i\,\varphi_j\,\cdot$$

$$\sinh k_j\,(\zeta + h)\,dx\,dy\}\,\dot{\theta} - \tfrac{1}{2}\,\dot{\theta}^2\,\{\iint\limits_S \varphi_i\,[(1 + \zeta)^2 - 3\,y^2]\,dx\,dy$$

$$- 4\sum_1^\infty \sum_1^\infty \frac{(y, \varphi_j)(y, \varphi_l)}{k_j\,k_l\,\|\varphi_j\|^2\,\|\varphi_l\|^2\cosh k_j\,h\cosh k_l\,h} \iint\limits_S \varphi_i\,[\nabla\varphi_j\,\nabla\varphi_l \sinh k_j\,\zeta\,\sinh k_l\,\zeta$$

$$+ k_j\,k_l\,\varphi_j\,\varphi_l \cosh k_j\,\zeta\,\cosh k_l\,\zeta\,]\,dx\,dy + 8\sum_1^\infty \frac{(y, \varphi_j)}{\|\varphi_j\|^2\cosh k_j\,h} \iint\limits_S \varphi_i\,\varphi_j \cosh k_j\,\zeta\,dx\,dy\} = 0$$

---

[3]See notes.

Likewise, substituting the expansions for φ and ζ in expression (3.8) and integrating the resulting equation over S, we obtain

$$A\,\{\dot{\alpha}_0 + (a_z' - g_z')(1+\xi_0)\} + \sum_1^\infty \dot{\alpha}_i \iint_S \varphi_i \cosh k_i\,(\zeta + h)\,dx\,dy \tag{3.24}$$

$$+ \tfrac{1}{2}\sum_1^\infty \sum_1^\infty \alpha_i\,\alpha_j \iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_i\,(\zeta + n)\cosh k_j\,(\zeta+h) + k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_i\,(\zeta+h)\,\cdot$$

$$\sinh k_j\,(\zeta+h)\,dx\,dy + \{2\sum_1^\infty \frac{(y,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S \varphi_i \sinh k_i\;\zeta\;dx\,dy - 1\,A$$

$$- \sum_1^\infty (y,\varphi_i)\,\xi_i\,\}\,\ddot{\theta} + 2\,\{\tfrac{1}{2}\sum_1^\infty\sum_1^\infty \alpha_j \frac{(y,\,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_j\,(\zeta+h)\sinh k_i\;\zeta$$

$$+ k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_j\,(\zeta+h)\cosh k_i\,\zeta]\,dx\,dy + \tfrac{1}{2}\sum_1^\infty\sum_1^\infty \alpha_i\;\frac{(y,\varphi_j)}{k_j\,\|\varphi_j\|^2\cosh k_j\;h}\;\cdot$$

$$\iint_S [\nabla\varphi_i\,\nabla\varphi_j \cosh k_i\,(\zeta+h)\sinh k_j\,\zeta + k_i\,k_j\,\varphi_i\,\varphi_j \sinh k_i\,(\zeta+h)\cosh k_j\,\zeta\,]\,dx\,dy$$

$$- 2\sum_1^\infty \alpha_i\,k_i \iint_S y\,\varphi_i \sinh k_i\,(\zeta+h)\,dx\,dy\}\,\dot{\theta} - \tfrac{1}{2}\,\{\iint_S [(1+\zeta)^2 - 3\,y^2]\,dx\,dy$$

$$- 4\sum_1^\infty\sum_1^\infty \frac{(y,\varphi_i)\,(y,\varphi_j)}{k_i\;k_j\,\|\varphi_i\|^2\|\varphi_j\|^2 \cosh k_i\,h\,\cosh k_j\,h}\iint_S [\nabla\varphi_i\,\nabla\varphi_j \sinh k_i\;\zeta \sinh k_j\;\zeta$$

$$+ k_i\,k_j\,\varphi_i\,\varphi_j \cosh k_i\;\zeta\,\cosh k_j\,\zeta]\,dx\,dy + 8\sum_1^\infty \frac{(y,\,\varphi_i)}{\|\varphi_i\|^2\cosh k_i\,h}\iint_S \varphi_i \cosh k_i\,\zeta\;dx\,dy\}\,\dot{\theta}^2 = 0\,.$$

Introducing the expansion for φ into (3.4, 5, 6) gives

$$M\,(a_y' - g_y') + \rho\,(a_y' - g_y')\iint_S \zeta\;dx\,dy - \rho\;\ddot{\theta}\,\{\iint_S (1\,\zeta + \zeta^2/2)\,dx\,dy \tag{3.25}$$

$$+ 2\sum_1^\infty \frac{(y,\,\varphi_i)}{k_i\,\|\varphi_i\|^2\cosh k_i\,h}\iint_S y\,\nabla\,[\frac{\cosh k_i\,\zeta - \cosh k_i\,h}{k_i}\,\nabla\,\varphi_i\,]\,dx\,dy\} - \rho\,\dot{\theta}^2\iint_S y\;\zeta\;dx\,dy$$

$$- \rho\sum_1^\infty \dot{\alpha}_i \iint_S y\,\nabla\,[\frac{\sinh k_i\,(\zeta+h)}{k_i}\,\nabla\varphi_i]\,dx\,dy + \tfrac{1}{2}\rho\iiint_{\tau(t)} \frac{\partial v^2}{\partial y}\,d\tau + [\ddot{\theta}\,(\cos\beta + 1)$$

$$+ (\dot{\theta} - \dot{\beta})^2 \sin\beta - \ddot{\beta}\cos\beta\,]\,S_{o_e y_e}^{e} + 2\,l_1\,\ddot{\theta}\,M_e = F_{ext_y}'\,,$$

$$M\,(a_z' - g_z') + \rho\,(a_z' - g_z')\iint_S \zeta\;dx\,dy - \rho\;\ddot{\theta}\,\{\iint_S y\;\zeta\,dx\,dy \tag{3.26}$$

$$- 2 \sum_{1}^{\infty} \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \iint_S (\sinh k_i \zeta + \sinh k_i h) \varphi_i \, dx \, dy - \rho \dot{\theta}^2 \iint_S (1 \zeta + \zeta^2/2) dx \, dy$$

$$+ \rho \sum_{1}^{\infty} \dot{\alpha}_i \iint_S [\cosh k_i (\zeta + h) - 1] \varphi_i \, dx \, dy + \tfrac{1}{2} \rho \iiint_{\tau(t)} \frac{\partial v^2}{\partial z} d\tau - [(\ddot{\theta} - \ddot{\beta}) \sin \beta$$

$$- (\dot{\theta} - \dot{\beta})^2 \cos \beta - \dot{\theta}^2] S_{o_e^\circ y_e} + 2 \, l_1 M_e \dot{\theta}^2 = F_{ext_z'} ,$$

and

$$\{ I_{o'x'} - 2 I_{o_z^1} + \rho \iint [(1^2 - y^2) \zeta + \zeta^2 + \zeta^3/3] dx \, dy + 2 \rho \sum_{1}^{\infty} \frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \cdot \tag{3.27}$$

$$\iint_S y \, \{ (1 + \zeta) \nabla [\frac{\cosh k_i \zeta}{k_i} \nabla \varphi_i] + k_i (1 - h) \varphi_i \cosh k_i h + 2 \, (\sinh k_i \zeta$$

$$+ \sinh k_i h) \varphi_i \} dx \, dy \} \ddot{\theta} + \rho (a_{z'} - g_{z'}) \iint_S y \, \zeta \, dx \, dy - \rho (a_{y'} - g_{y'}) \iint_S (1 \zeta + \zeta^2/2) \, dx \, dy$$

$$+ \rho \sum_{1}^{\infty} \dot{\alpha}_i \iint_S y \, \{ (1 + \zeta) \nabla [\frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i] + 2 [\cosh k_i (\zeta + h) - 1] \varphi_i \} \, dx \, dy$$

$$+ \tfrac{1}{2} \rho \iiint [y \frac{\partial v^2}{\partial z} - (1 + z) \frac{\partial v^2}{\partial y}] d\tau - I_{o_e^\circ x_e} \, \ddot{\beta} + \{ (a_{y'} - g_{y'}) (\cos \beta + 1)$$

$$- (a_{z'} - g_{z'}) \sin \beta + l_1 [2 \, \ddot{\theta} \, (\cos \beta - 1) - 2 \, \dot{\beta} \, \dot{\theta} \, \sin \beta + \dot{\beta}^2 \sin \beta - \ddot{\beta} \cos \beta] \} S_{o_e^\circ y_e}$$

$$+ 2 \, l_1 \, (a_{y'} - g_{y'}) M_e = L_{ext_{x'}} ,$$

respectively.

Formulae (3.7, 9_1, 10, 19, 21, 22, 23, 24, 25, 26, 27) together with the appropriate control system, flight path data and aerodynamic data are sufficient to describe the motion of the system. The dependent variables occurring in these expressions are ξ_i, $\dot{\theta}$, u_y, u_z and β. Note that the body rates u_y, u_z, $\dot{\theta}$ are not generalized coordinates; therefore, to determine the true orientation of the vehicle, it is necessary to express these rates in terms of three independent coordinates such as Euler angles. However, stability analyses of closed loop vehicle attitude control systems treat vehicle rigid body motions as a summation of perturbations from a reference motion and motions in which vehicle body axes remain coincident with reference axes. Moreover, the perturbation quantities are presumed to be infinitesimals so that products of infinitesimals and their derivatives can be neglected. In this case the perturbation equations, when referred to vehicle fixed axes, can be integrated to yield the orientation of the perturbed state with respect to the reference state.

The perturbation equations of motion are obtained by taking the vectorial difference between the perturbed equations of motion and reference equations of motion The resulting expressions are equated to the appropriate perturbed external forces and control system forces.

4/ PERTURBATION EQUATIONS FOR THE PLANAR MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

PERTURBATION EQUATIONS FOR THE PLANAR MOTION OF A LIQUID PROPELLANT LAUNCH VEHICLE

For simplicity suppose the vehicle to be moving in the direction of constant acceleration

$$(0,\ 0,\ a_{z'}) = (0,\ 0,\ a_r) \ ,$$

under the action of the force field

$$(0,\ 0,\ g_{z'}) = (0,\ 0,\ -g_r) \ ,$$

the only external force being the thrust T directed along the longitudinal axis of the missile (Fig. 4). The free surface of the liquid forms a plane perpendicular to $a_{z'}$, i.e., $\zeta = 0$, and the engine is aligned with the longitudinal axis of the vehicle $(\beta = 0)$. We shall call this the reference state.

A small disturbance from the reference state is effected by letting

$$a_{y'} = (a_{y'})_r + \delta a_{y'} = a_r \sin\theta + \delta a_{y'} \ , \tag{4.1}$$

$$a_{z'} = (a_{z'})_r + \delta a_{z'} = a_r \cos\theta + \delta a_{z'} \ ,$$

$$g_{y'} = -g_r \sin\theta \ ,$$

$$g_{z'} = -g_r \cos\theta \ ,$$

in formulae (3.7, 23, 24, 25, 26, 27), and considering $\delta a_{y'}$, $\delta a_{z'}$, θ, β, ζ, ... initially infinitesimal so that when product terms are neglected the resulting equations become linear. These equations are the perturbed equations of motion.

To obtain the perturbation equations, project the reference state equation of motion onto the instantaneous position of the body axes and subtract them from the perturbed equations of motion. Thus, we get perturbation force equation along axis $o'y'$

$$M\ \delta a_{y'} + \rho \sum_{1}^{\infty} (y,\ \varphi_1)\ \ddot{\xi}_1 + 2\ (S_{o_e y_e}^{e} + l_1 M_e)\ \ddot{\theta} - S_{o_e y_e}^{e}\ \ddot{\beta} = \delta F_{ext_{y'}} \ , \tag{4.2}$$

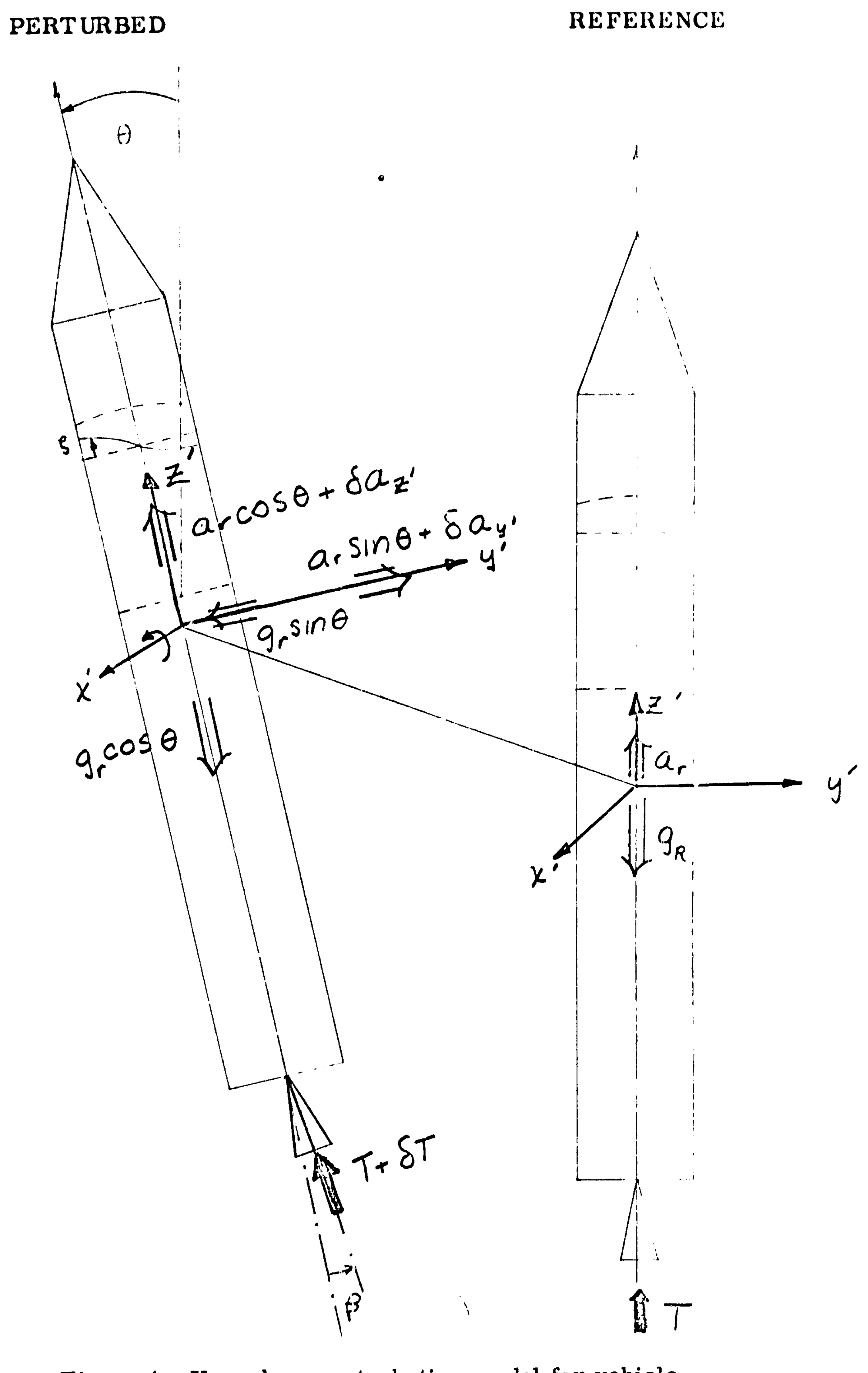

Figure 4. Yaw plane perturbation model for vehicle.

Perturbation force equation along axis $o'z'$

$$M\ \delta a_z' = \delta F_{ext_z}' \ , \tag{4.3}$$

Perturbation moment equation about axis $o'x'$

$$\{ I_{o'x'} - 4 I^1_{o_2} + 8\rho \sum_1^{\infty} \frac{\cosh k_1 h - 1}{k_1 \sinh k_1 h} \frac{(y, \varphi_1)^2}{\|\varphi_1\|^2} \} \ddot{\theta} + \rho \sum_1^{\infty} [\frac{2(\cosh k_1 h - 1)}{k_1 \sinh k_1 h} - 1](y, \varphi_1)\ddot{\xi}_1$$

$$+ \rho(a_r + g_r) \sum_1^{\infty} (y, \varphi_1)\, \xi_1 + 2(S_{o_e y_e}^{e} + l_1 M_e)\, \delta a_y' - (I_{o_e y_e}^{e} + l_1 S_{o_e y_e}^{e}) \ddot{\beta} \tag{4.4}$$

$$+ 2(a_r + g_r)(S_{o_e y_e}^{e} + l_1 M_e)\, \theta = \delta L_{ext_x}' \ ,$$

Perturbation engine equation

$$I_{o_e x_e}^{e} \ddot{\beta} - (I_{o_e y_e}^{e} + l_1 S_{o_e y_e}^{e}) \ddot{\theta} - S_{o_e y_e}^{e}\, \delta a_y' - (a_r + g_r) S_{o_e y_e}^{e}\, \theta + (a_r + g_r) S_{o_e y_e}^{e}\, \beta = Q_\beta \ , \tag{4.5}$$

Constancy of pressure at liquid free surface

$$\frac{\rho\|\varphi_1\|^2}{k_1 \tanh k_1 h} \ddot{\xi}_1 + \rho\|\varphi_1\|^2 (a_r + g_r)\, \xi_1 + \rho[\frac{2(\cosh k_1 h - 1)}{k_1 \sinh k_1 h} - 1](y, \varphi_1)\ddot{\theta} \tag{4.6}$$

$$+ \rho(y, \varphi_1)\, \delta a_y' + \rho(a_r + g_r)(y, \varphi_1)\, \theta = 0 \ ,$$

in which $\delta F_{ext_y}'$, $\delta F_{ext_z}'$, $\delta L_{ext_x}'$, Q_β are perturbation forces and moments.

Also

$$\alpha_1 = \frac{\dot{\xi}_1}{k_1 \sinh k_1 h} + 2\dot{\theta} \frac{(y, \varphi_1)}{k_1 \|\varphi_1\|^2 \cosh k_1 h} \frac{(\cosh k_1 h - 1)}{\sinh k_1 h} \ .$$

Denote by Y, Z the components of the displacement vector from the origin of the reference state to that of the perturbed state, measured along the instantaneous position of the body axes; then, to the same order of smallness, we have

$$\delta a_{y'} = \ddot{Y} \tag{4.7}$$

$$\delta a_{z'} = \ddot{Z}$$

With the definitions given previously, we see that the action of the liquid in formulae (4.2, 3, 4) is represented by the forces and moment

$$- \delta F_y^{1\prime} = M_1 \ddot{Y} + \rho \iint_S y\, \zeta_{tt}\, dx\, dy \qquad (4.8)$$

$$- \delta F_z^{1\prime} = M_1 \ddot{Z}$$

$$- \delta L_x^{1\prime} = \rho\, \ddot{\theta} \iiint_T (\nabla \phi^*)^2\, d\tau + \rho \iint_S \phi^* \zeta_{tt}\, dx\, dy + \rho\, (a_r + g_r) \iint_S y\, \zeta\, dx\, dy$$

in which

$$\phi^* = \varphi^* - y\,(1 + z) \ , \qquad (4.9)$$

$$\varphi^* = 2 \sum_i^{\infty} \frac{(y,\ \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \left\{ \frac{\cosh k_i h - 1}{\sinh k_i h}\ \varphi_i \cosh k_i\,(z + h) + \varphi_i \sinh k_i z \right\} ,$$

$$\rho \iiint_T (\nabla \phi^*)^2\, d\tau = \rho \iiint_T [y^2 + (1 + z)^2]\, d\tau - 4\rho \iiint_T y^2\, d\tau + \rho \iiint_T (\nabla \varphi^*)^2\, d\tau \ ,$$

$$\rho \iiint_T (\nabla \varphi^*)^2 = 8\rho \sum_1^{\infty} \frac{\cosh k_i h - 1}{k_i \sinh k_i h}\ \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2} \ ,$$

$$\zeta = \sum_1^{\infty} \xi_i\, \varphi_i + \xi_0$$

and ϕ^*, φ^* satisfy

$$\Delta \phi^* = \Delta \varphi^* = 0,\ P \epsilon\, T \qquad (4.10)$$

$$\frac{\partial \phi^*}{\partial n} = y \cos\,(n,\ z) - (1 + z) \cos\,(n,\ y),\ P \epsilon\, \Sigma,\ S,$$

$$\frac{\partial \varphi^*}{\partial n} = 2\, y \cos\,(n,\ z)\ ,\ P \epsilon\, \Sigma\ ,\ S\ .$$

From (2.34) et. seq., we see that ϕ^*, φ^* are modified Stokes potentials which are determined solely from the geometry of the vessel, the free surface being "capped".

With the introduction of potential ψ, where

$$\Delta \psi = 0 \quad ,\ P \epsilon\, T\ , \qquad (4.11)$$

$$\frac{\partial \psi}{\partial n} = \begin{cases} 0 & ,\ P \epsilon\, \Sigma\ , \\ \zeta_t & ,\ P \epsilon\, S\ , \end{cases}$$

formula (4.6) may be written as

$$\rho \frac{\partial \psi}{\partial t} + \rho\, \ddot{\theta}\, \varphi^{*} + \rho\, y\, \ddot{Y} + \rho\,(a_r + g_r)\,\zeta + \rho\,(a_r + g_r)\,\theta\, y = 0 \ , \tag{4.12}$$

$$\psi = \sum_1^{\infty} \dot{\xi}_i(t) \frac{\cosh k_i (z + h)}{k_i \sinh k_i h} \varphi_i \ ,$$

$$y = \sum_1^{\infty} \frac{(y,\ \varphi_i)}{\| \varphi_i \|^2} \varphi_i \ .$$

Thus, it follows from formulae (4.8, 9, 10, 11, 12) that the action of the liquid can be determined also from a computation of forces and moment (along $o'y'$, $o'z'$ and about $o'x'$ respectively) resulting from the solution of the linear problem shown in Fig. 5, Y, Z, θ, ζ being taken as infinitesimals.

Formula (4.9_3) represents the moment of inertia of the "capped" liquid about the center of rotation, and obviously does not equal the moment of inertia of the "frozen" liquid about that point. Indeed, if we denote by $I_0^{*\prime}$ the moment of inertia of the "capped" liquid about $0'$, and by $\tilde{I}_0{}'$ that of the "frozen" liquid about the same point, then

$$\frac{I_0^{*\prime}}{\tilde{I}_0{}'} = 1 - \frac{4\rho}{\tilde{I}_0{}'} \iiint_{\tau} y^2\, d\tau + \frac{8\rho}{\tilde{I}_0{}'} \sum_1^{\infty} \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2} \ , \tag{4.13}$$

$$\tilde{I}_0{}' = \rho \iiint_{\tau} [y^2 + (l + z)^2]\, d\tau \ ,$$

a ratio which can be shown to be less than unity. In particular,

$$\frac{I_{cg}^{*}}{\tilde{I}_{cg}} = 1 - \frac{4\rho}{\tilde{I}_{cg}} \iiint_{\tau} y^2\, d\tau + \frac{8\rho}{\tilde{I}_{cg}} \sum_1^{\infty} \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2} \ , \tag{4.14}$$

$$\tilde{I}_{cg} = \rho \iiint_{\tau} [y^2 + (z + h/2)^2]\, d\tau \ ,$$

at the center of gravity of the liquid $(l = h/2)$. Moreover,

$$I_0^{*\prime} = I_{cg}^{*} + M_l\, (l - h/2)^2 \ , \tag{4.15}$$

$$\tilde{I}_0{}' = \tilde{I}_{cg} + M_l\, (l - h/2)^2 \ .$$

The behavior of (4.13) may be illustrated effectively by considering a rectangular tank such as shown in Fig. 6. We have, for this example,

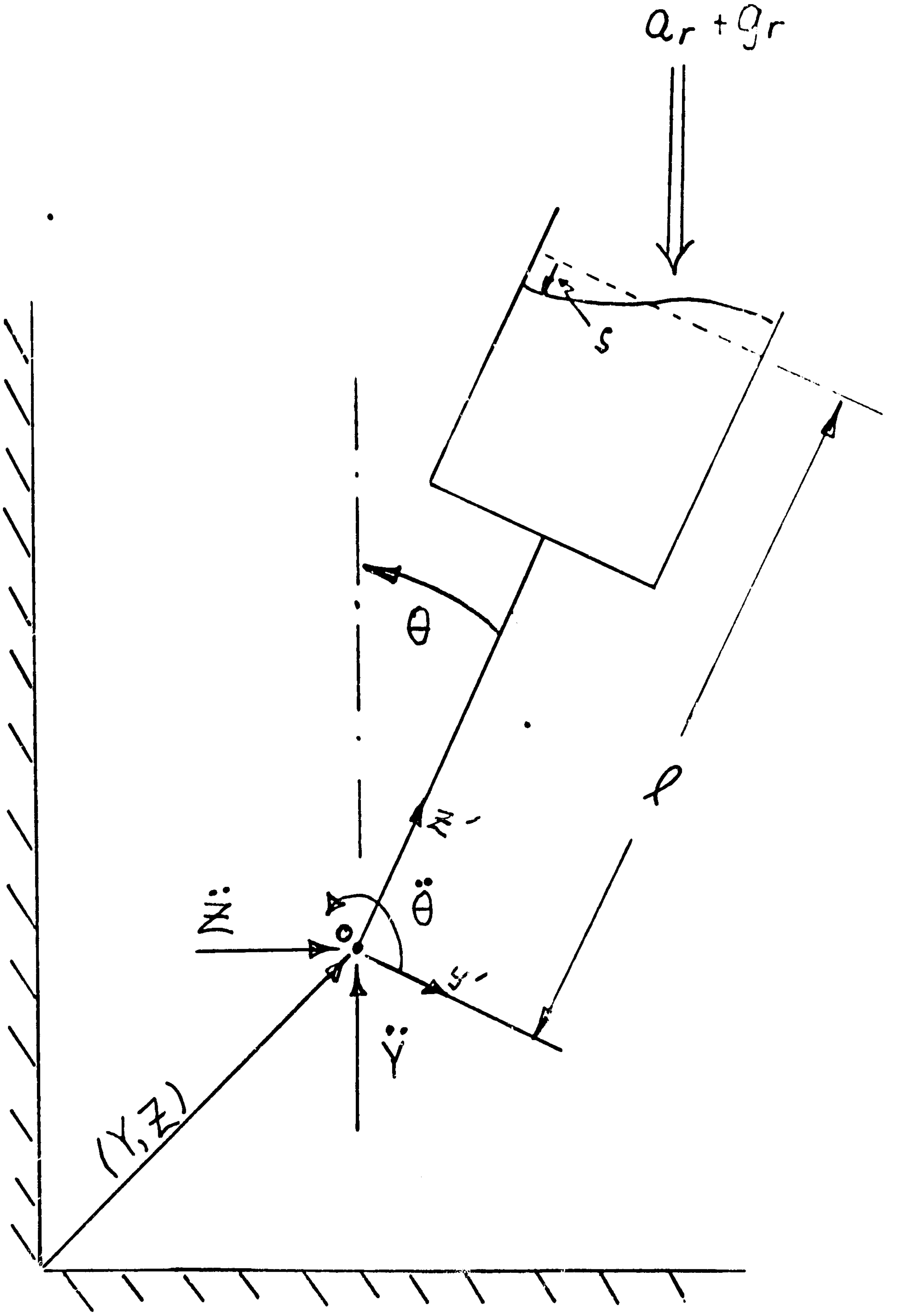

Figure 5. Linear model for simulating action of liquid in perturbation equations.

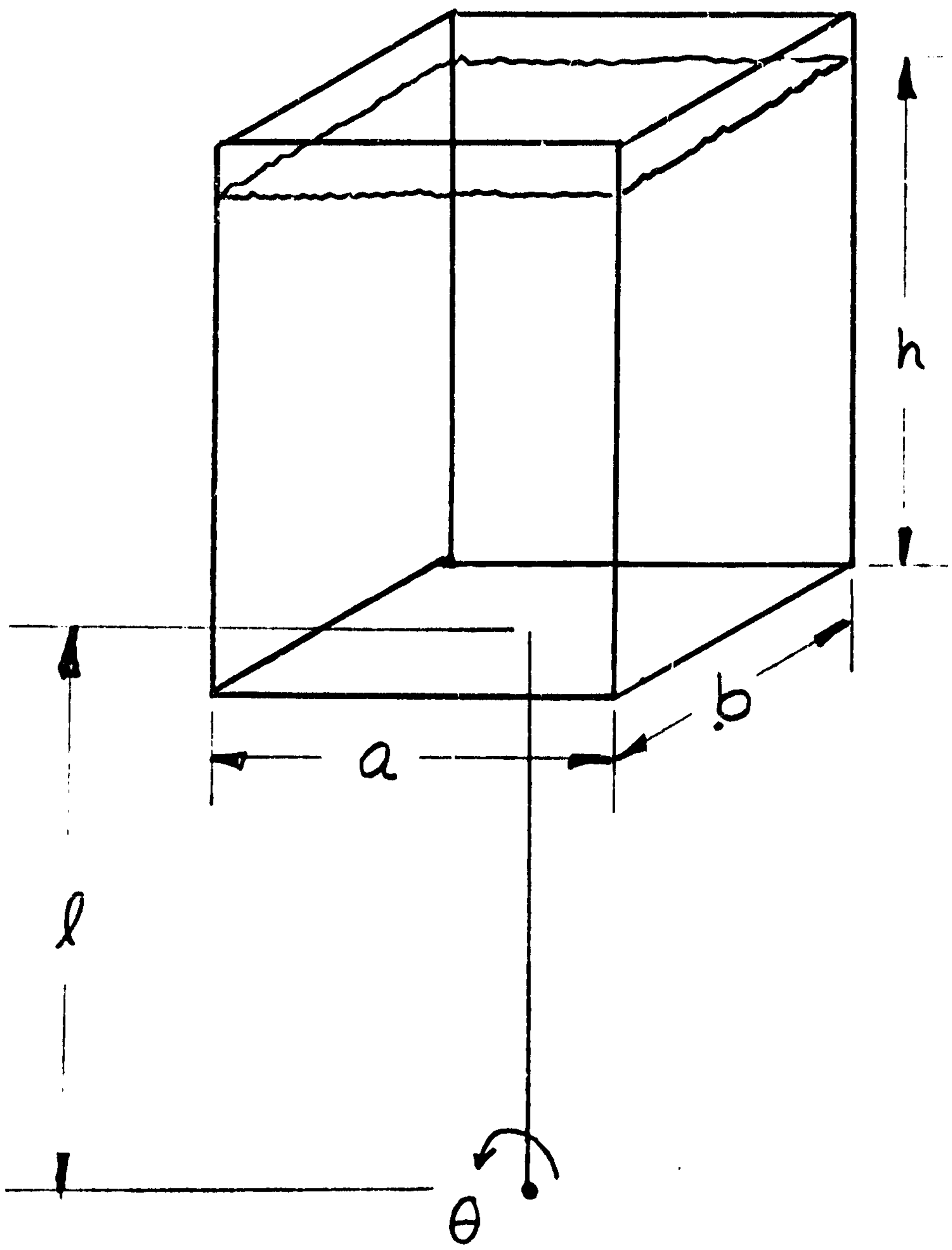

Figure 6. Liquid filled rectangular tank.

$$\frac{I_0^{*\prime}}{\tilde{I}_0{}'} = 1 - \frac{(1/3)}{\{(l/a)^2 - (l/a)(h/a) + 1/3\,(h/a)^2 + 1/12\}} + \frac{(64/\pi^5)\,(a/h)}{\{(l/a)^2 - (l/a)(h/a) + 1/3\,(h/a)^2 + 1/12\}} \cdot$$

$$\sum_{1,3,\ldots} \{1/i^5 \tanh i\pi h/2a\} \;. \qquad (4.16)$$

This ratio is plotted in Fig. 7 as a function of l/h with a/h as parameter. Note that for any given a/h the "capped" moment of inertia of the liquid is less for rotation about the center of gravity of the liquid ($l = h/2$) than for any other position. Similar results may be obtained for other configurations.

To put formulae (4.8, 12) into symmetrical form, we introduce the transformation

$$\xi_i = \frac{(y,\ \varphi_i)}{\|\varphi_i\|^2}\ q_i \ , \qquad (4.17)$$

getting

$$- \delta F_y^{1\prime} = M_1\ \delta a_y{}' + \rho \sum_1^\infty \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ q_i \ ,$$

$$- \delta F_z^{1\prime} = M_1\ \delta a_z{}' \ ,$$

$$- \delta L_x^{1\prime} = I_0^*\ \ddot\theta + \rho \sum_1^\infty \left[\frac{2\,(\cosh k_i\ h - 1)}{k_i \sinh k_i\ h} - 1\right] \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ \ddot q_i \qquad (4.18)$$

$$+ \rho\ (a_r + g_r) \sum_1^\infty \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ q_i \ ,$$

$$\frac{\rho\ (y,\ \varphi_i)^2}{k_i\ \|\varphi_i\|^2 \tanh k_i\ h}\ \ddot q_i + \rho\ (a_r + g_r) \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ q_i + \rho \left[\frac{2\,(\cosh k_i\ h - 1)}{k_i \sinh k_i\ h} - 1\right] \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ \ddot\theta$$

$$+ \rho\,(a_r + g_r) \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ \theta + \rho\ \frac{(y,\ \varphi_i)^2}{\|\varphi_i\|^2}\ \delta a_y{}' = 0 \ .$$

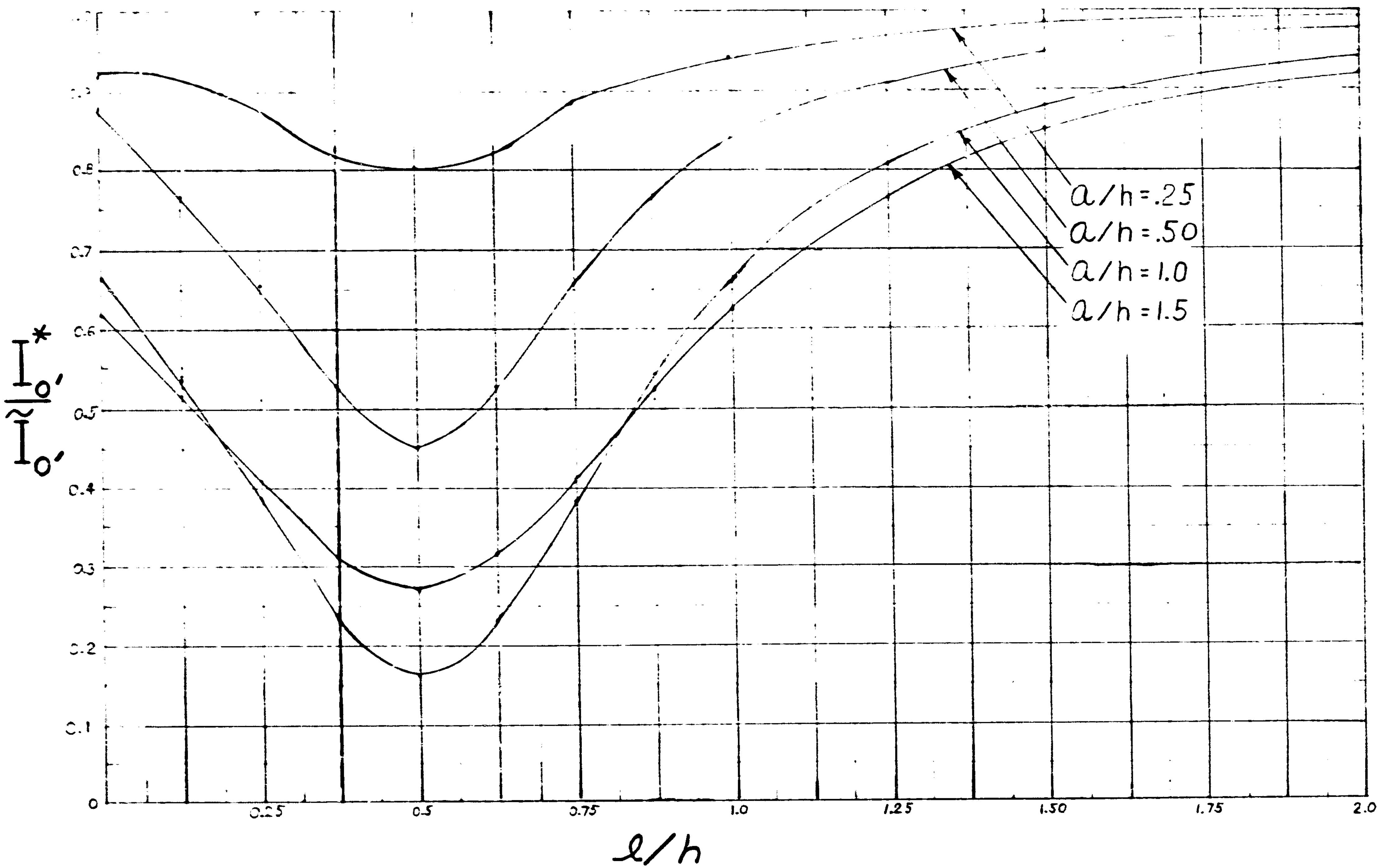

Figure 7. Ratio of "capped" moment of inertia to "frozen" moment of inertia for rectangular liquid filled tank.

5/ MECHANICAL ANALOG FOR REPRESENTING THE ACTION OF THE LIQUID IN THE PERTURBATION EQUATIONS OF MOTION

MECHANICAL ANALOG FOR REPRESENTING THE ACTION OF THE LIQUID IN THE PERTURBATION EQUATIONS OF MOTION

We enquire how to duplicate the action of the liquid, represented by formulae (4.18), with a mechanical system. Accordingly, consider the system of pendula attached to the vehicle shown in Fig. 8, where

M_{pi} ~ Mass of i th pendulum,

L_{si} ~ Distance from center of rotation to hinge point of i th pendulum,

L_{pi} ~ Length of i th pendulum arm,

I_O ~ Rigid moment of inertia,

M_O ~ Rigid mass,

L_O ~ Distance from center of rotation to point at which M_O is situated.

The velocity of the mass of the i th pendulum is simply

$$\bar{V}_i = \{ 0, [(u_y' - L_{si}\,\dot{\theta}) + (\dot{\theta} + \dot{q}_i)\,L_{pi}\cos q_i], [u_z' + L_{pi}(\dot{\theta} + \dot{q}_i)\sin q_i] \} \quad , \tag{5.1}$$

and that of mass M_O is

$$\bar{V}_O = \{ 0, (u_y' - L_O\,\dot{\theta}), u_z' \} \ . \tag{5.2}$$

It follows that the kinetic energy of the mechanical system is

$$\begin{aligned} T_{ms} = {} & \tfrac{1}{2}(u_y'^2 + u_z'^2)\{M_O + \sum_1^N M_{pi}\} + \tfrac{1}{2}\dot{\theta}^2\{I_O + M_O L_O^2 + \sum_1^N M_{pi}(L_{si} - L_{pi})^2\} \\ & - \dot{\theta}\,u_y'\{M_O L_O + \sum_1^N M_{pi}(L_{si} - L_{pi})\} + \tfrac{1}{2}\sum_1^N M_{pi}L_{pi}^2\,\dot{q}_i^2 + \dot{\theta}\sum_1^N M_{pi}L_{pi}^2\,\dot{q}_i \\ & + \sum_1^N M_{pi}L_{pi}\,\dot{q}_i\{u_z'\sin q_i + (u_y' - L_{si}\,\dot{\theta})\cos q_i\} \end{aligned} \tag{5.3}$$

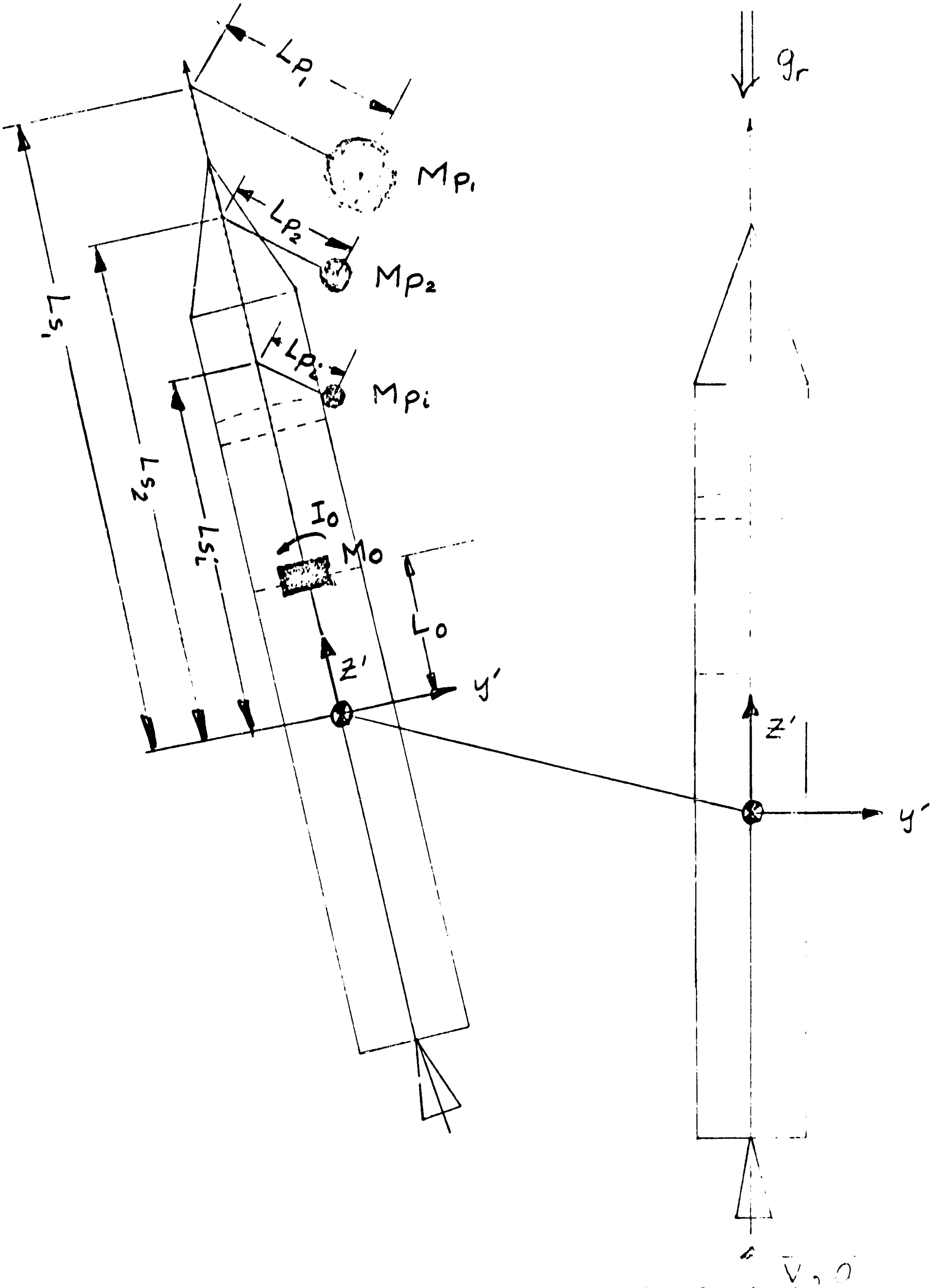

Figure 8. Mechanical analog for simulating liquid action in yaw plane perturbation equations.

$$+ \dot{\theta} \sum_{1}^{N} M_{pi} L_{pi} \{ u_z' \sin q_i + (u_y' - L_{si} \dot{\theta}) (\cos q_i - 1) \} \; .$$

The equations of motion may be obtained in manner similar to (3.2),

$$\frac{d}{dt} \left(\frac{\partial T}{\partial u_y'}\right) - \dot{\theta} \frac{\partial T}{\partial u_z'} = F_{ext_y}' - \iiint_{\tau^*} g_y' \, dm - \iiint_{\tau_e} g_y' \, dm - g_y' [M_0 + \sum_{1}^{N} M_{pi}] \; , \tag{5.4}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial u_z'}\right) + \dot{\theta} \frac{\partial T}{\partial u_y'} = F_{ext_z}' - \iiint_{\tau^*} g_z' \, dm - \iiint_{\tau_e} g_z' \, dm - g_z' [M_0 + \sum_{1}^{N} M_{pi}] \; ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}}\right) + u_y' \frac{\partial T}{\partial u_z'} - u_z' \frac{\partial T}{\partial u_y'} = L_{ext_x}' - \iiint_{\tau^*} (y' g_z' - z' g_y') \, dm$$

$$- \iiint_{\tau_e} [(y_e \cos \beta - z_e \sin \beta) g_z' + (l_1 + z_e \cos \beta + y_e \sin \beta) g_y'] \, dm + M_0 L_0 \, g_y'$$

$$- \sum_{1}^{N} M_{pi} \{ g_z' L_{pi} \sin q_i - g_y' (L_{si} - L_{pi} \cos q_i) \} \; ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = - \iiint_{\tau_e} (g_z' \sin \beta - g_y' \cos \beta) z_e \, dm + \iiint_{\tau_e} (g_z' \cos \beta + g_y' \sin \beta) y_e \, dm + Q_\beta \; ,$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i}\right) \quad \frac{\partial T}{\partial q_i} = - M_{pi} L_{pi} (g_y' \cos q_i + g_z' \sin q_i) \; ,$$

where

$$T = T_{mb} + T_e + T_{ms} \; .$$

T_{mb} and T_e are given in (3.1_1) and (3.1_2) .

Repeating the same simplifications and arguments used to obtain the perturbation equations for the motion of the liquid propellant vehicle, (5.4) gives, after considerable work,

$$(M_{mb} + M_e) \delta a_y' + 2 (S_{o_e y_e}^{e} + l_1 M_e) \ddot{\theta} - S_{o_e y_e}^{e} \ddot{\beta} = \delta F_{ext_y}' - \{[M_0 + \sum_{1}^{N} M_{pi}] \delta a_y' \tag{5.5}$$

$$+ \sum_{1}^{N} M_{pi} L_{pi} \ddot{q}_i \; ,$$

$$(M_{mb} + M_e) \delta a_z' = \delta F_{ext_z}' - \{ [M_0 + \sum_{1}^{N} M_{pi}] \delta a_z' \} \; ,$$

$$M_{pi} L_{pi} (L_{pi} - L_{si}) = \rho \left[\frac{2(\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} ,$$

$$I_0 + M_0 L_0^2 + \sum_1^N M_{pi} (L_{si} - L_{pi})^2 = I_0^{*\prime} .$$

Relations (5.7) contain six unknowns M_{pi}, L_{pi}, L_{si}, L_0, M_0, I_0. One more equation is needed to make the system determinate, namely, the first moment of mass about the center of rotation,

$$M_0 L_0 + \sum_1^N M_{pi} (L_{si} - L_{pi}) = M_1 (1 - h/2) . \tag{5.8}$$

solving (5.7, 8) simultaneously, we get

$$L_{pi} = \frac{1}{k_i \tanh k_i h} , \tag{5.9}$$

$$M_{pi} = \rho \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h ,$$

$$L_{si} = \frac{1}{k_i \tanh k_i h} + \left[1 - \frac{2(\cosh k_i h - 1)}{k_i \sinh k_i h} \right] ,$$

$$M_0 = M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h ,$$

$$L_0 = \frac{M_1 (1 - h/2) + \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[\frac{2(\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] k_i \tanh k_i h}{M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h} ,$$

$$I_0 = I_0^{*\prime} - \rho \sum \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[1 - \frac{2(\cosh k_i h - 1)}{k_i \sinh k_i h} \right]^2 k_i \tanh k_i h$$

$$- \frac{\left\{ M_1 (1 - h/2) + \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} \left[\frac{2(\cosh k_i h - 1)}{k_i \sinh k_i h} - 1 \right] k_i \tanh k_i h \right\}^2}{M_1 - \rho \sum_1^N \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} k_i \tanh k_i h} ,$$

in which

$$I_0^{*\prime} = \tilde{I}_0' - 4 \rho \iiint_\tau y^2 \, d\tau + 8 \rho \sum_1^N \frac{\cosh k_i h - 1}{k_i \sinh k_i h} \frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} .$$

$$(I_{O}{}^{Fb}_{x'} + I_{O}^{e}{}_{x'})\ddot{\theta} + 2(S_{O_{e}y_{e}}^{e} + l_1 M_e)\,\delta a_{y'} - (I_{O_{e}y_{e}}^{e} + l_1 S_{O_{e}y_{e}}^{e})\ddot{\beta}$$

$$+ 2(a_r - g_r)(S_{O_{e}y_{e}}^{e} + l_1 M_e)\,\theta - (a_r + g_r)S_{O_{e}y_{e}}^{e}\,\beta = \delta L_{ext_{x'}} - \{[I_O + M_O L_O^2$$

$$+ \sum_1^N M_{pi}(L_{si} - L_{pi})^2]\ddot{\theta} + \sum_1^N M_{pi} L_{pi}(L_{pi} - L_{si})\,\ddot{q}_i + (a_r + g_r)\sum_1^N M_{pi} L_{pi}\,q_i\} ,$$

$$M_{pi} L_{pi}^2\,\ddot{q}_i + (a_r + g_r) M_{pi} L_{pi}\,q_i + M_{pi} L_{pi}(L_{pi} - L_{si})\,\ddot{\theta} + (a_r + g_r) M_{pi} L_{pi}\,\theta$$

$$+ M_{pi} L_{pi}\,\delta a_{y'} = 0 ,$$

$$I_{O_{e}y_{e}}^{e}\,\ddot{\beta} + (a_r + g_r)\,S_{O_{e}y_{e}}^{e}\,\beta - (I_{O_{e}y_{e}}^{e} + l_1 S_{O_{e}y_{e}}^{e})\ddot{\theta} - (a_r + g_r)\,S_{O_{e}y_{e}}^{e}\,\theta - S_{O_{e}y_{e}}^{e}\,\delta a_{y'} = Q_{\beta} ,$$

in which the bracketed terms on the right hand members of $(5.5_{1,2,3})$ and (5.5_4) represent the action of the mechanical system. Explicitly,

$$- \delta F_{y'}^{ms} = [M_O + \sum_1^N M_{pi}]\,\delta a_{y'} + \sum_1^N M_{pi} L_{pi}\,\ddot{q}_i , \tag{5.6}$$

$$- \delta F_{z'}^{ms} = [M_O + \sum_1^N M_{pi}]\,\delta a_{z'} ,$$

$$- \delta L_{x'}^{ms} = [I_O + M_O L_O^2 + \sum_1^N M_{pi}(L_{si} - L_{pi})^2]\ddot{\theta} + \sum_1^N M_{pi} L_{pi}(L_{pi} - L_{si})\,\ddot{q}_i$$

$$+ (a_r + g_r)\sum_1^N M_{pi} L_{pi}\,q_i ,$$

$$M_{pi} L_{pi}^2\,\ddot{q}_i + (a_r + g_r) M_{pi} L_{pi}\,q_i + M_{pi} L_{pi}(L_{pi} - L_{si})\ddot{\theta} + (a_r + g_r) M_{pi} L_{pi}\,\theta$$

$$+ M_{pi} L_{pi}\,\delta a_{y'} = 0 .$$

The forces, moment and surface wave height terms in formulae (4.18) will match formulae (5.6) for a finite (or infinite) number of pendula if the following associations are made:

$$M_O + \sum_1^N M_{pi} = M_1 , \tag{5.7}$$

$$M_{pi} L_{pi} = \rho\,\frac{(y, \varphi_i)^2}{\|\varphi_i\|^2} ,$$

$$M_{pi} L_{pi}^2 = \frac{\rho(y, \varphi_i)^2}{k_i\,\|\varphi_i\|^2 \tanh k_i h} ,$$

The amplitudes of the wave height are related to the angular displacements of the pendula by

$$\xi_i = \frac{(y, \varphi_i)}{\|\varphi_i\|^2} q_i .$$

Thus, the action of the liquid represented by formulae (4.18) may be duplicated with a mechanical system--a system of pendula plus a concentrated mass and moment of inertia. A similar analogy may be effected with a system of springs and masses.

6/CORRELATION OF ANALYSIS OF VESSEL OF GENERAL SHAPE POSSESSING ROTATIONAL SYMMETRY WITH EXISTING MSFC ANALYSIS

CORRELATION OF ANALYSIS OF VESSEL OF GENERAL SHAPE POSSESSING ROTATIONAL SYMMETRY WITH EXISTING MSFC ANALYSIS

Formulae (4.8, 12_1) hold for containers of general shape possessing rotational symmetry about the z-axis, Figure 9, if we approximate the various volume integrals occurring in the analysis as

$$\iiint_{\tau(t)} (\ldots)\, d\tau = \iiint_{\tau} (\ldots)\, d\tau + \iint_{S} ds \int_0^{\zeta} (\ldots)\, dz \quad ,$$

and if we evaluate directional derivatives over the undisturbed surfaces Σ , S. Such simplifications can be justified. Thus, for the action of the liquid contained in the vessel of Figure 9, we have

$$- \delta F_y^{1\prime} = M_1\, \delta a_{y'} + \rho \iint_S y\, \zeta_{tt}\, ds \quad , \qquad (6.1)$$

$$- \delta F_z^{1\prime} = M_1\, \delta a_{z'} \quad ,$$

$$- \delta L_x^{1\prime} = \rho\, \ddot{\theta} \iiint_{\tau} (\nabla \phi^*)^2 d\tau + \rho \iint_S \phi^* \zeta_{tt}\, ds + \rho (a_r + g_r) \iint_S y\, \zeta\, ds \quad ,$$

$$\rho \frac{\partial \psi}{\partial t} + \rho\, \ddot{\theta}\, \phi^* + \rho\, y\, \delta a_{y'} + \rho (a_r + g_r)\, \zeta + \rho (a_r + g_r)\, \theta\, y = 0 \quad ,$$

where

$$\Delta \phi^* = 0 \;, \quad P \in \tau \qquad\qquad \Delta \psi = 0 \;, \quad P \in \tau \;,$$

$$\frac{\partial \phi^*}{\partial n} = y \cos(n, z) - (1 + z) \cos(n, y), \; P \in \Sigma, S, \qquad \frac{\partial \psi}{\partial n} = \begin{cases} 0 \;, & P \in \Sigma \;, \\ \zeta_t \;, & P \in S \;. \end{cases}$$

This system may be brought into a form suitable for comparison with the analysis of [2] , if we substitute

$$z - L \text{ for } z \quad ,$$

$$L + L_1 \text{ for } l \quad ,$$

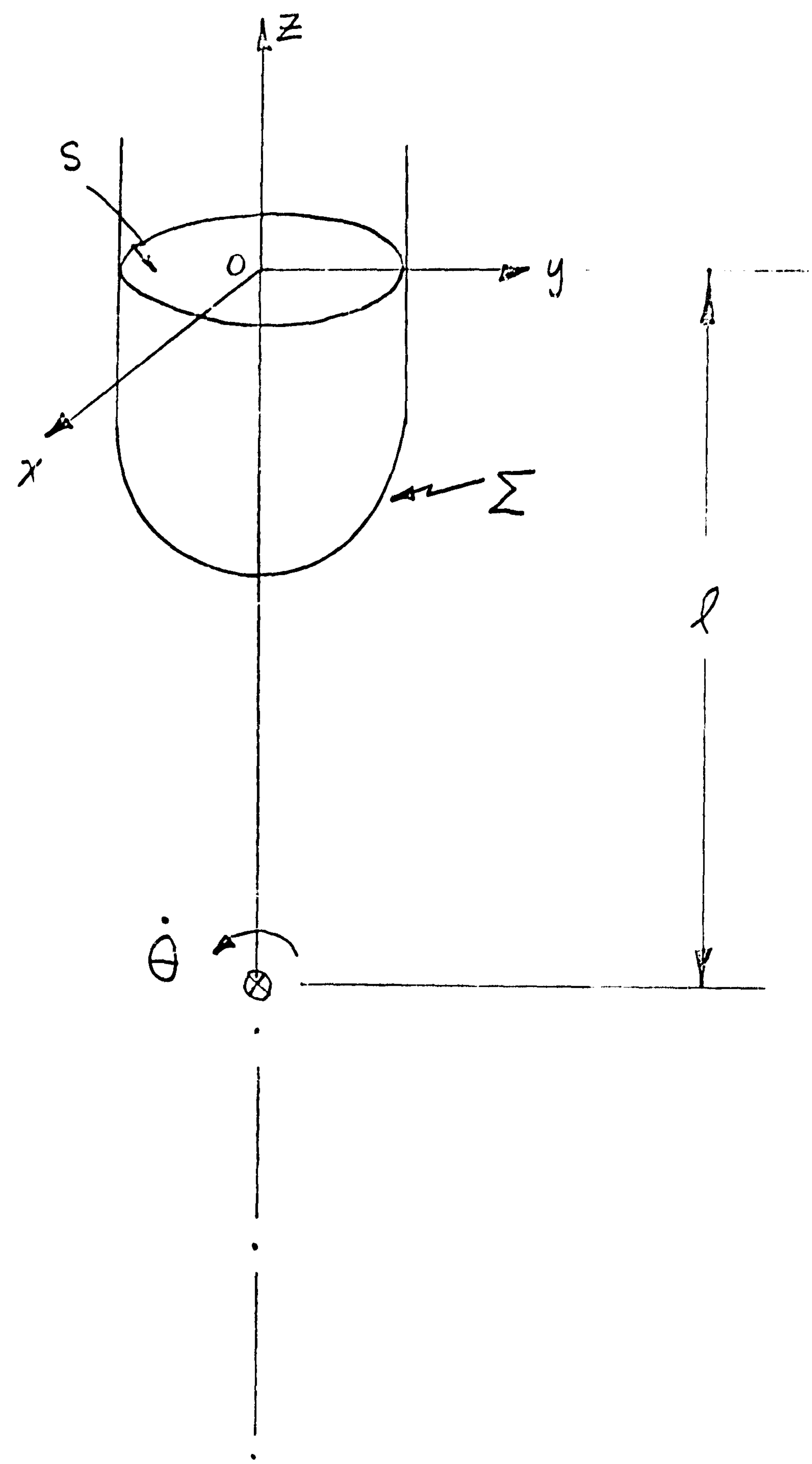

Figure 9. Arbitrary shaped container with rotational symmetry.

and let

$$\varphi^* = y\,(z + L_1) - L^2\,\psi^*\ ,$$

where the coordinates are now reckoned from the center of gravity of the undisturbed liquid; L is the distance from the center of gravity of the undisturbed liquid to the quiescent free surface; L_1 is the distance from the center of rotation of the vehicle to the center of gravity of the undisturbed liquid (Figure 10). These substitutions give

$$-\,\delta F^{1}_{y'} = M_1\,\delta a_{y'} + \rho \iint_S y\,\zeta_{tt}\,ds \tag{6.2}$$

$$-\,\delta F_{z'} = M_1\,\delta a_{z'}\ ,$$

$$-\,\delta L^{1}_{x'} = I^{*}_{0}\,\ddot{\theta} + \rho \iint_S \left[L\,(y - L\,\psi^*) + L_1\,y\right]\zeta_{tt}\,ds + \rho\,(a_r + g_r)\iint y\,\zeta\,ds\ ,$$

$$\rho\,\frac{\partial \psi}{\partial t} + \rho\,\ddot{\theta}\left[L(y - L\psi^*) + L_1 y\right] + \rho y\,\delta a_{y'} + \rho\,(a_r + g_r)\zeta + \rho\,(a_r + g_r)\,\theta y = 0\ ,$$

in which

$$\Delta\psi^* = 0 \qquad ,\quad P\epsilon\,\tau \tag{6.3}$$

$$\frac{\partial \psi^*}{\partial n} = \frac{2\,(L_1 + z)\cos(n,\,y)}{L^2}\ ,\quad P\epsilon\,\Sigma_1\ S\ ,$$

$$I^{*}_{0'} = \rho \iiint_\tau (y^2 + z^2)\,d\tau - 4\,\rho \iiint_\tau z^2\,d\tau + 2\,\rho\,L^2 \iint_{S+\Sigma} \psi^* z\cos(n,\,y)\,ds + M_1 L_1^2\ .$$

We can express the solution for ψ and ζ as a generalized fourier series of eigenfunctions ψ_i,

$$\zeta = \sum_1 \xi_i\,\psi_i\ , \tag{6.4}$$

$$\psi = \sum_1 \frac{L}{K_i}\,\dot{\xi}_i\,\psi_i\ ,$$

where ψ_i satisfies

$$\Delta\psi_i = 0\ ,\quad P\epsilon\,\tau\ , \tag{6.5}$$

$$\frac{\partial \psi_i}{\partial n} = \begin{cases} 0\ , & P\epsilon\,\Sigma\ ,\\ \equiv \dfrac{\partial \psi_i}{\partial z} = \dfrac{K_i}{L}\,\psi_i\ , & P\epsilon\,S\ , \end{cases}$$

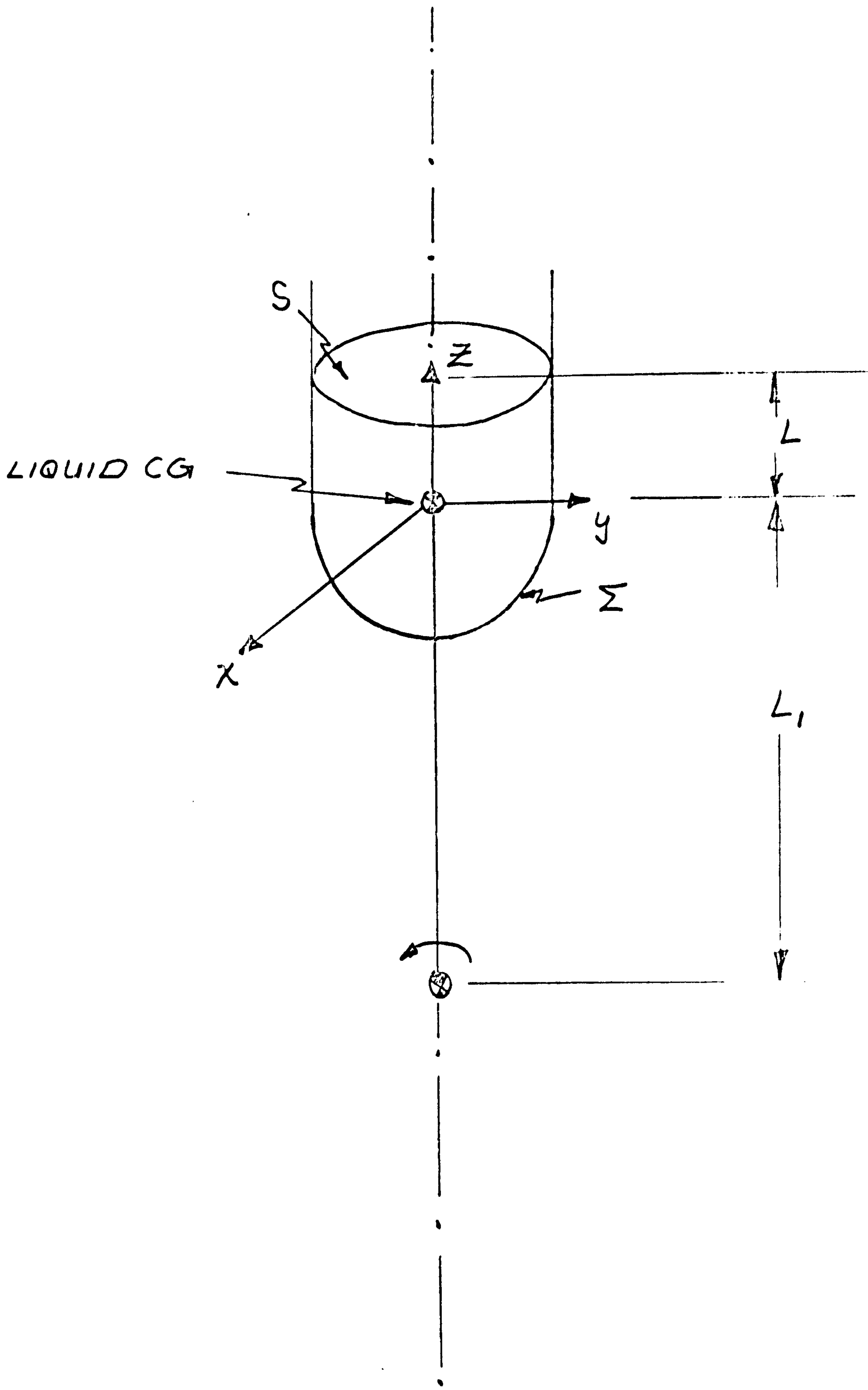

Figure 10. Arbitrary shaped container with coordinate system located at undisturbed liquid center-of-gravity.

$$\iint_S \psi_i \psi_j = 0 , \quad i \neq j .$$

Introducing (6.4) into (6.5), applying integral transformations, using boundary conditions (6.3_2, 5_2) and orthogonality condition (6.5_3), we get

$$- \delta F_y^{1\prime} = M_1 \ \delta a_y' + M_1 \sum_1^{\infty} \| \psi_i \|^2 b_i \ddot{\xi}_i \qquad (6.6)$$

$$- \delta F_z^{1\prime} = M_1 \delta a_{y'}$$

$$- \delta L_x^{1\prime} = I_0^{*\prime} \ddot{\theta} + M_1 \sum_1^{\infty} \| \psi_i \|^2 \{ (a_r + g_r) b_i \xi_i + [L (b_i - h_i) - L_i b_i] \ddot{\xi}_i \} ,$$

$$\ddot{\xi}_i + \frac{(a_r + g_r)}{L} K_i \xi_i + K_i [L (b_i - h_i) - L_i b_i] \ddot{\theta} + K_i (a_r + g_r) b_i \theta + K_i b_i \delta a_y' = 0 ,$$

where

$$\frac{\| \psi_i \|^2 V}{L} = \iint_S \psi_i^2 \, ds , \quad V = \iiint_\tau d\tau , \qquad (6.7)$$

$$b_i V \| \psi_i \|^2 = \iint_S y \psi_i \, ds , \quad K_i h_i V \| \psi_i \|^2 = 2 \iint_{S+\Sigma} z \psi_i \cos (n, y) \, ds .$$

Formulae ($6.6_{1,2,3}$) are equivalent to formulae (3.47, 46, 48) of [2], except for terms involving the first moment of mass about the center of rotation of the vehicle. In our analysis, such terms have been included in the rigid vehicle dynamics. Formula (6.6_4) differs from (3.49) of [2] by the term $K_i (a_r + g_r) b_i \theta$. The reason for this is that the equations for the action of the liquid in [2] were not perturbed but arbitrarily linearized.

On the other hand if we compare (6.6) with (5.6), we get

$$M_0 + \sum_1^N M_{pi} = M_1 , \qquad (6.8)$$

$$M_{pi} L_{pi} = M_1 L \| \psi_i \|^2 b_i^2 ,$$

$$M_{pi} L_{pi}^2 = M_1 \frac{L^2 b_i^2}{K_i} \| \psi_i \|^2 ,$$

$$M_{pi} L_{pi} (L_{pi} - L_{si}) = M_1 L b_i \| \psi_i \|^2 [L (b_i - h_i) - L_1 b_i] ,$$

$$I_0 + M_0 L_0^2 + \sum_1^N M_{pi} (L_{si} - L_{pi})^2 = I_0^{*\prime} ,$$

and the first moment of mass about the center of rotation yields

$$M_O L_O + \sum_{1}^{N} M_{pi} \left(L_{si} - L_{pi} \right) = M_1 L_1 \quad , \tag{6.9}$$

where

$$q_i = \frac{\xi_i}{Lb_i} \quad . \tag{6.10}$$

Formulae (6.8, 9, 10) are equivalent in all respects to formulae (3.83) of [2] . In other words either analysis yields the same mechanical model even though they differ by a term in the free surface equation.

7/ DISCUSSION AND CONCLUDING REMARKS

DISCUSSION AND CONCLUDING REMARKS

The planar equations of motion for a launch vehicle having a single liquid propellant tank and engine were derived. For simplicity, the tank was taken to be a prismatic cylinder. These equations were then used to obtain the perturbation equations of motion. The vehicle motion was treated as a summation of perturbations from a known reference motion and motion in which vehicle body axes remain coincident with reference axes. The role of the liquid motions in the perturbation equations were then identified.

The liquid propellant tank of the launch vehicle was replaced by a simple mechanical system (system of pendula plus discrete mass and moment of inertia), and planar perturbation equations of motion were derived. The effects of the mechanical system motions in the perturbation equations were isolated and identified.

The role of the liquid motions and mechanical system motions were then compared; it was shown that the mechanical system did exactly duplicate the action of the liquid.

The role of the liquid motions in the perturbation equations were extended to vessels of arbitrary shape having rotational symmetry. These were compared with the role of the mechanical system motions; again, it was found that the mechanical system duplicated the action of the liquid. Moreover, the mechanical system was shown to be equivalent to that of [2], which is essentially the one used at MSFC. However, on comparing the equation expressing the constancy of pressure at the free surface to that of [2], it was found to contain an additional term proportional to the angular displacement of the body axes with respect to the reference axes. This anomaly was ascribed to the fact that in [2] the equations were not perturbed but arbitrarily linearized.

Thus, we conclude that arbitrarily linearizing the equations of motion of the vehicle does not yield the same results obtained by perturbing the vehicle about its nominal trajectory. Body rates are not generalized coordinates; therefore, it is not valid to linearize the equations of motion with respect to them.

It should be noted that the perturbation equations of motion derived herein are strictly applicable to a yaw-plane stability analysis. To perturb the equations of motion from an initial constant pitch rate poses difficulties of a fundamental nature. However, a preliminary approach to the problem indicates that the effective specific body forces $(a_r + g_r)$ are diminished by $\frac{1}{2}\dot{\theta}^2_{pitch}\, l$. In most applications this effect is meaningless.

In addition it was noted that in some instances, dynamics personnel were using the "frozen" moment of inertia of the liquid rather than the "capped" moment of inertia. The difference between these quantities is appreciable whenever the center of rotation of the vehicle is near the center of gravity of the undisturbed liquid.

In conclusion, if the liquid propellant responses noted in flight data are at variance with predicted theoretical analyses and, if these anomalies are mainly due to the treatment of the liquid, then these errors must stem from:

1. Improperly linearizing the equations of motion, e.g. arbitrarily linearizing vehicle body rates;

2. Failure to use the proper effective moment of inertia of the liquid.

An actual stability analysis using the perturbation equations should bring out these points.

8/ NOTES PERTAINING TO TEXT

NOTES PERTAINING TO TEXT

1 $\left(\frac{d\bar{q}}{dt}\right)_{\text{inertial space}} = \left(\frac{d\bar{q}}{dt}\right)_{\text{moving space}} + \bar{\omega} \times \bar{q}$

2 When S(t) is a free surface consisting of material particles moving with velocity $\bar{v}$. If $z - \zeta = 0$ is the equation of the surface, we must have $d(z - \zeta)/dt = 0$ so that

$$\zeta_t = - v_x \frac{\partial \zeta}{\partial x} - v_y \frac{\partial \zeta}{\partial y} + v_z$$

or

$$\zeta_t = (v_x, v_y, v_z) \cdot \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) .$$

But

$$\left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) = \nabla (z - \zeta) ,$$

and since

$$\nabla (z - \zeta) = (\cos(n, x), \cos(n, y), \cos(n, z)) / \cos(n, z)$$

we get

$$v_n = \zeta_t \cos(n, z) ,$$

where

$$v_n = v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z)$$

is the velocity along the normal to S(t) .

Similarly

$$q_n = V_n + \zeta_t \cos(n, z) .$$

[3] This could also be anticipated from the divergence theorem,

$$\iiint\limits_{T(t)} \nabla \cdot \bar{v}\, d\tau = 0 = \iint\limits_{S(t)+\Sigma(t)} v_n\, ds = \iint\limits_{S(t)} \zeta_t \cos(n,\ z)\, ds = \iint\limits_{S} \zeta_t\, dx\, dy\ .$$

9/ REFERENCES

REFERENCES

1. Morse, M. P., and H. Feshbach. Methods of Theoretical Physics, in Part I, McGraw-Hill Book Company, Inc., New York, (1953).

2. Lomen, D. O. Liquid Propellant Sloshing in Mobile Tanks of Arbitrary Shape, Technical Report GD/A-DDE64-061, General Dynamics/Astronautics, (15 October 1964).