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# CULLEN COLLEGE OF ENGINEERING

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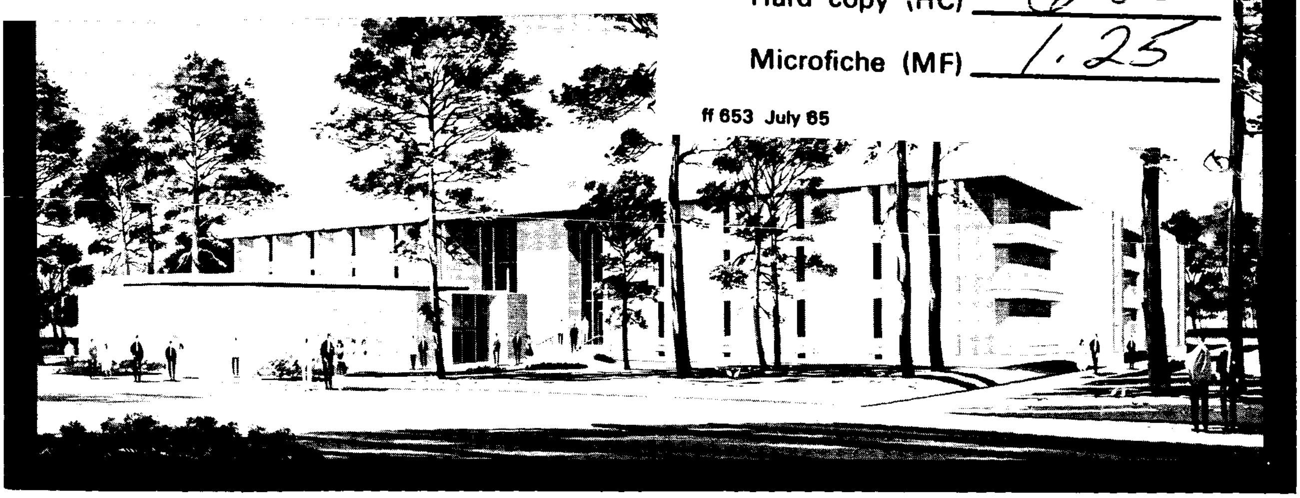

HOUSTON, TEXAS

ANALYTIC SIGNALS AND ZERO-LOCUS IN
MULTIPLEXING SYSTEMS

NGR-44-005-039

A Thesis
Presented to
The Faculty of the Department of Electrical Engineering
The University of Houston

In Partial Fulfillment
of the Requirement for the Degree
Master of Science in Electrical Engineering

by
Louis C. Puigjaner
August 1966

ANALYTIC SIGNALS AND ZERO-LOCUS IN
MULTIPLEXING SYSTEMS

APPROVED:

Chairman

Dean, Cullen College of Engineering

## DEDICATION

To my parents Louis and Caroline with much love and respect.

To the gentle mankind that lives and truly loves.

## ACKNOWLEDGMENT

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## SUMMARY

The theory of analytic functions is used as a new approach to the study of conventional multiplexing schemes. The multiplexing process involves double and triple modulation techniques. The most general form of modulated waves exhibit simultaneous phase and envelope fluctuation.

Phase and envelope relationships described in mathematical formulation have a strong dependence on the real and complex zeroes of the actual wave. The properties of analytic signals are studied, yielding a general theory of zero-representation of the original signal that contains its informational attributes. The mathematical representation of zeroes of a periodic real signal has physical significance when related to the spectra, phase, and envelope fluctuations. The zero-pattern representation offers the advantage of showing the phase envelope fluctuation in the time domain, and a frequency domain representation can be based on the frequency spectrum given by the Fourier series coefficient.

The zero representation is used to study different forms of modulated waves based on the principle of factorization in terms of the Fourier series expansion and in the property of zero-pattern superposition resulting from the product of two signals. Multiplicative processes are the most amenable to a zero-based description, and special

attention is devoted to the study of the characteristics of this type of modulated wave from its zero-pattern configuration. The zero-manipulation property makes mathematical representation easier to handle.

A qualitative comparison between different multiplexing systems in Frequency Division Multiplexing (FDM) and Time Division Multiplexing (TDM) is also possible in terms of zeroes of zero manipulation. Special attention is given to study of crosstalk effects in multiplexing systems whose mathematical analysis has not been developed so far. Tensor analysis is employed in the multi-space in which the zeros of multiplexed signals are located. The evaluation of crosstalk in terms of tensor forms results in simplification when automatic computing techniques are used.

A complete computer program has been developed in two different versions to yield the "zero-locus" of the modulated waveform. Some guidelines to compute the inter-channel crosstalk were also developed.

The thesis concludes with a comparison of different multiplexing systems.

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## 1.0 INTRODUCTION

### 1.1 Background

In radio telemetry applications it is necessary to transmit more than one channel of information simultaneously. It is usually impractical and inefficient to use a separate radio link for each channel. The saving of space, weight, and power is an important consideration in the design of a multichannel system. Such a process of transmitting more than one channel of information over a single link is called multiplexing.

Although there are other multiplexing schemes feasible, only two general methods of multiplexing have been commonly used, namely frequency division multiplexing (FDM) and time division multiplexing (TDM).

A simplified block diagram of an FDM transmitter is given in Fig. 1.1-1. Frequency division multiplexing uses a separate "subcarrier" frequency for each channel with enough spacing to provide non-overlapping frequency intervals for modulation of each subcarrier. The subcarriers are modulated by the information signal in each channel and are linearly added to form one electrical signal which again modulates the "carrier" frequency. The modulated carrier is transmitted. At the receiving end, as is shown in Fig. 1.1-2, the modulated carrier is demodulated to recover the transmitted form at point D'. Then the modulated subcarrier signals are selected by linear band-pass filters. After demodulation of the subcarriers, a final low-pass filter will give the transmitted message. In general, the waveforms at locations marked with primed letters in Fig. 1.1-2 correspond to those locations marked with the same letters (unprimed) in the transmitter, except for noise and distortion.

A time division multiplexing (TDM) system works in a completely different way as is shown in Fig. 1.1-3. Samples of the information waveforms are taken by the commutator, which rotates rapidly enough so that these waveforms change only slightly between samples. The sequence of sample pulses next modulates a carrier frequency, and the modulated carrier is transmitted. The receiver performs a reciprocal operation as is shown in Fig.1.1-4. The modulated subcarrier pulses are routed to the appropriate output channel by another decommutator. The selected pulses are then integrated and interpolated. The interpolated wave is then passed through a low-pass filter that reconstructs the original signal.

The duality between the FDM and TDM systems lies in the fact that the FDM system combines multiple information channels in the frequency domain, whereas a TDM system combines several message sources serially in the time domain and transmits them in parallel in the frequency

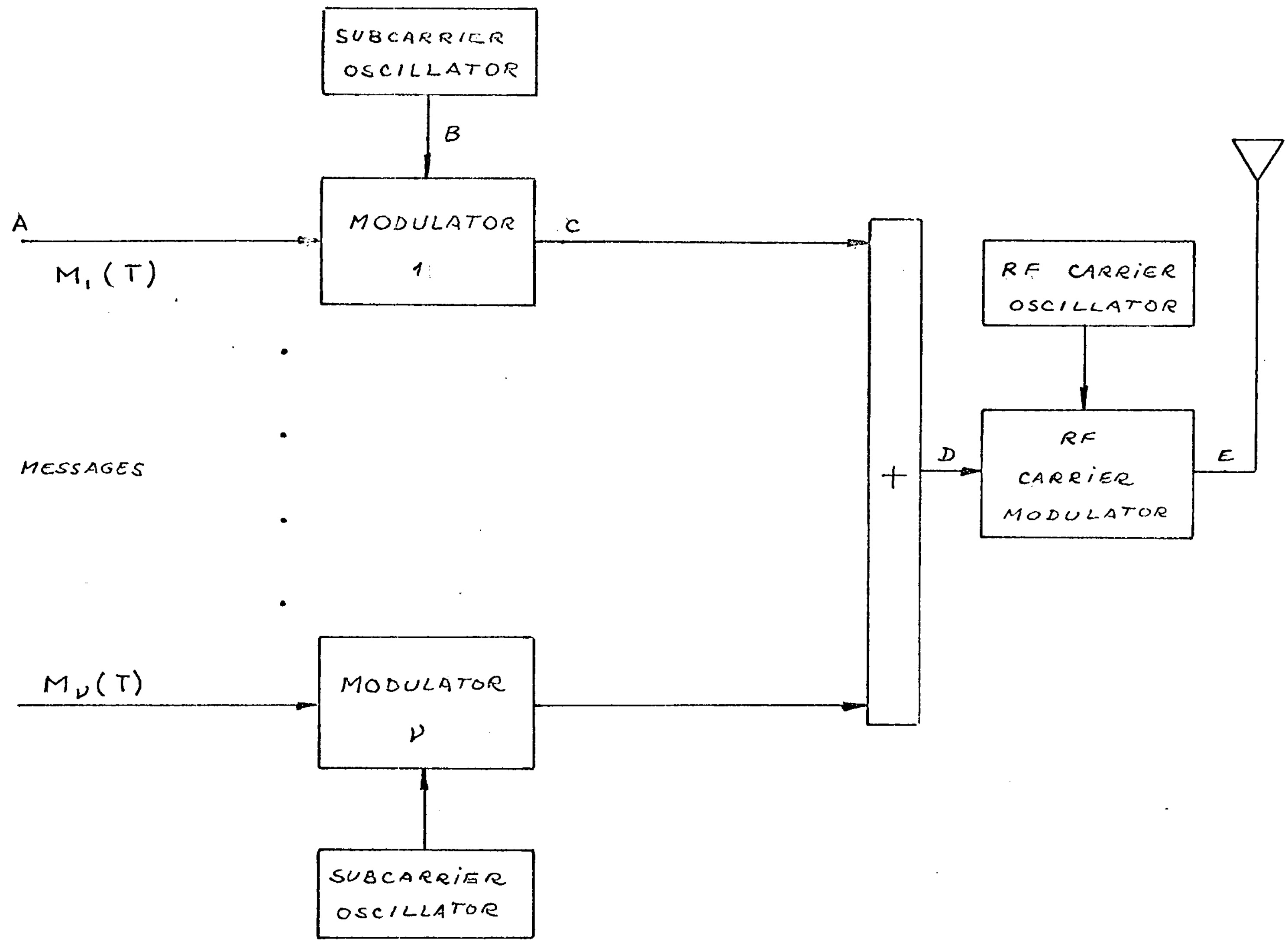



Figure 1.1-1 Frequency division multiplexing transmitter system.

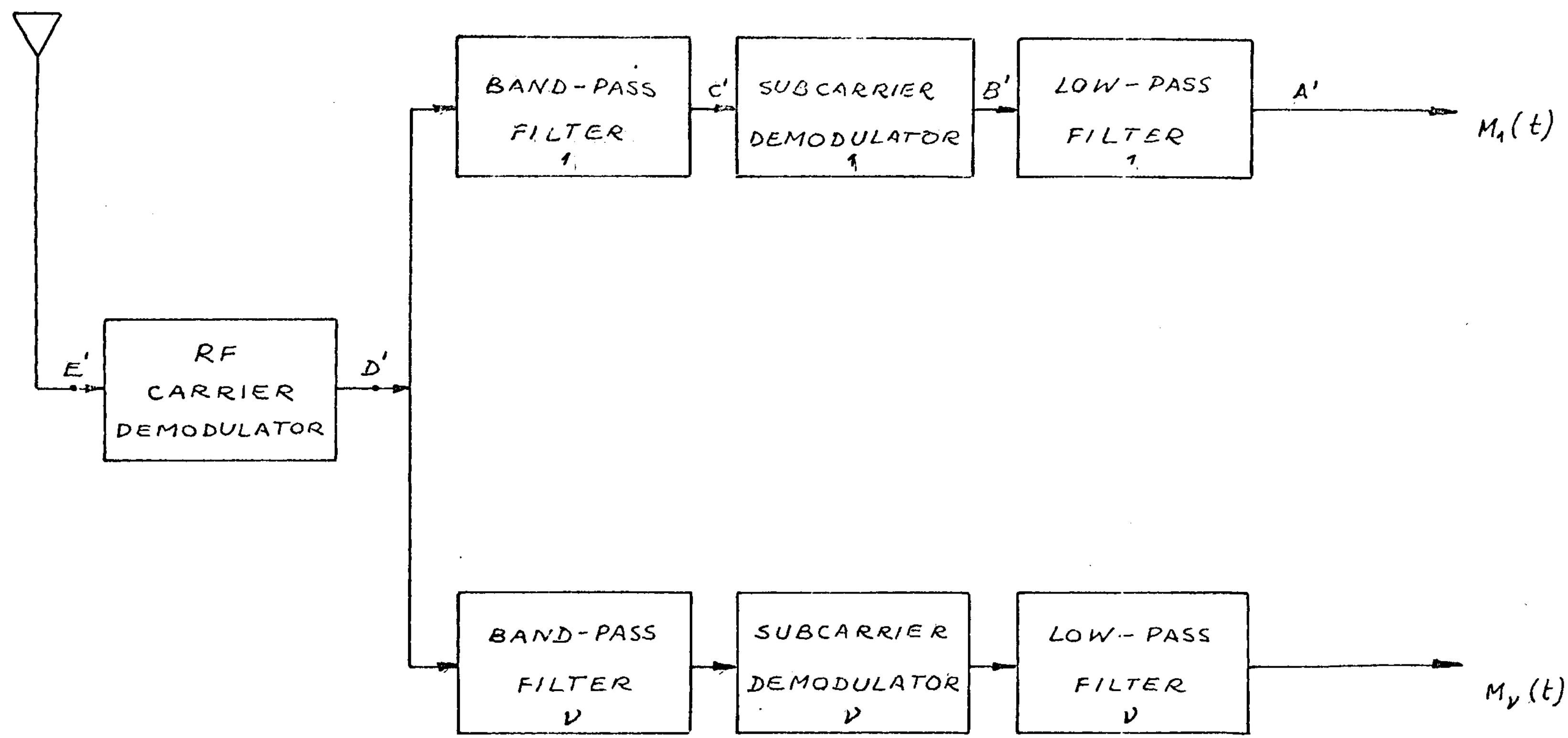



Figure 1.1-2 Frequency division multiplexing receiver system.

domain. Frequency division multiplexing and time division schemes will be analyzed and discussed in some detail later.

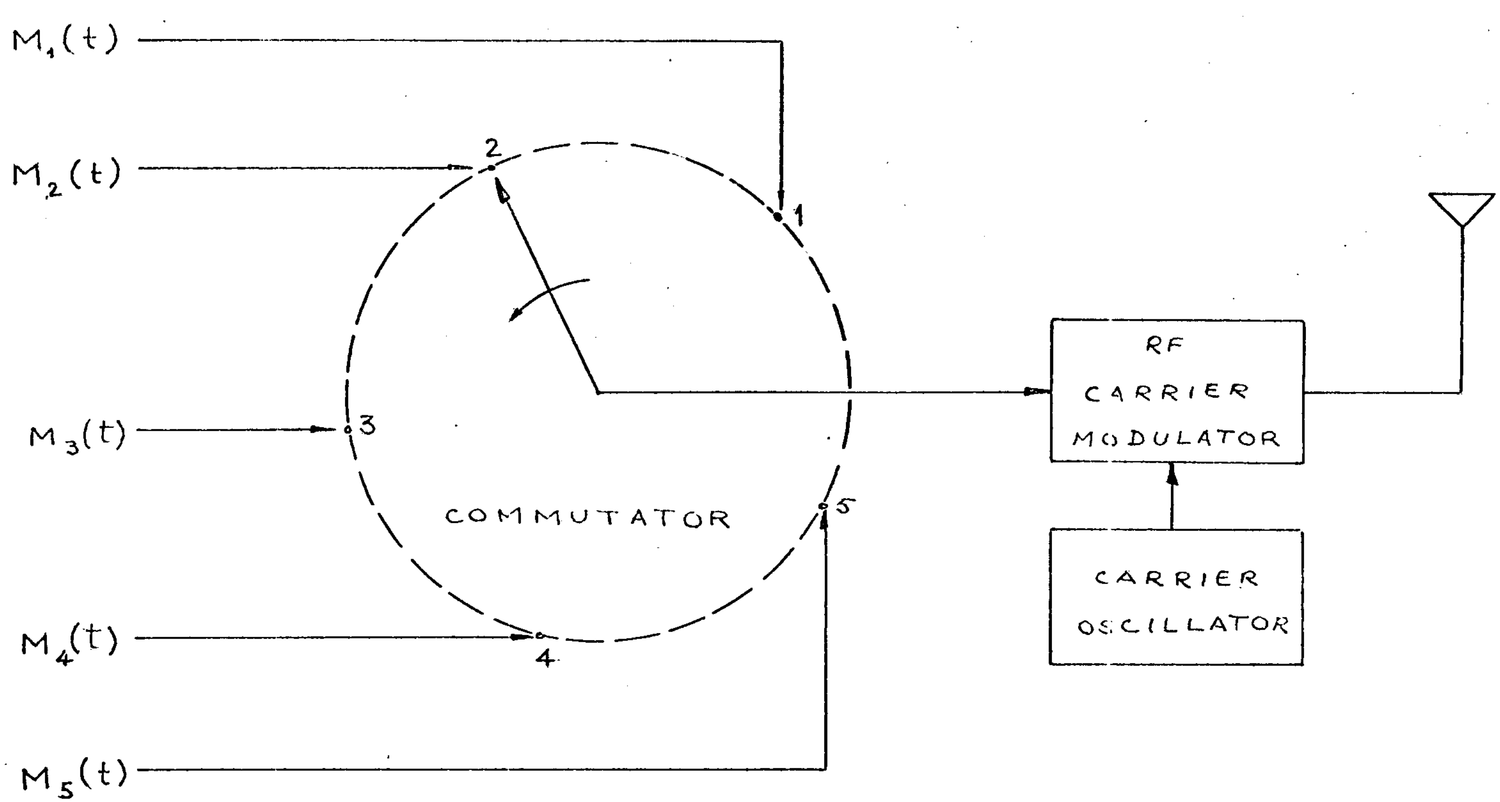



Figure 1.1-3 Time division multiplexing transmitter system.

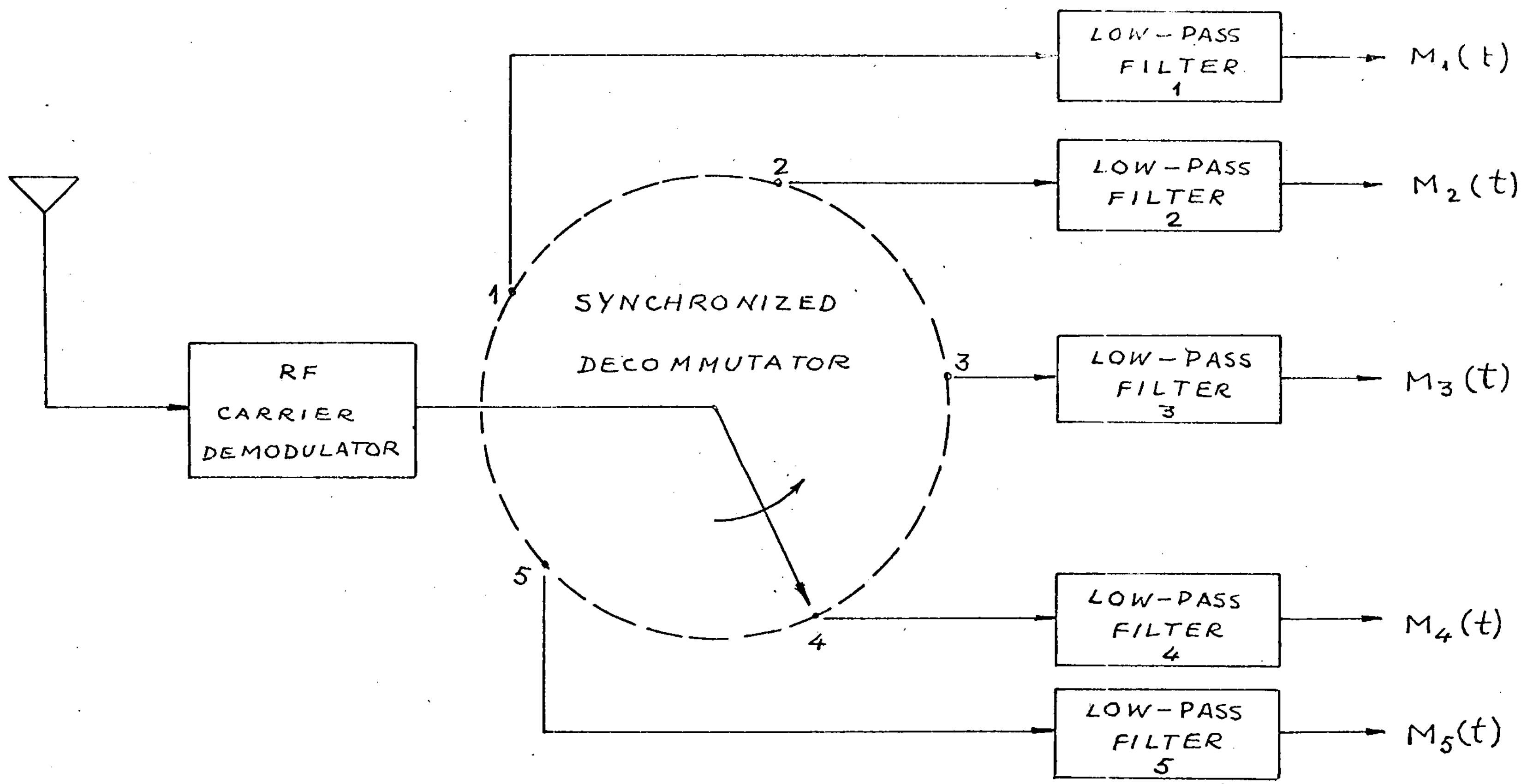



Figure 1.1-4    Time division multiplexing receiver.

## 1.2 Systems Specification

The FDM and TDM systems occur in many forms, according to the type of modulation used for subcarriers and carriers. An unspecified system is symbolized by XX-XX, where the first group of letters refers to the type of subcarrier modulation, and the second group of letters refers to the type of carrier modulation. For example, AM-FM means that the messages amplitude modulate the subcarriers, and the sum of the modulated subcarriers frequency modulates the carrier. Similarly, "triple" modulation systems can be specified. In such systems, there is a group of double modulation waves with spaced carrier frequencies which may be used to modulate a still higher frequency carrier.

A critical study of the different systems was first made by Landon in 1948. The efficiencies of the different conventional schemes were compared on the basis of:

a) Wide-band gain: signal-to-noise ratio of one single channel of a given multiplexing system is compared with that of one AM channel,

b) Threshold: average power required to obtain the wide-band gain,

c) Minimum received signal power required for a specific signal-to-noise ratio,

d) Signal-to-noise ratio on impulse noise (only for some types of TDM),

e) Interchannel cross modulation,

f) Interchannel cross talk,

g) Mutual interference between multiplexing systems operating on adjacent frequency channels.

The classical theory of multiplexing had little change in more than eighteen years. A systematic mathematical model has not yet been developed for the conventional multiplexing systems. During recent years new mathematical techniques have become available. One of the new signal models is called the analytic signal. The use of analytic signals provides a new and powerful tool for analysis of conventional multiplexing systems.

Future aerospace communications systems will require newly developed systems with increased channel capacity and high efficiency. Most of the new multiplexing techniques use hybrid models, and hence they feature simultaneous phase and amplitude modulation. Thus a systematic mathematical study of phase-envelope relationships by use of analytic signals offers the possibility of a general theory of multiplexing. This thesis is concerned with the application of modern signal and noise models to multiplexing systems in order to study their properties and limitations.

# 2.0 THE THEORY OF ANALYTIC SIGNALS

## 2.1 Definition

The analytic signal representation is very useful as a theoretical technique for various types of signal and noise waveforms. The analytic signal can be defined as a complex function of a real variable whose real and imaginary parts form a Hilbert transform pair. The general form of the analytic signal is given:

$$A_s(t) = s(t) + j\hat{s}(t) \qquad (2.1-1)$$

an alternate form of which is:

$$A_s(t) = |A_s(t)| \exp j\,\phi(t) \qquad (2.1-2)$$

where

$s(t)$ = a Fourier-transformable real signal

$\hat{s}(t)$ = a Hilbert transform of $s(t)$.

Periodic signals are considered as Fourier-transformable, and impulses are also allowed. The actual waveform is usually associated with the real or imaginary part of the analytic signal. In other words, the analytic signal $A_s(t)$ representation is used for analysis in a way similar to that of using complex exponential in A.C. circuit analysis.

## 2.2 Hilbert transforms

$$\hat{s}(t) \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau}\, d\tau \qquad (2.2-1)$$

where P is the Cauchy principal value and is defined as

$$P = \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{t-\varepsilon} \frac{s(\tau)}{t-\tau} d\tau + \int_{t+\varepsilon}^{\infty} \frac{s(\tau)}{t-\tau} d\tau \right].$$

Since s(t) and $\hat{s}(t)$ form a transform pair

$$s(t) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\hat{s}(t)}{t-\tau} d\tau \qquad (2.2-2)$$

with an interesting property first observed by Hilbert, namely

$$\hat{\hat{s}}(t) = -s(t).$$

Equivalent formulas for (2.2-1) are:

$$\hat{s}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(t+\tau)}{-\tau} d\tau \qquad (2.2-3)$$

$$\hat{s}(t) = \frac{1}{\pi} \int_{0}^{\infty} \frac{s(t+\tau) - s(t-\tau)}{-\tau} d\tau \qquad (2.2-4)$$

Where Equation (2.2-3) is obtained by letting $t-\tau = \tau'$ Equation (2.2-4) is easily obtained from (2.2-3) as shown below:

$$\hat{s}(t) = \frac{1}{\pi} \int_{-\infty}^{0^-} \frac{s(t+\tau)}{-\tau} d\tau + \frac{1}{\pi} \int_{0^+}^{\infty} \frac{s(t+\tau)}{-\tau} d\tau$$

$$= \frac{1}{\pi} \int_{-0}^{-\infty} \frac{-s(t-\tau)}{-\tau} d\tau + \frac{1}{\pi} \int_{0^+}^{\infty} \frac{s(t+\tau)}{-\tau} d\tau$$

$$= \frac{1}{\pi} \int_{0^+}^{\infty} \frac{s(t+\tau) - s(t-\tau)}{-\tau} d\tau .$$

If

$$s_1(t) = \cos(\omega t + \varphi)$$

then the Hilbert transform of $s_1(t)$ is given by

$$\hat{s}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos[\omega(t+\tau)+\varphi]}{-\tau}\, d\tau$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega t+\varphi)\cos\omega\tau - \sin(\omega t+\varphi)\sin\omega\tau}{\tau}\, d\tau$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega t+\varphi)\cos\omega\tau}{\tau}\, d\tau + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega t+\varphi)\sin\omega\tau}{\tau}\, d\tau$$

The integrand in the first integral is an odd function of $\tau$ ; therefore, the integral is zero. Hence,

$$\hat{s}_1(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega t+\varphi)}{\tau}\sin\omega\tau\; d\tau$$

$$= \frac{1}{\pi}\sin(\omega t+\varphi)\int_{-\infty}^{\infty} \frac{\sin\omega\tau}{\tau}\; d\tau$$

$$= \frac{1}{\pi}\sin(\omega t+\varphi)\,\pi = \sin(\omega t+\varphi)$$

which leads to

$$\hat{s}_1(t) = \sin(\omega t+\varphi)\,. \qquad (2.2\text{-}6)$$

Another similar transform pair is:

$$s_2(t) = \sin(\omega t + \varphi)$$

$$\hat{s}_2(t) = -\cos(\omega t + \varphi) . \qquad (2.2-7)$$

## 2.3 Properties of analytic signals

Whenever s(t) is Fourier transformable, its frequency spectrum is given by

$$F_s(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt .$$

Equation (2.2-1) suggests that $\hat{s}(t)$ may be the convolution of s(t) and $\frac{1}{\pi t}$ . Therefore, the Fourier transform of $\hat{s}(t)$ is the product of the Fourier transform of s(t) and that of $\frac{1}{\pi t}$ as shown below:

$$F_{\hat{s}}(\omega) = F_s(\omega) \int_{-\infty}^{\infty} \frac{1}{\pi t}\, e^{-j\omega t}\, dt$$

and, since $\frac{1}{\pi t}$ is an "odd" function, the last integral reduces to

$$F_{\hat{s}}(\omega) = F_s(\omega) \int_{-\infty}^{\infty} \frac{1}{\pi t}\, j \sin \omega t\, dt .$$

Hence for

$$\omega = 0 \qquad F_{\hat{s}}(\omega) = 0$$

$$\omega > 0 \qquad F_{\hat{s}}(\omega) = -j F_s(\omega)$$

$$\omega < 0 \qquad F_{\hat{s}}(\omega) = j F_s(\omega)$$

and these results can be expressed in terms of the "signum function" defined as

$$\operatorname{sgn} \omega = 1 \qquad \text{for} \quad \omega > 0$$

$$\operatorname{sgn} \omega = -1 \qquad \text{for} \quad \omega < 0$$

$$\operatorname{sgn} \omega = 0 \qquad \text{for} \quad \omega = 0$$

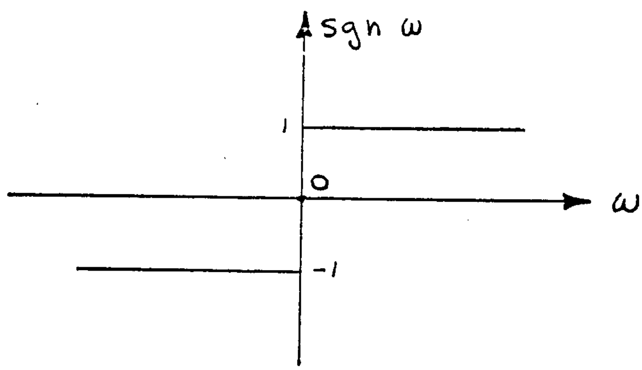



Figure 2.3-1 The function sgn $\omega$.

Thus:

$$F_{\hat{s}}(\omega) = -j \operatorname{sgn} \omega \, F_s(\omega). \qquad (2.3-1)$$

For positive frequencies, the spectrum of $\hat{s}(t)$ is identical to that of s(t) except for the multiplying factor (-j) that represents a phase lag of 90°. It can be seen from Equation (2.3-1) that the Hilbert transform of s(t) can be obtained by passing s(t) through a filter whose system function is:

$$H(\omega) = -j \operatorname{sgn}(\omega)$$

and whose impulse response is:

$$h(t) = \frac{1}{\pi t}.$$

This observation permits a simple proof of the property that $\hat{\hat{s}}(t) = -s(t)$. Suppose the output of the Hilbert transforming filter $\hat{s}(t)$ is passed through another Hilbert transforming filter. Then the output of the second filter would be $\hat{\hat{s}}(t)$. The overall transfer function of the two filters in cascade is:

$$H^2(\omega) = [-j \operatorname{sgn} \omega]^2.$$

Since $\operatorname{sgn}^2(\omega) = +1$,

$$F_{\hat{\hat{s}}}(\omega) = H^2(\omega) F_s(\omega) = -F_s(\omega)$$

or $\hat{\hat{s}}(t) = -s(t)$. The Fourier transform of the analytic signal becomes, using the results above:

$$F_{A_s}(\omega) = F_s(\omega) + j F_{\hat{s}}(\omega)$$

$$= F_s(\omega) + j[-j \operatorname{sgn} \omega \, F_s(\omega)]$$

$$= F_s(\omega)[1 + \operatorname{sgn} \omega]$$

$$F_{A_s}(\omega) = 2 F_s(\omega) \quad \text{for } \omega > 0$$

$$F_{A_s}(\omega) = 0 \quad \text{for } \omega < 0. \qquad (2.3-2)$$

This is an important result in that the Fourier transform of the analytic signal has no negative components, so consequently its complex conjugate vanishes for positive frequencies.

$$F^*_{A_s}(\omega) = 0 \quad \text{for } \omega > 0 \qquad (2.3-3)$$

Therefore the analytic signal is an analytic function of a complex variable in the upper half complex plane. Thus when the Fourier spectrum of the actual signal is band-limited $(-\omega_0 < \omega < +\omega_0)$, the Fourier transform of the analytic signal vanishes for all $\omega$ except $0 < \omega < \omega_0$, and the signal energy is calculated by using Parseval's theorem as

$$E_{A_s} \equiv \int_{-\infty}^{\infty} |A_s(t)|^2 dt = \int_0^{\omega_0} |F_{A_s}(\omega)|^2 d\omega \,.$$

The real and imaginary parts of the analytic signal have identical autocorrelation functions and power spectra, which is derived below for a signal s(t):

$$P_s(\omega) \equiv F_s(\omega) \cdot F_s^*(\omega) = |F_s(\omega)|^2$$

Now the power spectrum of $\hat{s}(t)$ is:

$$P_{\hat{s}}(\omega) \equiv F_{\hat{s}}(\omega) \cdot F_{\hat{s}}^* = [-j \operatorname{sgn} \omega\, F_s(\omega)][+j \operatorname{sgn} \omega\, F_s^*(\omega)] = |F_s(\omega)|^2$$

and thus:

$$P_{\hat{s}}(\omega) = P_s(\omega)$$

The autocorrelation function is given by:

$$R_s(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_s(\omega)\, e^{j\omega t}\, d\omega \,.$$

Since the autocorrelation function is the Fourier transform of the power spectrum:

$$R_s(\tau) = R_{\hat{s}}(\tau).$$

The magnitude and phase of the analytic signal mathematically define the magnitude and phase of the actual signal. Equation (2.4-1) gives:

$$A_s(t) = s(t) + j\hat{s}(t)$$

$$= \sqrt{s^2(t) + j\hat{s}(t)}\; e^{-j\varphi(t)}$$

$$= |A_s(t)|\, e^{-j\varphi(t)} \qquad (2.3-4)$$

where $\varphi(t) = \arctan\left[\frac{\hat{s}(t)}{s(t)}\right]$ (2.3-5)

The quantity $|A_s(t)|$ corresponds to the modulus or "envelope." The phase of an analytic signal $\varphi(t)$ is mathematically defined as the phase of the actual signal as shown by Equation (2.3-4).

It was pointed out (Bedrosian 1962) that the envelope and phase of an actual waveform have physical significance only for narrow-band signals and cannot be measured precisely except for pure sinusoids. It was also shown that the correspondence of this mathematical representation and that of the "rotating vector" or "phase diagram" are employed to visualize a mathematical abstraction. The analytic signal is a means of generalizing the phasor diagram for use with modulated signals instead of simple sine waves.

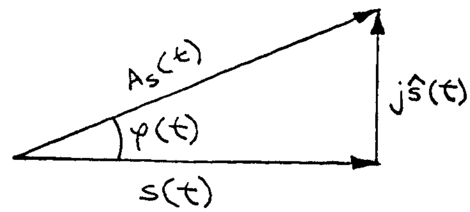



Figure 2.3-2 Phasor diagram representing the analytic signal

A definition of the analytic signal $A_s(t)$ will now be made in terms of the analyticity of the complex function $A_s(z)$ of a complex variable z. A complex function is analytic at a point if its derivative exists at the point. Such a function has all order of derivatives at this point and within some neighborhood of the point (Churchill 1960). A function is analytic in a region R if it is analytic at every point in R. The Cauchy-Riemann equations are necessary conditions for analyticity of a function.

Now let $A_s(z)$ be a function of the complex variable

$$z = t + j\sigma$$

The function $A_s(t)$ is called an analytic signal if:

a) $A_s(z)$ is defined everywhere on the real axis, except at the discontinuities of s(t), the real part of $A_s(t)$,

b) $A_s(z)$ is bounded in the upper half plane (for any value of $\sigma > 0$), and

c) $A_s(z)$ may have poles in the lower half plane

That the given function $A_s(z)$ is analytic in the upper half plane is shown by applying Cauchy's theorem to the path of integration shown in Fig. 2.3-3, where $t=t_o$ is an arbitraty point on the real axis, but is distinct from any discontinuity of s(t).

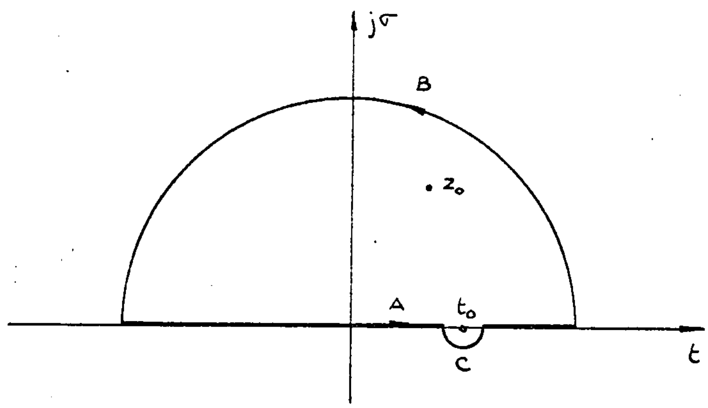



Figure 2.3-3 Contour for an analytic signal

The Cauchy formula gives:

$$A_s(z) = \frac{1}{2\pi j}\oint_{A+B} \frac{A_s(\xi)}{z-\xi}\, d\xi \qquad (2.3\text{-}6)$$

Now evaluate (2.3-6) at the point $z=t_o$.

$$A_s(t_o) = \frac{1}{\pi j}\, P\int_{-\infty}^{\infty} \frac{A_s(t)}{t-t_o}\, dt \qquad (2.3\text{-}7)$$

because as the radius of the B circle approaches infinity, the integral along B tends to zero and the integral along the small half circle C is equal to

By taking the real and imaginary parts of Equation (2.3-7), the Hilbert transform pair is obtained:

$$s(t_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{s}(t)}{t - t_0}\, dt \qquad (2.3\text{-}8)$$

$$\hat{s}(t_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s(t)}{t - t_0}\, dt \qquad (2.3\text{-}9)$$

It follows that the given signal in Equation (2.2-1) is analytic as defined above. This important result shows that the alternate approach to the definition of the analytic signal yields the same result. The analytic signal is defined in section 2.1 as a complex function whose real and imaginary parts form a Hilbert transform pair. Now from the definition of the analytic function of a complex variable the property that describes the analytic signal as a Hilbert transform pair is obtained. Furthermore, it may be noted that

$$I_m\left[A_s(t)\right] = H\left\{R_e\left[A_s(t)\right]\right\} \qquad (2.3\text{-}10)$$

$$R_e\left[A_s(t)\right] = -H\left\{I_m\left[A_s(t)\right]\right\} \qquad (2.3\text{-}11)$$

The function s(t) and $\hat{s}(t)$ form an orthogonal pair

$$\int_{-\infty}^{\infty} \hat{s}(t)\, s(t)\; dt = 0$$

and by Parseval's theorem, these are orthonormal as well, since

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |F_s(f)|^2 df = \int_{-\infty}^{\infty} |F_{\hat{s}}(f)|^2 df = \int_{-\infty}^{\infty} |\hat{s}(t)|^2 dt$$

The phase and envelope form of the analytic signal is defined

$$A_s(t) = \sqrt{s^2(t) + \hat{s}^2(t)}\; e^{-j\varphi(t)} \qquad (2.3\text{-}12)$$

This representation is particularly useful when comparing the analytic signal to the modulated real signal. The modulated signal appears as the product of two time functions:

$$A(t) = A_c(t)\, M\left[s(t)\right].$$

The symbol $A_c(t)$ represents the "carrier," and $M[s(t)]$ represents the modulating signal. Assuming the general form as

$$A_c(t) = e^{j\omega_o t} = \cos \omega_o t + j \sin \omega_o t,$$

the modulated signal becomes, (using SSB as an example):

$$A(t) = M[s(t)]e^{j\omega_o t} = A_s(t)\, e^{j\omega_o t} = \left[s(t) + j\hat{s}(t)\right] e^{j\omega_o t},$$

and

$$\mathrm{Re}\left[A(t)\right] = s(t) \cos \omega_o t - \hat{s}(t) \sin \omega_o t.$$

For other types of modulation, the $M[s(t)]$ is different.

Now, according to Equation (2.3-4),

$$\mathrm{Re}\left[A(t)\right] = \mathrm{Re}\left[|A_s(t)|\, e^{j\varphi(t)+j\omega_0 t}\right]$$

$$= |A_s(t)| \cos\left[\omega_0 t - \phi(t)\right] \qquad (2.3-13)$$

This equation demonstrates that the actual signal s(t) represented by (2.3-13) has an envelope and phase defined by the analytic signal properties. The carrier frequency $\omega_0$ is usually much greater than the bandwidth of the modulated signal. The phase and envelope representation of the actual signal are wholly describable in terms of its zeroes. Thus the zeroes of a band-limited wave can be viewed as its informational attributes (Voelcker 1966). That can be shown as follows:

Taking the natural logarithm of the expression (2.3-4), one obtains:

$$\ln\left[A_s(t)\right] = \ln|A_s(t)| + j\,\varphi(t) \qquad (2.3-14)$$

and if $\ln\left[A_s(t)\right]$ is an analytic signal,

$$s(t) = \mathrm{Re}\left[A_s(t)\right] = |A_s(t)| \cos\left\{H\left[\ln|A_s(t)|\right]\right\} \qquad (2.3-15)$$

since from Equation (2.3-14)

$$\varphi(t) = H\left\{\ln|A_s(t)|\right\} \qquad (2.3-16)$$

where H symbolizes the Hilbert transform. The Equation (2.3-15) shows that s(t) depends only on the envelope $|A_s(t)|$.

The conditions for $\ln[A_s(t)]$ to be an analytic function require that $\ln[A_s(z)]$ cannot have singularities in the upper half plane. In other words, $A_s(z)$ cannot have zeroes there.

One way to eliminate the zeroes of $A_s(t)$ is to add a positive constant to the real part of $A_s(t)$, so that it will never be negative. Thus the modified function whose logarithm is an analytic signal is given as

$$A_{MP}(t) = c + A_s(t) = c + s(t) + j\hat{s}(t)$$

where $[c + s(t)] > 0$ for all t. The subscript MP refers to "minimum phase", that is, the function for which there are no zeroes in the upper half plane. The second condition for analyticity is also satisfied by the modified function

$$\ln A_{MP}(t) = \ln c + \ln\left[1 + \frac{A_s(t)}{c}\right] \qquad (2.3\text{-}17)$$

$$\lim_{z \to \sigma} \ln A_{MP}(z) = \ln c \qquad \text{for } \sigma \geq 0 \qquad (2.3\text{-}18)$$

where $z = t + j\sigma$ in z-plane, because $A_s(t)$ is an analytic signal.

The real part of the analytic signal can be recovered from the <u>envelope</u> if the logarithm of the signal $A_s(t)$ is analytic, and if the signal is free of "zeroes" in the upper half plane. These conditions are met by the normalized function (Voelcker 1966)

$$\Lambda + s(t) \cdot \frac{|A_{MP}(t)|}{c} \cos\left\{ H\left[ \ln \frac{|A_{MP}(t)|}{c} \right] \right\} \qquad (2.3-19)$$

Equation (2.3-16) shows that when A(z) is zero-free in the upper half plane, phase information is contained in $\varphi(z)$.

Some general phase-envelope relationships can be extended to the cases of "non-minimum phase functions" (Voelcker 1966) by noting that

$$\sum n(0) = \sum_{U} n(0) + \sum_{L} n(0) . \qquad (2.3-20)$$

The symbol $\sum n(0)$ symbolizes the total number of zeroes with subscripts U and L denoting the upper and lower half plane. This general expression can be applied without the requirement of analyticity and therefore is useful for a large class of functions.

"One or more zeroes of A(t) can be conjugated without affecting the envelope fluctuations." (Voelcker 1966) This is important in multiplexing, there can be different A(t) with the same phase-envelope as a whole. The phase or the envelope can be manipulated to produce different signals without one affecting the other. Thus a "common envelope set" will be a set containing signals that differ in phase but not in envelope or bandwidth.

From Equation (2.3-20) it can be seen that the minimum phase case corresponds to the first term of the sum on the right side. The "maximum phase function" corresponds to the second term:

$$A_{MAX}(t) = |A(t)| \exp j\left[kt - \varphi_{MP}(t)\right] \qquad (2.3-21)$$

Clearly $A_{MAX}(t)$ is the translated conjugate of $A_{MP}(t)$. The same equation shows that grouping zeroes is an additive operation meaning multiplication of the functions themselves, since Equation (2.3-20) refers to $\ln A(t)$. Thus; an analytic signal can be factored into as many multiplicative components as there are zeroes in the signal. The above consideration may be summarized as below:

a) The real part of the analytic signal can be recovered from the envelope if the logarithm of the signal is analytic.

b) One or more zeroes of $A_s(t)$ can be conjugated without affecting the envelope fluctuations.

c) An analytic signal can be factored into as many components as there are zeroes in the signal.

## 2.4 Analytic Signal Representation

The properties of analytic signals are illustrated by a few examples. A band-limited periodic signal has a finite number of zeroes in a given time interval, and the zeroes are related to its frequencies. However, band-limited aperiodic signals have an infinite number of zeroes and an assymtotically finite density of zeroes.

a) The periodic signal:

Let s(t) be a periodic signal defined by

$$s(t) = K - B\cos\omega t \qquad (2.4-1)$$

The "phasor" representation of the periodic signal given above is:

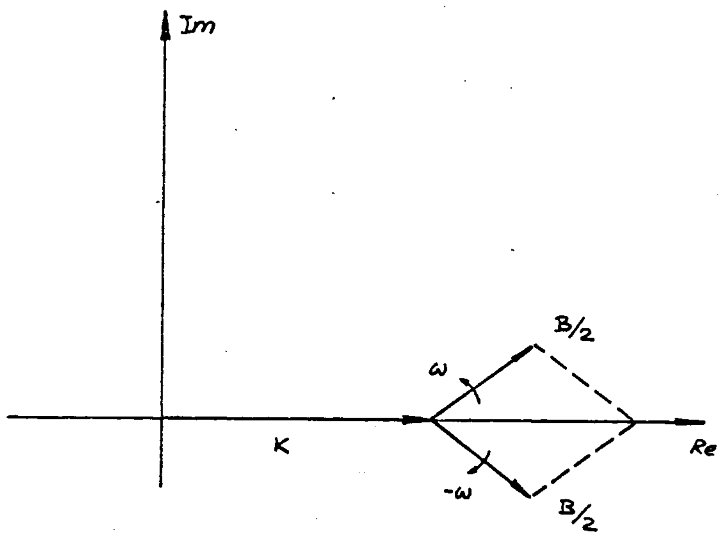



Figure 2.4-1 Phasor representation of $s(t) = K - B\cos\omega t$

because

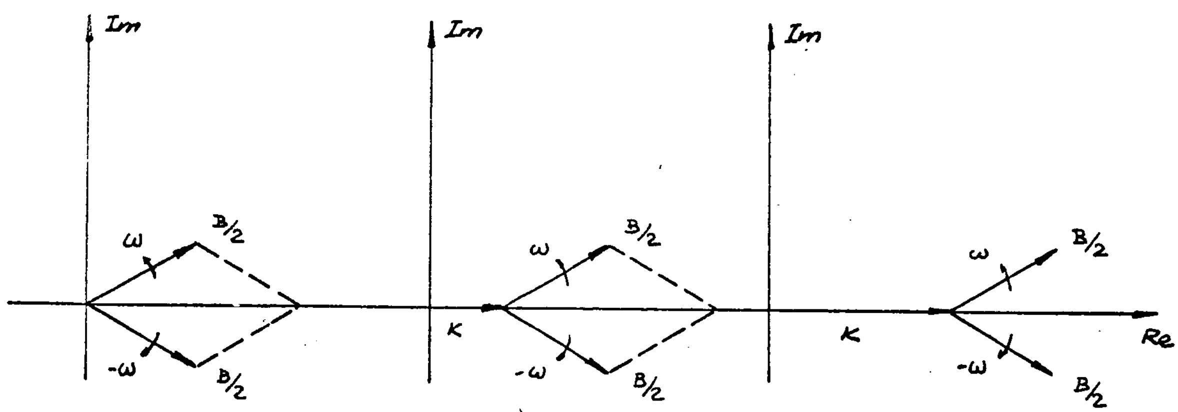
$$s(t) = K - \frac{B}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] \qquad (2.4\text{-}2)$$

The s(t) phasor is reproduced for the following cases:

a) $K = 0$

b) $0 < K < B$

c) $K > B$




a) $K=0$ b) $0<K<B$ c) $K>B$

Figure 2.4-2 Phasor representation for $s(t) = K - B\cos\omega t$

The same signal may be studied by use of its "zero" representation by replacing t with z in Equation (2.4-2) as:

$$z = t + j\sigma$$

$$s(z) = K - \frac{B}{2}\left[e^{j\omega z} + e^{-j\omega z}\right]$$

$$s(r) = K - \frac{B}{2}\left[r + \frac{1}{r}\right] = -\frac{B}{2r}\left[r^2 - \frac{2K}{B}r + 1\right] \quad (2.4\text{-}3)$$

where $r = \exp j\omega z$

The roots of this last equation are:

$$r = \frac{1}{B}\left[K \pm \sqrt{K^2 - B^2}\right] = \exp j\omega\,(t + j\sigma) \quad (2.4\text{-}4)$$

The result of this expression is:

$$z_n = t + j\sigma = \frac{2\pi n}{\omega} - \ln\left[\frac{1}{B}\left(K \pm \sqrt{K^2 - B^2}\right)\right] \quad (2.4\text{-}5)$$

$$= \frac{2\pi n}{\omega} \pm j \,\mathrm{Arccosh}\left(\frac{K}{B}\right) \qquad \text{for} \quad K > B$$

because

$$\mathrm{arccosh}\left(\frac{K}{B}\right) = \ln\left[\frac{1}{B}\left(K \pm \sqrt{K^2 - B^2}\right)\right] \quad (2.4\text{-}6)$$

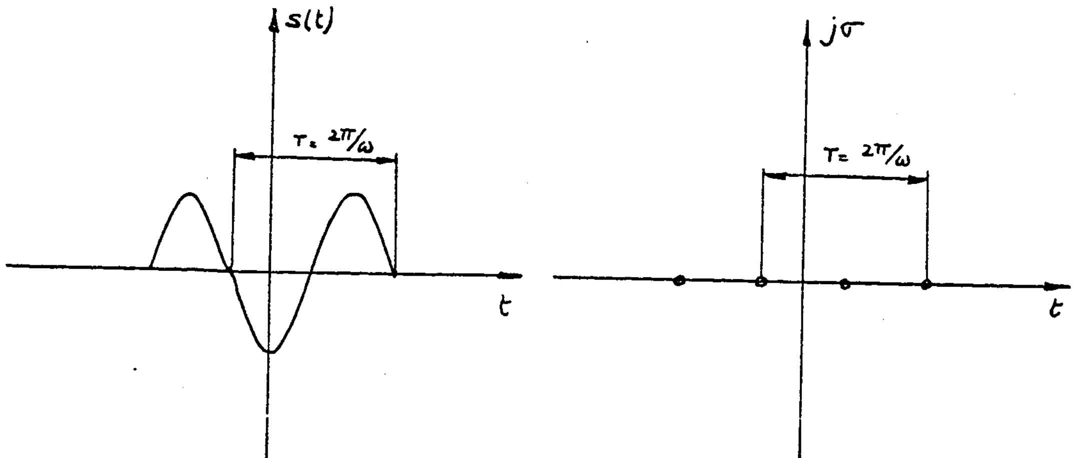
Thus in this form the signal can be described in terms of its zeroes.

The roots of Equation (2.4-5) give the location of "zeroes" in the complex z-plane as is shown in the right half Figures 2.4-3, 2.4-4, and 2.4-5, and their conventional representation is given on the left half of these figures for

a) $K = 0$

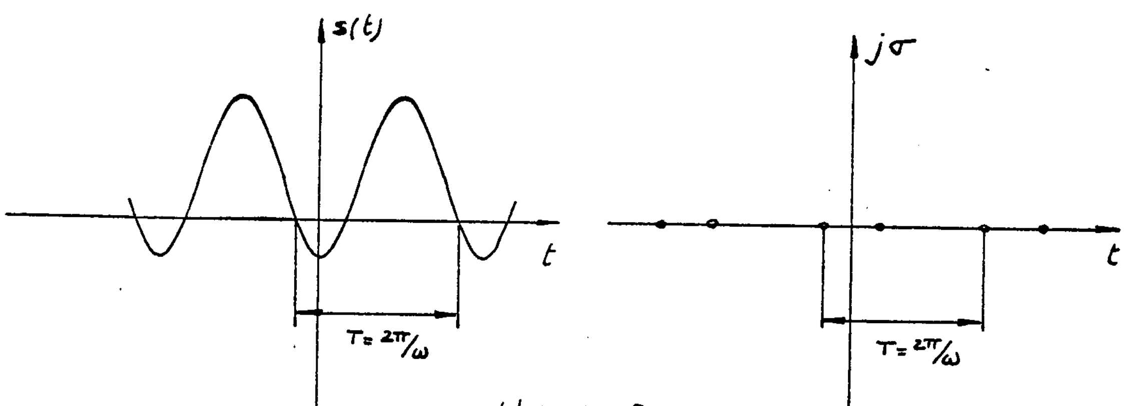
b) $0 < K < B$

c) $K > B$



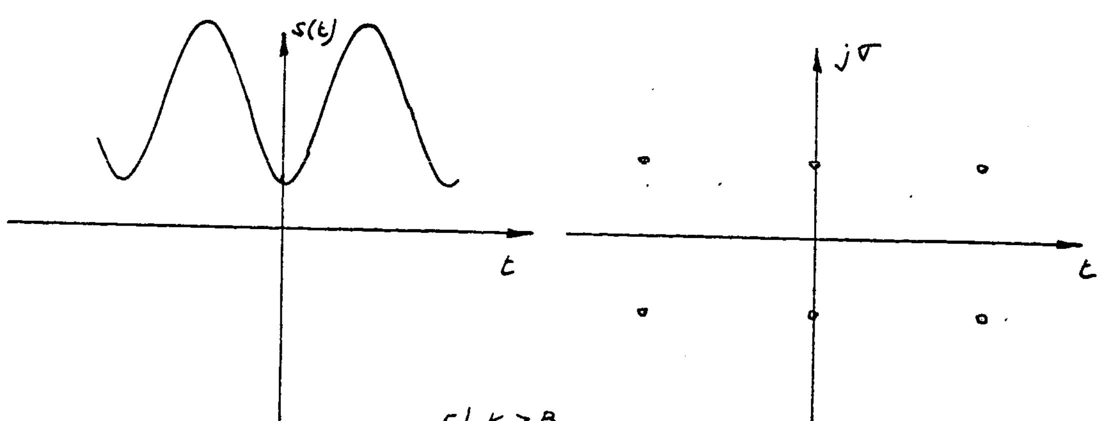

a) K = 0

Figure 2.4-3 Sine wave of conventional representation
b) zero array




b) 0 < K < B

Figure 2.4-4 Zero array representation as K parameter varies




c) K > B

Figure 2.4-5 Zero array for K > B

A single frequency periodic signal

$$s(t) = K\cos\omega t = K/2\left[\exp(j\omega t) + \exp(-j\omega t)\right]$$

has a second order polynomial representation in the complex z-plane, and there are two zeroes in each period. These results can be generalized in the following way: A band-limited $(|\omega| < n\omega_0)$ periodic function s(t) can be expanded in a Fourier series

$$s(t) = \sum_{k=-n}^{n} c_k \exp jk\omega_0 t \qquad (2.4-7)$$

where $T = 2\pi/\omega_0$ = period

The characteristic polynomial of this function in parametric form will be of second order. That is, its representation on the z-plane will give 2n zeroes in conjugate pairs. This double symmetry about the real and imaginary axis follows from the properties of a polynomial with real coefficients as it is in Equation (2.4-6).

Multiplicative Models for Analytic Signals

The above theory is generalized now to demonstrate how the signal-factorization principle can be used to find the zeroes of the analytic function for more complex signals.

Given an analytic signal of some complicated form, the complex z-plane will contain a zero-array which is periodic with a period of $T = \frac{2\pi}{\omega}$ and has 2n zeroes per period.

Consider the $i$th zero in the pattern of T; it is located at

$$z_i = t_i + j\sigma_i$$

of an analytic signal $A_i(t)$

$$A_i(t) = 1 - \alpha_i \exp j\omega_o t \qquad (2.4-8)$$

A solution for the zeroes can be obtained as follows:
With $\alpha_i = \exp(-j\omega_o z_i)$, Equation (2.4-8) becomes:

$$A_i(t) = 1 - e^{j\omega_o(t - z_i)}$$

which reduces to zero for $t = z_i$, and

$$\alpha_i = \exp(-j\omega_o z_i) = |\alpha_i| \exp(-j\omega_o t_i) \qquad (2.4-8a)$$

or $$|\alpha_i| = \exp \omega_o \sigma_i \qquad (2.4-8b)$$

where $$\sigma_i = \frac{1}{\omega_o} \ln |\alpha_i|$$

and

$$t_i = \frac{2\pi n}{\omega_o} \qquad n = \pm 1, \pm 2, \pm 3 \ldots \qquad (2.4-8c)$$

Multiplicative Property

The "multiplicative property" associated with the zero decomposition of a signal can be described as

$$A(t) = \prod_{i=1}^{n} A_i(t)$$

where $A_i(t)$ is an "elementary signal" of the type described above in (2.4-8); then

$$A(t) = \prod_{i=1}^{n} \left[1 - \alpha_i \exp j\omega_0 t\right] \qquad (2.4-9)$$

where $\alpha_i = \alpha[z_i]$ is associated with the $ith$ zero, and also

$$A(t) = \sum_{k=0}^{n} c_k \exp jk\omega t \qquad (2.4-10)$$

where $c_k$ are the coefficients of the expanded product. The last expression describes the model of an analytic signal associated with the n-zero pattern.

Summary of Characteristics and Rules for Analytic Signals

The following characteristics will be used frequently in the present work:

a) The zeroes of the analytic signal are determined by the factorization of its Fourier series as shown in Equations (2.4-9) and (2.4-10):

$$A(t) = \prod_{i=1}^{n} \left[1 - \alpha_i \exp j\omega_0 t\right]$$

$$A(t) = \sum_{k=0}^{n} c_k \exp jk\omega t$$

b) If the bandwidth of one elementary signal is defined as $W = \frac{\omega}{2\pi} = \frac{1}{T}$ where T is the period of the corresponding analytic signal, then the bandwidth of the analytic signal A(t) is $n \cdot W$.

c) The common envelope set associated with A(t) can be generated by systematic zero conjugation; that is, by replacing $z_i$ with $z_i^*$ or $|\alpha_i|$ with $|\alpha_i|^{-1}$.

d) If all zeroes are in the lower half plane, then $|\alpha_i| < 1$ for all $i$, A(t) is a "minimum phase" function (MP), and its phase and logarithm envelope are a Hilbert pair as are their derivatives.

e) If all zeroes are in the upper half plane, then $|\alpha_i| > 1$ for all $i$ and A(t) is a "maximum phase" function with $k \geqslant n\omega$. The coefficients of the Fourier series are reversed and conjugated. $|A(t)|$ is held invariant.

f) If K of the zeroes are temporally periodic within the pattern and have the same ordinate, then the product of the corresponding K elementary signals can be reduced by z-plane scaling to a single elementary signal of form:

$$1 - \exp jk\omega(t - z_k),$$

g) If complex zeroes occur only in conjugate pairs and n is an even number, then

$$n = 2(n_R + n_c)$$

where $n_R$ is half the number of real zeroes, and $n_c$ is the number of pairs of complex conjugate zeroes. Then the analytic signal can be represented as

$$A_s(t) = s(t) \exp j\left[\frac{n}{2}\omega t + \theta_0\right].$$

The real signal s(t) is periodic in T and band-limited to $\pm\frac{n}{2}W$, and the coefficients in Equation (2.4-10) must exhibit real-even, imaginary-odd symmetry about the $\frac{n}{2}$th coefficient.

h) The signal s(t) can be factored into the product of two band-limited real signals

$$s(t) = s_{cz}(t) \cdot s_{Rz}(t)$$

where $cz$ and $Rz$ denote the wholly complex parts and real parts respectively. If $n_c$ or $n_R$ is zero then $s_{cz}$ or $s_{Rz}$ is taken as unity. The values of $s_{cz}$ and $s_{Rz}$ are given by

$$s_{cz}(t) = \prod_{i=1}^{n_c}\left[\cosh \omega(\sigma_i) - \cos\omega(t - \tau_i)\right]$$

where $z_i = t_i \pm j|\sigma_i|$ is the location of the $i$th conjugate pair:

$$s_{RZ}(t) = \prod_{(k,l)=1}^{n_R} \left[\cos\omega\left(\frac{\tau_k - \tau_l}{2}\right) - \cos\omega\left(t - \frac{\tau_k - \tau_l}{2}\right)\right].$$

the $2n_R$ real zeroes are arbitrarily arranged into $n_R$ pairs $(z_k\ z_l)$.

i) If all zeroes are of even order, then

$$|A(t)| = s(t) \geq 0$$

and $|A(t)|$ is band limited.

## 2.5 Analytic Signals Applications in Modulation Systems

So far the theory of analytic signals has been developed to show how the "zero-locus" of such signals contains their fundamental informational attributes. Now a manipulation of zeroes of the analytic signal is shown to modify the signal properties.

Classical modulation processes are based simply on making some parameter of the carrier (such as amplitude or angle) directly proportional to some linear function of the modulation signal. A new approach to modulation can be made through the theory of analytic signals, which permits the development of additional useful modulation methods. Toward this end, it will be said that a modulation process is admissible if the zeroes of the modulating signal can be exactly recovered from the zeroes of the modulated wave. The modulated wave need not have the same zeroes, or even the same number of zeroes as the modulating signal. The zeroes can be moved about, inverted, or deleted for convenience, provided that such manipulation can be reversed to obtain the original zeroes in a demodulation process.

An application of the analytic signal theory to modulation is illustrated in the cases of three classical modulation techniques, which are Amplitude Modulation (AM), Single Side Band (SSB), and Frequency Modulation (FM). Analagous to the "root-locus" method in servomechanisms, a so-called "zero locus" can be drawn, showing how the zeroes of the analytic function vary as a function of a parameter of the modulating signal.

## 2.6 Amplitude Modulation (AM)

Amplitude modulation is conventionally defined as

$$s_{AM}(t) = s(t) \cos \omega_0 t \qquad \text{with} \quad \omega_0 \geq \pi W_0$$

where $\omega_0$ is the carrier frequency, and s(t) is a real modulating signal of bandwidth $\pm \frac{W_0}{2}$. In analytic form, where $\hat{s}_{AM}(t)$ is the Hilbert transform of $s_{AM}(t)$, from

$$s_{AM}(t) = s(t) \cos \omega_0 t \tag{2.6-1}$$

then

$$\hat{s}_{AM}(t) = s(t) \sin \omega_0 t \tag{2.6-2}$$

and

$$A_{AM}(t) = s(t) \exp j\omega_0 t.$$

The zeroes of $A_{AM}(t)$ are those of s(t), because s(t) is real. The zeroes either must be real or appear in complex conjugate pairs.

Introducing the property (g) of analytic signals (from Section 2.4):

$$s(t) = s_{cz}(t) \cdot s_{Rz}(t) \tag{2.6-3}$$

The number of zeroes per period n is:

$$n = 2(n_c + n_r) = W_0 T. \tag{2.6-4}$$

For example, let the modulating signal s(t) be the two-component square wave approximation shown in Fig. 2.6-1.

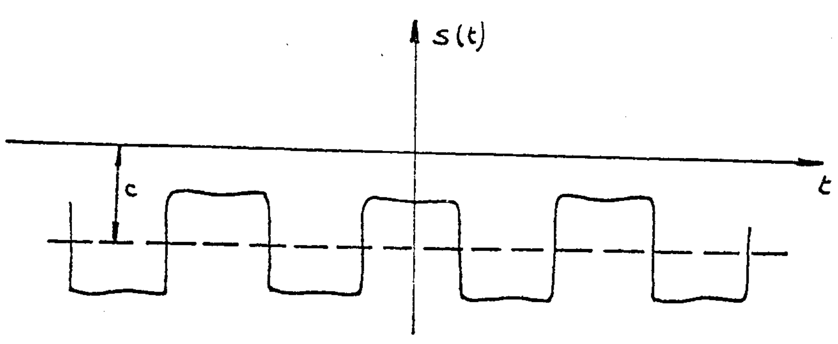



Figure 2.6-1 The two component square wave representation

$$s(t) = c + \cos t - \cos 3t \tag{2.6-5}$$

It follows that the zero count is six from bandwidth considerations as shown below in Equation (2.6-6). The real signal can be expressed as

$$s(t) = c + \frac{e^{j\omega t} + e^{-j\omega t}}{2} - \frac{e^{j3\omega t} + e^{-j3\omega t}}{6}$$

In order to find the zeroes of the analytic signal m(t) corresponding to s(t), replace the variable "t" by

$$z = t + j\sigma$$

to obtain:

$$s(x) = -\frac{1}{6x^3}\left(x^6 - 3x^4 - 6cx^3 + 3x^2 - 1\right)$$

where

$$e^{j\omega z} = x \tag{2.6-6}$$

Thus the zeroes of the analytic signal are given by the six roots of the equation

$$x^6 - 3x^4 - 6cx^3 + 3x^2 - 1 = 0. \tag{2.6-7}$$

The roots of this polynomial are a function of the parameter modulation factor c, and it is a variation of c that produces the "zero-locus."

The analytic signal corresponding to s(t) is $A_s(t)$

$$A_s(t) = s(t) + j\,\hat{s}(t)$$

where $\hat{s}(t)$ is the Hilbert transform of s(t). From Equation (2.2-6),

$$H\{\cos \omega t\} = \sin \omega t$$

$$H\{\cos 3\omega t\} = \sin 3\omega t,$$

and when this is substituted in Equation (2.6-7), it results in

$$A_s(t) = c + \cos \omega t + j \sin \omega t - \frac{1}{3} \cos 3\omega t - j \frac{1}{3} \sin 3\omega t$$

$$= c + e^{j\omega t} - \frac{1}{3} e^{j3\omega t}. \qquad (2.6-9)$$

Now it is desired to find the zero trajectories as the parameter c is varied. The technique previously indicated can be used here as follows:

a) substitute $t$ for $z = t + j\sigma$

b) let $x = \exp j\omega z$

c) therefore, $A_s(z) = c + \exp(j\omega z) - \frac{1}{3} \exp(j\omega z)$

The zeroes of the analytic signal are given by

$$x^3 - 3x - 3c = 0 \qquad (2.6-10)$$

The simplification is apparent. The analytic signal model provides a simpler way to find the zero trajectories of the original signal. In the previous sample, the original modulated signal had a zero-pattern with a total number of six zeroes, but it is found that the corresponding analytic signal has a zero-pattern of three zeroes given by the expression (2.6-10). A cubic equation results instead of the sixth order polynomial of the original signal. The simplified calculations needed for classical modulation techniques is one of the main advantages of the use of the analytic signal representation. In fact it is shown later that a more efficient mathematical description of the classical models will allow development of further results.

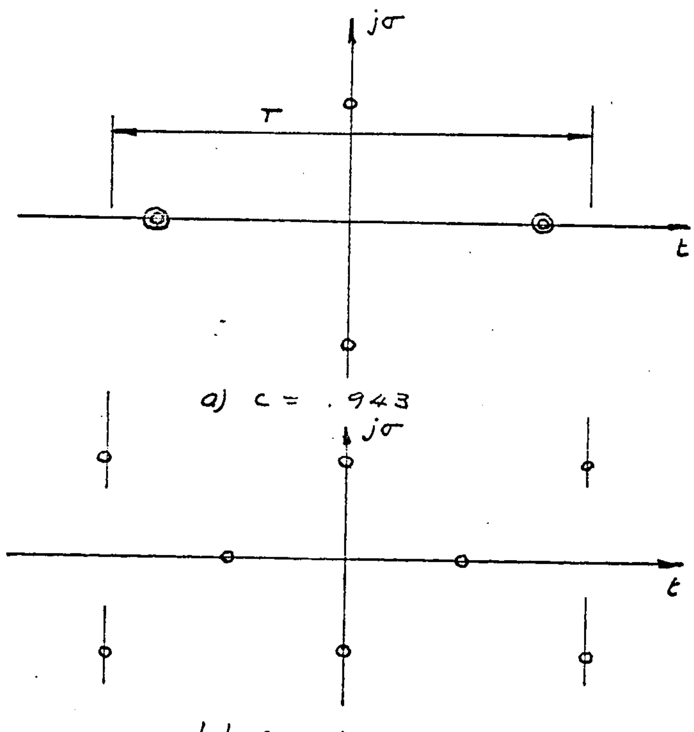



Figure 2.6 Zero pattern for the real signal: a) for 100% modulation b) for suppressed carrier operation

## 2.7 Single-Side Band Systems (SSB)

Amplitude modulation systems are characterized by a symmetrical zero pattern along the real axis in the z-plane and envelope detectability in case of a strong carrier.

It is known (Property c of analytic signals) that the zeroes of a modulated wave can be conjugated without its affecting the envelope or the bandwidth of the wave. There will be a common envelope "set" of signals for each modulating signal. The following examples show how different zero-pattern configurations may represent the same modulation wave; in this case, the square wave signal of the previous example Equation (2.6-5) is used.

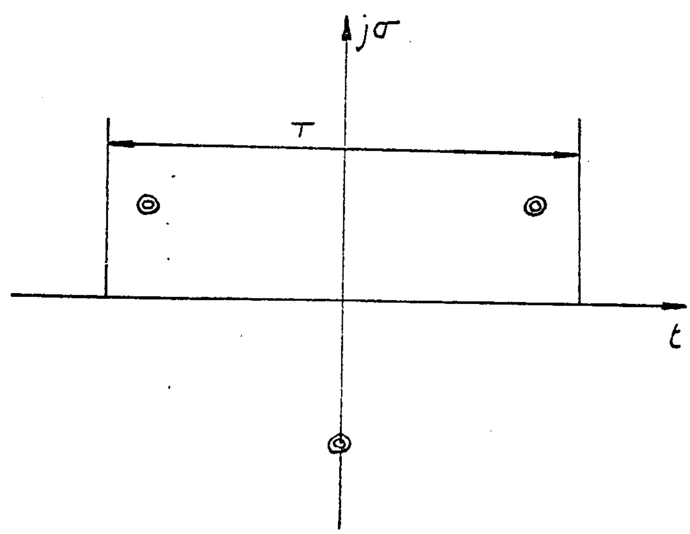



Figure 2.7-2 Zeroes of the non-minimum phase signal

A non-minimum phase signal (NMP) is arbitrarily chosen in Figure 2.7-1 to represent the square wave signal. It has zeroes in the upper and lower z-plane. The same square wave signal is represented in Figure 2.7-2 by an arbitrarily chosen minimum phase signal (MP) that has all its zeroes in the lower half z-plane. Various experiments (Voelcker 1966) show that the NMP pattern with zeroes in both UHP and LHP yields a more symmetrical spectrum than the MP pattern.

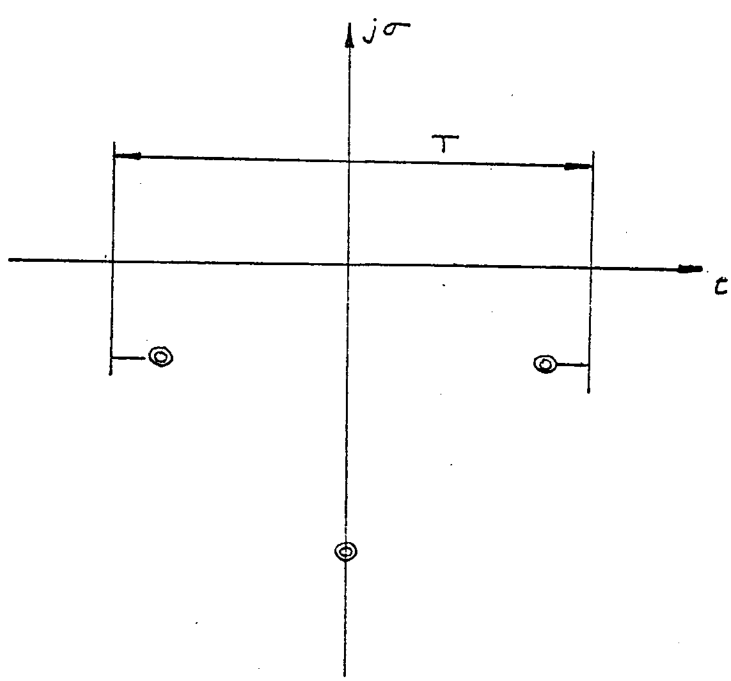



Figure 2.7-1 Zeros of minimum phase signal

The structure of both zero patterns are now asymmetrical in contrast to the symmetry found in the previous examples (2.4). The envelope and bandwidth characteristics are not changed, however, since the zero count remains the same, and they both represent the signal of Figure 2.6-11, as can be shown by following the procedure indicated in Section 2.4, Multiplicative Models For Analytic Signals.

Consider the pattern of Figure 2.7-3 below where the zeroes are given:

$$z_i = \begin{cases} -\dfrac{3\pi}{\omega} \\ +\dfrac{3\pi}{\omega} \\ 0 \end{cases} + j\,\frac{1}{\omega}\,\text{arc cosh}\,\sqrt{2} \qquad (2.7-1)$$

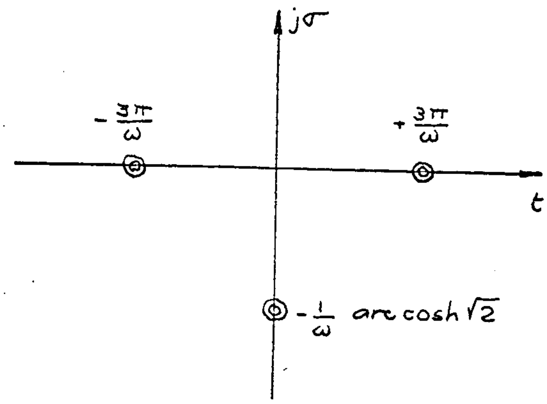



Figure 2.6-3 Minimum phase signal

A substitution of the zero coordinates in

$$\alpha_i = exp\,(-j\omega z_i)$$

and an expansion of s(t)

$$s(t) = \prod_{i=1}^{n} \left[1 - \alpha_i \, exp\, j\omega t\right]$$

where

$$x = exp\, j\omega t$$

results in

$$s(t) = \left[1 - exp(j3\pi x)\right]\left[1 - exp(j3\pi x)\right]\left[1 - exp(arccosh\sqrt{2}\; x)\right]^2$$

$$= \left[1 + x + (\sqrt{2} - 1)x^2 - (\sqrt{2} - 1)\, x^3\right]^2 \qquad (2.7-2)$$

where the corresponding modulated signal will be:

$$s_{MP,AM}(t) = Re\left\{ s(t)\, exp\, j\omega_0 t\right\} \qquad (2.7-3)$$

where $\omega_0$ is the carrier frequency.

The previous discussion demonstrated that envelope and bandwidth are preserved so long as the zero count is not changed, and these zeroes can be recovered by rearranging the zeroes of the modulated wave into conjugate pairs.

If the zero pattern is modified by decreasing the zero count, the bandwidth of the actual signal is reduced. One of the most interesting cases of reduced bandwidth is that of SSB. In single sideband modulation, half of the zeroes are suppressed. Since all zeroes are in conjugate pairs, only one zero from each pair can be specified in order to determine the complete pattern, remembering that deleted zeroes will be replaced during demodulation via conjugation of the transmitted zeroes.

Let it be supposed that the suppressed zeroes are located in the UHP. The remaining zeroes correspond to an MP signal which is the "square root" of the asymmetric AM signal (AAM). The envelope of the half-suppressed zeroes signal must be the square root of the MP-AAM signal with half phase excursions with respect to those of the MP-AAM signals. For this reason, the reduced -zero MP signals are termed "Square law single-side band" (SQ-SSB). This means that their bandwidth corresponds to that of conventional SSB, and the envelope can be recovered by a square-law envelope detector. A practical transmitter for SQ-SSB is shown in Fig. 2.7-4. Experimental tests of this circuit are described by Von Urff and Zoni (Von Urff and Zoni 1962).

Now consider the case in which the modulating signal contains an odd number of real zeroes. The reduced zero pattern must contain an integral number of zeroes. The procedure which reduces the zero count to a "sideband-and-a-fraction" is called "partial SQ-SSB." An example

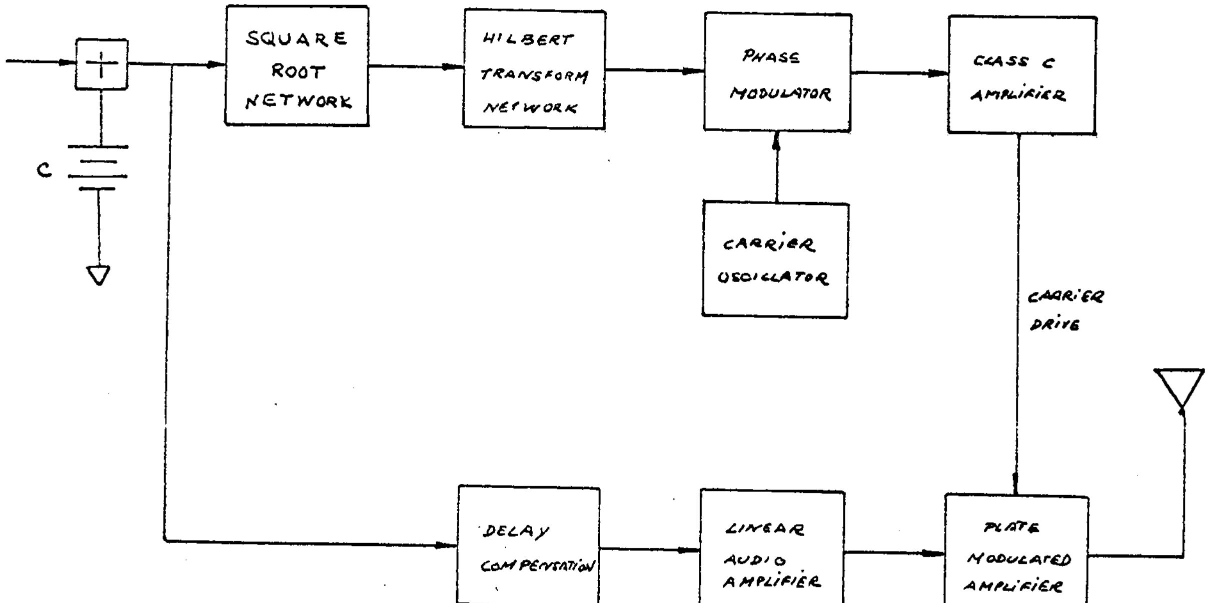



Figure 2.7-4 An SQ-SSB modulator transmitter (Voelcker 1966)

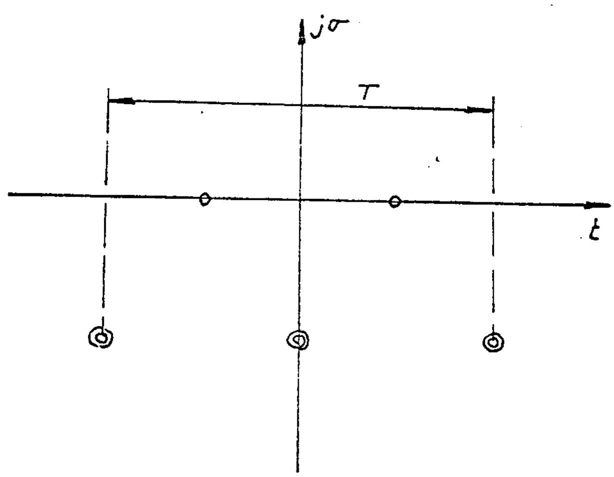



Figure 2.7-5 Zero pattern for SQ-SSB with odd number of zeroes

of partial SQ-SSB is shown in Fig. 2.7-5.

Since the process of modulation and demodulation for partial SQ-SSB is neither simple nor practical, no further discussion of it is included. However, it is instructive to examine the relationships of zero reduction, bandwidth reduction, and efficiency at this point. It has been shown in Section 2.3 how it is always possible to force a real signal into a wholly conjugate zero form by adding a convenient constant. The addition of the constants, however, results in a loss of efficiency. Another solution is to process the complex zeroes of s(t) through the SQ-SSB system and the real zeroes of s(t) through SSB in separate processes. This solution requires synchronism and conditions of sufficiency.

In order to examine the details of SSB, let the analytic signal representation of the message be given as

$$A_s(t) = s(t) + j\hat{s}(t) \tag{2.7-4}$$

and let the messages be modulated by the carrier

$$A_c(t) = e^{j\omega_0 t} \tag{2.7-5}$$

thus resulting in

$$A_{s,SSB}(t) = A_s(t) \cdot A_c(t) \tag{2.7-6}$$

$$= s(t)\cos\omega_o t - \hat{s}(t)\sin\omega_o t +$$

$$+ j\left[\hat{s}(t)\cos\omega_o t + s(t)\sin\omega_o t\right] \tag{2.7-7}$$

The envelope is then given by

$$|A_{s,SSB}(t)| = \sqrt{2s^2(t) + 2\hat{s}^2(t)} \tag{2.7-8}$$

Equation (2.7-7) represents a simple sideband modulated signal, which can be seen by noting that $\sin\omega_o t$ and $\cos\omega_o t$ form a Hilbert pair and writing the Fourier expansion of s(t) and $\hat{s}(t)$ as

$$s(t) = \sum_{n=0}^{\infty} c_n \sin(\omega_n t + \varphi_n) \tag{2.7-9}$$

$$\hat{s}(t) = \sum_{n=0}^{\infty} c_n \cos(\omega_n t + \varphi_n) \tag{2.7-10}$$

Equation (2.7-7) can then be rewritten as

$$A_{s,SSB}(t) = \sum_{n=0}^{\infty} c_n \cos\left[(\omega_n + \omega_o)t + \varphi_n\right] +$$

$$+ j\sum_{n=0}^{\infty} c_n \sin\left[(\omega_n + \omega_o)t + \varphi_n\right] \tag{2.7-11}$$

The last expression shows how the frequency spectrum of $s(t)$ is translated by the amount $\omega_0$ , the carrier frequency.

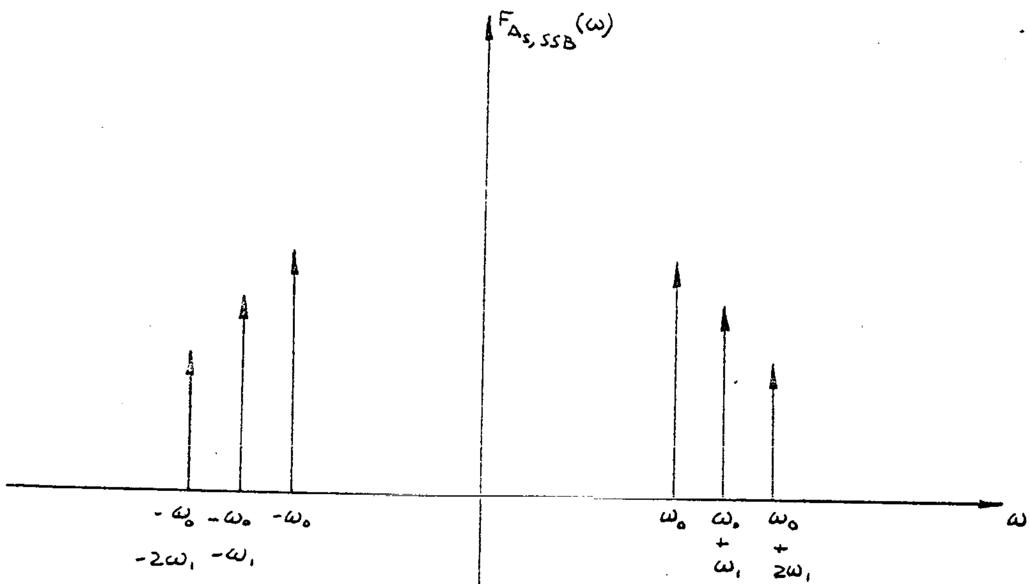



Figure 2.7-6  Single-side band spectrum of $A_s(t)$

For  $\omega_0 > 0$

$$F_{s,USB}(\omega) = \begin{cases} F_s(\omega - \omega_0) & \omega > \omega_0 \\ 0 & |\omega| < \omega_0 \\ F_s(\omega + \omega_0) & \omega < -\omega_0 \end{cases} \qquad (2.7-12)$$

where the USB is the upper sideband of the SSB modulated signal, note that the lower sideband is given by

$$A^{*}_{s,SSB}(t) = s(t) - j\hat{s}(t) \qquad (2.7\text{-}13)$$

since the spectrum of $A^{*}_{s,SSB}(t)$ vanishes for positive frequencies as previously indicated (Section 2.2). SSB can then be characterized by the half band-width technique in frequency domain and the linear operation of frequency translation. this linear operation <u>does not imply a linear change</u> in the zero pattern of the actual signal as it appears in Fig. 2.6-2 when the linear modulating parameter vanishes. Although there is no correspondence between the linear operation observed in the frequency domain, and the zero patterns change (non-linear) in the z-plane, the zero-locus provides new and useful information as is shown later.

Let $A_s(t)$ be the analytic form of the square wave used in the example (2.6-5).

$$A_s(t) = s(t) + j\hat{s}(t)$$

$$= c + \exp j\omega t - \frac{1}{3}\exp 3j\omega t \qquad (2.7\text{-}14)$$

As the modulating parameter c decreases, the zero pattern shows the following representation:

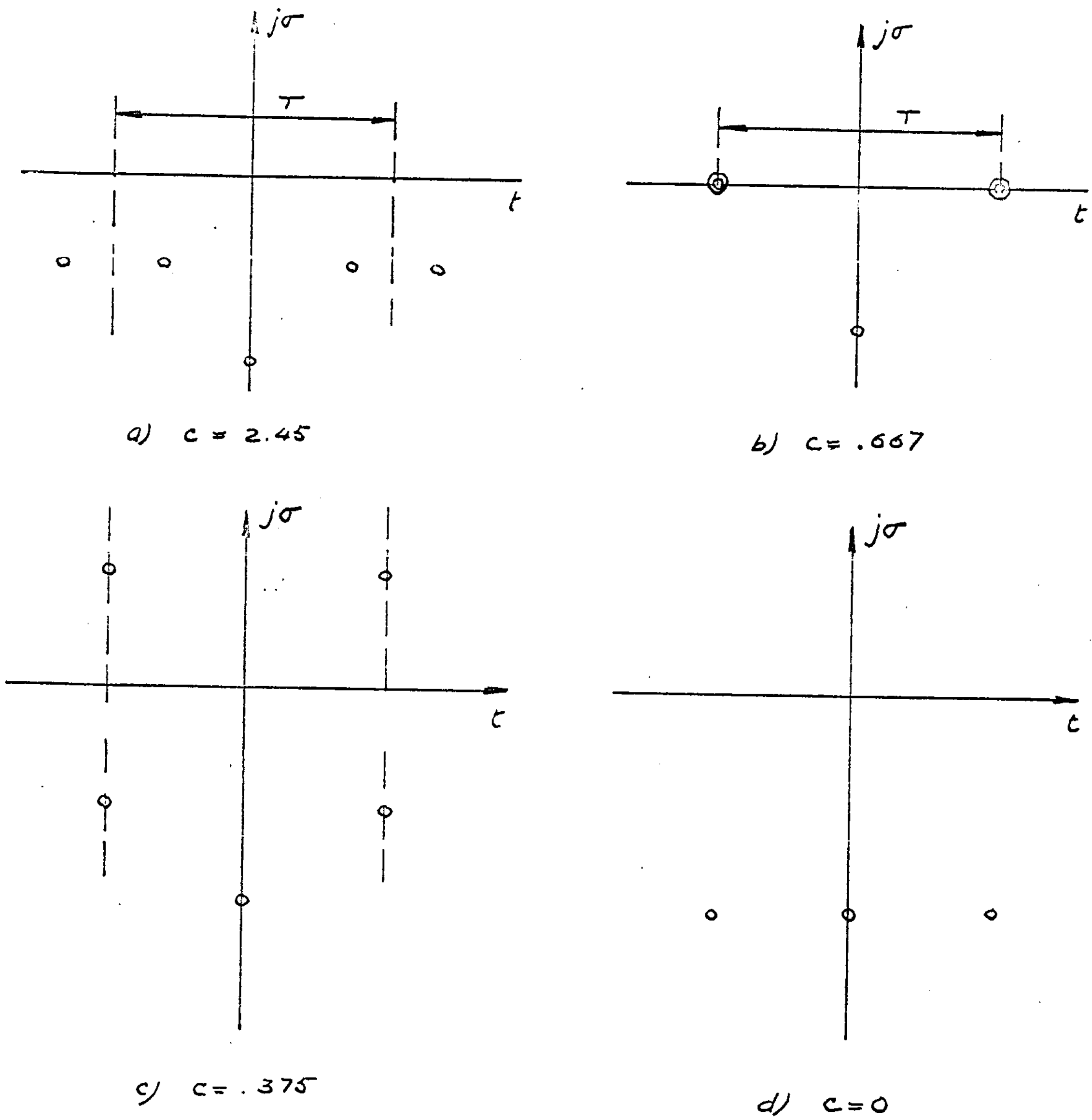



Figure 2.7-7 Zero pattern representation of the analytic signal

As c decreases, the zeroes curve up and when zeroes appear in the UHP for c = 0, corresponding to the case of the suppressed carrier, only a pair of complex zeroes exist in the LHP.

The example can be generalized by writing, again, the analytical signal of Equation (2.7-11) in the Fourier series expansion

$$A_{SSB}(t) = c_0 + \sum_{k=n_1}^{n} c_k \exp kj\omega t \qquad \text{for} \quad n_1 < n$$

where (Section 2.6 "Multiplicative Models for Analytic Signals") c is the modulating parameter and $c_1 \ldots c_k$ are the coefficients of the expanded product is a result of the signal factorization principle.

Whether or not SSB is a "coherent" modulation system, or, in other words, if the zeroes of the real modulating signal can be recovered from those of the analytic signal is the question that will be answered next. Looking at the transformation itself, the answer must be positive, because an analytic function of an analytic function is itself analytic, and its properties of analyticity are retained. Looking at the zero count criterion, the only condition imposed is that the zero count must be preserved in order that the recovery of $Re\left[A_s(t)\right]$ be possible.

According to Voelcker (Voelcker 1966), this is possible when the carrier is not suppressed completely. The reason given is that if the carrier is suppressed completely, the zero count and dimensionality are reduced according to the preceding criterion. SSB-SC can, however, be an admissible process if one can replace the lost zeroes by virtue of prior knowledge or "side-information." Therefore, SSB-SC is an insufficient process which requires exceedingly sophisticated techniques.

The recovery process of SSB is based on generating $A^*_{s,SSB}(t)$ by a complete conjugation of the transmitted zero pattern. The requirements for recovering the modulating signal without distortion are found in Voelcker (Voelcker 1966).

## 2.8 Wideband Modulation

The symbol PM will be used for angle modulation in general, except when otherwise specified for phase modulation.

Narrow band processes are characterized by preserving or decreasing the zero-count. The next logical step is to deal with processes which increase the zero count. Classical analysis of wideband processes involves a differential equation and Fourier expansion approach which yields a complicated series of Bessel functions whose physical significance is difficult to understand. An analytic signal representation along with a zero-locus manipulation, however, provides a new insight into the characteristics of the wideband process.

Wideband processes can be viewed in either DSB or SSB. Angle modulation in general is conventionally defined as

$$s_{M,\omega_0}(t) = \cos\left[\omega_0 t + \varphi_0 + \psi(t)\right] \qquad (2.8-1)$$

where $\omega_0$ is the carrier frequency, and $\psi(t)$ is defined as the instantaneous phase angle. Where

$$\psi(t) = M_{PM}\, s(t) \quad \text{for phase modulation (PM)} \qquad (2.8-2a)$$

$$\psi(t) = M_{FM}\int_0^t s(t)\,dt \quad \text{for frequency modulation (FM)} \qquad (2.8-2b)$$

and where s(t) is the modulating signal, Equation (2.7-2a,b) can be expressed by a common modulating function $M\{s(t)\}$, so that the new expression for Equation (2.8-1) becomes:

$$s_{M,\omega_0}(t) = \cos\left[\omega_0 t + M\{s(t)\}\right] \qquad (2.8-3)$$

The complex form of Equation (2.7-1) is:

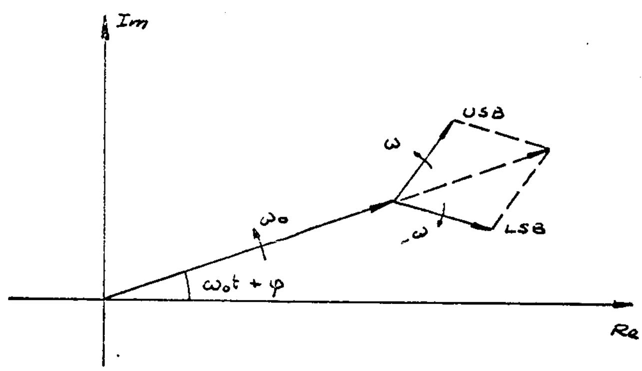
$$s_M(t) = Re\left[\mu_M(t)\exp j\omega_0 t\right] \qquad (2.8-4)$$

where

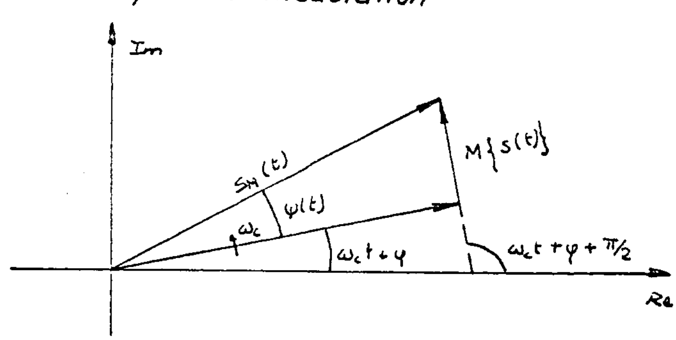
$$\mu_M(t) = \exp j M\{s(t)\} \qquad (2.8-5)$$

It turns out that $|\mu_M(t)|$ is constant; therefore, $\mu_M(t)$ must have an infinite number of zeroes, and they are located at infinity. In other words, the expression (2.8-4) is not analytic, and zero manipulation is not possible. It is necessary, therefore, to evolve band-limited analytic models that can converge to the conventional expression (2.8-4) under suitable conditions.

An approach to the analytical representation of angle modulation is a basis of Armstrong's method (Armstrong 1936). The conventional phasor diagram for AM and PM is shown in Fig. 2.8-1.




a) Amplitude modulation




b) Frequency Modulation

Figure 2.8-1 Phasor diagrams comparing narrow-band AM with FM

The classical approach to FM suggested above yields the expression:

$$s_{FM}(t) = \underbrace{A_0 J_0(D) \sin \omega_c (t)}_{\text{carrier}} + A_0 \sum_{n=1}^{\infty} J_{2n}(D) \underbrace{\left[ \underbrace{\sin(\omega_0 + 2n\omega_m)t}_{USB} + \underbrace{\sin(\omega_0 - 2n\omega_m)t}_{LSB} \right]}_{\text{Even Order}} +$$

$$+ A_0 \sum_{n=1}^{\infty} J_{2n-1}(D) \underbrace{\left\{ \underbrace{\sin\left[\omega_c + (2n-1)\omega_m\right]t}_{USB} - \underbrace{\sin\left[\omega_0 - (2n-1)\omega_m\right]t}_{LSB} \right\}}_{\text{Odd order}}$$

where $J_p(D)$ is the Bessel function of the first kind and order of p with argument D as the modulation index. If Fig. 2.8-1 is redrawn utilizing the different components of the above expression, Fig. 2.8-2. results.

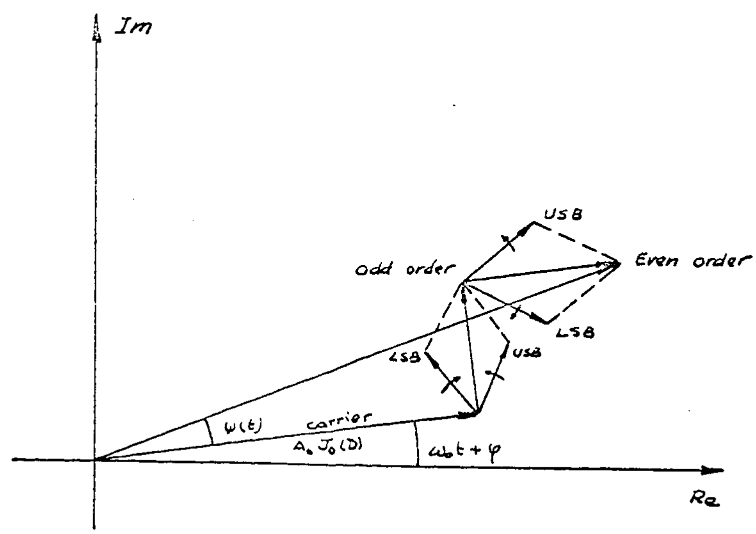



Figure 2.8-2 Frequency modulation

The analytic signal approach to FM suggests that for small deviations, angle modulation can be represented by quadrature modulation (QM) where QM can be derived from AM by a 90° relative phase shift between the constant carrier component and the modulated carrier component. Fig. 2.8-3 shows the phasor diagram for the QM signal.

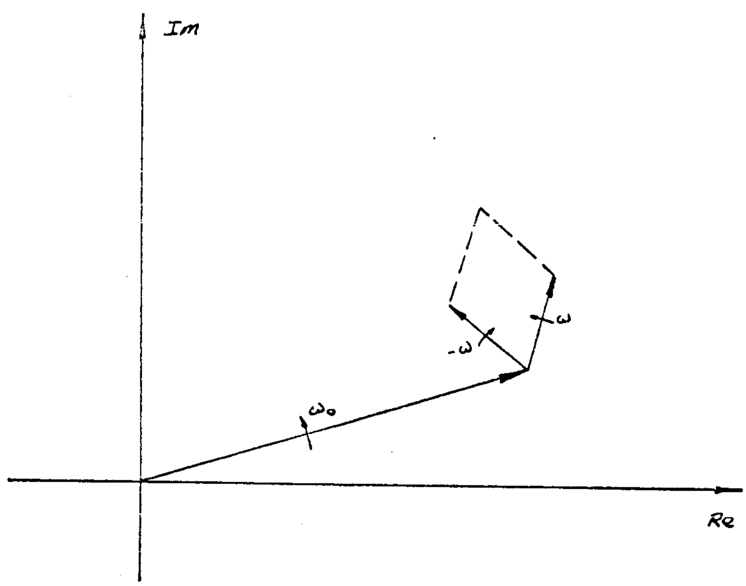



Figure 2.7-3 Quadrature modulation

This phasor representation corresponds to the case of angle modulation for small deviation which is given in conventional form by

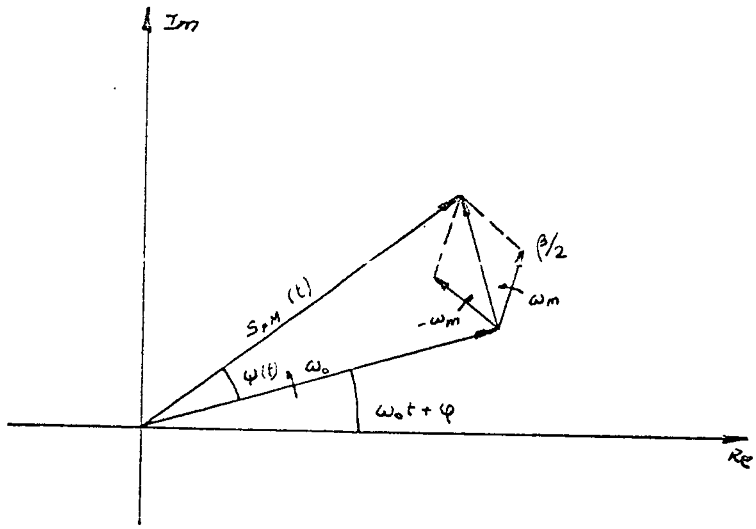



Figure 2.8-4 Phasor diagram for small deviation PM

The AM of Fig. 2.8-1a is described in analytic form by

$$A_{s,AM}(t) = \left[1 - B \cos \omega t\right] \exp j\omega_0 t \qquad (2.8-8)$$

with $0 < B < 1$ and $\omega \leq \omega_0$

The zeroes of Equation (2.8-8) are easily found by the techniques explained previously in Fig. 2.8-5.

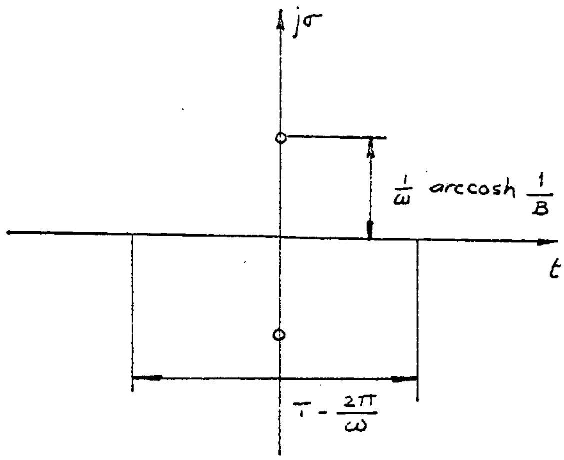



Figure 2.8-5 Zero pattern of $A_{s,AM}(t)$

The AM analytic form is converted to QM by the 90° relative phase shift shown above.

$$A_{s,QM}(t) = \left[1 + jB\cos\omega t\right] \exp j\omega_0 t$$

and its zero pattern is:

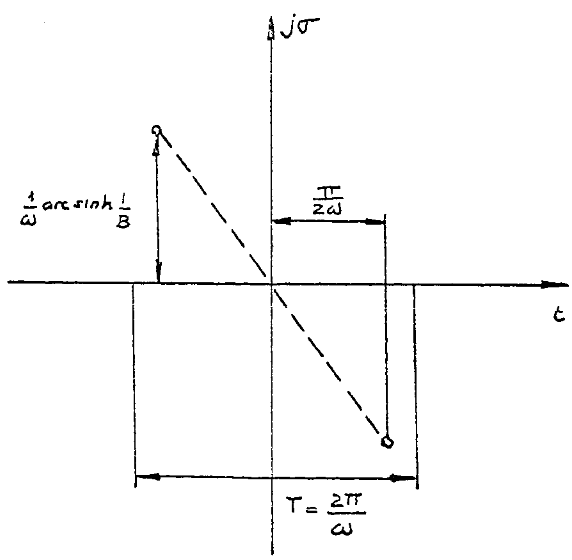



Figure 2.8-6 Zero pattern of $A_{s,QM}(t)$

where the parameter B is given by("Multiplicative Models for Analytic Signals" 2.4):

$$B = \operatorname{cosech} \omega |\sigma|$$

For $B \ll 1$

$$1 + jB \cos \omega t \approx \exp \left[ jB \cos \omega t \right] \tag{2.8-9}$$

gives a good approximation to FM in case of small deviations.

The procedure for constructing zero patterns when the modulation signal is an L component real part of the Fourier expansion for the modulating signal is given by Voelcker (Voelcker 1966) as

$$M\left\{ s(t) \right\} = M\left\{ \sum_{\ell=1}^{L} c_\ell \cos \ell\omega \left( t - \tau_\ell \right) \right\}$$

$$= \sum_{\ell=1}^{L} c'_\ell \cos \ell\omega \left( t - \tau'_\ell \right) \quad for \quad 0 < c'_\ell \ll 1 \tag{2.8-10}$$

The steps are as follows:

1. Assign $2\ell$ zeroes per period T to the $\ell$ th component of $M\{s(t)\}$. Let the zeroes be located in periodic conjugate pairs within imaginary component

$$|\sigma_\ell| = \frac{1}{\ell\omega} \operatorname{arccosech} c'_\ell \tag{2.8-11}$$

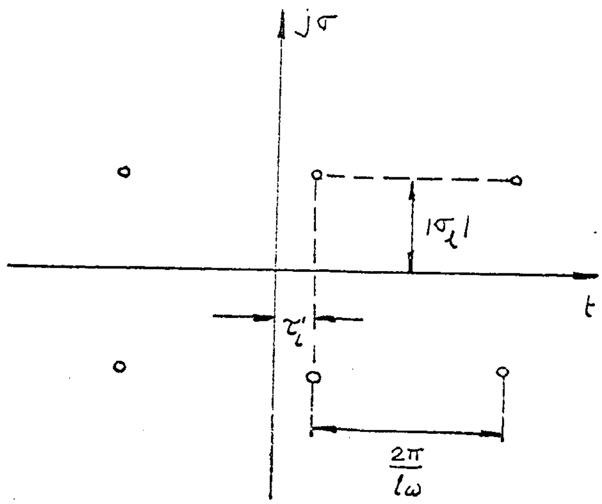



Figure 2.8-7 Construction of a zero pattern of narrow band angle modulation, step 1

2. Shift the LHP and UHP zeroes $\frac{\pi}{2\omega l}$ seconds right and left respectively. The analytic signal produced by this shift is:

$$A_{s_l}(t) = \left[1 + jc_l' \cos l\omega\,(t-\tau')\right] \exp j\,l\omega t \qquad (2.8\text{-}12)$$

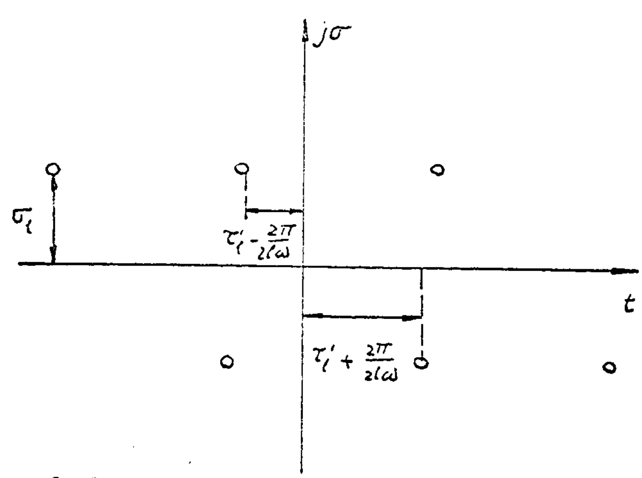



Figure 2.8-8 Construction of a zero pattern of narrow angle modulation, step 2

3. Repeat steps 1 and 2 for each component of $M\left\{s(t)\right\}$ . Superimpose the zero patterns thus constructed and multiply the $A_{s_\ell}(t)$ to obtain:

$$A_{s,PM}(t) = \prod_{\ell=1}^{L} A_{s_\ell}(t) \qquad (2.8-13)$$

with a zero count of

$$n = 2\sum_{\ell=1}^{L} \ell = L(L+1) \qquad (2.8-14)$$

and with signal deviation $0 < c_\ell' \leqslant 1$

Thus the expression that allows the calculation of zero-patterns is:

$$A_{s,PM} \cong \left\{\prod_{\ell=1}^{L} \exp\left[j c_\ell' \cos(\omega(t-\tau_\ell')\right)\right\} \exp j\frac{n\omega t}{2} \qquad (2.8-15)$$

$$\cong \exp j\left[\frac{n\omega t}{2} + M\left\{s(t)\right\}\right]$$

It turns out that the last expression is analytic under the condition that the phase deviation of the actual signal approaches zero. Each factor $A_{s_\ell}(t)$ is a QM elementary signal. $A_{s,PM}(t)$ , however, does not represent a quadrature carrier modulated by $M\left\{s(t)\right\}$ , but, since the QM modulating process is a zero preserving process, $A_{s,PM}(t)$ is an zero expanding process.

Wideband Angle Modulation

The conventional angle modulation in analytic form is defined as

$$A_s(t) = e^{jM\{s(t)\}} \qquad (2.8\text{-}16)$$

where the message $A_s(t)$ is frequency modulated by the carrier

$$A_c(t) = e^{j\omega_o t}$$

the modulated signal will be

$$A_{s,DSB,PM}(t) = A_s(t) \cdot A_c(t)$$

$$= e^{j\omega_o t} \, e^{jM\{s(t)\}}$$

$$= \cos\left[\omega_o t + M\{s(t)\}\right] + j\sin\left[\omega_o t + M\{s(t)\}\right] \qquad (2.8\text{-}18)$$

The last expression represents conventional PM, but it is designated as DSB to emphasize that the spectrum is two-sided about $\omega_o$ .

The expression (2.8-18) shows that the spectrum of such a signal extends to infinity in both directions; therefore, the shift of $\omega_o$ in the frequency domain cannot produce a spectrum containing only positive frequencies

(Fig. 2.8-9), and the above representation (Eq. 2.8-18) in itself cannot be analytic.

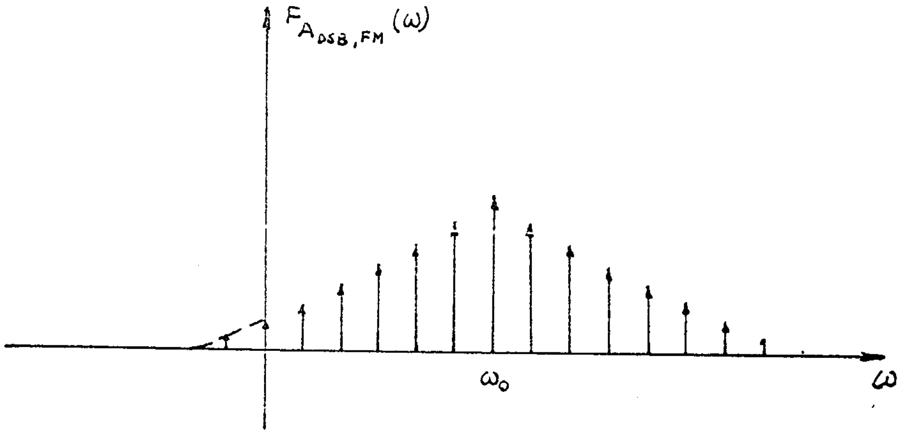



Figure 2.8-9 Spectrum of $A_{DSB,FM}(t)$

It is necessary at this point to apply the results obtained in the preceding section (2.7). Using QM elementary signals of the the type of Equation (2.8-12), one redefines:

$$A_{s_{\ell,N}}(t) = \left[1 + j\,\frac{c_\ell}{N}\cos \ell\omega\,(t-\tau'_\ell)\right] \exp j\ell\omega t$$

and using the multiplicative property (Sec. 2.4) of associating the elementary signal

$$A^N_{DSB,PM}(t) = \prod_{\ell=1}^{L} \left[A_{s_{\ell,N}}\right]^N \tag{2.8-17}$$

It can be shown (Davenport and Root 1958, Downing 1964) that by applying the techniques of the Central Limit Theorem

$$\lim_{N\to\infty} c\left(j\frac{s}{\sqrt{N}}\right) = \lim_{N\to\infty}\left(1-\frac{s^2}{2N}\right)^N = e^{-s^2/2}$$

in Equation (2.7-19) the above property yields:

$$\lim A^N_{DSB,PM}(t) = \left(\exp j\,\frac{NL(L+1)}{2}\,\omega t\right)\left(\exp\left[\sum_{l=1}^{L} c_l' \cos l\omega\,(t-\tau_l')\right]\right)$$

$$= \exp j\left[\frac{nN\omega t}{2} + M\{s(t)\}\right] \qquad (2.8-20)$$

where $nN = NL(L+1)$ is the zero count and $\sigma_l$ the ordinate of complex zero in the z-plane is:

$$|\sigma_l| = \frac{1}{l\omega}\ \text{arc cosech}\ \frac{c_l'}{N} \qquad (2.8-21)$$

Equation (2.8-20) exhibits infinite zero count (infinite frequency), infinite carrier frequency, and infinite zero ordinates. Again some limitation is necessary, since according to Dugundji's squared envelope theorem (Dugundji 1958), whenever an ideal angle modulation is forced to be band-limited, envelope fluctuation must appear.

The requirement for finite bandwidth forces N to be finite, and the distortion problem arises. The zero count can be minimized to yield allowable distortion, or

$$A^{(N)}_{DSB,PM}(t) = \prod_{l=1}^{L} \left[A^{(N)}_{s_l}(t)\right]^{(N)}$$

$$= \exp j\left[\omega t \sum_{l=1}^{L} l(N) + M\{s(t)\}\right]$$

where $(N) \equiv$ *smallest integer* $\geqslant L\left(\frac{c_l'}{B}\right)^2$ (2.8-22)

and $B^2$ is the specified distortion parameter which is less than unity. The envelope distortion associated with Equation (2.8-22) is:

$$\left|A^{(N)}_{DSB,PM}(t)\right| - 1 \leqslant \frac{B^2}{2L} \sum_{l=1}^{L} \cos^2 l\omega\left(t - \tau_l'\right) + O(A^4) \qquad (2.8\text{-}23)$$

and the angular distortion is of the order

$$\frac{B^2}{3L} \sum_{l=1}^{L} (N_l)^{-1} \qquad (2.8\text{-}24)$$

(Voelcker 1966).

Single Sideband Angle Modulation

The analytic form describing a SSB phase modulation system for the modulating signal

$$A_s(t) = e^{j[M\{s(t)\}] + jM\{\hat{s}(t)\}]} \qquad (2.8-25)$$

is

$$A_{SSB,FM}(t) = A_s(t) \cdot e^{j\omega_0 t} \qquad (2.8-26)$$

$$= \left[\exp\left(-M\{\hat{s}(t)\}\right)\right] \exp j\left[M\{s(t)\} + \omega_0 t\right]$$

$$= e^{-M\{\hat{S}(t)\}}\left[\cos\omega_0 t + M\{s(t)\} + j\sin\omega_0 t + M\{s(t)\}\right]$$

Since an analytic function of an analytic function is also analytic, $A_{SSB,FM}(t)$ will be analytic with

$$|A_{SSB,FM}| = \exp\left[-M\{\hat{s}(t)\}\right] \qquad (2.8-27)$$

The spectrum of $A_{FM,SSB}(t)$ must vanish for $\omega < \omega_0$. Since the modulation function is now analytic, and thus it contains no negative frequencies. The spectrum still extends to infinity in the positive direction. As further proof of the same property, let:

$$\frac{d\varphi}{dt} = \omega_0 + f_1'(t) \qquad (2.8-28)$$

where $f_1(t) = M\{s(t)\}$ is used to simplify the calculation. Now let

$$f_1'(t) = \Omega \cos \omega t$$

and neglecting the constant

$$f_1(t) = \frac{\Omega}{\omega} \sin \omega t \qquad (2.8-29)$$

$$\hat{f}_1(t) = -\frac{\Omega}{\omega} \cos \omega t \qquad (2.8-30)$$

where $\Omega$ is the peak frequency deviation. The analytic function in this case will be:

$$A_{f_1}(t) = e^{j[f(t) + j\hat{f}_1(t)]}$$

$$= \exp\left[\frac{\Omega}{\omega}\left(\sin \omega t - j \cos \omega t\right)\right]$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!}\left[\frac{\Omega}{\omega} \exp j\omega t\right]^K$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{\Omega}{\omega}\right)^K \exp K\omega t$$

or

$$A_{f_1}(t)\, e^{j\omega_0 t} = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{\Omega}{\omega}\right)^K \exp j(K\omega + \omega_0)t \qquad (2.8-31_a)$$

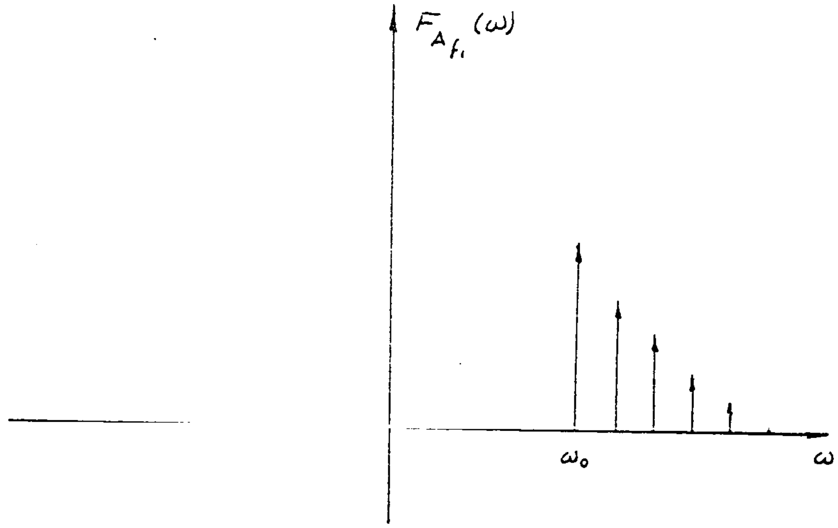



Figure 2.8-10 Frequency spectrum of $A_{f_1}(t)$ modulated wave

The single-sided property of the analytic signal is therefore established. The sketch of Fig. 2.8-10 illustrates Equation (2.8-31a).

Now let:

$$f_2'(t) = -\Omega \cos \omega t$$

and

$$f_2(t) = -\frac{\Omega}{\omega} \sin \omega t$$

thus

$$\hat{f}_2(t) = \frac{\Omega}{\omega} \cos \omega t$$

and

$$A_{f_2}(t) = \exp\left[-\frac{\Omega}{\omega} \exp(-j\omega t)\right] = \sum_{k=0}^{\infty} (-1)^k \frac{\Omega}{K!\omega} e^{-jK\omega t}$$

Hence

$$A_{f_2}(t)\, e^{j\omega_0 t} = \sum_{K=0}^{\infty} \frac{(-1)^K}{K!} \left(\frac{\Omega}{\omega}\right)^K \exp\left[j(\omega_0 - K\omega)\right] t \qquad (2.8\text{-}31b)$$

Finally, combining the real part of Equation (2.8-31a) and Equation (2.8-31b),

$$Re\left[A_{f_1}(t)\, e^{j\omega_o t}\right] = \sum_{K=0}^{\infty} (-1)^K \frac{1}{K!}\left(\frac{\Omega}{\omega}\right)^K \cos(\omega_o - K\omega)t \qquad (2.8\text{-}32a)$$

$$Re\left[A_{f_2}(t)\, e^{j\omega_o t}\right] = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{\Omega}{\omega}\right)^K \cos(\omega_o + K\omega)t \qquad (2.8\text{-}32b)$$

$$A_{SSB,FM}(t) = Re\left[A_{f_1}(t)\, e^{j\omega_o t} + A_{f_2}(t)\, e^{j\omega_o t}\right]$$

$$= \sum_{K=0}^{\infty} a_n \cos(\omega_o - K\omega)t + \sum_{K=0}^{\infty} b_n \cos(\omega_o + K\omega)t \qquad (2.8\text{-}33)$$

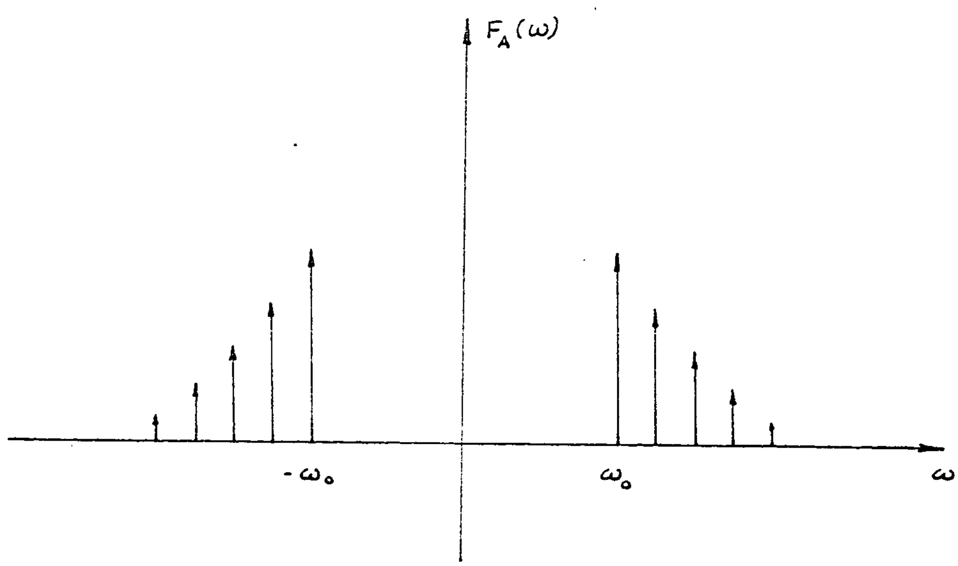



Figure 2.8-11 Frequency spectrum of $A_{SSB,FM}(t)$

For SSB with suppressed carrier (SSB-SC), note that by replacing

$$\omega_{sc} + \theta(t) = \omega t \qquad (2.8-34)$$

Equation (2.8-28) becomes:

$$A_{SSBSC,FM}(t) = \sum_{K=0}^{\infty} a_n \cos\left[\left(\omega_{sc} - K\omega_c\right)t + \theta(t)\right] +$$

$$+ \sum_{K=0}^{\infty} b_n \cos\left[\left(\omega_{sc} + K\omega_c\right)t + \theta(t)\right] \qquad (2.8-35)$$

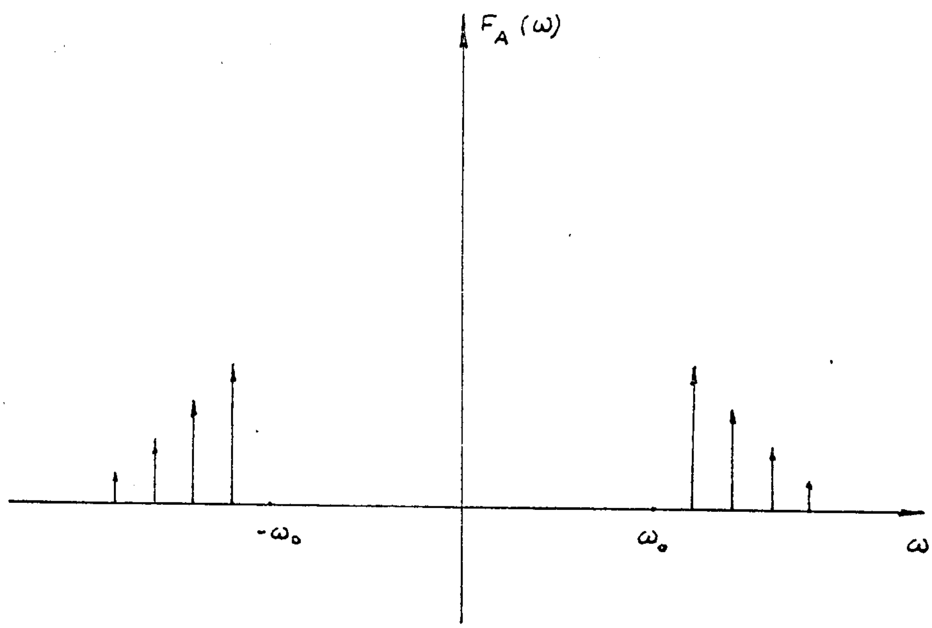



Figure 2.8-12 Frequency spectrum of SSBSC angle modulation

The zero modil for SSB angle modulation will now be analyzed. By observation of Equations (2.8-6) and (2.8-7), $A_{SSB,PM}$ is a MP function, because its phase and the logarithm form a Hilbert pair. Therefore, all the zeroes must be in the LHP. Also note that

$$\exp j2M\{s(t)\} = \left[\exp jM\{m(t)\}\right]\left[\exp jM\{m^*(t)\}\right] \qquad (2.8\text{-}36)$$

and the spectrum of the righthand expression must vanish for UHP and for $\omega > 0$. The two factors of Equation (2.8-32) can be interpreted as the USB and the LSB. Then angle modulation is given by multiplication in the time domain or by convolution in the frequency domain. The multiplication in the time domain of the analytic signals means the superposition of their zero-patterns in the z-plane (see the properties of analytic signals in Sec. 2.4). It follows that the zero patterns of the SSB angle modulation is the superposition of the zeroes of the sidebands; therefore, they lie in the LHP portion of the z-plane.

By following the same steps of calculating the zeroes of DSB, the following type of formulation is obtained (Voelcker 1966):

$$A_{SSB,PM}(t) = \prod_{l=1}^{L}\left[1 + j\,\frac{c_l'}{N_l}\,\exp j\left(\omega\left(t - \tau_l'\right)\right)\right]^{(N_l)} \qquad (2.8\text{-}37_b)$$

$$= \left[\exp\left(-M\{\hat{s}(t)\}\right)\right]\exp jM\{s(t)\} \qquad (2.8\text{-}37_b)$$

The smallest number $N_l$ determining the zero count with tolerable distortion is given as

$$N_l \geqslant L\left(\frac{c'_l}{A}\right)^2$$

and the ordinates are given by

$$|\sigma_l| = \frac{1}{l\omega} \ln \frac{N_l}{c'_l} \qquad (2.8-38)$$

The bandwidth requirements of SSB angle modulation are half that of conventional angle modulation. This favorable feature is diminished in practice by the fact that experimental data does not verify this compression of two-to-one because of factors not considered above. Furthermore, current methods of generating SSB angle modulation are quite expensive. For instance, if s(t) is a bandlimited signal as discussed previously,

$$s(t) = \cos\omega t - \frac{1}{3}\cos 3\omega t$$

The modulating functions are:

$$M_{PM}\{s(t)\} = \gamma\left[\cos\omega t - \frac{1}{3}\cos 3\omega t\right] \qquad (2.8-39)$$

$$M_{FM}\{s(t)\} = \frac{\beta}{\omega}\left[\sin\omega t - \frac{1}{9}\sin 3\omega t\right] \qquad (2.8-40)$$

The parameters $\gamma$, $\beta$ are defined

$\gamma$ = maximum phase deviation

$\beta$ = froquency deviation

and $\frac{\beta}{\omega}$ = modulation index or FM deviation ratio

For conventional PM, the zero pattern can be constructed in the following manner. Recall that

$$A_{DSB,PM}(t) = \prod_{l=1}^{L} \left[ A_{s_l}^{(N_l)}(t) \right]^{(N_l)} \qquad (2.8\text{-}41)$$

where

$N_l$ is the smallest integer $\geq L\left(\frac{c'_l}{B}\right)^2$ (2.8-42)

$c'_l$ is the coefficient of "$l$" component real Fourier series of modulating signal

B is the specified distortion.

L is the number or real component of Fourier series

$$A_{s_l}^{(N_l)}(t) = \left[1 + j\frac{c'_l}{N_l} \cdot \cos l\omega(t - \tau'_l)\right] \exp jl\omega t \qquad (2.8\text{-}43)$$

Now for the square wave Equation (2.8-35), let:

$$B^2 = 0.4 \qquad \gamma = 2$$

and for $l = 1$, the following parameters are found:

$$L = 2 \qquad c_l = 1$$

$$\tau_l = 0 = \tau'_l \qquad c'_l = 2 \cdot 1 = 2$$

$$N_l = N_1 \geq L\left(\frac{c'_l}{B}\right)^2 = 2\,\frac{4}{0.4} = 20$$

and

$$A_{S_1}^{(N_1)}(t) = \left[1 + \frac{c_1'}{N_1} j \cos \omega t\right] \exp j\omega t$$

$$= \left[1 + \frac{2}{20} j \cos \omega t\right] \exp j\omega t$$

$$\left[A_{S_1}^{(N_1)}(t)\right]^{N_1} = \left[1 + 0.1 j \cos \omega t\right]^{20} \exp j\, 20\, \omega t \qquad (2.8\text{-}44)$$

For $l = 3$ , the parameters are found to be:

$$L = 2 \qquad c_l = \frac{1}{3} = c_3$$

$$\tau_l = -\frac{\pi}{3\omega} \qquad c_l' = \gamma \cdot c_l = \frac{2}{3}$$

$$N_l = N_2 \geqslant L\left(\frac{c_l'}{A}\right)^2 = 2\left(\frac{2/3}{0.2}\right)^2 = 2$$

and

$$A_2^{(N_2)}(t) = \left[1 + \frac{c_2'}{N_2} j \cos 3\omega \left(t - \frac{\pi}{3\omega}\right)\right] \exp j 3\omega t$$

$$\left[A_2^{(N_2)}(t)\right]^{N_2} = \left[1 + 0.33 j \cos 3\omega \left(t - \frac{\pi}{3\omega}\right)\right]^2 \exp 6\omega t \qquad (2.8-45)$$

Finally,

$$A_{SSB,PM}^{(N)}(t) = \left(1 + 0.1 j \cos \omega t\right)^{20} \left[1 + 0.33 j \cos 3\omega \left(t - \frac{\pi}{3\omega}\right)\right] \exp 26\omega t$$

(2.8-46)

The zero pattern is shown below in the z-plane:

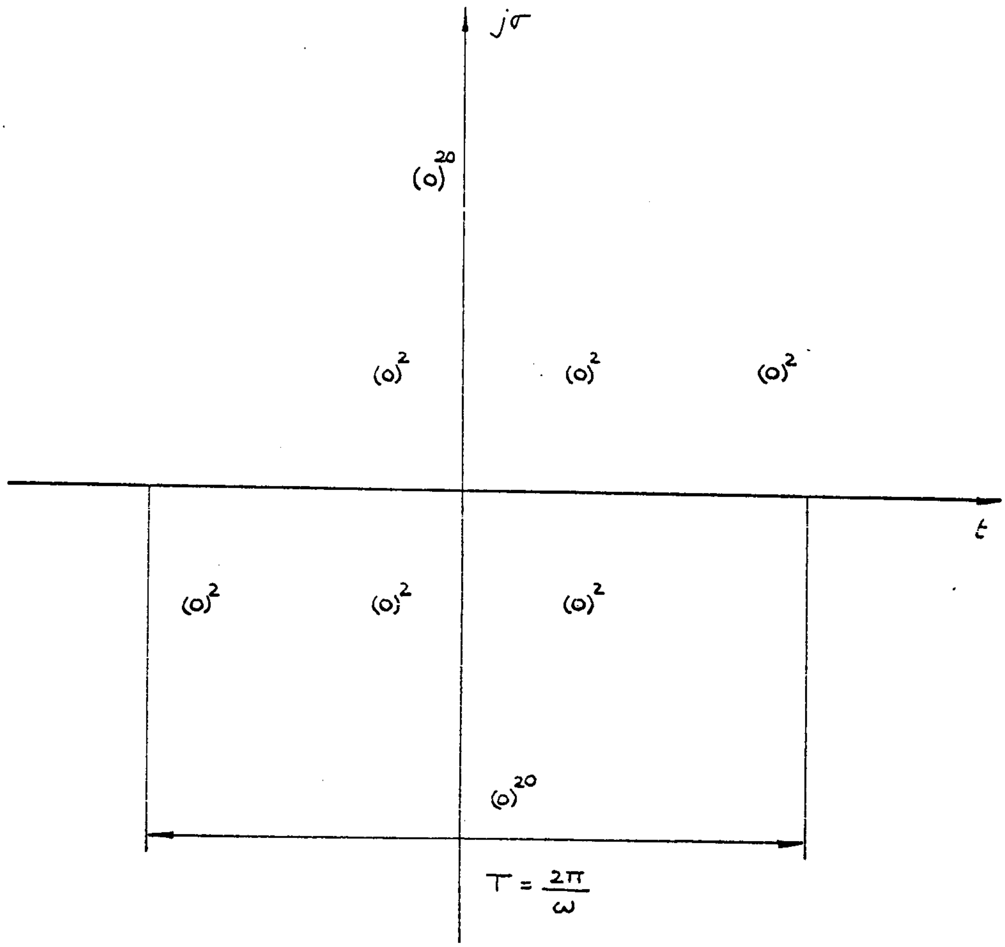



Figure 2.8-13 Zero pattern for conventional phase modulation

For SSB the zero pattern is the LHP portion of the conventional angle modulation. Therefore, the pattern of Fig. 2.7-14 is obtained.

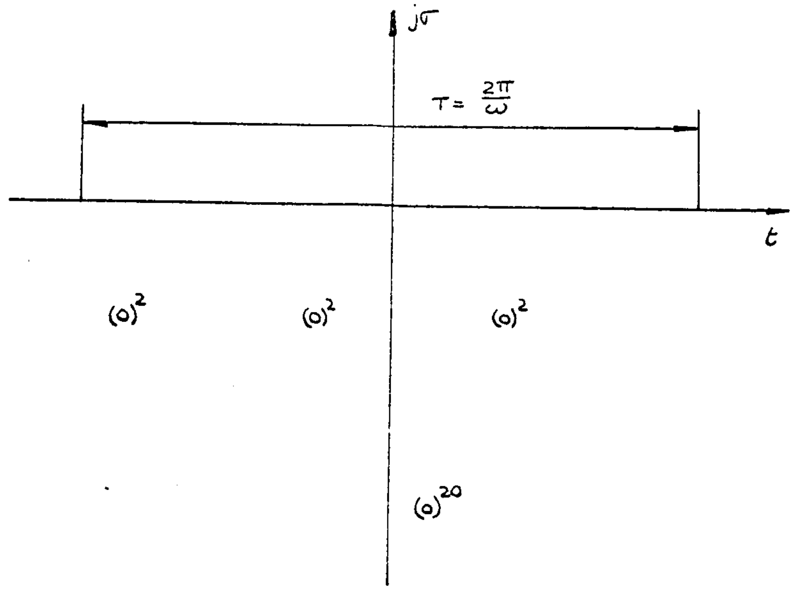



Figure 2.8-14  Zero pattern for SSB phase modulation

Note that the same result could be obtained by calculations made using formula (2.8-33a) and formula 2.8-34).

It is simple to follow the same procedure to obtain the zero pattern for FM in both cases, conventional and SSB, FM. It is also interesting to point out that the mapping of real zeroes of function Equation (2.8-46) is easily obtained by representing first the zeroes of Equations (2.8-45) and (2.8-44), and then shifting to the right the zero pattern in LHP and the left those in the UHP by the quantity:

$$\frac{\pi}{2l\omega}$$

The next step is the expansion of zeroes to the $N_l$ degree given by distortion limitations. These two steps can be reversed in a similar process to recover the modulating signal (its zeroes) from the angle-modulated zero pattern signal.

## 2.9 Computation Techniques for Finding the "Root-locus" of an Analytical Signal

The development of a modulation process from zero manipulation involves the solution of polynomial of high order as was shown in the last section and can be handled easily by the use of automatic computing techniques. Bairstowe's method is used to write programs for the roots (see Appendix A). However, even though it is considered one of the best, Bairstowe's method lacks efficiency when the parameters of the quadratic factors P and Q are initially chosen too far from the actual values. An attempt has been made to improve the efficiency of the method by using "Routh Criterion" and arriving at a unique way of approaching the actual values of P and Q (Appendix B).

This useful and efficient method is applied to find the zero locus of the analytic signal representing amplitude modulation as described previously. The real and/or complex zeroes of the equation for x are used in "plot functions" to obtain the graphic representation of the zero-locus

$$z = \tau + j\sigma$$

by varying the modulating parameter c. The computer program results are given in Appendix C, including an explanation of the efficiency this method provides.

```
LUIS PUIGJANER                           S031                  005    100    400
$COMPILE MAD,EXECUTE,PRINT OBJECT,DUMP
                R
                  PROGRAM FINDING THE ZERO LOCUS REPRESENTATION
                R
                 DIMENSION A(100),AA(100),B(100),C(100)
                 INTEGER I,J,M,N,COUNT,NN
START            READ DATA
                 NN=N
                 W'R A.NE.1.
                 T'H T3, FOR I=N   ,-1,I.L.0
T3               A(I)=A(I)/A(0)
                 E'L
                 THROUGH SAVE,FOR I=0,1,I.G.N
     SAVE        AA(I)=A(I)
                 T'H MANAUS,FOR LL=-2,.1,LL.G.2
                 N=NN
                 T'H LISBOA,FOR I=0,1,I.G.N
LISBOA           A(I)=AA(I)
                 A(3)=A(3)*LL
                 P'S LL,A(3)
                 P'S A(0)...A(N)
                 L=1.
                 Y=1.
                 M=N-1
    LOOP1        COUNT=0
                 PRINT FORMAT TITL
                 OP=1.E20
                 OQ=1.E20
                 P=0.
                 Q=-1.0
                 B(0)=A(0)
                 C(0)=B(0)
    LOOP2        B(1)=A(1)-P*B(0)
                 C(1)=B(1)-P*C(0)
                 COUNT=COUNT + 1
                 W'R COUNT .G.25
                 PRINT COMMENT $0 NO CONVERGENCE*$
                 L=L*2.0
                 Y=Y*L
                 T'H T1, FOR I=1,1,I.G.N
T1               A(I)=A(I)/(L.P.I)
                 T'O LOOP1
                 END OF CONDITIONAL
                 T'H UPDATE, FOR I=2,1,I.G.M
                 B(I)=A(I)-P*B(I-1)-Q*B(I-2)
                 C(I)=B(I) - P*C(I-1)-Q*C(I-2)
    UPDATE       CONTINUE
                 B(I)=A(I)-P*B(I-1)-Q*B(I-2)
                 C(M)=-C(N-2)*P-C(N-3)*Q
                 DENOM=C(N-2)*C(N-2)-C(M)*C(N-3)
                 PRINT FORMAT TABLE,COUNT,P,Q,DELTAP,DELTAQ
                 DELTAP=(B(N-1)*C(N-2)-B(N)*C(N-3))/DENOM
                 DELTAQ=(B(N)*C(N-2)-C(M)*B(N-1))/DENOM
```

```
                P=P+DELTAP
                Q=Q+DELTAQ
                W'R .ABS.(OP-P).LE.1.E-7 .AND..ABS.(OQ-Q).LE.1.E-7
                PRINT COMMENTS NO CHANGE IN P OR Q$
                T'O FOUND
                O'E
                OP=P
                OQ=Q
                E'L
                W'R .ABS.P.LE.1.E-6.OR..ABS.P.GE.1.E6.P=10.
                W'R .ABS.Q.LE.1.E-6.OR..ABS.Q.GE.1.E6,Q=10.
                W'R .ABS. B(N).LE.1.E-7.AND..ABS. B(N-1).LE.1.E-7,T'O FOUND
                TRANSFER TO LOOP2
         FOUND  D=P*P - 4.*Q
                WHENEVER D .L.0.
                DD=SQRT.(-D)*Y*0.5
                P'T COMPLX,-P*Y*0.5,DD  ,-P*Y*0.5,DD
                PUNCH FORMAT COMPLX,-P*Y*0.5,DD,-P*Y*0.5,DD
                OTHERWISE
                DD=SQRT.(D)
                P'T REAL,0.5*(-P-DD)*Y,0.5*(-P+DD)*Y
                PUNCH FORMAT REAL,0.5*(-P+DD)*Y,0.5*(-P-DD)*Y
                END OF CONDITIONAL
                N=N-2
                W'R N.E.0
      T5        PRINT COMMENT $0 ALL ROOTS FOUND$
                A(3)=A(3)/LL
      MANAUS    CONTINUE
                TRANSFER TO START
                END OF CONDITIONAL
                WHENEVER N .E. 2
                P=B(1)/B(0)
                Q=B(2)/B(0)
                T'O FOUND
                OR WHENEVER N .E. 1
                P=B(1)/B(0)
                P'T SINGRT,-P*Y
                PUNCH FORMAT SINGRT.-P*Y
                T'O T5
                OTHERWISE
                THROUGH BB,FOR I=1,1,I.G.N
         BB     A(I)=B(I)
                END OF CONDITIONAL
                TRANSFER TO LOOP1
                V'S TITL=$ 1H .S10,5HCOUNT,S12,1HP,S19.1HQ,S16,
               17HDELTA-P,S13,7HDELTA-Q/ *$
                VECTOR VALUES TABLE=$S11,I3,S6,4(E15.8.S5)*$
                V'S SINGRT=$1H0,S30,H*        SINGLE ROOT * / S25.3HX1=  .F15.
               1*$
                V'S COMPLX=$1H0,S20.          H*          COMPLEX ROOTS */
               1S18,3HX1=,F15.8,S3,1H+,S2,F15.8,3H I /S4.3HX2=.F15.8,S2.1H-,
               2S2,F15.8,3H I * $
                V'S REAL=$1H0,S30.        H*          REAL ROOTS*/ S18.3HX1=
               1F15.8,S15,4HX2 = ,F15.8 *$
                END OF PROGRAM
     $DATA
     N=6,A(0)=1,0,0,-3,-6,3,-1*
```

```
LUIS PUIGJANER                         S031                  005    100    400
$COMPILE MAD,EXECUTE
               R
                PROGRAM PLOTTING THE ZERO LOCUS
               R
                R'A N,XMAX,XMIN,YMAX,YMIN
                INTEGER I,N
                O=1.
                DIMENSION IMAGE(867)
                EXECUTE PLOT2.(IMAGE,XMAX,XMIN,YMAX,YMIN)
                T'H L1,FOR I=1,1,I.G.N
                R'T IMGY,A,B
                VECTOR VALUES IMGY=$S21,F15.8,S5,F15.8*$
                S=-(ELOG.(SQRT.(A*A+B*B)))/O
                T=(ATAN .(B/A))/O
                EXECUTE PLOT3.($*$,S,T,1)
                TNEG=-T
L1              EXECUTE PLOT3.($*$,S,TNEG,1)
                R'A N
                T'H L2,FOR I=1,1,I.G.N
                R'T REAL,X1,X2
                V'S REAL=$S21,F15.8,S19,F15.8*$
                S1=-(ELOG.(X1))/O
                PI=-X2
                S2=-(ELOG.(PI))/O
                T2=3.14159/O
                EXECUTE PLOT3.($*$,S1,T2,1)
L2              EXECUTE PLOT3.($*$,S2,T2,1)
                EXECUTE PLOT4.(32,ORD)
                PRINT FORMAT ABS
                V'S ABS=$1HO,S55,14H THE ABCISSA T *$
                V'S ORD=$                        THE ORDINATE JS*$
                END OF PROGRAM
$DATA
N=82,XMAX=2.,XMIN=-2.,YMAX=2.,YMIN=-2.*
                      X1=         .19185834   +          .48148299 I
                      X1=         .56741712   +         1.60250433 I
                      X1=         .20273931   +          .47703678 I
                      X1=         .53435994   +         1.58582518 I
                      X1=         .21345539   +          .47126492 I
                      X1=         .49977447   +         1.57042836 I
                      X1=         .22370775   +          .46412701 I
                      X1=         .46380153   +         1.55660279 I
                      X1=         .23316449   +          .45568262 I
                      X1=         .42666864   +         1.54461052 I
                      X1=         .24150679   +          .44611115 I
                      X1=         .38868144   +         1.53465144 I
                      X1=         .24848632   +          .43570359 I
                      X1=         .35019678   +         1.52683235 I
                      X1=         .25397285   +          .42482099 I
                      X1=         .31158289   +         1.52115557 I
                      X1=         .25797121   +          .41383217 I
                      X1=         .27317772   +         1.51752062 I
                      X1=         .26060144   +          .40305528 I
                      X1=         .23525789   +         1.51575399 I
                      X1=         .26205568   +          .39272369 I
                      X1=         .19802422   +         1.51563908 I
```

| | | | |
|---|---|---|---|
| X1= | .26255220 | + | .38298046 I |
| X1= | .16160315 | + | 1.51694717 I |
| X1= | .26230115 | + | .37389228 I |
| X1= | .12605784 | + | 1.51945966 I |
| X1= | .26148585 | + | .36547096 I |
| X1= | .09140385 | + | 1.52298152 I |
| X1= | .26025648 | + | .35769379 I |
| X1= | .05762371 | + | 1.52734651 I |
| X1= | .25873111 | + | .35051914 I |
| X1= | .02467947 | + | 1.53241782 I |
| X1= | .25700023 | + | .34389712 I |
| X1= | -.00747806 | + | 1.53808576 I |
| X1= | .25513195 | + | .33777602 I |
| X1= | -.03890284 | + | 1.54426408 I |
| X1= | .25317695 | + | .33210597 I |
| X1= | -.06964927 | + | 1.55088630 I |
| X1= | .25117271 | + | .32684077 I |
| X1= | -.09976948 | + | 1.55790213 I |
| X1= | .24914666 | + | .32193861 I |
| X1= | -.12931170 | + | 1.56527436 I |
| X1= | .24711885 | + | .31736213 I |
| X1= | -.15831949 | + | 1.57297603 I |
| X1= | .24510375 | + | .31307827 I |
| X1= | -.18683114 | + | 1.58098863 I |
| X1= | .24311172 | + | .30905788 I |
| X1= | -.21487969 | + | 1.58929986 I |
| X1= | .24115006 | + | .30527528 I |
| X1= | -.24249293 | + | 1.59790245 I |
| X1= | .23922380 | + | .30170792 I |
| X1= | -.26969358 | + | 1.60679260 I |
| X1= | .23733627 | + | .29833583 I |
| X1= | -.29649953 | + | 1.61596923 I |
| X1= | .23548956 | + | .29514166 I |
| X1= | -.32292412 | + | 1.62543283 I |
| X1= | .23368480 | + | .29210959 I |
| X1= | -.34897642 | + | 1.63518479 I |
| X1= | .23192248 | + | .28922620 I |
| X1= | -.37466170 | + | 1.64522666 I |
| X1= | .23020251 | + | .28647895 I |
| X1= | -.39998169 | + | 1.65555964 I |
| X1= | .22852453 | + | .28385722 I |
| X1= | -.42493511 | + | 1.66618396 I |
| X1= | .22688782 | + | .28135112 I |
| X1= | -.44951808 | + | 1.67709848 I |
| X1= | .22529173 | + | .27895239 I |
| X1= | -.47372473 | + | 1.68830043 I |
| X1= | .22373466 | + | .27665201 I |
| X1= | -.49754721 | + | 1.69978493 I |
| X1= | .22221611 | + | .27444401 I |
| X1= | -.52097720 | + | 1.71154521 I |
| X1= | .22073477 | + | .27232185 I |
| X1= | -.54400545 | + | 1.72357216 I |
| X1= | .21928946 | + | .27027961 I |
| X1= | -.56662276 | + | 1.73585454 I |
| X1= | .21787901 | + | .26831203 I |
| X1= | -.58882037 | + | 1.74837913 I |
| X1= | .21650226 | + | .26641439 I |

| | | | |
|---|---|---|---|
| X1= | -.61059032 | + | 1.76113077 I |
| X1= | .21515806 | + | .26458219 I |
| X1= | -.63192587 | + | 1.77409281 I |

N=41*

| | | | |
|---|---|---|---|
| X1= | .60621893 | X2 = | -2.12476987 |
| X1= | .63127445 | X2 = | -2.10547292 |
| X1= | .65946873 | X2 = | -2.08592841 |
| X1= | .69111332 | X2 = | -2.06613180 |
| X1= | .72641242 | X2 = | -2.04607865 |
| X1= | .76538841 | X2 = | -2.02576485 |
| X1= | .80782020 | X2 = | -2.00518639 |
| X1= | .85322817 | X2 = | -1.98433964 |
| X1= | .90092358 | X2 = | -1.96322137 |
| X1= | .95011024 | X2 = | -1.94182883 |
| X1= | 1.00000003 | X2 = | -1.92015983 |
| X1= | 1.04990226 | X2 = | -1.89821286 |
| X1= | 1.09926929 | X2 = | -1.87598728 |
| X1= | 1.14770389 | X2 = | -1.85348327 |
| X1= | 1.19494185 | X2 = | -1.83070222 |
| X1= | 1.24082549 | X2 = | -1.80764666 |
| X1= | 1.28527641 | X2 = | -1.78432076 |
| X1= | 1.32827193 | X2 = | -1.76073013 |
| X1= | 1.36982699 | X2 = | -1.73688236 |
| X1= | 1.40998057 | X2 = | -1.71278702 |
| X1= | 1.44878618 | X2 = | -1.68845606 |
| X1= | 1.48630516 | X2 = | -1.66390386 |
| X1= | 1.52260233 | X2 = | -1.63914755 |
| X1= | 1.55774315 | X2 = | -1.61420718 |
| X1= | 1.59179161 | X2 = | -1.58910587 |
| X1= | 1.62480949 | X2 = | -1.56386992 |
| X1= | 1.65685537 | X2 = | -1.53852884 |
| X1= | 1.68798459 | X2 = | -1.51311545 |
| X1= | 1.71824892 | X2 = | -1.48766565 |
| X1= | 1.74769677 | X2 = | -1.46221830 |
| X1= | 1.77637324 | X2 = | -1.43681489 |
| X1= | 1.80432026 | X2 = | -1.41149908 |
| X1= | 1.83157681 | X2 = | -1.38631627 |
| X1= | 1.85817909 | X2 = | -1.36131288 |
| X1= | 1.88416080 | X2 = | -1.33653569 |
| X1= | 1.90955316 | X2 = | -1.31203100 |
| X1= | 1.93438536 | X2 = | -1.28784402 |
| X1= | 1.95868441 | X2 = | -1.26401784 |
| X1= | 1.98247564 | X2 = | -1.24059294 |
| X1= | 2.00578251 | X2 = | -1.21760643 |
| X1= | 2.02862707 | X2 = | -1.19509146 |

# 3.0 FREQUENCY DIVISION MULTIPLEXING SYSTEMS

Frequency division multiplexing systems have been defined and briefly described in the first section of this thesis. The theory of analytic signals presents a new approach to the mathematical description of conventional multiplexing systems. This section describes in greater detail the different FDM systems by applying the models and techniques developed previously. Section 2.0 covers the new concept of "Zero-locus" and its significance when an analytic model is used to describe different modulation techniques. Conventional multiplexing systems are based on ordinary modulation techniques with additional limitations imposed by the complexity of the multiplexing process. Therefore, the results obtained previously can be applied, with certain restrictions, to multiplexing systems, such as the widely sued systems SSC-FM which are discussed in detail.

## 3.1 Subcarrier Modulation Process

The subcarrier modulation in FDM systems consists of a modulating section in which a sequence of sinusoidal subcarriers produced by local oscillators, one for each channel, is modulated by the individual messages. The modulation process depends upon the specific type of modulator chosen for each subcarrier. In the single sideband-frequency- modulation multiplexing system (SSC-FM), for instance, each subcarrier is single-sideband-amplitude-modulated by a message. A conventional SSC-FM system is represented in the block diagram of Fig. 3.1-1, where the subcarrier modulator and local oscillator is indicated for each channel. Then the modulated subcarrier is passed through an amplifier $K_i$ which is determined by signal-to-noise requirements. The resulting waveforms are linearly multiplexed into a single composite waveform, which, in turn, frequency-modulates the radio frequency carrier which is generated also by a local oscillator.

To explain the subcarrier modulation, suppose that a message s(t) is given as

$$s(t) = \sum_{-\infty}^{\infty} c'_n \exp jn\omega t \tag{3.1-1}$$

In suppressed carrier transmission, $c'_0$ is zero. Let s(t) be a two-component square wave which was used previously

$$s(t) = c + \cos\omega t - \frac{1}{3}\cos 3\omega t \tag{3.1-2}$$

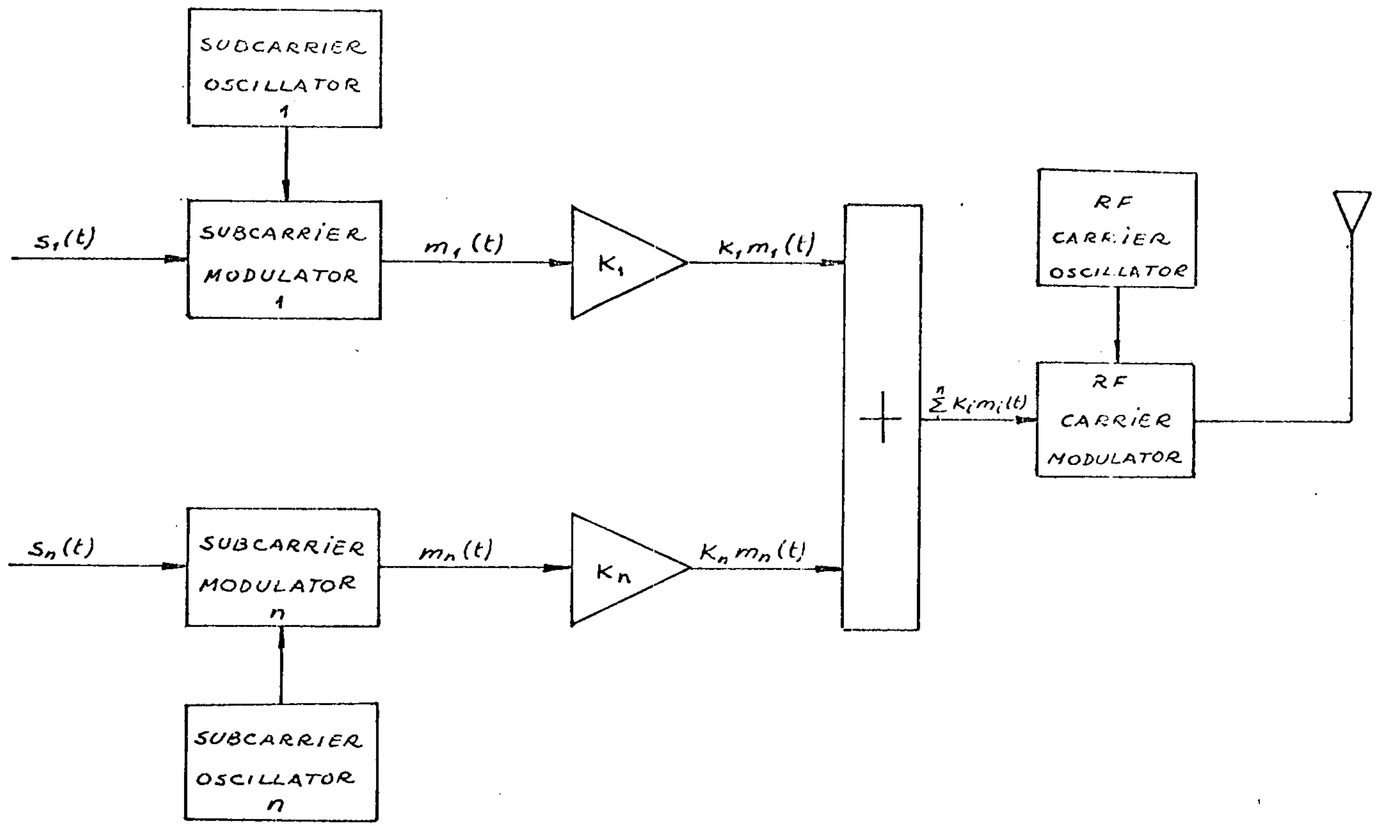



Figure 3.1-1 The multiplexing system SSC-FM

and therefore its analytic form is given by

$$A_s(t) = s(t) + j\hat{s}(t)$$
$$= c + \exp j\omega t - \frac{1}{3}\exp 3j\omega t \qquad (3.1\text{-}3)$$

Equation (3.1-1), which is the expression for single sideband, may be represented in analytic signal form by

$$A_S = c_0 + \sum_{k=n_1}^{n} c_k \exp kj\omega t \qquad \text{for } n_1 < n \qquad (3.1\text{-}4)$$

where $c_0 = 0$ for the carrier is suppressed.

The zero-locus of the analytic model as a function of the c parameter can be obtained by using techniques of Section 2.0. The analytic form of the actual signal again offers the advantage of halving the number of zeroes in the z-plane. This approach to multiplexing is possible if the zeroes of the modulated signal can be recovered in exact numbers, because the modulated wave is required to have the same number of zeroes in the modulating signal. They can be moved about, inserted, or deleted if such manipulations can be reversed in a demodulation process.

The modulated signal in analytic form is:

$$A_{SSB}(t) = A_s(t)\exp j\omega_o t = |A_s(t)|\exp j\theta(t) \qquad (3.1\text{-}5)$$

where

$$\theta(t) = \arctan\left[\frac{\hat{s}(t)}{s(t)}\right] \quad \text{Phase modulation component} \qquad (3.1\text{-}6)$$

$$|A_s(t)| = \text{Envelope of } s(t) \qquad (3.1\text{-}7)$$

Modulation does not affect the number of zeroes of the original analytic signal, which implies a spectrum translation along the frequency scale to a side-band of the subcarrier $\omega_0$ . The single-sided nature of the actual signal is preserved in a synchronous demodulation process.

## 3.2 The Linear Multiplexing Process

Multiplexing communication systems may be divided into basic categories: linear systems and non-linear systems. A multiplex system is linear or non-linear according to whether the separation of signals belonging to different channels is affected by linear or non-linear filters. The conventional frequency and time-division multiplex systems fall into the category of linear systems.

Multiplexing is a linear operation in frequency division systems, and it reduces to simple addition of the different channels in the frequency domain. There is no overlapping of adjacent messages in the frequency domain if appropriate selections of subcarrier frequencies for the individual channels are made. The multiplexing signal resulting from this linear operation is:

$$\sum_{i=1}^{n} m_i(t) = m_1(t) + m_2(t) + \ldots m_n(t) \qquad (3.2-1)$$

A power amplifier $K_i$ is included for the $i^{th}$ channel, because of the parabolic noise spectrum that appears at the receiving end of the modulated carrier when frequency modulation is used for the carrier (SSC-FM) (Fig. 3.1-1). The parabolic form of the noise spectrum $S_{n,FM}(f)$ in FM systems is given by (Downing 1964)

$$S_{n,FM}(f) = \begin{cases} \dfrac{2K^2 f^2}{A^2} & 0 \leq f \leq B \\ 0 & f > B \end{cases} \qquad (3.2-2)$$

where $K^2$ is the IF noise spectral density for $f>0$ and A is the signal amplitude. In SSC-FM multiplexing, each of the message-channel bandwidths $B_i$ is ordinarily the same $B_m$ and is much smaller than the total baseband bandwidth as indicated in Fig. 3.2-1.

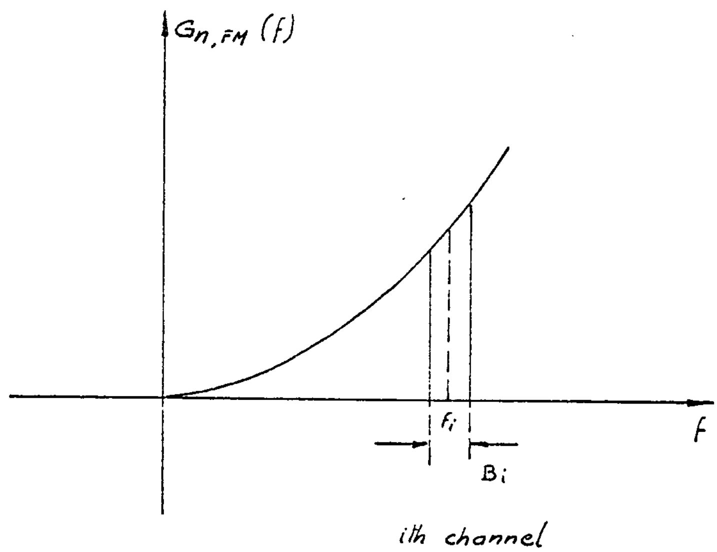



Figure 3.2-1 FM baseband noise power spectrum

The power amplifier $K_i$ gives the necessary pre-emphasis to the $i$th channel for an equal per channel signal-to-noise ratio over the entire baseband of the carrier. Thus for SSC-FM systems, the linear combination of the signals is of the form

$$m(t) = K_1 m_1(t) + \dots + K_n m_n(t)$$
$$= \sum_{i=1}^{n} K_i m_i(t). \qquad (3.2-3)$$

The non-overlapping condition imposed by the definition of the linear operation preserves the zero-pattern, so that the total number of zeroes is preserved. The density spectrum appears over the entire waveband as in Fig. 3.2-2.

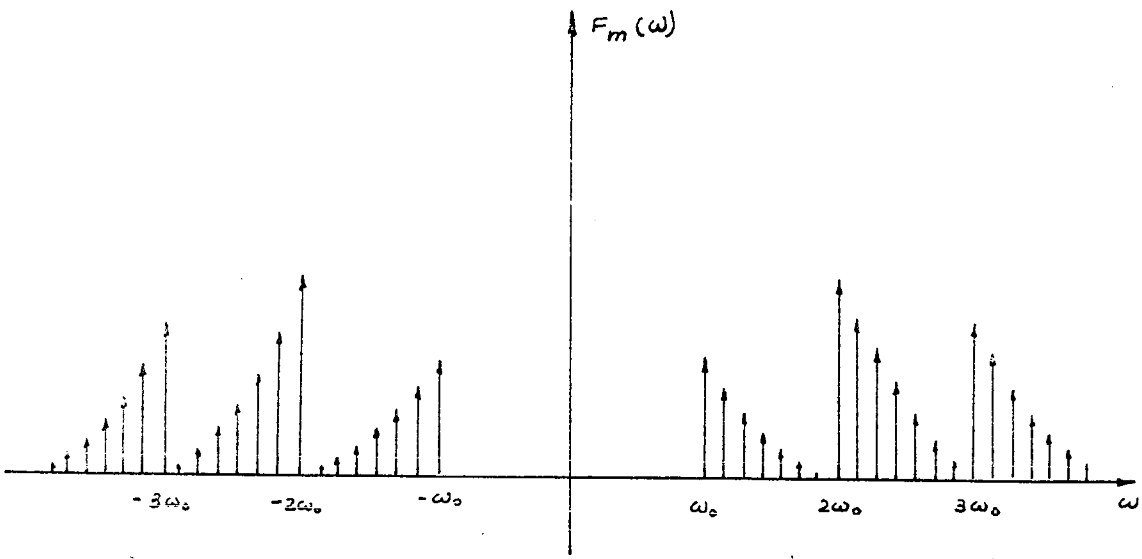



Figure 3.2-2 Density sprectrum at the output of the mixer

Let an individual z-plane be defined for each signal with its zero-locus theoretically independent (crosstalk and channel interference will be considered later). After multiplexing, each modulated subcarrier has its own z-plane containing the zero-pattern information of its original signal. The relative position for each zero-pattern along the multiplexed channel is given by the phasors $v_1$, $v_2 \ldots v_n$ that rotate counter clockwise with phase angles $\omega_o t$ , $2\omega_o t$ , ... $n\omega_o t$ respectively.

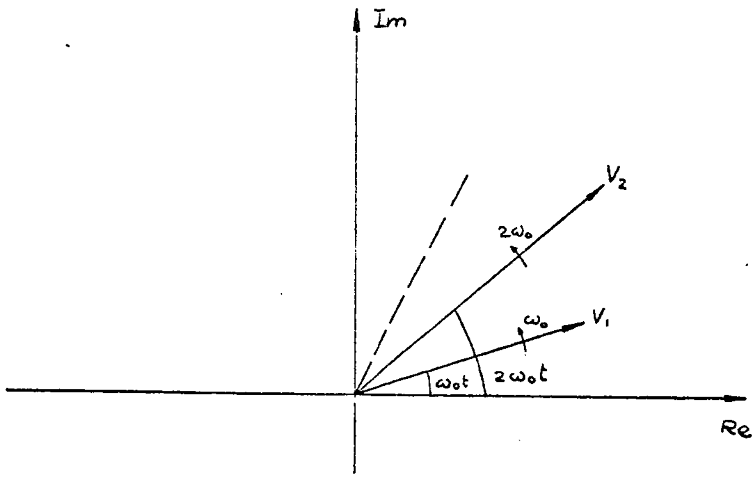



Figure 3.2-3 Phasor diagram for multiplexed signals

Each phasor $V_i$ occupies a sub-space $\Sigma_i$, a subset of a multi-dimensional signal space $\Sigma$.

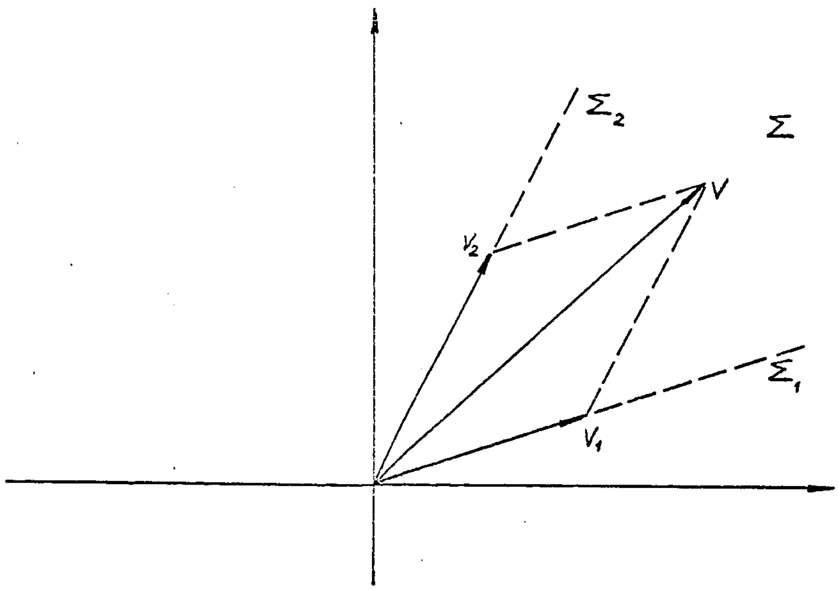



Figure 3.2-4 Geometrical representation for linear signal-space

In applying the geometrical approach, a linear set of signals such as $V_i$ corresponds in the signal space to a linear manifold $\Sigma_i$ , which is a linear subspace of $\Sigma$ . Straight lines and planes are linear manifolds in three-dimensional space. It is impossible to visualize a multi-dimensional linear manifold, but its properties can be inferred from those of a three-dimensional space.

Fig. 3.2-4 shows the geometrical representation of two sets of signals, $\Sigma_1$ and $\Sigma_2$. If the subsets $\Sigma_i$ are linear and disjoint, their representations will be straight lines passing through the origin of the system of co-ordinates defining the $\Sigma$-plane. Assume that the plane $\Sigma_1$ is specified in terms of two vectors with components $(a_{11}, a_{21}, a_{31}, a_{41})$ and $(a_{12}, a_{22}, a_{32}, a_{42})$, and that $\Sigma_2$ is similarly specified by two vectors with components $(a_{13}, a_{23}, a_{33}, a_{43})$ and $(a_{14}, a_{24}, a_{34}, a_{44})$. Form the matrix

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (3.2\text{-}4)$$

whose columns represent the vectors in question. The first two column vectors may define the z-plane of complex zeroes for the signal $v_1(t)$ contained in the signal space $\Sigma_1$ , as the third and fourth vector columns do for $v_2(t)$. The zero-pattern combination of the components $v_1(t)$ and $v_2(t)$ lies along the vector space $V$. The zeroes of the components $v_1(t)$ and $v_2(t)$ in the composite z-plane yield information about channel limitations, to include crosstalk as is shown in Section 3.5.

In geometrical terms the linear filtering required in FDM systems is represented by projecting the received signal v(t) on the plane $\Sigma_1$ along $\Sigma_2$, thus yielding the signal $v_1(t)$ in channel one. This transformation is given in matrix form by

$$V_1 = W_1 \cdot V \qquad (3.2\text{-}5)$$

where $W_1$ is the matrix representation of the impulse respone of the filter. The elements of the matrix are obtained by the general expression.

$$W_1(i,j) = \sum_{k=1}^{2} a_{ik}\, a^{-1}_{kj} \qquad (3.2\text{-}6)$$

Here $a_{ij}$ is the general element of the matrix $\mathbf{a}$, and $a^{-1}_{ij}$ is that of the inverse of $\mathbf{a}$. The operation (3.1-5) can be performed, and the message recovered if the signal spaces $\Sigma_1$ and $\Sigma_2$ are equally dimensioned. Fig. 3.2-5 shows the case in which v cannot be extracted from $v_1 + v_2$ because the sum of the dimensions of $\Sigma_1$ and $\Sigma_2$ exceeds that of $\Sigma$ .

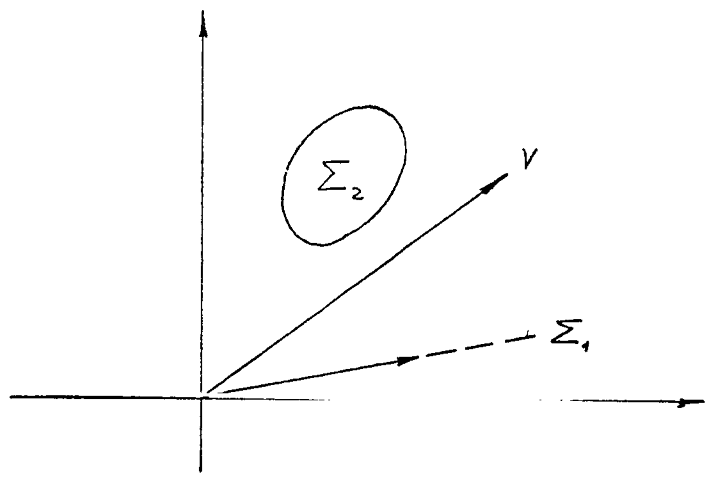



Figure 3.2-5 Case in which signal spaces $\Sigma_1$ and $\Sigma_2$ are not equally dimensioned

The conclusion that stems from the above discussion (other important considerations such as distortion and crosstalk are considered later) is that the bandwidth of a linear multiplex system cannot be compressed. In other words, if the n channels have the same bandwidth, and the bandwidth is equal to $W_o$, then the total bandwidth necessary for the common channel cannot be less than the total sum of the bandwidths of the components $nW_o$.

## 3.3 Carrier Modulation

The composite signal obtained after multiplexing

$$m(t) = \sum_{i=1}^{n} K_i \, m_i(t) \tag{3.3-1}$$

is now modulated by a carrier of higher frequency than that of subcarriers produced by a local oscillator. Fig. 3.3-1 shows the carrier modulator for the SSC-FM system where frequency modulation is used.

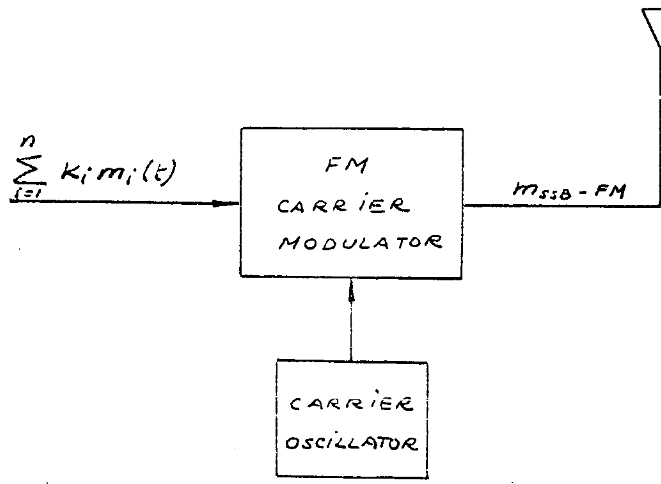



Figure 3.3-1 Carrier modulation section

The analytic form for frequency modulation was described as

$$A_s(t) = \exp j\left[M\{s(t)\} + jM\{\hat{s}(t)\}\right] \tag{3.3-2}$$

$$A_{SSB,FM}(t) = A_s(t) \cdot e^{j\omega_0 t}$$

$$= e^{-M\{\hat{s}(t)\}}\left[\cos\omega_0 t + M\{s(t)\} + j\sin\omega_0 t + M\{s(t)\}\right] \tag{3.3-3}$$

for single-sided frequency modulation. The zero model for SSB angle modulation developed previously (Sec. 2.8) yields

$$\dot{A}^{(N)}_{SSB,FM}(t) = \prod_{l=1}^{L} \left[ A^{(N)}_{s,l} \right]^{(N)} \qquad (3.3\text{-}4)$$

where

$$A^{(N)}_{s_l}(t) = \left[1 + j\frac{c_l'}{N}\cos l\omega\left(t-\tau_l'\right)\right\} \exp jl\omega_c t \qquad (3.3\text{-}5)$$

represents a QM elementary signal. Fig. 3.3-2 shows a scheme that uses quadrature modulation to produce $A_{SSB,FM}$.

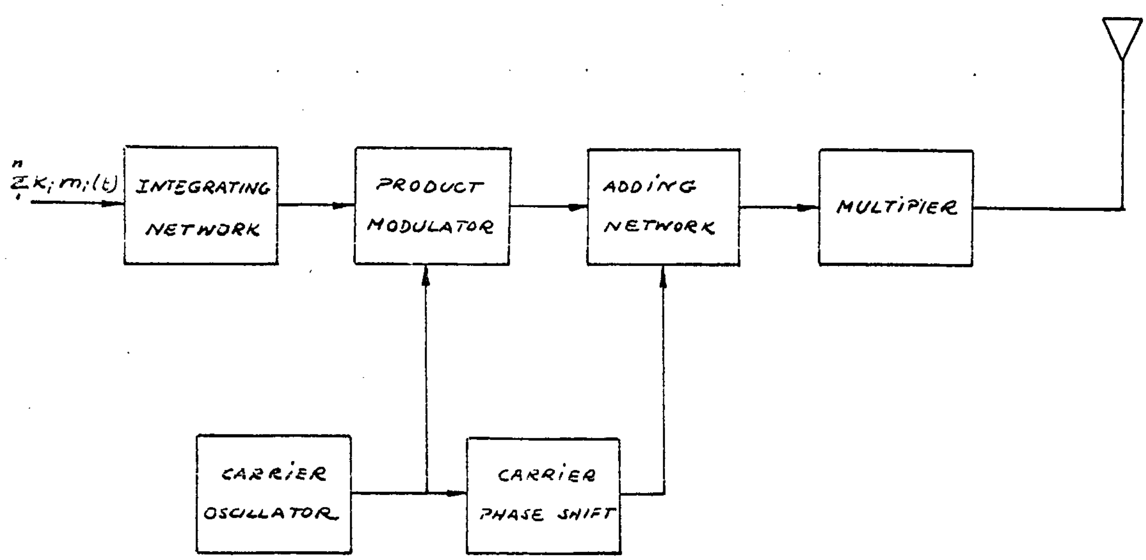



Figure 3.3-2 FM modulation network

The modulated carrier is finally transmitted, but the transmitted signal is not received in its original shape. The effects of noise and distortion are considered in the next section.

## 3.4 Noise and Distortion

Consider the receiver system. The signal transmitted plus added noise form its input (Fig. 3.4-1). To evaluate the effects of noise is a difficult task, but it is easier to estimate the performance of the various systems in the important limiting case of high signal-to-noise ratio. Thus the efficiencies of various schemes of multiplexing are compared on the basis of their signal-to-noise ratio (SNR) in reference to that of amplitude modulation systems. The SSB systems are chosen for a detailed study of distortion.

SSB systems require synchronous demodulation, which is merely downward spectrum translation along the frequency scale before passing on to envelope detection. If the local oscillator output is:

$$q(t) = \exp\left\{-j\left[(\omega_o - \varepsilon)t - \theta\right]\right\} \qquad (3.4\text{-}1)$$

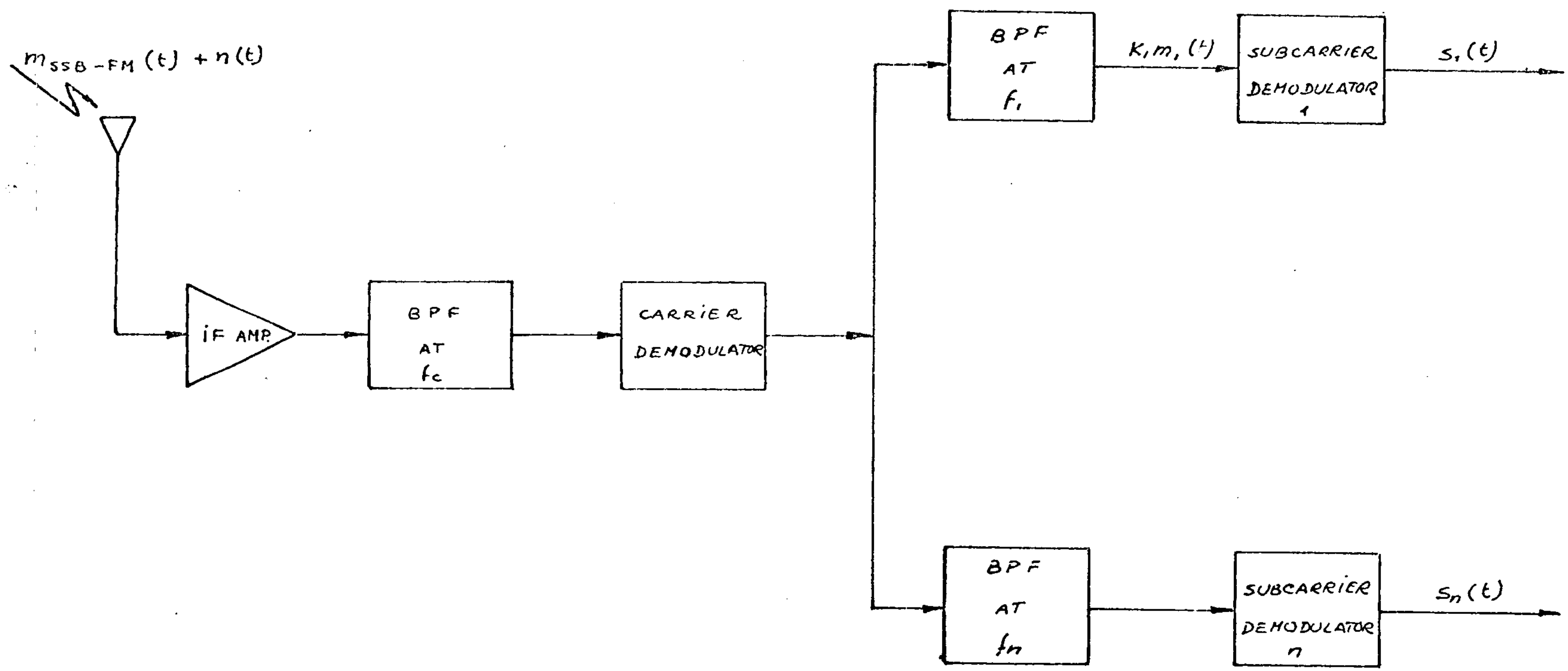



Figure 3.4-1 SSB-FM Multiplexing receiver

where $\varepsilon, \theta$ are frequency and phase errors, the demodulator output would be:

$$s_o(t) = Re\left[m_{\omega_o}(t) \cdot q(t)\right] =$$

$$= s(t)\cos\left[\varepsilon t + \theta\right] - \hat{s}(t)\sin\left[\varepsilon t + \theta\right] \qquad (3.4\text{-}2)$$

and the demodulation is perfect when $\varepsilon$ and $\theta$ are both zero. If $\varepsilon$ only is zero,

$$F_s(\omega)_{in} = F_s(\omega)_{out}$$

but the shape of $s_o(t)$ can be very different from that of s(t). If $\varepsilon$ and $\theta$ are not both zero, phase and envelope distortion are introduced.

The envelope of an SSB modulating signal in analytic form is:

$$|A_s(t)| = \sqrt{s^2(t) + \hat{s}^2(t)}$$

and the analytic expression for the modulated signal is:

$$A_{SSB}(t) = c + s(t) + j\,\hat{s}(t) \qquad (3.4\text{-}3)$$

where c represents the added carrier to the SSB signal, or a binomial expansion after normalizing results in

$$\frac{A_{SSB}(t)}{c} = 1 + \frac{s(t)}{c} + \frac{\hat{s}^2(t)}{2c^2} + \cdots \qquad (3.4\text{-}4)$$

The expansion (3.1-12) is valid for (Voelcker 1966)

$$\left[2c\,s(t) + |A_{SSB}(t)|^2\right]^2 < c^2$$

Therefore distortion is represented by the term$\left(\frac{\hat{s}^2(t)}{2c^2}\right)$ and the following equation holds for distortion and large c:

$$\frac{A_{SSB}(t)}{c} \simeq 1 + \frac{s(t)}{c} \tag{3.4-5}$$

Distortion reduction is achieved at the cost of power.

The distortionless signal should be

$$s(t) = Re\left[A_s(t)\right] = |A_s(t)| \cos\varphi(t) \tag{3.4-6}$$

and its distortionless envelope $|A_s(t)|$ can be recovered at the receiver if $\varphi(t)$ can be generated at the receiver from $|A_s(t)|$.

The single-sided nature of the analytic signal given by the Equation (3.4-3) representing the information signal requires the absence of zeroes in the UHP; moreover, the function must vanish as $z \to \infty$ for $\sigma \geqslant 0$ .

The logarithm of Equation (3.4-6) is

$$\ln\left(A_s(t)\right) = \ln|A_s(t)| + j\varphi(t) \tag{3.4-7}$$

and

$$\varphi(t) = H\left\{\ln|A_s(t)|\right\} \tag{3.4-8}$$

When $\ln|A_s(t)|$ is analytic, and s(t) is a minimum phase function of the form

$$s(t) = |A_s(t)| \cos\left\{H\left[\ln|A_s(t)|\right]\right\} \tag{3.4-9}$$

The function $\ln|A_s(z)|$ is also analytic when $A_s(z)$ does not have zeroes in the UHP. This obstacle is easily removed for SSB by adding a constant c to insure a balance of zeroes in UHP. This yields

$$A_{MP}(t) = c + s(t) + j\hat{s}(t) \tag{3.4-10}$$

and

$$\ln A_{MP}(t) = \ln c + \ln\left[1 + \frac{A_s(t)}{c}\right] \tag{3.4-11}$$

Since $A_s(t)$ is Fourier transformable, it therefore vanishes for $z \to \infty$ and,

$$\lim_{z\to\infty} \ln A_{MP}(z) = \ln c \qquad \text{for} \quad \sigma \geq 0 \tag{3.4-12}$$

The above expression is normalized as

$$\lim_{z\to\infty} \ln\left[\frac{A_{MP}(z)}{c}\right] = 0 \qquad \sigma \geq 0 \tag{3.4-13}$$

and all of the conditions of analyticity are now satisfied.

- The results obtained above are summarized below:

1. The real part of an analytic signal can be recovered solely from the envelope of the logarithm of the signal. This satisfies the condition of analyticity in that the function is free from zeroes in the UHP, and it vanishes as $z \to \infty$ for $\sigma \geq 0$.

2. The signal must meet the conditions given in the item above.

3. The logarithm of the signal can be made to satisfy the condition in 1 by normalizing it with respect to a constant c.

The normalized analytic signal that satisfies the three conditions and is therefore suitable for a distortionless envelope detection is

$$1 + \frac{s(t)}{c} = \frac{A_{MP}}{c} \cos\left\{ H\left[ \ln \frac{|A_{MP}(t)|}{c} \right]\right\} \qquad (3.4\text{-}14a)$$

The above procedure defines envelope detection plus post detection processing and has been called "asynchronism" SSB (ASSB).

The phase function associated with Equation (3.4-3) is

$$\varphi_{MP}(t) = \arctan \frac{\hat{s}(t)}{c + s(t)} \qquad (3.4\text{-}14b)$$

and requires that $c + s(t) \geqslant 0$ Equation (3.4-15) and thus $\varphi_{MP}(t)$ varies for

$$0 \leqslant \varphi \leqslant \pi/2 \qquad (3.4\text{-}16)$$

Voelcker (Voelcker 1966) gave some practical details of an ASSB, including an AASB "converter" to process the envelope output of a conventional high quality receiver.

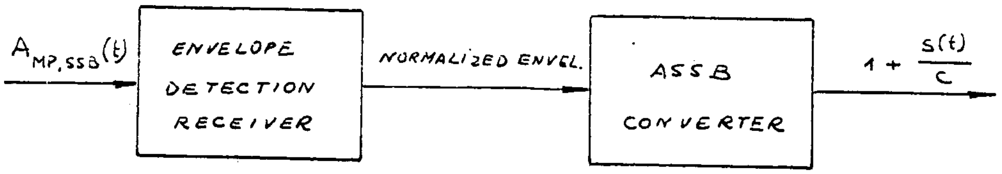



Figure 3.4-2 ASSB receiver with ASSB converter

The ASSB converter is shown in Fig. 3.4-3 and performs the operation indicated by Equation (3.4-13).

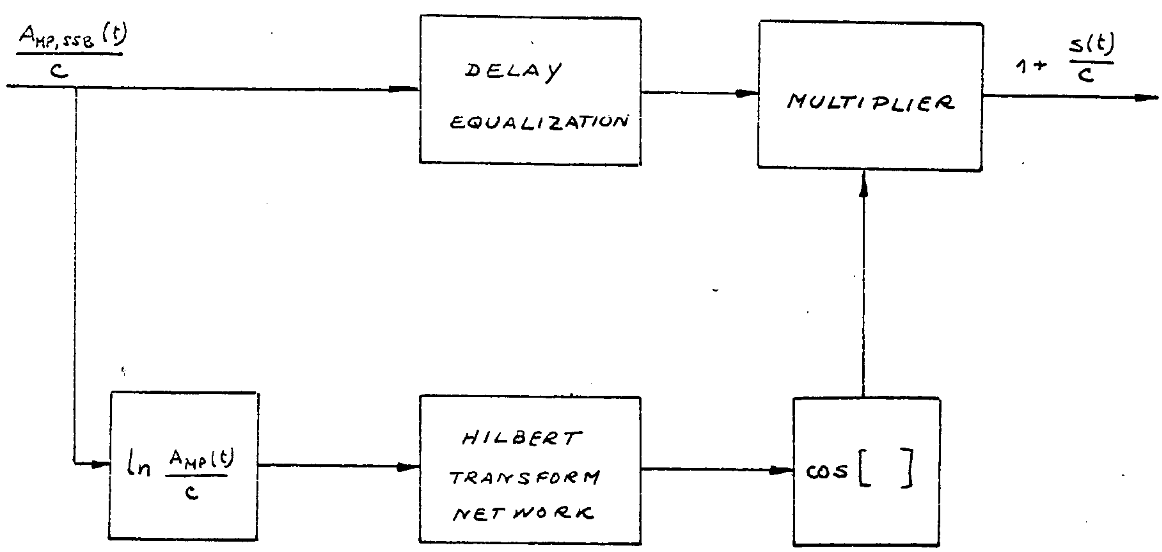



Figure 3.4-3 ASSB converter

The delay equalization compensates for the inherent time delay of the Hilbert transform network in the lower branch. The theoretical implementation of Equation (3.4-14) in block-diagram form is quite simple, but the experimental realization is more difficult. The same author (Voelcker 1966) gives an experimental ASSB converter that makes use of:

a) SSB filtering to effect the Hilbert transformation.

b) Strong carrier injection to guarantee low deviation phase modulation.

c) Frequency multiplication to expand the phase deviation.

d) Sychronous demodulation to effect both multiplication and cosine function generation.

The cost of distortionless detection as mentioned above is quite expensive. Once more it is necessary to determine the minimum distortion required and to indicate how this must be achieved. This problem is solved by using optimization techniques. For example, to approximate the ASSB converter of Fig. 3.4-3, a square law envelope is used to eliminate the term $\frac{\hat{s}^2(t)}{2c^2}$ in Fig. 3.4-4.

Note that

$$H\left\{ \ln \frac{|A_{MP}(t)|}{c} \right\} \cong H\left\{ \frac{1}{2} \frac{|A_{MP}(t)|^2}{c^2} \right\} = \frac{\hat{s}(t)}{c} + \cdots$$

and therefore

$$\frac{A_{MP}(t)}{c} - \frac{1}{2}\left[ H\left\{ \frac{|A_{MP}(t)|^2}{c^2} \right\}\right]^2 = 1 + \frac{s(t)}{c} + \left\{ c^{-3} \ldots c^{-n} \text{ order terms} \right\}$$

so that the approximate network (similar to that of Fig. 3.4-3) that gives a reasonable distortion is:

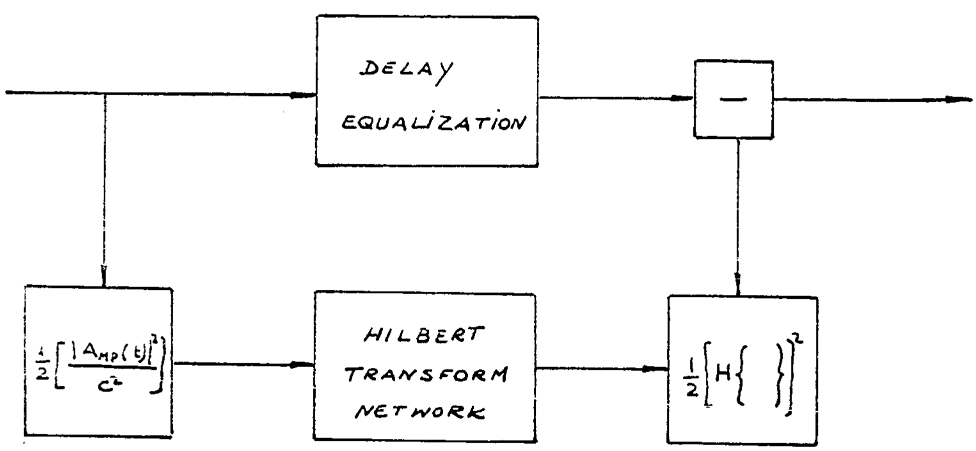



Figure 3.4-4 Distortionless envelope detection schema

The efficiency of AASB is defined by

$$\eta = \frac{\text{sideband power}}{\text{total power}} \tag{3.4-17}$$

If $\overline{s^2(t)}$ is the average power of s(t) across $1\Omega$ resistor

$$E\left[s^2(t)\right] \equiv \overline{s^2(t)} = \frac{1}{T}\int_0^T s^2(t)\, dt$$

and the sideband power is:

$$E\left[A_s^2(t)\right] = E\left[\left(s(t) + j\hat{s}(t)\right)^2\right]$$

$$= E\left[s^2(t)\right] + E\left[\hat{s}^2(t)\right] + 2E\left[s(t)\cdot\hat{s}(t)\right]$$

but, since $\hat{s}(t) = H\{s(t)\}$, from the properties of Hilbert transform (see Section 2.1)

$$E\left[\hat{s}^2(t)\right] = E\left[s^2(t)\right]$$

$$E\left[s(t)\cdot\hat{s}(t)\right] = 0, \tag{3.4-18}$$

Therefore, one obtains:

$$E\left[A_s^2(t)\right] = 2E\left[s^2(t)\right] = 2\overline{s^2(t)} \tag{3.4-19}$$

The total power is given by the power carrier plus that of $A_s(t)$:

$$P(t) = \frac{1}{T}\int_0^T c^2 dt + \overline{A_s^2(t)} = c^2 + 2\overline{s^2(t)}$$

and

$$\eta_{A_{SSB}} = \frac{2\overline{s^2(t)}}{c^2 + 2\overline{s^2(t)}} \tag{3.4-20}$$

For conventional AM systems, the efficiency is given by

signal power = $\overline{A_s^2(t)} = \overline{s^2(t)}$

total power = $\overline{A_s^2(t)} + c^2 = \overline{s^2(t)} + c^2$

therefore,

$$\eta_{A_{AM}} = \frac{\overline{s^2(t)}}{c^2 + \overline{s^2(t)}} \qquad (3.4-20)$$

On a comparative basis for the same s(t) in both systems, the following expression holds:

$$\eta_{A_{SSB}} = \frac{2\,\eta_{A_{AM}}}{1 + \eta_{A_{AM}}} \qquad (3.4-21)$$

Thus ASSB has an efficiency ranging between one-to-two times that of conventional AM.

Noise consideration in ASSB involves complex nonlinear operations when noise is present in the demodulation operation. For the case of high SRN: $\frac{S}{N} \gg 1$ , consider the additive noise confined to the spectral band of the ASSB signal. The signal-plus-noise combination will be:

$$A_{MP,n}(t) = \left[c + s(t) + n(t)\right] + j\left[\hat{s}(t) + \hat{n}(t)\right] \qquad (3.4-22)$$

If

$$c + s(t) + n(t) \geqslant 0 \qquad \text{for } \frac{S}{N} \gg 1 \qquad (3.4-23)$$

then for high SNR, the received ASSB signal is essentially zero-free in the UHP. At the output of the ASSB converter, supposed linear demodulation will give:

$$A_{s_0}(t) = 1 + \frac{\left[s(t) + n(t)\right]}{c}$$

and the signal-to-noise ratio at the output in terms of the analytic form will be:

$$\left(\frac{S}{N}\right)_{OUT,\,SSB} = \frac{\overline{\left[1+\frac{s(t)+n(t)}{c}\right]^2}}{\overline{n^2(t)}} = \frac{\overline{c+\left[s(t)+n(t)\right]^2}}{c^2\,\overline{n^2(t)}} = \frac{c^2+2c\overline{s(t)}+\overline{s^2(t)}}{c^2\overline{n^2(t)}} \qquad (3.4\text{-}24)$$

When the noise is strong enough to cause $Re\left[A_{MP,n}(t)\right]$ to be negative for a significant portion of the time, UHP zeroes will occur in significant numbers, and non-linear distortion will appear in the output. The distortion will fall within the s(t) bandwidth, and it will tend to concentrate in the lower region of the band. Note that although a qualitative description of weak SNR is mathematically and theoretically clear, it is difficult to evaluate it quantitatively, and only an experimental study yields definitive results.

In FM systems the same technique is used to calculate the efficiency. The AM system is usually chosen to be the reference comparative system. For FM systems,

$$s_{FM}(t) = A\cos\left[\omega_o(t) + M\left\{f(t)\right\}\right]$$

and the noise affecting FM is the $n_c(t)$ component of its

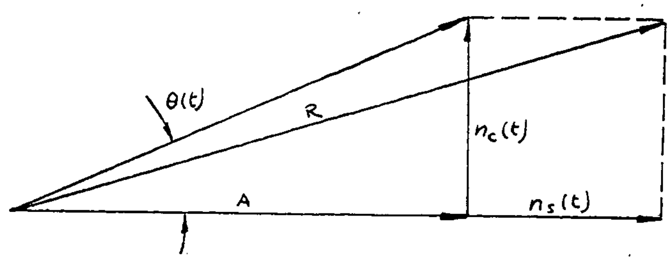



Figure 3.4-5 Quadrature carrier noise representation

quadrature-carrier noise representation in phasor-diagram form (Fig. 3.4-5), and its contribution to output signal is:

$$s_n(t) = A \sin\left[\omega_0 t + \theta(t)\right] \tag{3.4-25}$$

where

$$\theta(t) = \tan^{-1} \frac{n_c(t)}{A} \tag{3.4-26}$$

The signal at the output will be:

$$s_{FM,n}(t) = \sin\left[\omega_0 t + M\{f(t)\} + \theta(t) + \varphi\right] \tag{3.4-27}$$

There the same difficulty as discussed previously appears when the analytic representation of FM systems is calculated. If the method previously presented is applied, a well-known formula is obtained:

$$\left(\frac{S}{N}\right)_{OUT,\,FM} = 3\beta^2 \left(\frac{S}{N}\right)_{OUT,\,AM} \tag{3.4-28}$$

where $\beta$ is the deviation ratio.

## 3.5 Crosstalk

Quantitative theoretical studies of crosstalk using analytic functions have not been done until recently. When two or more information channels are multiplexed, crosstalk results. The input-output relationship in a system is given by

$$e_o = a_1 e_i + a_2 e_i^2 + \quad . . . \quad + a_m e_i^m \tag{3.5-1}$$

in which case the system will be linear if $a_1 \ldots a_m$ are constants and all but $a_1$ are zero. The non-linear Equation (3.5-1) generates new frequencies, as is shown for a single input of the form

$$e_1 = \cos\omega_1 t + \cos\omega_2 t + \cdots \cos\omega_n t \tag{3.5-2}$$

where $\omega_o \ldots \omega_n$ are the subcarriers of a multiplex. The output of Equation (3.5-1) is:

$$\begin{aligned} e_o = {} & a_1\left[\cos\omega_1 t + \cos\omega_2 t + \ldots + \cos\omega_n t\right] + a_2\Big[\cos^2\omega_1 t + \cos^2\omega_2 t + \ldots + \\ & + \cos^2\omega_n t + 2\cos\omega_1 t\cos\omega_2 t + 2\cos\omega_1\cos\omega_3 t + \ldots + \\ & + 2\cos\omega_{n-1} t\cos\omega_n t\Big] + a_3\Big[\cos^3\omega_1 t + \ldots + \cos\omega_n^3 t + \\ & + 3\cos^2\omega_1 t\cos\omega_2 t + \ldots + 6\cos\omega_1 t\cos\omega_2 t\cos\omega_3 t + \ldots\Big] + \\ & + a_4\left[\ldots .\right] + \ldots . + a_n\left[\ldots .\right] \end{aligned} \tag{3.5-3}$$

The zero-pattern representation is much more useful for analysis of crosstalk than the other systems which require use of the complex expression (3.5-3). The problem of crosstalk is reduced to the problem of finding the interference of zeroes between adjacent channels. The location of zeroes gives a measure of crosstalk, because a relative measure of how much one channel affects the adjacent channel for a specified bandwidth of the subcarriers, and the effect on each subcarrier is of interest.

Crosstalk will appear in zero-pattern representation as an invasion of zeroes from signal space $\Sigma_1$, as is shown in signal space $\Sigma_2$ in Fig.3.5-1. The case

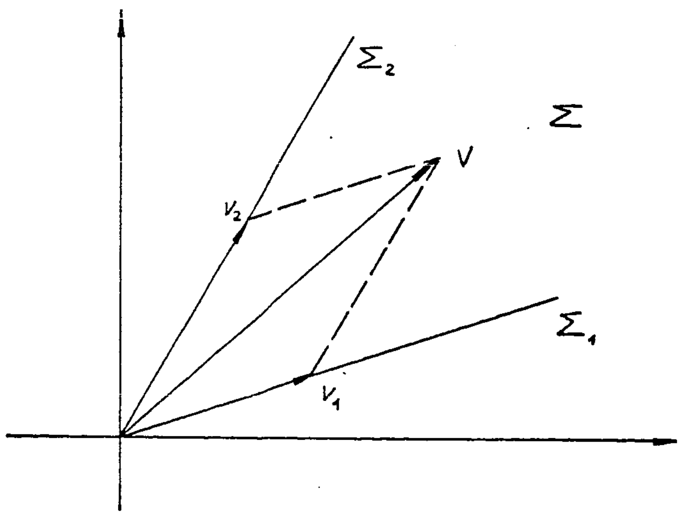



Figure 3.5-1 Linear signal-space representation for adjacent channels

presented in Fig. 3.5-1 cannot give crosstalk, since the straight lines representing the subsets $\Sigma_1, \Sigma_2$ mean that they form linear and disjoint signal-spaces. But when the non-linearity of a filter is introduced in one space signal $\Sigma_1, \Sigma_2$ this will no longer be the case, and $\Sigma_1$ and $\Sigma_2$ are not disjoint, or crosstalk results.

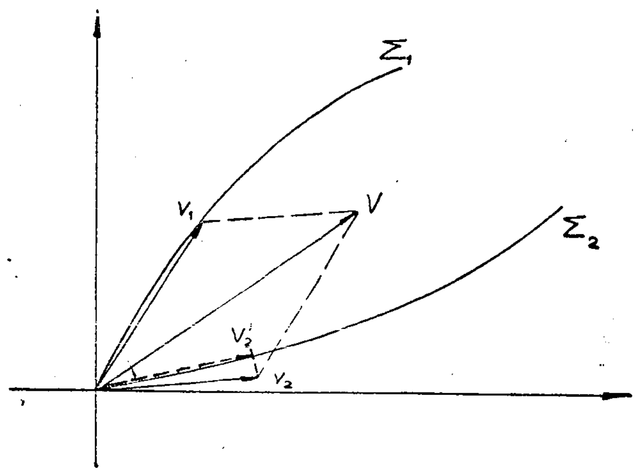



Figure 3.5-2 Effects of distortion in signal-space representation

The points (zeroes) in the z-plane defined for each subset $\Sigma_1, \Sigma_2$ are determined by a tensor or "psuedo tensor," since this is the "space-temps" of Minkowski (Lichnerowicz 1962).

According to the fundamental theorem of the tensor calculus, if the components for the tensor considered are all null for a point in the defined space, the <u>same</u>

components remain null in any other space. This theorem suggests a solution. The signal defined in its signal space and by its time vector space will have a definite shape with real and complex zeroes where zero denotes a point of zero components in its space. Therefore, it should be zero in a new system of coordinates defining a new space. For example, if $A'^{ij}$ is a tensor twice contravariant

$$A'^{ij} = \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} A^{\alpha\beta} \qquad \text{for} \quad \alpha, \beta, i, j = 1, 2, \ldots n$$

if

$$A^{\alpha\beta} = 0 \qquad \text{for} \quad \text{all } \alpha, \beta$$

and then

$$A'^{ij} = 0 \qquad \text{for} \quad \text{all } i, j$$

Then once all the zeroes of a message in analytic form are found, their transformation into a new space (next message) must preserve their characteristic of null coordinates. Since a psuedo tensor is under discussion, the change of coordinates into a new space is realized through the Jacobian of the transformation. For example, the tensor $A'^i$, once contravariant in a system of coordinates y, is changed into $A^\alpha$, that of x coordinates through

$$A'^i (y^1, y^2, \ldots y^n) = J^W \frac{\partial y^i}{\partial x^\alpha} A^\alpha (x^1, x^2, \ldots x^n)$$

$$\text{for } \alpha, i = 1, 2 \ldots n$$

where $J^W$ is the Jacobian of the transformation

$$J^W = \left| \frac{\partial x}{\partial y} \right|^{(w)} \qquad w \equiv \text{weight of Jacobian} = \pm 0, 1, \ldots n$$

If the Jacobian of such a transformation (say from $\Sigma_1$ space with $\Sigma_2$ ) is the same and coincides with the Jacobian of the transformation $J^{W'}$ for some zero from $\Sigma_2$ space into $\Sigma_1$ , it means that both zeroes were in the same signal space, and therfore an "intersection" or "crosstalk" (in terms of communication) is found. Repeating the operation for each zero of $\Sigma_1$ into $\Sigma_2$ and vice versa, the common zeroes are detected and their "relative density" gives a good index of crosstalk.

The theory is more difficult than the realization. A summary of the method is given below:

1. Only complex zeroes must be tested, since real zeroes do not contribute to any amount of power; therefore, no noise is caused by the "inoffensive" zeroes.
2. The testing of complex zeroes is reduced to calculating the Jacobian of transformation, that in the case of specific psuedo-tensors under consideration, it consists simply of a matrix formed by elements of sines and cosines, is easy to carry out.

3. The calculation of the matrix of transformation is performed easily by using calculating techniques that are really efficient.

The method proposed here can be favorably compared with that carried out by conventional techniques once the abstract concepts of "signal space" and "tensor of $n$ dimensions" have been understood, the application of the more general theory always yields simplified results.

# 4.0 TIME DIVISION MULTIPLEXING (TDM)

Time division multiplexing accomplishes a multi-channel transmission by representing the continuous input messages in sampled form. Each sampled-message channel occupies the entire base bandwidth rather than a designated part of it as in the FDM system.

Time division multiplexing, therefore, combines several message sources serially in the time domain and transmits them in parallel in the frequency domain. This "combination" of messages in the time domain is accomplished by a commutator. The commutator performs the functions of sampling and time sequencing. Only one carrier is needed to transmit all messages on one channel. At the receiving end, the "decommutator" reverses the operation of combination in a synchronized manner with the commutator, and the sampled messages are separated again into different channels. Each message waveform is then processed by an interpolating device that reconstructs a continuous waveform from the sampled messages (Fig. 4.0-1).

A wide variety of message representation schemes can be used for sampled-data TDM transmission: pulse amplitude modulation (PAM), pulse duration modulation (PDM), pulse code modulation (PCM), pulse phase modulation (PPM), etc.

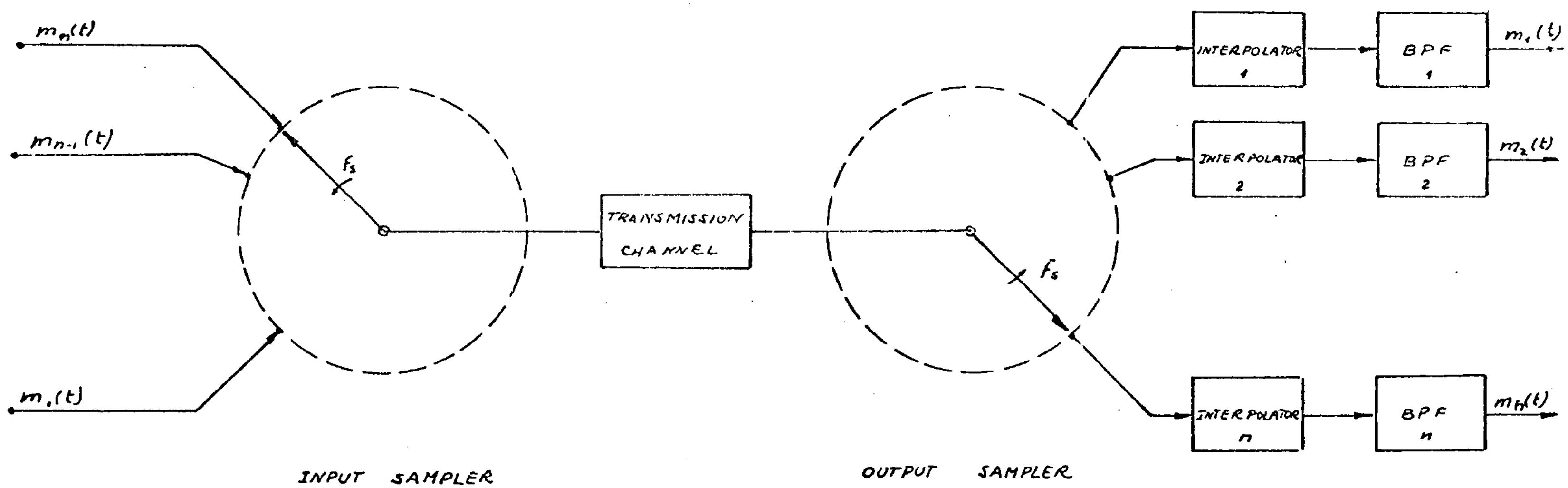



Figure 4.0-1 Time division multiplexing

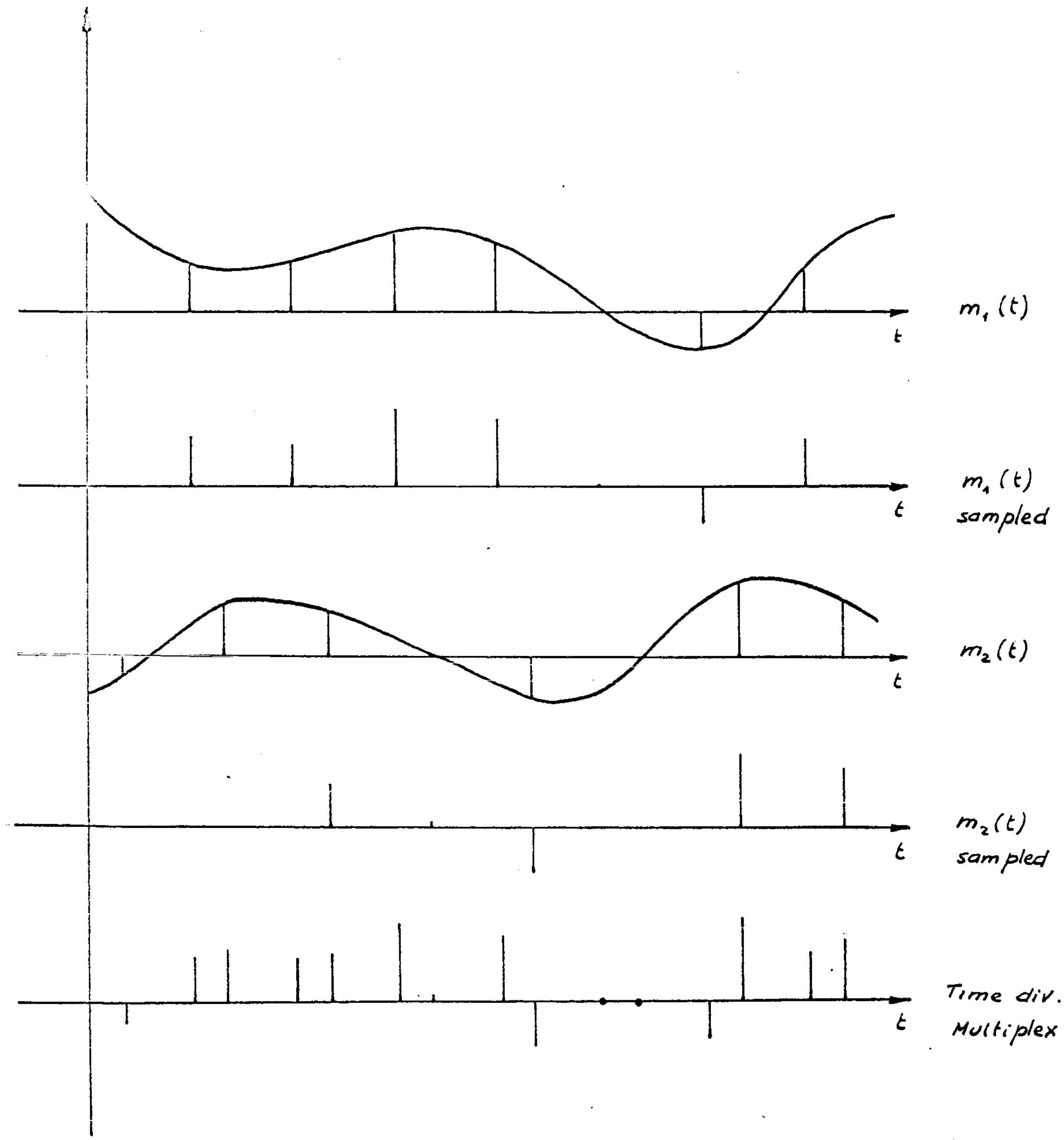



Figure 4.1-1 Sampling operation verified by commutator

The conventional TDM systems commonly used are included in this work. The variety of schemes is, however, almost unlimited.

## 4.1 Sampling

The continuous message waveform is "sampled" at discrete intervals generated by a switch or sampling gate. The sampling gate is introduced in series with the continuous message. It is usually open but is closed at short, regularly spaced intervals (Fig. 4.1-0).

The maximum sampling period, or maximum time between two switch closures so that no information is lost, is given by the Shannon sampling theorem, which states that no information is lost if the sampling period is at least equal to or less than twice the highest period present in the message (Downing 1964).

Consider a time function x(t) that has a Fourier transform $F_x(\omega)$ . The relationship between x(t) and its Fourier transform is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_x(\omega)\, e^{j\omega t}\, d\omega \qquad (4.1-1)$$

or

$$x(t) = \int_{-\infty}^{\infty} F_x(f)\, e^{j2\pi f t}\, df \qquad (4.1-2)$$

where $F_x(f)$ has been written for brevity as $F_x(2\pi f)$

If Fx(f) is zero outside the interval $-W \leq f \leq W$

$$x(t) = \int_{-W}^{W} F_x(f)\, e^{j2\pi f t}\, df \qquad (4.1-3)$$

now for $t = \frac{n}{2W}$ , the period of sampling

$$x\left(\frac{n}{2W}\right) = \int_{-W}^{W} F_x(f)\, e^{j\pi n f/W}\, df \qquad (4.1-4)$$

Since $F_x(f)$ is a periodic function, it can be represented through a Fourier series expansion in the interval $-W \leq f \leq W$

$$F_x(f) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\pi f/W} \qquad (4.1-5)$$

where $c_n$ , the Fourier coefficient is defnined ·

$$c_n = \frac{1}{2W} \int_{-W}^{W} F_x(f)\, e^{-jn\pi f/W}\, df \qquad (4.1-6)$$

Therefore,

$$c_{-n} = \frac{1}{2W} \int_{-W}^{W} F_x(f)\, e^{jn\pi f/W}\, df \qquad (4.1-7)$$

and Equation (4.1-4) can be written

$$x\left(\frac{n}{2W}\right) = 2W\, c_{-n} \qquad (4.1-8)$$

Hence all the coefficients of the Fourier series expansion of the transform $F_x(f)$ are specified for the interval

$$-W \leq f \leq +W$$

Therefore $F_x(f)$ is itself specified, and x(t) is determined through the transformation

$$x(t) = \int_{-\infty}^{\infty} F_x(f)\, e^{-j2\pi n f t}\, df$$

It follows that sampling a signal at a rate of at least 2 $W$ samples per second does preserve all the information contained in the original continuous waveform.

## 4.2 The Process of Interpolation

The process of interpolation is defined as that of reconstructing a continuous function from a set of sample values. The process of sampling a continuous function yields a sequence of discrete instantaneous samples $x\left(\frac{n}{2W}\right)$ of the continuous function that is ideally bandlimited to $W$ cps. It is desired to reconstruct the original function x(t) with the information available by building up a continuous function $x_1(t)$ which passes through the sample values $x\left(\frac{n}{2W}\right)$ and represents the function x(t) between samples as accurately as possible.

$$x_1(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) u\left(t - \frac{n}{2W}\right) \qquad (4.2-1)$$

The interpolating generating function is $u(t)$, defined as follows:

$$u(0) = 1$$

$$u\left(\frac{n}{2W}\right) = 0 \qquad \text{for } n \neq 0 \qquad (4.2\text{-}2)$$

Through band limitation considerations

$$F_u(f) = 0 \quad \text{for} \quad |f| > W \qquad (4.2\text{-}2a)$$

The Fourier transform relationship for the interpolating generating function $S_1$ is:

$$u(t) = \int_{-W}^{W} F_u(f)\, e^{j2\pi f}\, df \qquad (4.2\text{-}3)$$

The expansion of $F_u(f)$ in a Fourier series in the interval $-W \leq f \leq W$ gives

$$F_u(f) = \sum_{k=-\infty}^{\infty} a_k\, e^{j\pi k f/W} \qquad (4.2\text{-}4)$$

therefore,

$$u\left(\frac{n}{2W}\right) = \sum_{k=-\infty}^{\infty} a_k \int_{-W}^{W} e^{j\pi(n+k) f/W}\, df \qquad (4.2\text{-}5)$$

the integral vanishes except for $n+k = 0$

$$u\left(\frac{n}{2W}\right) = 2W a_{-n} \qquad (4.2\text{-}6)$$

According to Equation (4.2-2), the values of the last expression for $n$ = 0, 1,2...are:

$$a_0 = \frac{1}{2W}$$

$$a_n = 0 \qquad \text{for} \quad n \neq 0$$

The Fourier expansion of $F_u(f)$ is thus given by

$$F_u(f) = \frac{1}{2W}$$

and the time domain u(t) is:

$$u(t) = \frac{1}{2W}\int_{-W}^{W} e^{j2\pi f t}\, df$$

$$= \frac{\sin 2\pi W t}{2\pi W t} \tag{4.2-7}$$

The specified function that satisfies the condition of interpolation is found by substituting Equation (4.2-7) for Equation (4.2-1):

$$x_I(t) = \sum_{n=-\infty}^{\infty} u\left(\frac{n}{2W}\right) \frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{2\pi W\left(t - \frac{n}{2W}\right)} \tag{4.2-8}$$

The representation of $x_I(t)$ and $u(t)$ are given below (Fig. 4.2-1). The method followed is usually called minimum bandwidth interpolation. Other interpolation functions are the step-function interpolation and polygonal interpolation. A detailed analysis for determination of

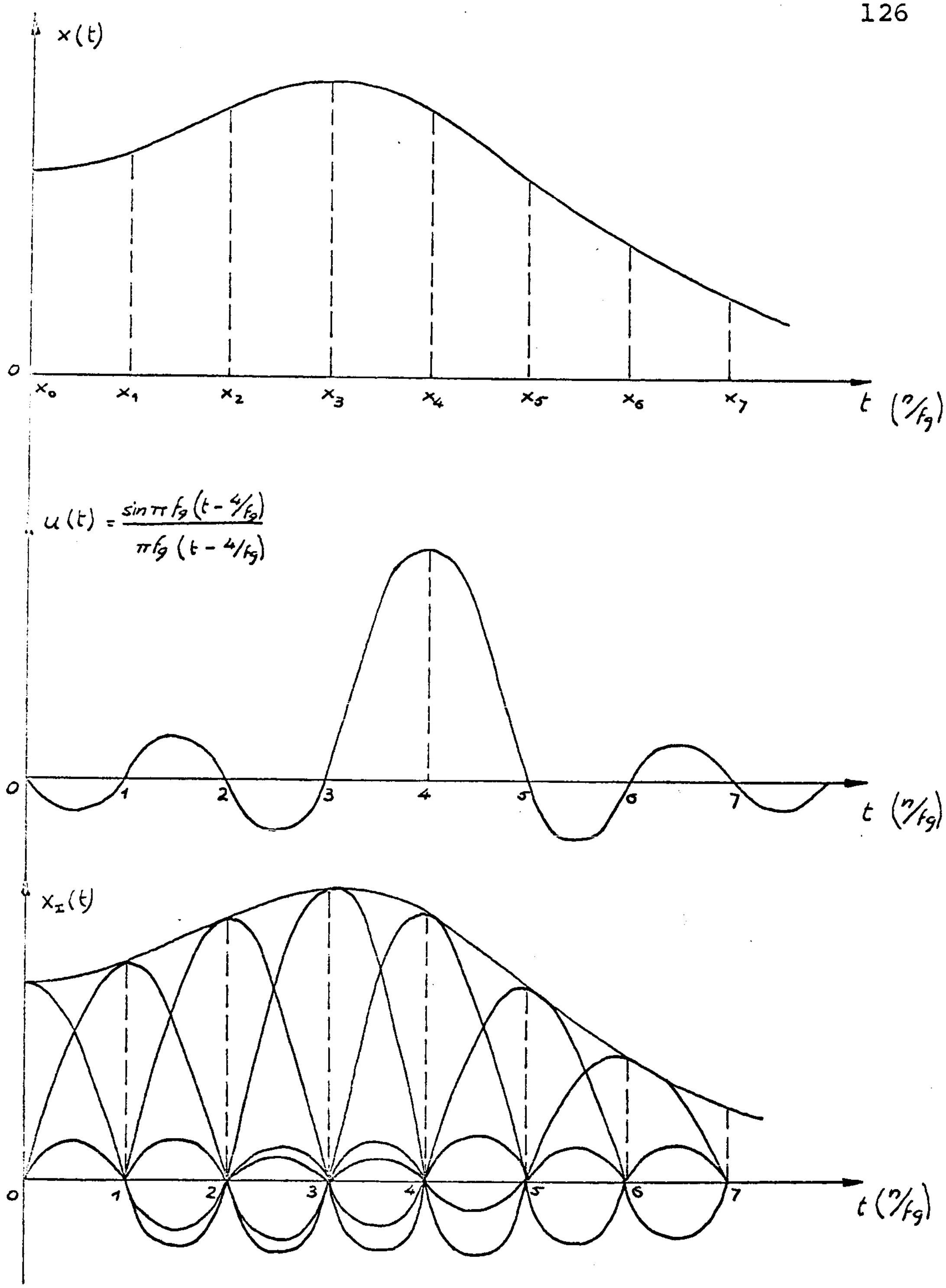



Figure 4.2-1 Minimum bandwidth interpolation

interpolation error can be found in Nichols (1956).

The interpolation-generating function can be regarded as the impulse response of an interpolation filter of bandwidth B.

## 4.3 Zero Representation of Pulse Amplitude Modulation

The fundamental operations involved in time division systems have already been discussed. Now the new approach to TDM systems will be introduced by means of analytic signal representation.

According to the symbols used in FDM to distinguish between the real and analytic signals, the actual signal in pulse amplitude modulation systems can be represented as

$$s_{PAM}(t) = s(t) \cdot p(t) \tag{4.3-1}$$

where s(t) is a bandlimited modulating signal, and p(t) is a periodic pulse-train carrier, $s_{PAM}(t)$ corresponds to the pulse amplitude modulated signal.

The zero representation technique has been discussed thoroughly in Section 2.4. Since pulse amplitude modulation is a product process, according to the properties of the zero representation for analytic signals (Section 2.4), its zero pattern is simply the superposition of the zero pattern of the information and of the zero pattern of the pulse. Consider the following example:

$$s(t) = \frac{5}{4} - \cos \omega t \qquad (4.3-2)$$

$$= \frac{5}{4} - \frac{1}{2} e^{j\omega t} - \frac{1}{2} e^{-j\omega t} \qquad (4.3-3)$$

Following the same procedure indicated in Section 2, a transformation is made to the complex z-plane by replacing

$$t \quad \text{by} \quad z = \tau + j\sigma \qquad (4.3-4)$$

$$s(z) = \frac{5}{4} - \frac{1}{2} e^{j\omega z} - \frac{1}{2} e^{-j\omega z} \qquad (4.3-5)$$

now let $x = \exp j\omega z$

$$s(x) = \frac{5}{4} - \frac{x}{2} - \frac{1}{2x} = \frac{5x - 2x^2 - 2}{4x} \qquad (4.3-6)$$

The zeroes of s(x) are found by solving:

$$-2x^2 + 5x - 2 = 0$$

The equation has two roots, which are:

$$x_1 = 2 \qquad\qquad x_2 = \frac{1}{2}$$

Now to produce the representation in the z-plane

$$x_1 = \exp j\omega z_1$$

$$x_2 = \exp j\omega z_2$$

so that

$$z_1 = -\frac{j}{\omega} \ln x_1 \qquad z_2 = -\frac{j}{\omega} \ln x_2$$

Replacing the values found for $x_1, x_2$

$$z_1 = -\frac{j}{\omega} \ln 2 \qquad z_2 = \frac{j}{\omega} \ln 2$$

The real signal s(t) introduced as an example presents two complete conjugate roots in the z-plane with imaginary components only.

Let p(t) represent a periodic pulse train with period $\frac{T}{4}$ where $T = \frac{2\pi}{\omega}$. The pulse consists of an ideal delta function $\delta(t)$ band-limited by an ideal filter. It is possible to obtain for p(t) a representation of zeroes distributed on the real axis (Woodward 1953). This representation is shown in Fig. 3.4-1b. Since the product of the two signals s(t), p(t) has as its zero-pattern, the superposition of the zero-patterns of each s(t) and p(t), the zero-pattern representation of the modulated signal $s_{PAM}(t)$ will have the distribution that appears in Fig. 4.3-1c.

The zero-pattern representation of PAM offers no difficulties, since it is based on the property of superposition of zero correspondence. The sampling function can be chosen according to the actual signal to be

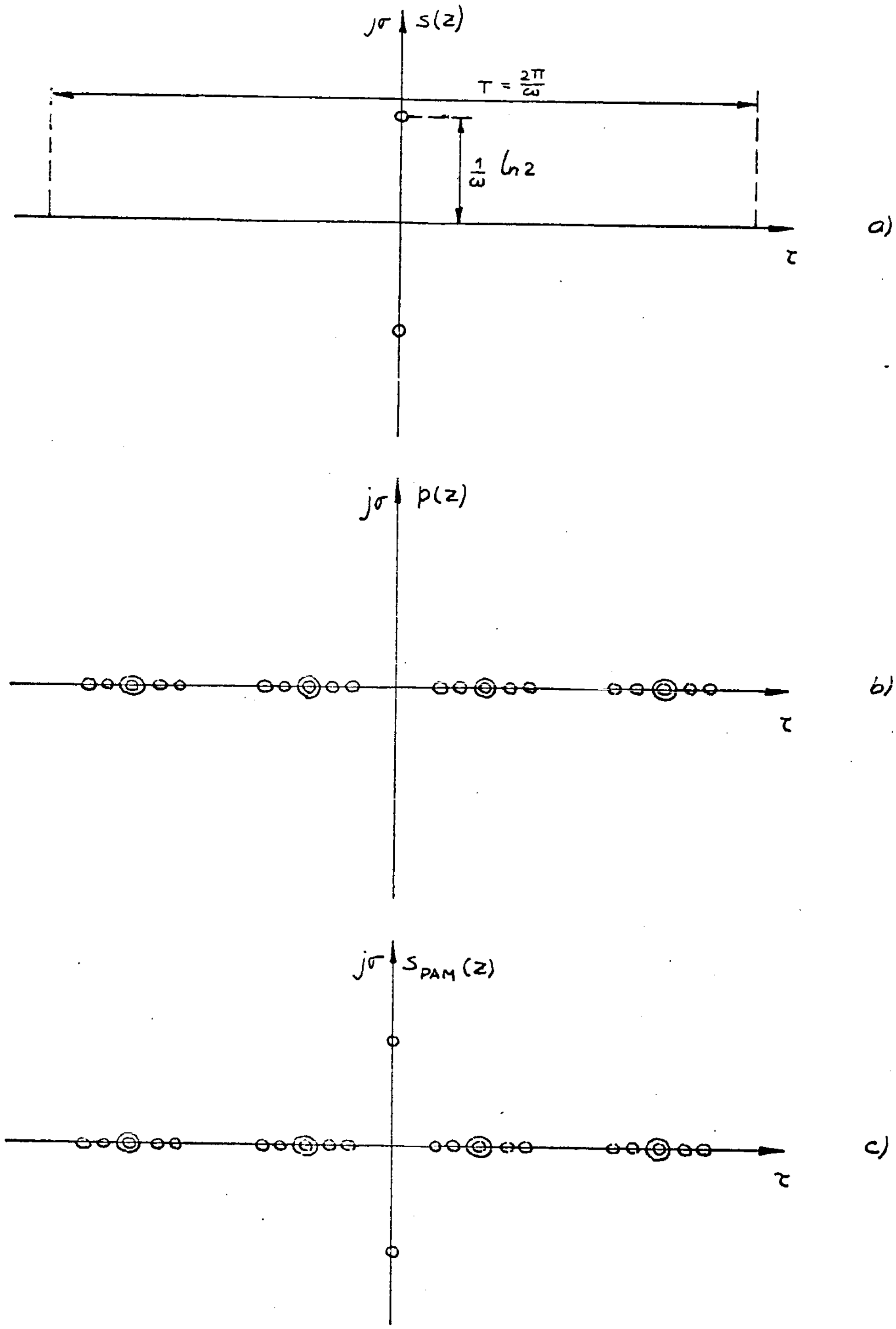



Figure 3.4-1 Zero pattern for PAM

transmitted. Different classes of pulses can be used, and different rates of sampling, and narrower pulses. The superposition property will give in each case the configuration of the zero pattern of the modulated signal by simple superposition of the zeroes of the specific pulse used with those of the modulating signal.

The problem associated with the zero-pattern representation of PAM systems consists of the difficult problem of identification of zeroes of the modulating signal from those of the carrier. No further study has been conducted in this sense, and the utility of such study is under discussion.

### 4.4 Crosstalk Considerations in TDM

Interchannel crosstalk arises in time division multiplexing systems when each sample is not confined to its assigned time slot. The limitation of bandwidth and the non-linearities of the transfer characteristics of the common transmission path will cause each sample pulse of the PAM time-multiplexed signal to overlap the neighboring time slots and interchannel interference will result. Only the interference caused by band-limiting is called crosstalk. The amount of this crosstalk is dependent upon the upper and lower cutoff frequencies of the transmission path.

a) At the upper cutoff frequency, alteration and phase distortion will prevent a pulse of one channel from decaying to zero before the time at which the gate of the next channel is opened to receive its pulse.

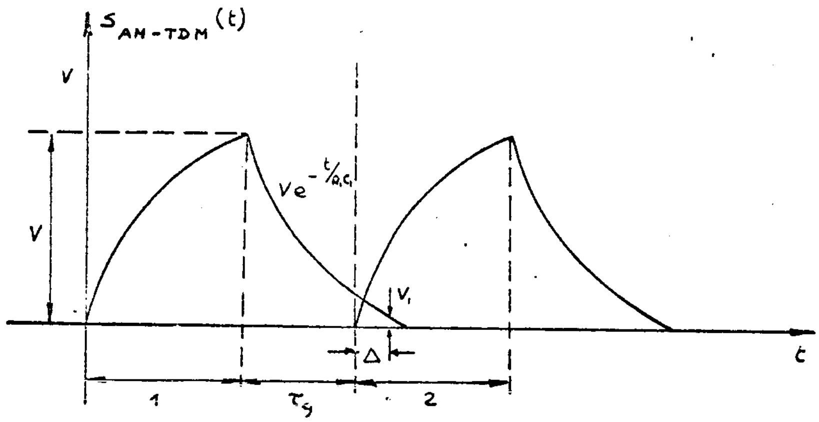



Figure 4.4-1 Interchannel crosstalk in TDM system

If a low pass filter is chosen as a model transmission path, at the end of the first slot the crosstalk voltage is:

$$V = V \exp\left(-\frac{t}{R_1C_1}\right) \qquad (4.4-1)$$

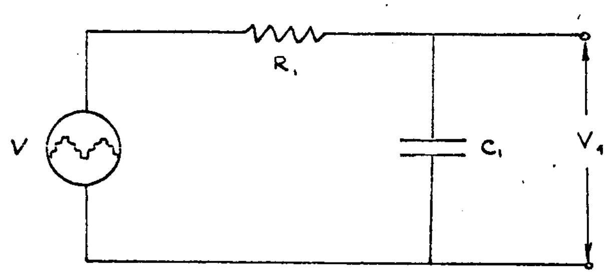



Figure 4.4-2 Low pass transmission model

where $R_1C_1$ is the time constant introduced by the characteristic function of the transmission path. Consider the adjacent channel number two; the cross voltage is found by letting

$$t = \tau_g + \Delta$$

where there is the assumption of some interval $\tau_g$ between channels, and $\Delta$ is the time interval of crosstalk. Therefore,

$$V_1 = V \exp\left(-\frac{\tau_g + \Delta}{R_1 C_1}\right) \tag{4.4-2}$$

The crosstalk ratio is defined by

$$CTR \equiv \frac{V_1}{V} = \exp\left(-\frac{\tau_g + \Delta}{R_1 C_1}\right) \tag{4.4-3}$$

or the crosstalk attenuation in db

$$(CTR)_{db} = 8.696 \frac{\tau_g + \Delta}{R_1 C_1} \tag{4.4-4}$$

b) At the lower cutoff frequency:

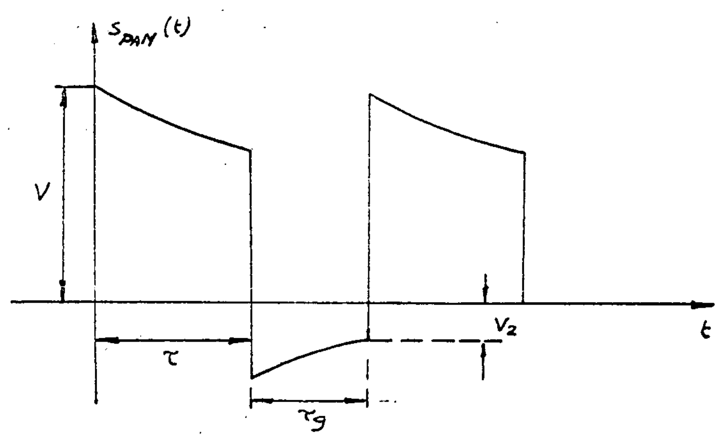



Figure 4.4-3 Crosstalk due to insufficient low frequency transmission

The same calculation, taking as a model the high pass filter as model of transmission path, yields for $0<t<(\tau+\tau_g)$

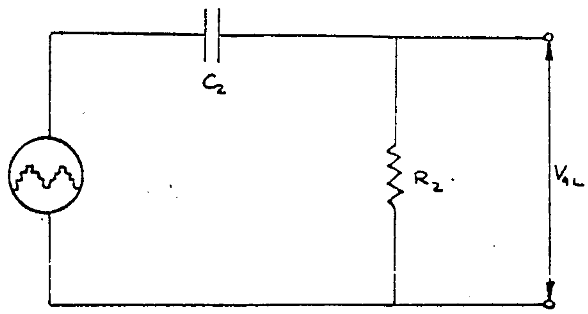



Figure 4.4-4 High pass transmission model

$$V_{1L} = V\exp\left(-\frac{t}{R_2C_2}\right)\left[u(t) - u(t-\tau)\right] + (-V)\exp\left(-\frac{t-\tau}{R_2C_2}\right) \qquad (4.4\text{-}5)$$

This equation (see Fig. 4.4-3) expresses the value of $V_{1L}$ when applied as step voltage at t = 0 and a negative step voltage of the same amount at t = $\tau_g$ . Equation (4.4-5) can be expressed as follows:

$$V_{1L} = V\left(1 - \exp\left(\frac{\tau}{R_2C_2}\right)\right)\exp\left(-\frac{t}{R_2C_2}\right) \qquad \text{for} \quad t<\tau$$

Let t = $\tau + \tau_g$ to evaluate the crosstalk in the interval of overlapping slots

$$V_2 = V\left(1 - \exp\left(\frac{\tau}{R_2C_2}\right)\right)\exp\left(-\frac{\tau+\tau_g}{R_2C_2}\right) \qquad (4.4\text{-}6)$$

c) Crosstalk due to the non-linearities of the transfer characteristics of the transmission path:

Consider the Fourier series representation of the unmodulated pulse train

$$p(t) = \frac{\tau}{T}\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \cos n\left(\omega_o t - \frac{\pi\tau}{T}\right)\right] \qquad (4.4\text{-}7)$$

Let $s(t) = 1 + M_{AM}\cos\omega_m t$ be the modulating signal of frequency $\omega_m$. The modulated signal is

$$s_{PAM}(t) = \frac{\tau}{T}\Bigg[1 + M_{AM}\cos\omega_m t + 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \cos n\left(\omega_o t - \frac{\pi\tau}{T}\right) +$$

$$+ M_{AM}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \Bigg\{\cos\left[\left(n\omega_o + \omega_m\right)t - \frac{n\pi\tau}{T}\right] +$$

$$+ \cos\left[\left(n\omega_o - \omega_m\right)t - \frac{n\pi\tau}{T}\right]\Bigg\}\Bigg) \qquad (4.4.\text{-}8)$$

where $M_{AM}$ is the modulation factor.

When the train modulated pulses are applied to a network with the transfer function

$$H(\omega) = A(\omega)\, e^{-j\theta(\omega)}$$

the output will be:

$$g_s(t) = \frac{\tau}{T}\Bigg\{ A_o + M_{An}\, A(\omega_m) \cos\left[\omega_m t + \theta(\omega_m)\right] +$$

$$+ 2\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \cos\left[n\omega_o t - \frac{n\pi\tau}{T} + \theta(n\omega_o)\right] +$$

$$+ M_{AM} \sum_{n=1}^{\infty} A(n\omega_o + \omega_m) \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \cos\Bigg[(n\omega_o + \omega_m)t -$$

$$- \frac{n\pi\tau}{T} + \theta(n\omega_o + \omega_m)\Bigg] +$$

$$+ M_{AM} \sum_{m=1}^{\infty} A(n\omega_o - \omega_m) \frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}} \cos\Bigg[(n\omega_o - \omega_m)t -$$

$$- \frac{n\pi\tau}{T} + \theta(n\omega_o - \omega_m)\Bigg]\Bigg\} \qquad (4.4-9)$$

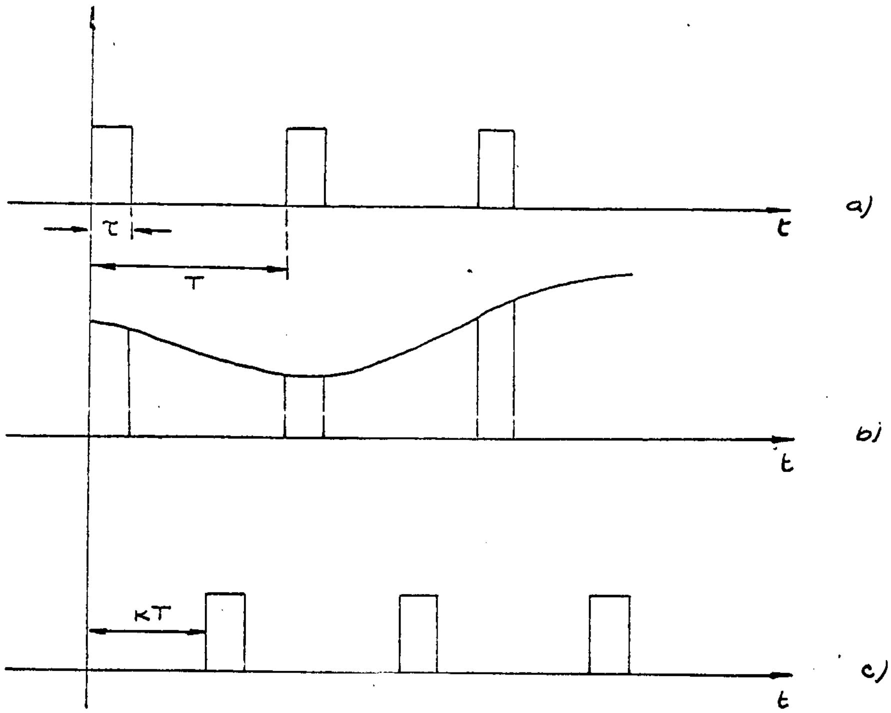



Figure 4.4-5  Pulse amplitude modulation

Figure 4.4-5 shows the modulating pulse (a), the AM pulse (b), and the demodulating pulse (c). Note tha the output pulse is demodulated starting at a time t = kT after the start of each modulated pulse. The "gating" process is defined by

$$g_g(t) = g_s(t)\, f_g(t) \tag{4.4-10}$$

where

$$f_g(t) = 0 \quad \text{for} \quad kT > t > 0 \quad \text{and} \quad T > t > (kT + \tau)$$

$$f_g(t) = 1 \quad \text{for} \quad kT + \tau > t > kT$$

Therefore,

$$f_g(t) = \frac{\tau}{T}\left[1 + 2\sum_{q=1}^{\infty} \frac{\sin\frac{q\pi\tau}{T}}{\frac{q\pi\tau}{T}} \cos\left(q\omega_o t - \frac{q\pi\tau}{T} - 2q\pi k\right)\right] \qquad (4\text{-}4\text{-}11)$$

The gated signal $g_g(t)$ is demodulated by passing the signal through a low pass filter of cutoff frequency $\omega_m \leq \omega_o/2$ . The expression for the demodulated frequency will be:

$$\begin{aligned} g_g(t) = {} & A_o\left(\frac{\tau}{T}\right)^2 + M_{AM}\left(\frac{\tau}{T}\right)^2 \cos\omega_m t \Bigg( A(\omega_m)\cos\theta(\omega_m) + \\ & + \sum_{n=1}^{\infty}\left(\frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}}\right)^2 \Big\{ A(n\omega_o + \omega_m)\cos\left[2\pi nk + \theta(n\omega_o + \omega_m)\right] + \\ & + A(n\omega_o - \omega_m)\cos\left[2\pi nK + \theta(n\omega_o - \omega_m)\right]\Big\}\Bigg) - \\ & - M_{AM}\left(\frac{\tau}{T}\right)^2 \sin\omega_m t\Bigg(A(\omega_m)\sin\theta(\omega_m) + \\ & + \sum_{m=1}^{\infty}\left(\frac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}}\right)^2 \Big\{ A(n\omega_o + \omega_m)\sin\left[2\pi nK + \theta(n\omega_o + \omega_m)\right] - \\ & - A(n\omega_o - \omega_m)\sin\left[2\pi nK + \theta(n\omega_o - \omega_m)\right]\Big\}\Bigg) \end{aligned} \qquad (4\text{-}4\text{-}12)$$

The final expression for crosstalk calculated by Bennett (Bennett 1961) for a specified bandwidth is:

$$\frac{1 + 2\sum_{n=1}^{M}\left[\dfrac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}}\right]^2 \cos 2\pi n K}{1 + 2\sum_{n=1}^{M}\left[\dfrac{\sin\frac{n\pi\tau}{T}}{\frac{n\pi\tau}{T}}\right]^2} \tag{4.4-13}$$

## 4.5 Signal Noise Ratio in TDM Systems

Pulse amplitude modulation provides a poorer signal-to- noise ratio than conventional amplitude modulation. This is due to the fact that the receiver is not blocked exactly before and after the pulse duration time where the falling edge of the pulse continues.

Let $P$ denote the average power of the unmodulated RF pulse train; after modulation the power is increased by $\frac{1}{2} M_{AM}^2 P$, where $M_{AM}$ is the modulation index due to the energy in the sidebands. If the bandwidth of the RF signal is $W$, then the ratio of the unmodulated input carrier power to noise power is:

$$\left(\frac{C}{N}\right)_{input} = \frac{P}{n_o W} \tag{4.5-1}$$

where $n_o$ = noisepower density. Since it is used as amplitude modulation, the power spectral density of a periodic waveform is:

$$G_s(f) = \sum_{-\infty}^{\infty} |c_n|^2 \delta(f - nf_o) \qquad (4.5-2)$$

and the total power spectrum in the bandwidth $\pm f_1$ is:

$$\int_{-f_1}^{f_1} G_s(u)\,du - \frac{M_{AM}^2}{2}\, U(f - nf_o) \qquad (4.5-3)$$

Therefore the modulated signal waveform has the power

$$\frac{M_{AM}^2}{2} P$$

and

$$\left(\frac{S}{N}\right)_{input} = \frac{M_{AM}^2}{2} \frac{P}{n_o W} \qquad (4.5-4)$$

Assuming a perfect video detector, the output of the first detector contains undirectional video pulse of mean power not yet modulated. After modulation

$$v(t) = \sqrt{P\frac{\tau}{T}}\,(1 + M_{AM} \sin \omega_s t)\left[1 + 2\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi\tau}{T}\right)}{\frac{n\pi\tau}{T}} \cos n\omega t\right] \qquad (4.5-4a)$$

and the signal-to-noise ratio at the output of the video detector is:

$$\left(\frac{S}{N}\right)_{OUT\,(V)} = \frac{1/2\, M_{AM}^2 P}{n_o\, W/2} \qquad (4.5-5)$$

From Equation (4.5-4a) it follows that the low frequency signal amplitude is:

$$M_{AM} \sqrt{P \frac{\tau}{T}}$$

and the low-frequency signal power is equal to

$$\frac{1}{2T} M_{AM}^2 \tau P$$

If the bandwidth of the low pass filter is $f_s$ , where $f_s$ is the top frequency of the message function, the SNR in the output is:

$$\left(\frac{S}{N}\right)_{OUT} = \frac{1/2\, M_{AM}^2 \left(\frac{\tau}{T}\right) P}{n_o f_s} \qquad (4.5\text{-}6a)$$

To eliminate the noise in the interpulse, the noise power is reduced in the ratio $\tau/T$ , and the SNR at the output of the low pass fulter is:

$$\left(\frac{S}{N}\right)_{OUT\,(LPF)} = \frac{1/2\, M_{AM}^2 \left(\tau/T\right) P}{n_o f_s\ \tau/T} = \frac{1/2\, M_{AM}^2 P}{n_o f_s} \qquad (4.5\text{-}6b)$$

The result obtained is the same as for conventional carrier amplitude modulation systems for a continuous wave. But only a part of the actual signal power is used when the low-frequency band is filtered out of the video-frequency spectrum. Each harmonic in the pulse spectrum has a pair of sidebands associated with it; any such side-band can be separated with a bandpass filter, and the message can be detected with a normal amplitude modulation detector, but the process described involves a deterioration of the signal-to-noise ratio.

The advantages of time division multiplexing with an improved SNR are found in pulse-position modulation (PPM) and pulse-duration modulation (PDM).

## 4.6 Optimum Interpolation Filter

Disregarding noise and other pertubations, the primary function of the interpolation filter is to minimize the total error.

Denote the lowest frequency component of the sampled-message power spectrum by $S_m(f)$. The higher order components $S_1(f)$, $S_2(f)$ will be collectively designated as $S_n(f)$, so that

$$S_n(f) = \sum_{i=1}^{\infty} S_i(f)$$

$$= \sum_{i=1}^{\infty} S_m(f - i f_s) \qquad (4.6\text{-}1)$$

disregarding the effect of the spectral envelope of $S_s(t)$. The model used for the analysis is shown in Fig. 4.6-1.

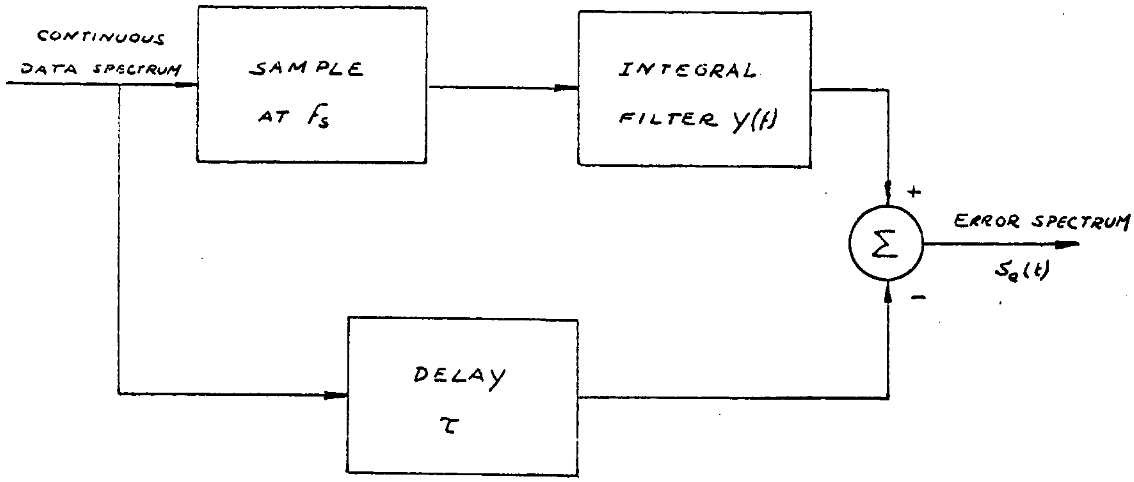



Figure 4.6-1 Circuit finding the error spectrum

The delay is necessary for final synchronism requirements so the waveform may be compared. By inspection of the diagram, the power spectrum of the interpolation is:

$$S_e(f) = \left| y(f) - \exp(-j2\pi f\tau) \right|^2 S_m(f) + \left| y(f) \right|^2 S_m(f) \qquad (4.6-2)$$

Make the substitution

$$y(f) = \left| y(f) \right| \exp(-j\theta(f)) \qquad (4.6-3)$$

Thus

$$S_e(f) = \left| \left| y(f) \right| e^{-j\theta(f)} - e^{-j2\pi f\tau} \right|^2 S_m(f) + \left| y(f) \right|^2 S_n(f) \qquad (4.6-4)$$

The value of the first member of the above expression is:

$$A = \left| |Y(f)| e^{-j\theta(f)} - e^{-2\pi f \tau} \right|^2$$

$$= \left| |Y(f)| \cos\theta - j|Y(f)| \sin\theta - \cos 2\pi f\tau + j\sin 2\pi f\tau \right|^2$$

$$= \left| \left\{ |Y(f)| \cos\theta - \cos 2\pi f\tau \right\} - j\left\{ |Y(f)| \sin\theta - \sin 2\pi f\tau \right\} \right|^2$$

Calculate the absolute value of the expression under the square as

$$= |Y(f)|^2 + 1 - 2|Y(f)| \cos(\theta(f) - 2\pi f\tau)$$

Therefore,

$$S_e(f) = \left\{ |Y(f)|^2 + 1 - 2|Y(f)| \cos\left[\theta(f) - 2\pi f\tau\right] \right\} S_m(f) +$$

$$+ |Y(f)|^2 \; S_n(f) \qquad (4.6-5)$$

The interpolation error power $P_e$ is defined:

$$P_e = \int_0^\infty S_e(f)\, df \qquad (4.6-6)$$

Minimum interpolation error will be for

$$\frac{\partial P_e}{\partial Y(f)} = 0 \qquad (4.6-7)$$

Differentiating first under the integral

$$\frac{\partial P_e}{\partial y(f)} = \int_0^{\infty} \frac{\partial S_e(f)}{\partial y(f)}\, df$$

From the expression (4.6-2)

$$\frac{\partial S_e(f)}{\partial y(f)} = \frac{\partial S_e(f)}{\partial |y(f)|}\,\frac{\partial |y(f)|}{\partial y(f)} + \frac{\partial S_e(f)}{\partial \theta(f)}\,\frac{\partial \theta(f)}{\partial y(f)} \qquad (4.6\text{-}8)$$

the evaluation of each term gives:

$$\frac{\partial S_e(f)}{\partial |y(f)|} = 2\left| y(f) S_m(f) - 2\cos\left[\theta(f) - 2\pi f \tau\right] S_m(f) + 2|y(f)|\, S_n(f)\right.$$

$$\frac{\partial |y(f)|}{\partial y(f)} = \exp j\theta(f)$$

$$\frac{\partial S_e(f)}{\partial \theta(f)} = 2|y(f)| \sin\left[\theta(f) - 2\pi f \tau\right] S_m(f)$$

To calculate $\dfrac{\partial \theta(f)}{\partial y(f)}$, consider:

$$y(f) = |y(f)|\, e^{-j\,\theta(f)}$$

$$\exp j\theta(f) = \frac{|y(f)|}{y(f)}$$

$$j\theta(f) = \ln \frac{|y(f)|}{y(f)} = \ln |y(f)| - \ln y(f)$$

$$\theta(f) = j \ln y(f) - j \ln |y(f)|$$

therefore,

$$\frac{\partial \theta(f)}{\partial y(f)} = j \frac{1}{y(f)}$$

Now the expression (4.6-8) becomes:

$$\frac{\partial S_e(f)}{\partial y(f)} = \Big[ 2|y(f)|\, S_n(f) - 2\cos[\theta(f) - 2\pi f \tau]\, S_m(f) +$$

$$+ 2|y(f)|\, S_n(f) \Big] \exp j\,\theta(f) +$$

$$+ 2|y(f)| \sin[\theta(f) - 2\pi f \tau]\, S_n(f)\, j \frac{1}{y(f)}$$

Let $y(f) = |y(f)| \exp(-j\theta(f))$

$$\frac{\partial S_e'(f)}{\partial y(f)} = 2e^{j\theta} \Big\{ |y(f) \, S_m(f) - \cos[\theta(f) - 2\pi f \tau]\, S_m(f) +$$

$$+ |y(f)|\, S_n(f) + j \sin[\theta(f) - 2\pi f \tau]\, S_m'(f) \Big\}$$

(4.6-8a)

The condition $\frac{\partial S_e(f)}{\partial y(f)}$ is found for

$$\left\{ |y(f)| - \cos\left[\theta(f) - 2\pi f \tau\right] \right\} S_m(f) +$$

$$+ |y(f)| S_n(f) + j \sin\left[\theta(f) - 2\pi f \tau\right] S_m(f) = 0$$

The above expression holds for

Imaginary part $= 0$ for $\sin\left[\theta(f) - 2\pi f \tau\right] = 0$

Real part $= 0$ for:

$$\left\{ |y(f)| - \cos\left[\theta(f) - 2\pi f \tau\right] \right\} S_m(f) + |y(f)| S_n(f) = 0 \quad (4.6\text{-}9)$$

with $\theta(f) = 2\pi f \tau$ from Equation (4.6-9) yields to

$$|y(f)| S_m(f) - S_m(f) + |y(f)| S_n(f) = 0$$

$$|y(f)| = \frac{S_m(f)}{S_m(f) + S_n(f)} \quad (4.6\text{-}10)$$

Thus, since

$$y(f) = |y(f) \exp(-j\theta(f)) = |y(f)| \exp(-j2\pi f \tau)$$

$$y(f) = \frac{S_m(f)}{S_m(f) + S_n(f)} e^{-j2n\tau} \quad (4.6\text{-}11)$$

This is the transfer function of the optimum interpolation filter in the sense of minimum error power.

Now introduce the value obtained into Equation (4.6-5)

$$S_e(f) = \left\{ |Y(f)|^2 + 1 - 2|Y(f)|\cos\left[\theta(f) - 2\pi f \tau\right] \right\} S_n(f) + |Y(f)|^2 S_n(f)$$

$$\theta(t) = 2\pi f \tau$$

$$S_e(f) = \frac{S_n^2(f)\, S_m(f) + S_m^2(f)\, S_n(f)}{\left[S_m(f) + S_n(f)\right]^2}$$

$$S_e(f) = \frac{S_m(f) \quad S_n(f)}{S_m(f) \quad + \quad S_n(f)} \qquad (4.6\text{-}12)$$

and therefore the minimum interpolation error power is given by

$$P_{e\,min} = \int_0^\infty \frac{S_m(f)\, S_n(f)}{S_m(f) + S_n(f)}\, df \qquad (4.6\text{-}13)$$

# 5.0 CONCLUSION

The theory of analytic functions has been used as a new approach to the study of conventional multiplexing schemes. The multiplexing processes involve double and triple modulation techniques, and the most general form of modulated waves exhibits simultaneous and envelope fluctuations. Modulation processes can be described as different techniques for varying the two parameters of a sinusodial wave (carrier) - magnitude and angle, according to the message wave to be transmitted. Modulation processes, therefore, can be described by envelope-phase relationships.

The phase-envelope relationships described in mathematical representation prove to have a close dependence on real and complex zeroes of the actual wave. The concept of real zeroes is easy to understand and visualize, but the concept of complex zeroes is more difficult to visualize. However, the mathematical representation of zeroes is easy in terms of Fourier series, and they show a physical meaning when related to the spectra, phase, and envelope functions. The representation of signals in terms of zeroes gives useful information about the attributes of the signal, using the principle of factorization of the terms of the Fourier series expansion of the signal. Thus the zero pattern offers the advantage of a time domain significance which shows the phase-envelope temporal fluctuations, and a frequency domain interpretation as a result of Fourier series coefficients that describe the spectrum of the signal.

The zeroes of the signal are therefore the informational attributes of modulated waves. It is shown how zero manipulation yields to the representation of the different forms of modulated waves. The difficulties inherent in complex-zero representation are diminished when a qualitative study is intended rather than a quantitative one. It is not necessary to know the location of zeroes with complete precision so long as the total zero count and the relative position of zeroes are known. Futhermore, it is always possible to recover the original wave from the modulated signal whenever the zero count is preserved. The important principle of a zero manipulation allows the zero conjugation of complex pairs, and the possibility of obtaining a modulated waveform carrier to handle mathematically, but which shows the same characteristics as long as the zero count has been preserved in the zero manipulation.

Another useful principle in zero operation is that the zero pattern of a product process is obtained simply bysuperposition of the zeroes of each product member. This means that multiplicative processes are the most amenable to a zero-based description. Fortunately most of the modulation processes are multiplicative. Non-multiplicative processes, such as SSB, because of the linear operations involved, are more difficult to handle.

A qualitative study of a comparison between different multiplexing sustems is also possible in terms of zero representation. Special attention has been given to crosstalk effects in multiplexing where mathematical analysis has barely been developed up to now. Multiplexing schemes expressed in terms of zero operation yields a multidimensional space where each zero can be mathematically described in terms of a tensor or a "psuedo tensor" as is shown. Tensor analysis is considered to be very helpful in this new approach to multiplexing schemes. In the application of this new representation of multiplexing schemes, difficulties are to be expected, but a strong possibility exists that these representations will prove to be a powerful analytical tool. In fact it has been shown that the mathematical expression for crosstalk in multiplexing schemes given by tensor analysis yields to a more practical evaluation qualitatively and quantitatively. Further work is needed to develop a comprehensive and compact theory expressed in terms of the new tools proposed here.

## 5.1 Quantitative Results for Multiplexing Schemes

As a complement to this thesis, the results of a library research are offered here on the comparative studies made by many authors of the performance for the different multiplexing schemes.

Table I Specifications

B = Effective noise bandwidth

fdm = Frequency deviation of the main carrier caused by modulated carriers

fsem = The frequency of the subcarrier

$K_1$ = modulation parameter

$M_1$ = Modulation index

$K_2$ = Constant associated with the second modulation

$M_2$ = Modulation index for each subcarrier =

Table I which is due to Landon (Landon 1948) gives a comparison between different multiplexing systems and modulation based on signal-to-noise ratio compared to single channel AM as standard.

TABLE I (Landon 1948)

| DENOMINATION OF SYSTEMS | LINEAR SUMMATION | | | | | R-M-S SUMMATION | | ADDITIONAL FACTOR DUE TO PRE-E. |
|---|---|---|---|---|---|---|---|---|
| | $K_1$ | $M_1$ | $M_2$ | $K_2$ | $\frac{SNR\,(XX-XX)}{SNR\,(AM)}$ | $M_2$ | $\frac{SNR\,(XX-XX)}{SNR\,(AM)}$ | |
| AM | | | | | 1 | | | |
| FM | | | | | $2^{-1}\sqrt{3}\,f_m^{-1}\,B$ | | | |
| AM - AM | 1 | 1 | $\frac{1}{2n}$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}\;4^{-1}\;n^{-1}$ | $\sqrt{2}/4n^{1/2}$ | $4^{-1}\;n^{-1/2}$ | 1.38 |
| AM - PM | 1 | 1 | $\frac{B}{4n^2 f_0}$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}\;8^{-1}\;B^{n-2}\,f_c^{-1}$ | $\frac{1}{8}\sqrt{\frac{3}{2}}\;\frac{B}{n^{3/2} f_0}$ | $3^{1/2}\;16^{-1}\;B\,n^{3/2}\,f_c^{-1}$ | 1.38 |
| FM - AM | $\sqrt{3}$ | $\frac{B}{4nf_m} = \frac{f_d}{f_m}$ | $\frac{1}{n}$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{6}\;8^{-1}\;B^{n-2}\;f_m^{-1}$ | $\frac{\sqrt{2}}{4\,n^{1/2}}$ | $3^{1/2}\;16^{-1}\;B\,m^{-3/2}\,f_m^{-1}$ | 1.86 |
| FM - PM | $\sqrt{3}$ | $\frac{B}{4nf_m} > \frac{f_d}{f_m} > 1$ | $\frac{B}{2n^2 f_m\,M_1}$ | $\frac{\sqrt{2}}{2}$ | $3^{1/2}\;2^{-3/2}\;B\,n^{-2}\;f_m^{-1}$ | $\frac{1}{8}\sqrt{\frac{3}{2}}\;\frac{B}{M_1\,m^{3/2} f_0}$ | $3 \cdot 16^{-1}\;B\,n^{-3/2}\,f_m^{-1}$ | 1.86 |
| SS - AM | | 1 | $\frac{1}{n}$ | 1 | $n^{-1}$ | $\frac{1}{\sqrt{n}}$ | $n^{1/2}$ | 1.38 |
| SS - PM | | 1 | $\frac{B}{n^2 f_0}$ | 1 | $B\,n^{-2}\;f_c^{-1}$ | $\frac{\sqrt{3}}{2}\;\frac{B}{n^{3/2} f_0}$ | $3^{1/2}\;2^{-1}\,B\,n^{-3/2}\,f_c^{-1}$ | 1.38 |
| SS - PM - FM | | | | | | | $n^{-1/2}$ | 1.38 |

| | | LINEAR SUMM. $\frac{SNR(XX-XX)}{SNR(AM)}$ | | RMS SUMM. $\frac{SNR(XX-XX)}{SNR(AM}$ | PRE-EMPHASIS FACTOR |
|---|---|---|---|---|---|
| PAM - FM slow | | | | $3^{1/2}\, 2^{-1}\, B n^{-1} f_m^{-1}$ | 1.38 |
| PAM - FM | | | | $\pi\, 12^{-1}\, B\, n^{-3/2} f_m^{-1}$ | 1.86 |
| PPM - AM | | | | $2 \cdot 9^{-1} B\, n^{-3/2} f_m^{-1}$ | 1.38 |
| PPM - FM | | | | $2 \cdot 3^{-1}\, B^{1/2} f_m^{-1/2} n^{-1}$ | 1.38 |
| PNM - FM | | $\infty$ | | | |
| PAM - PCM - AM | | $\infty$ | | | |
| PAM - FCM - AM | | $\infty$ | | | |
| PWM - AM | | | | $6^{-1/2}\, B^{1/2}\, n^{-1} f_m^{-1/2}$ | 1.38 |
| PWM - FM | | | | $\pi\, 3^{-1/2}\, B^{1/2} n^{-1} f_m^{-1/2}$ | 1.38 |
| PNM - AM | | $\infty$ | | | |

Tables II and III Specifications

$a_{oi}$ = unmodulated rms amplitude of subcarrier

$\alpha$ = positive number related to $P_{AM}$ as the passing of permissible time that an individual channel is switched on; (1-1/2) is used to denote guard space in PDM and PPM

$\beta$ = bandwidth required for the carrier

$\beta i$ = bandwidth required for the subcarriers

$D_{\iota}$ = deviation rates of a frequency modulated radio link

$D_i$ = deviation rates of FM subcarrier

F = number of samples for record per channel of a TDM

$F_e$ = video passband of a radio link

$f_D$ = max. frequency deviation of a frequency modulated radio carrier

$f_{di}$ = max. frequency deviation of the frequency modulated subcarrier

$f_{dh}$ = max. frequency deviation of the highest frequency modulated subcarrier

$f_{mi}$ = max. information frequency transmitted in channel

$K_2$ = rms fluctuation noise per unit bandwidth

$K_{Li}$ = constant characteristic of the type of modulation of the $i$th carrier

$M_{2i}$ = modulation index of the radio link due to the subcarrier

$M_{Li}$ = modulation index of the $i$th subcarrier

$r$ = number of wideband pairs in the video passband of a PAM multiplex

$\bar{\Omega}$ = max. phase deviation of a phase modulated radio carrier

$R_{oi}$ = wide band gain referred to the $i$th channel of a multiplex

$S$ = rms amplitude of the sinusoidal video output of the comparison single channel AM link under condition of full modulation

$S_r$ = rms amplitide of carrier required for video improvement threshold

$S_t$ = improvement threshold of a frequency or phase modulated radio carrier

$S_{ti}$ = rms amplitude of carrier required for threshold of $i$th FM subcarrier

$\left(\frac{S}{N}\right)_{it}$ = minimum acceptance signed to fluctuation noise ratio in the output of the $i$th channel

Table II which is due to Nichols (Nichols 1954) gives the "wideband gain and threshold expressions" comparing the various types of modulation and MPX.

Table III which is due to Nichols (Nichols 1954) gives the "bandwidth and improvement thresholds for minimum acceptable output SNR."

Table IV Specifications

The symbols used by the author are consistent with those used in Tables II and III. Note only,

$R_1$ = SNR in output channel

$R_2$ = SNR in carrier

E = information efficiency

The table due to Nichols and Rauch (Nichols 1956) gives another parameter for comparison between multiplex systems. The "information efficiency" defined as the ratio of the information capacity of the output signal channel divided by the information capacity of the modulated signal channel. The ideal value of information efficiency is equal to one.

$$E = \frac{W_1 (1 + R_1^2)}{W_2 (1 + R_2^2)}$$

TABLE II (Nichols 1954)

| SYSTEM | $K_{Li}$ | $M_{Li}$ | $M_{2i}$ | $\frac{S_{ti}}{K_2}$ | $\frac{S_t}{K_2}$ |
|---|---|---|---|---|---|
| AM - AM | 1 | $\frac{a_{oi} f_{di}}{S f_{mi}}$ | 1 | $2.8 \frac{S}{a_{oi}} (\beta_i f_{di})^{1/2}$ | |
| FM - AM | $\sqrt{3}$ | $\frac{a_{oi}}{S}$ | 1 | $2.8 \frac{S}{a_{oi}} (\beta_i f_{di})^{1/2}$ | |
| AM - FM | 1 | $\frac{a_{oi}}{S}$ | $\frac{f_D}{f_i}$ | $2.8 \frac{S f_i}{a_{oi} f_D} (\beta_i f_{di})^{1/2}$ | $2(\beta f_D)^{1/2}$ |
| FM - FM | $\sqrt{3}$ | $\frac{a_{oi} f_{di}}{S f_{mi}}$ | $\frac{f_D}{f_i}$ | $2.8 \frac{S f_i}{a_{oi} f_D} (\beta_i f_{di})^{1/2}$ | $2(\beta f_D)^{1/2}$ |
| AM - PM | 1 | $\frac{a_{oi} f_{di}}{S f_{mi}}$ | $\Phi_D$ | | $2(\beta f_D)^{1/2}$ |
| FM - PM | $\sqrt{3}$ | $\frac{a_{oi} f_{di}}{S f_{mi}}$ | $\Phi_D$ | $\frac{2.8}{\Phi_D} \frac{S'}{a_{oi}} (\beta_i f_{di})^{1/2}$ | $2(\beta f_D)^{1/2}$ |

| SYSTEM | $R_{oi}$ | CARRIER THRESHOLD FOR $\frac{S_t}{K_2}$ | CARRIER THRESHOLD FOR $S_t = S_v$ |
|---|---|---|---|
| PAM - AM | $\frac{S_t}{K_2} =$ | $R_{oi} = \frac{1}{n^{1/2}}$ | ADDITIONAL F. = |
| PAM - PM | $2(\beta f_D)^{1/2}$ | $\frac{f_D}{(\alpha n)^{1/2} r F}$ | |
| PAM - FM | $2(\beta f_D)^{1/2}$ | $\frac{\pi f_D}{\alpha n F r^{1/2}}$ | |
| PDM - AM | $4\left(\frac{F_c}{\alpha}\right)^{1/2}$ | $\frac{1}{n}\left(\frac{F_c}{\alpha F}\right)^{1/2}$ | |
| PDM - FM | $2(\beta F_c \Phi_D)$ | $\frac{\sqrt{6}\, f_D}{\alpha n (F_c F)^{1/2}}$ | $4 F_c^{1/2}$ |
| PPM - AM | $4(nF)^{1/2}$ | $\frac{5 F_c}{4 n^{3/2} \alpha F}$ | |
| PCM - AM | $2(2nMF)^{1/2}$ | $\sigma$ | |
| PDM - PM | $2(\beta F_c \Phi_D)$ | $\frac{\sqrt{2}\, f_D}{\alpha n (F_c F)^{1/2}}$ | $4.3 F_c^{1/2}$ |
| PCM - FM | $2(\beta F_D)^{1/2}$ | $\sigma$ | $4 F_c^{1/2}$ |
| PCM - PM | $2(\beta F_c \Phi_D)$ | $\sigma$ | $4.3 F_c^{1/2}$ |

TABLE III (Nichols 1954)

| TYPE | FORMULAE |
|---|---|
| AM - AM | $\left(\frac{S}{K_2}\right)_{min} = \frac{\sqrt{2}}{a_{oi}} S f_{mi}^{1/2} \left(\frac{S}{N}\right)$ |
| AM - PM | $f_D = \Phi_D f_h = \left[0.7 \frac{S f_h}{a_{oi}} \left(\frac{S}{N}\right)_{it}\right]^{2/3} \left(\frac{f_{mi}}{\beta}\right)^{1/3}$ <br> $\frac{S_t}{K_2} = 2 (\beta f_D)^{1/2} = *$ |
| FM - AM | $D_i = \frac{f_{di}}{f_{mi}} = \left[\frac{0.28}{\beta_i^{1/2}} \left(\frac{S}{N}\right)_{it}\right]^{2/3}$ <br> $\frac{S_v}{K_2} = \frac{2.83}{a_{oi}} (\beta_i f_{di})^{1/2}$ |
| FM - FM | $D_i = \left[\frac{0.28}{\beta_i^{1/2}} \left(\frac{S}{N}\right)_{it}\right]^{2/3}, \quad \frac{S_t}{K_2} = \frac{S_v}{K_2} = *$ <br> $f_D = \left(\frac{S f_i}{a_{oi}}\right)^{2/3} \left(\frac{2\beta_i f_{di}}{\beta}\right)^{1/3}$ |
| FM - PM | $D_i = \left[\frac{0.28}{\beta_i^{1/2}} \left(\frac{S}{N}\right)_{it}\right]^{2/3}, \quad \frac{S_t}{K_2} = \frac{S_v}{K_2} = *$ <br> $f_D = \Phi_D f_h = \left(\frac{S f_h}{a_{oi}}\right)^{2/3} \left(\frac{2\beta_i f_{di}}{\beta}\right)^{1/3}$ |

| TYPE | FORMULAE |
| --- | --- |
| PAM - AM | $\frac{S}{K_2}_{min} = (n f_m)^{1/2} \left(\frac{S}{N}\right)_{it}$ |
| PAM - FM | $f_D = \left[\frac{\alpha n F}{2\pi} \left(\frac{r f_m}{\beta}\right)^{1/2} \left(\frac{S}{N}\right)_{it}\right]^{2/3}$ <br> $\frac{S_t}{K_2} = 2(\beta f_D)^{1/2}$ * |
| PAM - PM | $f_D = \left[\frac{rF}{2} \left(\frac{\alpha n f_m}{\beta}\right)^{1/2} \left(\frac{S}{N}\right)_{it}\right]^{2/3}$ |
| PDM - AM | $F_c = 0.25\, n\, (\alpha F f_m)^2 \left(\frac{S}{N}\right)_{it}$ <br> $\frac{S}{K_2} = 4 F_c^{1/2}$ |
| PDM - FM | $F_c = 0.25 \alpha n\, (F f_m)^{1/2} \left(\frac{S}{N}\right)_{it}$ <br> $f_D = 0.4 F_c$ , $\beta = 10$ , $\frac{S_K}{K_2} =$ * |
| PDM - PM | $F_c = 0.25 \alpha\, n\, (F f_m)^{1/2} \left(\frac{S}{N}\right)_{it}$ <br> $\Phi_D = 0.6$ , $f_D = 0.6\, F_c$ , $\beta = 8$ , * |
| PPM - AM | $F_c = 0.2 \alpha\, n\, (F f_m)^{1/2} \left(\frac{S}{N}\right)_{it}$ <br> $S_K/K_2 = 4(nF)^{1/2}$ |

TABLE IV (Nichols 1954)

| SYSTEM | $W_1$ | $R_1$ | $W_2$ | $R_2$ | $E$ | REMARKS |
|---|---|---|---|---|---|---|
| AM - AM | $\frac{F_c}{2}$ | $\frac{S \cdot 0.61}{K_2 F_c^{1/2}}$ | $2F_c$ | $\frac{S}{K_2 (F_c)^{1/2}}$ | $\frac{1}{4} \frac{\log(1+R_1^2)}{\log(1+2.7R_1^2)}$ | $f^{1/2}$ taper |
| FM - AM | $\frac{F_c}{\beta_i D_i}$ | $5.7\left(\frac{f_{dj}}{f_{mj}}\right)^{3/2}$ | $2F_c$ | 4.7 | $\frac{0.45}{R_1^{2/3}} \log(1+R_1^2)$ | $f^{1/2}$ taper |
| AM - FM | $\frac{F_c}{2}$ | $3.8\ D^{3/2}$ | $R_1^{3/2} F_c$ | 2.8 | $\frac{0.53}{R_1^{2/3}} \log(1+R_1^2)$ | $f^{3/2}$ taper<br>$D$ 7.5, $\beta = 2.6$ |
| AM - PM | $\frac{F_c}{2}$ | $2\Phi\ D^{3/2}$ | $1.5 R_1^{2/3} F_c$ | 2.8 | $\frac{0.35}{R_1^{2/3}} \log(1+R_1^2)$ | $\Phi_D = 12.5$, $f^{1/2}$ taper<br>$\beta = 2.6$ |
| FM - FM | $\frac{F_c}{\beta_i D_i}$ | $5.7\left(\frac{f_{di}}{f_{mj}}\right)^{3/2}$ | $5.0\ F_c$ | 2.8 | $\frac{0.25}{R_1^{2/3}} \log(1+R_1^2)$ | $\beta = 12$, $D = 0.4$<br>$\beta_j = 2.6$, $f^{2/3}$ taper |
| FM - PM | $\frac{F_c}{\beta_i D_i}$ | $5.7\left(\frac{f_{di}}{f_{mj}}\right)^{3/2}$ | $6.5\ F_c$ | 2.8 | $\frac{0.20}{R_1^{2/3}} \log(1+R_1^2)$ | $\beta = 6.5$ $\Phi_D = 1$<br>$\beta_j = 2.6$ $f^{1/2}$ taper |
| PAM - AM | $n f_m$ | $\frac{S}{K^2} \frac{1}{(n f_m)^{1/2}}$ | $16 n f_m$ | $\frac{S}{K_2^2} \frac{1}{2(2n f_m)^{1/2}}$ | $\frac{1}{16} \cdot \frac{\log(1+R_1^2)}{\log(1+R_1^2/\beta)}$ | |
| PAM - FM | $n f_m$ | $\frac{4.2\ f_D^{3/2}}{n^{3/2} f_m^{3/2}}$ | $2.6\ f_D$ | 2.8 | $\frac{0.48}{R_1^{2/3}} \log(1+R_1^2)$ | $\beta = 2.6$ |

TABLE IV (Nichols 1954)

| | | | | | | |
|---|---|---|---|---|---|---|
| PAM - PM | $n f_m$ | $\frac{0.28}{n^{3/2}} \frac{f_D^{3/2}}{f_m^{3/2}}$ | $2.6 f_D$ | 2.8 | $\frac{0.18}{R_1^{2/3}} \log(1+R_1^2)$ | $\beta = 2.6$ |
| PDM - AM | $n f_m$ | $\frac{2.8\ F_c}{n f_m}$ | $2 F_c$ | 4 | $\frac{1.15}{R_1} \log(1+R_1^2)$ | $\alpha = 1$ |
| PDM - FM | $n f_m$ | $\frac{4.9\ f_D}{n' f_m}$ | $3.4 f_D$ | 2.8 | $\frac{1.51}{R_1} \log(1+R_1^2)$ | $D = 0.6,\ \alpha = 1$ $\beta = 3.4$ |
| PDM - PM | $n f_m$ | $\frac{3.0\ f_D}{n f_m}$ | $2.5 f_D$ | 2.8 | $\frac{1.25}{R_1} \log(1+R_1^2)$ | $\Phi_D = 0.9,\ \alpha = 1$ $\beta = 2.5$ |
| PPM - AM | $n f_m$ | $\frac{3.5\ F_c}{n f_m}$ | $2 F_c$ | $4\left(\frac{2n f_m}{F_c}\right)^{1/2}$ | $\frac{1.77}{R_1} \frac{\log(1+R_1^2)}{\log(1+\frac{112}{R_1})}$ | $\alpha = 1$ |
| PCM - AM | $n f_m$ | $1.2\,(2^M)$ | $2 F_c$ | 4 | $\frac{0.2 \log(1+R_1^2)}{\log R_1 - 0.09}$ | $F_c = n M f_m$ |
| PCM - FM | $n f_m$ | $1.2\,(2^M)$ | $3.4 f_D$ | 2.8 | $\frac{0.6 \log(1+R_1^2)}{\log R_1 - 0.09}$ | $F_c = n M f_m,\ D = 0.6$ $\beta = 3.4$ |
| PCM - PM | $n f_m$ | $1.2\,(2^M)$ | $2.5 f_D$ | 2.8 | $\frac{0.14 \log(1+R_1^2)}{\log R_1 - 0.09}$ | $F_c = n M f_m,\ \Phi_M = 0.9$ $\beta = 2.5$ |

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## APPENDIX A

### Bairstowe's Iteration to Find the Roots of a Polynomial

The purpose of this method is to find a quadratic form that divides exactly an nth polynomial. This quadratic form contains two roots of the polynomial that can be real, multiple, or conjugate complex, if it exactly divides the polynomial. The problem is to find the product and the sum of the roots p and q. Consider then the nth order polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

and a quadratic form

$$m = x^2 + px + q$$

where x and y are the two values picked as first approximations to the product and sum of the two roots. If the quotient $\frac{f(x)}{m}$ is formed:

$$\frac{f(x)}{m} = \frac{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n}{x^2 + px + q}$$

$$f(x) = (b_0 x^{n-2} + b_1 x^{n-3} + \dots + b_{n-3} x + b_{n-2})(x^2 + px + q) + Rx + S$$

the following relations will result:

$$b_0 = a_0$$

$$b_1 = a_1 p$$

$$b_2 = a_2 - pb_1 - q$$

$$b_3 = a_3 - pb_2 - qb_1$$

$$b_k = a_k - pb_{k-1} - qb_{k-2} \quad \text{for } k = 0,1\ 2\ldots n \quad (A-1)$$

with

$$b_{-1} = 0$$

$$b_{-2} = 0$$

p and q being only approximate values, there will be a remaining term $R_x + S$ with

$$R = b_{n-1} \qquad \text{and} \qquad S = b_n + pb_{n-1}$$

If R = S = 0, then the divisor is a factor of the f(x), and there are two roots of the polynomial. The basic plan is to find an adequate approximation of R and S. A Taylor Series expansion made and terminated the first derivative (Newton-Ralphson Approximation)

$$R = f(p + \ p,\ q + \ q) = f(p,q) + \Delta p \frac{\partial f(p,q)}{\partial p} + \Delta q \frac{\partial f(p,q)}{\partial q}$$

$$S = g(p + \ p,\ q + \ q) = g(p,q) + \Delta p \frac{\partial g(p,q)}{\partial p} + \Delta q \frac{\partial g(p,q)}{\partial p}$$

substituting

$$R = b_{n-1} \qquad \text{and} \qquad S = b_n + pb_{n-1}$$

for

$$f(b + \ p,\ q + \ q) = b_{n1-1} + \Delta p \frac{\partial b_{n-1}}{\partial p} + \Delta q \frac{\partial b_{n-1}}{\partial q} + \Delta q \frac{\partial(b_n + pb_{n-1})}{\partial q}$$

$$g(p + \quad p, \; q + \quad q) = b_n + pb_{n-1} + \Delta p\left[\frac{\partial b_n}{\partial p} + p\frac{\partial b_{n-1}}{\partial p} + b_{n-1} + \right.$$
$$\left.\Delta q\left[\frac{\partial b_n}{\partial g} + p\frac{\partial b_{n-1}}{\partial g}\right]\right.$$

Since the quantity wanted is:

$$R(p,q) = S(p,q) = 0,$$

the desired relation is:

$$b_{n-1} + \Delta p\frac{\partial b_{n-1}}{\partial p} + \Delta p\frac{\partial b_{n-1}}{\partial q} = o \qquad \text{(A-2)}$$

$$b_n + pb_{n-1} + \Delta p\left[\frac{\partial b_n}{\partial p} + p\frac{\partial b_{n-1}}{\partial p} + b_{n-1}\right] + \Delta g\left[\frac{b_n}{g} + p\frac{b_{n-1}}{g}\right]$$

then calculate the Equation (A-2) less p times Equation (A-3)

$$b_n + \Delta p\frac{\partial b_n}{\partial p} + b_{n-1} + \Delta g\frac{\partial b_n}{\partial g} = 0 \qquad \text{(A-4)}$$

and a new system consisting of the two Equations (A-2) and (A-4) results.

The derivative with respect ot p and q gives the necessary relationships between bk and ak. Therefore, the recursion formula for the coefficient bk is:

$$b_k = a_k - pb_{k-1} - qb_{k-2}$$

$$\frac{\partial b_k}{\partial p} = -p\frac{\partial b_{k-1}}{\partial p} - b_{k-1} - q\frac{\partial b_{k-2}}{\partial p} \qquad \text{(A-5)}$$

$$\frac{\partial b_{-1}}{\partial p} = \frac{\partial b_0}{\partial p} = \frac{\partial b_{-1}}{\partial q} = \frac{\partial b_0}{\partial q} = 0 \qquad \left\{ \begin{array}{l} \text{for } k = -1 \\ \text{and } k = 0 \end{array} \right.$$

$$\frac{\partial b_k}{\partial q} = -p\frac{\partial b_{k-1}}{\partial q} - b_{k-2} - q\frac{\partial b_{k-2}}{\partial q} \qquad \text{(A-6)}$$

Hence a new vector can be introduced whose components are related to those of vector b by the recurrence formula:

$$c_k = b_k - p\,c_{k-1} - q\,c_{k-2} \qquad \text{for} \quad k = 1, 2, \ldots n \qquad \text{(A-7)}$$

with $c_{-1} = 0$ and $c_0 = 1$

from Equations (A-4) and (A-5) follows:

$$\frac{\partial b_k}{\partial p} = -c_{k-1}$$
$$\frac{\partial b_k}{\partial q} = -c_{k-2} \qquad \text{for } k = 1, 2, \ldots n \qquad \text{(A-8)}$$

then

$$b x^{n-2} + b_1 x^{n-3} + \ldots\ b_{n-3}\, x + b_{n-2} =$$
$$= (x^2 + px + q)\,(c_0 x^{n-4} + c_1 x^{n-5} + \ldots + c_{n-5}\, x + c_{n-4}) + R'_x + S'$$

can be written where $R' = c_{n-3}$ and $S' = c_{n-2} + p\,c_{n-3}$

so that $\dfrac{\partial b_n}{\partial q} = -c_{n-1}$ , $\dfrac{\partial b_{n-1}}{\partial q} = -c_{n-2}$

$\dfrac{\partial b_n}{\partial q} = -c_{n-2}$ , $\dfrac{\partial b_{n-1}}{\partial q} = -c_{n-3}$

Substituting in Equation (A-4):

$$\frac{\partial b_n}{\partial p} + b_{n-1} = -c_{n-1} + b_{n-1} = -\bar{c}_{n-1}$$

$$\bar{c}_{n-1} = c_{n-1} - b_{n-1} = b_{n-1} - p c_{n-2} - q c_{n-3} - b_{n-1}$$

$$\bar{c}_{n-1} = -p c_{n-2} - q c_{n-3}$$

The Bairstowe iteration takes its final form in the system of two equations:

$$c_{n-2} \quad p + c_{n-3} \quad q = b_{n-1} \qquad \text{(A-9)}$$

$$\bar{c}_{n-1} \quad p + c_{n-2} \quad q = b_n \qquad \text{(A-10)}$$

where $\Delta p$ and $\Delta p$ are the increments to be given to p and q in order to get the next approximation

$$p^* = p + \Delta p \qquad \text{and} \qquad q^* = q + \Delta q$$

These increments are calculated by Cramer's rule:

$$\Delta p = \frac{\begin{vmatrix} b_{n-1} & c_{n-3} \\ b_n & c_{n-2} \end{vmatrix}}{\begin{vmatrix} c_{n-2} & c_{n-3} \\ \bar{c}_{n-1} & c_{n-2} \end{vmatrix}}$$

$$\Delta q = \frac{\begin{vmatrix} c_{n-2} & b_{n-1} \\ \bar{c}_{n-1} & b_n \end{vmatrix}}{\begin{vmatrix} c_{n-2} & c_{n-3} \\ \bar{c}_{n-1} & c_{n-2} \end{vmatrix}}$$

Thus,

$$p = \frac{b_{n-1} \quad c_{n-2} - b_n \quad c_{n-3}}{c^2_{n-2} - \bar{c}_{n-1} \cdot c_{n-3}} \qquad \text{(A-11)}$$

$$q = \frac{b_n \cdot c_{n-2} - b_{n-1} \cdot \bar{c}_{n-1}}{c^2_{n-2} - \bar{c}_{n-1} \cdot c_{n-3}} \qquad \text{(A-12)}$$

To summarize the method:

1) Choose values for p and q.

2) The vector A is given; from it get the components for vectors B and C with recurrence formulas (A-1), (A-11), and (A-12) respectively.

3) Calculate from Equations (A-11) and (A-12) values of the increments; and

4) if $|\Delta p|$ and $|\Delta q|$ are equal to or less than a very small quantity $\varepsilon$, adequate values of p and q have been found.

5) If not, the cycle is repeated with new values

$$p = p + \Delta p$$

$$q = q + \Delta q \qquad \text{(A-13)}.$$

# APPENDIX B

## The Routh Criterion

Bairstowe's method does not give any boundaries for p and q. The starting point of this method actually consists in choosing some trial values for p and q. Therefore, the efficiency of the method depends on the precision of the first trial. The usual techinque starts with reasonable values for p and q; for example, p = 0.1, q = 0.1 and follows an iterative proceudre to increase p and q until the conditions specified in Appendix A are satisfied.

The Routh Criterion establishes the limits of the real part of the roots of a polynomial of the nth degree. Given a polynomial of the nth degree

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0 \tag{B-1}$$

with real coefficients, the determinant

$$\Delta_n = \begin{vmatrix} a_1 & a_2 & \cdots & & 0 \\ a_3 & a_2 & a_1 & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{2n-1} & a_{2n-2} & \cdots & & a_{2n-n} \end{vmatrix}$$

is formed.

The Routh Criterion says that when the sequence of determinants for all n is positive, the roots of the polynomial are located in the half plane.

Let m(x) be a polynomial of the second degree

$$m(x) = x^2 + px + q$$

The relationship between p, q, and the roots of the equation is:

$$p = -(x_1 + x_2)$$

$$q = x_1 \cdot x_2$$

then if, and only if $x_1$ and $x_2$ are both negative,

$$p > 0$$
$$q > 0$$

The test for determining whether p and q are positive or not is to apply the Routh Criterion. If the determinant

$$\Delta_n = \begin{vmatrix} p & 1 \\ 0 & q \end{vmatrix}$$

is positive for all n, all the roots in their real parts will be negative or else will be located in the left half plane.

Suppose, however, that the original polynomial does not satisfy the Routh Criterion. A convenient shift of the axis of reference can solve the difficulty, and the Routh Criterion is used again as a test. The shift of the imaginary axis is repeated until the determinant of the shifted polynomial.

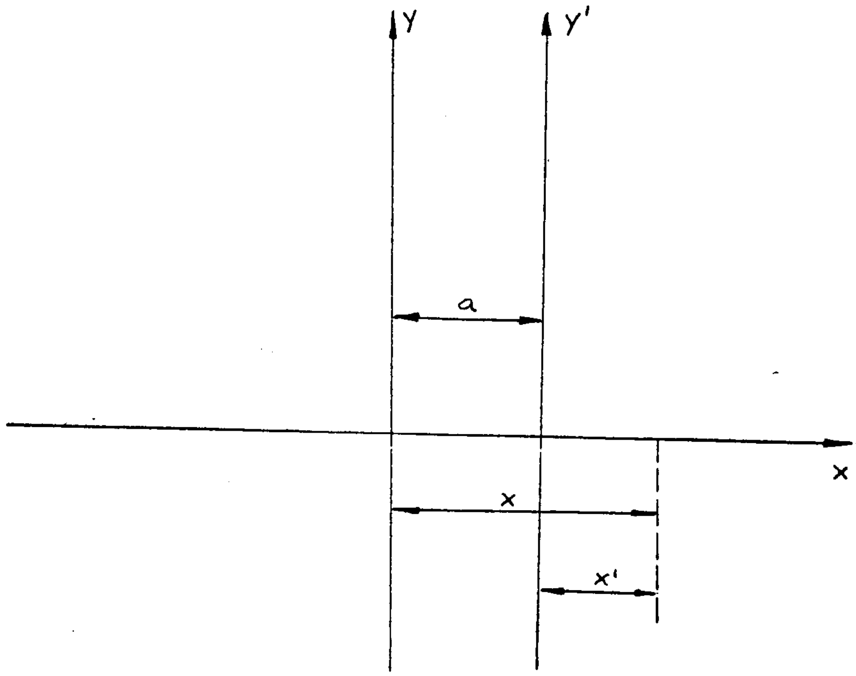


Figure B-1 Shift of the real part of the roots of a polynomial

exhibits the condition $\Delta_n > 0$ for all n.

The shift is performed by a new variable

$$x' = x - a$$

so that the new equation with the root-shift of "a" units will be:

$$f(x'-a) = a_0 (x'+a)^n + a_1 (x'+a)^{n-1} + \dots + a_n = 0$$

The new series of coefficients will be:

$$P_0 = a_0$$
$$P_1 = a_0 a + a_1$$
$$\vdots$$
$$P_n = a_0 a^n + a_1 a^{n-1} + \dots + a_n$$

and the new polynomial is

$$f(x') = P_0 x'^n + P_1 x'^{n-1} + \dots P_n = 0$$

or

$$f(x-a) = P_0 (x-a)^n + P_1 (x-a)^{n-1} + \dots + P_n = 0$$

In order to generate the coefficient of P, proceed by synthetic division. In this manner, obtain the roots with their real part shifted to the left by a given amount. There is an easy way to perform synthetic division by using computing methods. The shifted polynomial is tested again for the Routh Criterion as previously explained.

The Bairstowe's method can be applied now with greater efficiency, since the initial values of p and q are chosen as small positive numbers with the security that they are fairly close to the actual values of p and q of the quadratic factors and that the convergence of the method is that it will be rapid.

Note that the improved method for the estimation of p and q is correct in the two cases found which is shown in Figure B-2.

The rightmost root of $f_1(x)$ is real; however, $f_2(x)$ shows a complex root. In both cases the method proposed is correct, since the real part of the roots is observed.

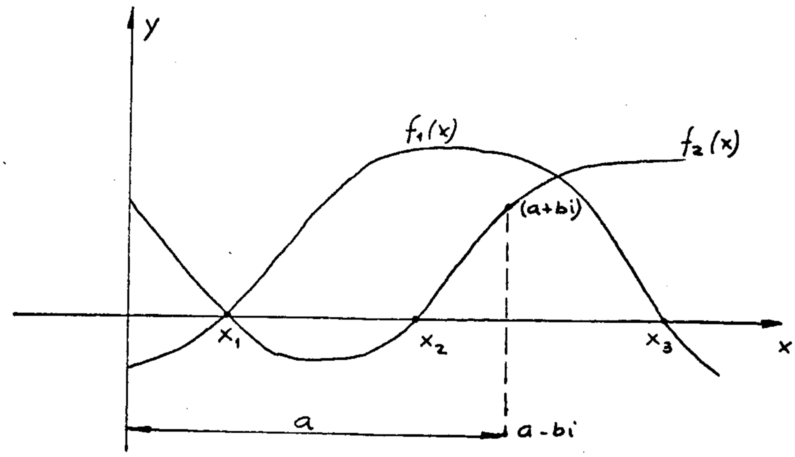


Figure B 2 Evaluation of p and q for (a) real roots only and (b) real and complex roots

The estimation of the error, however, is different in each case, since for $f_1(x)$

$$q = r_1 \cdot r_2 \qquad \text{B-}$$

and for $f_2(x)$

$$q = a^2 + b^2$$

When $a = r_2$, the following relation is obtained:

$$r_1 \cdot r_2 < a^2 + b^2$$

An improved method for the Routh Criterion has been devised. Suppose that $\Delta_{n-1} = 0$

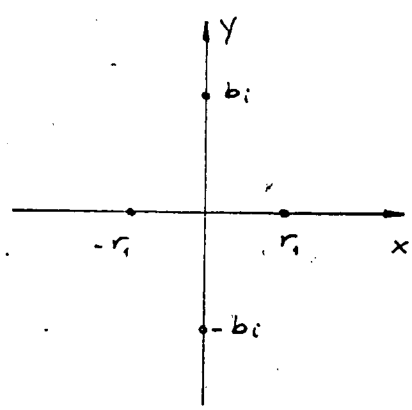


Figure B-3 Polynomial with real roots opposite or zero

The meaning is that the roots of the given polynomial have their real part opposite or zero. The two cases are shown in Fig. B-2, where

$$p = 0$$
$$q = r_1^2$$

One good approximation for the quadratic factor in this area can be:

$$m = x^2 + \varepsilon_1 x + \varepsilon_2$$

By choosing a small value for $\varepsilon$, the convergence can be rapid, and $r_1$ and b will be obtained immediately.

At this point the general case can be considered. By applying the same techniques which have been explained, and through successive shifts of the real part of the roots of the original polynomial obtained by synthetic division, the values obtained for p and q will be fairly close to the actual ones. Choose, for example, an interval of 0.5. After t shifts, the Routh Criterion test finally results in $\Delta_n \approx 0$. At this point the value of the rightmost root as estimated will be:

$$r_1^* = 0.5\,t$$

Now a rotation is made, pivoting on the "y" axis. In other words, x is replaced by (-x) in the polynomial. Applying this same technique, the estimated value will be obtained for the leftmost root after $t'$ shifts, as in

$$r_2^* = -0.5t$$

Then the estimated values for p and q will be:

$$p^* = r_1^* + r_2^*$$

$$q^* = r_1^* \cdot r_2^*$$

The p* and q* values can then be used in Bairstowe's method as intiial values to find the actual roots. The efficiency of the method will depend, of course, on the "rates" chosen for the shift. The value of 0.5 seems to be reasonable for most cases:

This method has only two inconveniences:

1) It fails in case a complex zero is found in one or both sides.

2) Furthermore, the accuracy of the method decreases notably (see Fig. B-2).

The solution to the first inconvenience is to apply the Descartes Rule of Signs or Sturm sequences if:

1) a complex zero is detected, the method remains satisfactory.

2) all roots are complex zeroes. Then

$$p^* = 2r_1^*$$

$$q^* = (r_1^*)^2 + \left[f(r_1^*)^2\right]$$

3) all of the roots except one are imaginary (complex), the lower limits of the negative roots are determined. Then, by rotating $f(x)$ into $f(-x)$ and applying the Routh Criterion, the rightmost or leftmost complex zeroes will be obtained and solution 2 (above) can be used.

If more than one root is present, the upper and lower limits for them are easily found by applying Descates's method, and a good estimation for the values of these roots is again available. The method is efficient and precise even when complex roots are present.

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0
COMPLEX ROOTS
X1= .19185834 + .48148299 I
X2= .19185834 - .48148299 I
0
REAL ROOTS
X1= .60621893 X2 = -2.12476987
0
COMPLEX ROOTS
X1= .56741712 + 1.60250433 I
X2= .56741712 - 1.60250433 I
0
COMPLEX ROOTS
X1= .20273931 + .47703678 I
X2= .20273931 - .47703678 I
0
REAL ROOTS
X1= .63127445 X2 = -2.10547292
0
COMPLEX ROOTS
X1= .53435994 + 1.58582518 I
X2= .53435994 - 1.58582518 I
0
COMPLEX ROOTS
X1= .21345539 + .47126492 I
X2= .21345539 - .47126492 I
0
REAL ROOTS
X1= .65946873 X2 = -2.08592841
0
COMPLEX ROOTS
X1= .49977447 + 1.57042836 I
X2= .49977447 - 1.57042836 I
0
COMPLEX ROOTS
X1= .22370775 + .46412701 I
X2= .22370775 - .46412701 I
0
REAL ROOTS
X1= .69111332 X2 = -2.06613180
0
COMPLEX ROOTS
X1= .46380153 + 1.55660279 I
X2= .46380153 - 1.55660279 I
0
COMPLEX ROOTS
X1= .23316449 + .45568262 I
X2= .23316449 - .45568262 I
0
REAL ROOTS
X1= .72641242 X2 = -2.04607865
0
COMPLEX ROOTS
X1= .42666864 + 1.54461052 I
X2= .42666864 - 1.54461052 I
0
COMPLEX ROOTS
X1= .24150679 + .44611115 I
X2= .24150679 - .44611115 I
0
REAL ROOTS
X1= .76538841 X2 = -2.02576485
0
COMPLEX ROOTS
X1= .38868144 + 1.53465144 I
X2= .38868144 - 1.53465144 I
0
COMPLEX ROOTS
X1= .24848632 + .43570359 I
X2= .24848632 - .43570359 I
0
REAL ROOTS
X1= .80782020 X2 = -2.00518638
0
COMPLEX ROOTS
X1= .35019678 + 1.52683285 I
X2= .35019678 - 1.52683285 I

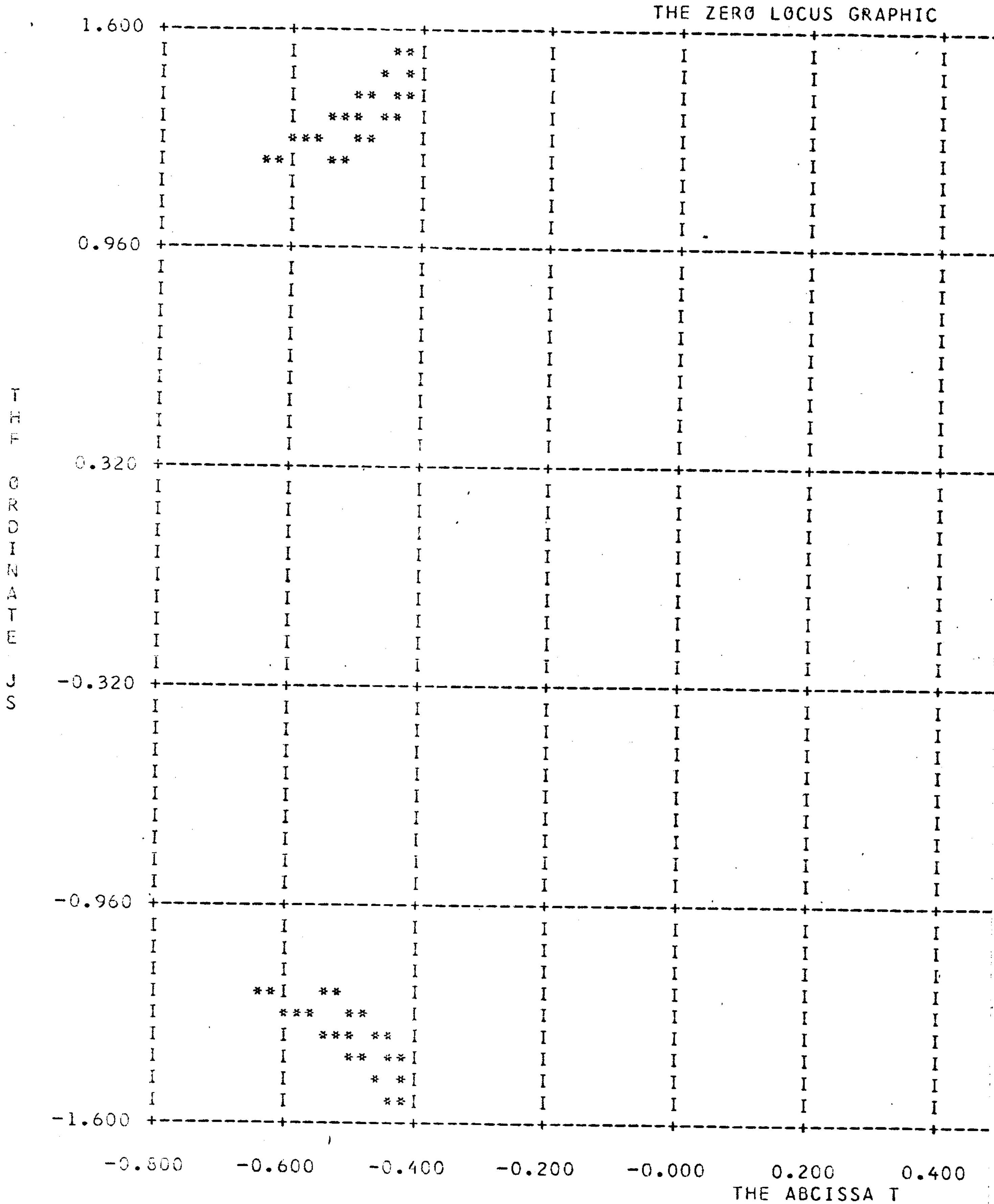
```
0            COMPLEX ROOTS
                    X1=  .25397285   +   .42482099 I
X2=  .25397285   -   .42482099 I
0            REAL ROOTS
                    X1=  .85322817           X2 =  -1.9843396
0            COMPLEX ROOTS
                    X1=  .31158289   +   1.52115557 I
X2=  .31158289   -   1.52115557 I
0            COMPLEX ROOTS
                    X1=  .25797121   +   .41383217 I
X2=  .25797121   -   .41383217 I
0            REAL ROOTS
                    X1=  .90092358           X2 =  -1.9632213
0            COMPLEX ROOTS
                    X1=  .27317772   +   1.51752062 I
X2=  .27317772   -   1.51752062 I
0            COMPLEX ROOTS
                    X1=  .26060144   +   .40305528 I
X2=  .26060144   -   .40305528 I
0            REAL ROOTS
                    X1=  .95011024           X2 =  -1.9418288
0            COMPLEX ROOTS
                    X1=  .23525789   +   1.51575398 I
X2=  .23525789   -   1.51575398 I
0            COMPLEX ROOTS
                    X1=  .26205568   +   .39272369 I
X2=  .26205568   -   .39272369 I
0            REAL ROOTS
                    X1=  1.00000003          X2 =  -1.9201598
0            COMPLEX ROOTS
                    X1=  .19802422   +   1.51563908 I
X2=  .19802422   -   1.51563908 I
0            COMPLEX ROOTS
                    X1=  .26255220   +   .38298046 I
X2=  .26255220   -   .38298046 I
0            REAL ROOTS
                    X1=  1.04990226          X2 =  -1.8982128
0            COMPLEX ROOTS
                    X1=  .16160315   +   1.51694717 I
X2=  .16160315   -   1.51694717 I
0            COMPLEX ROOTS
                    X1=  .26230115   +   .37389228 I
X2=  .26230115   -   .37389228 I
0            REAL ROOTS
                    X1=  1.09926929          X2 =  -1.8759872
0            COMPLEX ROOTS
                    X1=  .12605784   +   1.51945966 I
X2=  .12605784   -   1.51945966 I
0            COMPLEX ROOTS
                    X1=  .26148585   +   .36547096 I
X2=  .26148585   -   .36547096 I
0            REAL ROOTS
                    X1=  1.14770389          X2 =  -1.8534832
0            COMPLEX ROOTS
                    X1=  .09140385   +   1.52298152 I
X2=  .09140385   -   1.52298152 I
0            COMPLEX ROOTS
```

```
X1=  .26025648  +  .35769379 I
X2=  .26025648  -  .35769379 I
0
REAL ROOTS
X1=  1.19494185
X2 =  -1.8307022
0
COMPLEX ROOTS
X1=  .05762371  +  1.52734651 I
X2=  .05762371  -  1.52734651 I
0
COMPLEX ROOTS
X1=  .25873111  +  .35051914 I
X2=  .25873111  -  .35051914 I
0
REAL ROOTS
X1=  1.24082549
X2 =  -1.8076466
0
COMPLEX ROOTS
X1=  .02467947  +  1.53241782 I
X2=  .02467947  -  1.53241782 I
0
COMPLEX ROOTS
X1=  .25700023  +  .34389712 I
X2=  .25700023  -  .34389712 I
0
REAL ROOTS
X1=  1.28527641
X2 =  -1.7843207
0
COMPLEX ROOTS
X1=  -.00747806  +  1.53808576 I
X2=  -.00747806  -  1.53808576 I
0
COMPLEX ROOTS
X1=  .25513195  +  .33777602 I
X2=  .25513195  -  .33777602 I
0
REAL ROOTS
X1=  1.32827193
X2 =  -1.7607301
0
COMPLEX ROOTS
X1=  -.03890284  +  1.54426408 I
X2=  -.03890284  -  1.54426408 I
0
COMPLEX ROOTS
X1=  .25317695  +  .33210597 I
X2=  .25317695  -  .33210597 I
0
REAL ROOTS
X1=  1.36982699
X2 =  -1.7368823
0
COMPLEX ROOTS
X1=  -.06964927  +  1.55088630 I
X2=  -.06964927  -  1.55088630 I
0
COMPLEX ROOTS
X1=  .25117271  +  .32684077 I
X2=  .25117271  -  .32684077 I
0
REAL ROOTS
X1=  1.40998057
X2 =  -1.71278702
0
COMPLEX ROOTS
X1=  -.09976948  +  1.55790213 I
X2=  -.09976948  -  1.55790213 I
0
COMPLEX ROOTS
X1=  .24914666  +  .32193861 I
X2=  .24914666  -  .32193861 I
0
REAL ROOTS
X1=  1.44878618
X2 =  -1.68845606
0
COMPLEX ROOTS
X1=  -.12931170  +  1.56527436 I
X2=  -.12931170  -  1.56527436 I
0
COMPLEX ROOTS
X1=  .24711885  +  .31736213 I
```

```
            X2=       .24711885   -        .31736213 I
      0                                               REAL ROOTS
                        X1=      1.48630516                         X2 =     -1.6639038
      0                                   COMPLEX ROOTS
                        X1=      -.15831949    +          1.57297605 I
            X2=       -.15831949   -        1.57297605 I
      0                                   COMPLEX ROOTS
                        X1=       .24510375    +           .31307827 I
            X2=       .24510375   -         .31307827 I
      0                                               REAL ROOTS
                        X1=      1.52260233                         X2 =     -1.6391475
      0                                   COMPLEX ROOTS
                        X1=      -.18683114    +          1.58098863 I
            X2=       -.18683114   -        1.58098863 I
      0                                   COMPLEX ROOTS
                        X1=       .24311172    +           .30905788 I
            X2=       .24311172   -         .30905788 I
      0                                               REAL ROOTS
                        X1=      1.55774315                         X2 =     -1.6142071
      0                                   COMPLEX ROOTS
                        X1=      -.21487969    +          1.58929986 I
            X2=       -.21487969   -        1.58929986 I
      0                                   COMPLEX ROOTS
                        X1=       .24115006    +           .30527528 I
            X2=       .24115006   -         .30527528 I
      0                                               REAL ROOTS
                        X1=      1.59179161                         X2 =     -1.5891058
      0                                   COMPLEX ROOTS
                        X1=      -.24249293    +          1.59790245 I
            X2=       -.24249293   -        1.59790245 I
      0                                   COMPLEX ROOTS
                        X1=       .23922380    +           .30170792 I
            X2=       .23922380   -         .30170792 I
      0                                               REAL ROOTS
                        X1=      1.62480949                         X2 =     -1.5638699
      0                                   COMPLEX ROOTS
                        X1=      -.26969358    +          1.60679260 I
            X2=       -.26969358   -        1.60679260 I
      0                                   COMPLEX ROOTS
                        X1=       .23733627    +           .29833588 I
            X2=       .23733627   -         .29833588 I
      0                                               REAL ROOTS
                        X1=      1.65685537                         X2 =     -1.5385288
      0                                   COMPLEX ROOTS
                        X1=      -.29649953    +          1.61596923 I
            X2=       -.29649953   -        1.61596923 I
      0                                               REAL ROOTS
                        X1=      1.68798459                         X2 =     -1.5131154
      0                                   COMPLEX ROOTS
                        X1=       .23548956    +           .29514166 I
            X2=       .23548956   -         .29514166 I
      0                                   COMPLEX ROOTS
                        X1=      -.32292412    +          1.62543283 I
            X2=       -.32292412   -        1.62543283 I
      0                                               REAL ROOTS
                        X1=      1.71824892                         X2 =     -1.4876656
      0                                   COMPLEX ROOTS
```

```
                    X1=    .23368480    +      .29210959 I
       X2=   .23368480   -     .29210959 I
   0                      COMPLEX ROOTS
                    X1=   -.34897642    +     1.63518479 I
       X2=  -.34897642   -    1.63518479 I
   0                                     REAL ROOTS
                    X1=   1.74769677                X2 =   -1.46221830
   0                      COMPLEX ROOTS
                    X1=    .23192248    +      .28922620 I
       X2=   .23192248   -     .28922620 I
   0                      COMPLEX ROOTS
                    X1=   -.37466170    +     1.64522666 I
       X2=  -.37466170   -    1.64522666 I
   0                                     REAL ROOTS
                    X1=   1.77637324                X2 =   -1.43681489
   0                      COMPLEX ROOTS
                    X1=    .23020251    +      .28647895 I
       X2=   .23020251   -     .28647895 I
   0                      COMPLEX ROOTS
                    X1=   -.39998169    +     1.65555964 I
       X2=  -.39998169   -    1.65555964 I
   0                                     REAL ROOTS
                    X1=   1.80432026                X2 =   -1.41149908
   0                      COMPLEX ROOTS
                    X1=    .22852453    +      .28385722 I
       X2=   .22852453   -     .28385722 I
   0                      COMPLEX ROOTS
                    X1=   -.42493511    +     1.66618396 I
       X2=  -.42493511   -    1.66618396 I
   0                                     REAL ROOTS
                    X1=   1.83157681                X2 =   -1.38631627
   0                      COMPLEX ROOTS
                    X1=    .22688782    +      .28135112 I
       X2=   .22688782   -     .28135112 I
   0                      COMPLEX ROOTS
                    X1=   -.44951808    +     1.67709848 I
       X2=  -.44951808   -    1.67709848 I
   0                                     REAL ROOTS
                    X1=   1.85817909                X2 =   -1.36131288
   0                      COMPLEX ROOTS
                    X1=    .22529173    +      .27895259 I
       X2=   .22529173   -     .27895259 I
   0                      COMPLEX ROOTS
                    X1=   -.47372473    +     1.68830043 I
       X2=  -.47372473   -    1.68830043 I
   0                                     REAL ROOTS
                    X1=   1.88416080                X2 =   -1.33653569
   0                      COMPLEX ROOTS
                    X1=    .22373466    +      .27665201 I
       X2=   .22373466   -     .27665201 I
   0                      COMPLEX ROOTS
                    X1=   -.49754721    +     1.69978493 I
       X2=  -.49754721   -    1.69978493 I
   0                                     REAL ROOTS
                    X1=   1.90955316                X2 =   -1.31203100
   0                      COMPLEX ROOTS
                    X1=    .22221611    +      .27444401 I
```

```
X2=  .22221611  -  .27444401 I
0  COMPLEX ROOTS
X1=  -.52097720  +  1.71154521 I
X2=  -.52097720  -  1.71154521 I
0  REAL ROOTS
X1=  1.93438536  X2 =  -1.28784402
0  COMPLEX ROOTS
X1=  .22073477  +  .27232185 I
X2=  .22073477  -  .27232185 I
0  COMPLEX ROOTS
X1=  -.54400545  +  1.72357216 I
X2=  -.54400545  -  1.72357216 I
0  REAL ROOTS
X1=  1.95868441  X2 =  -1.26401784
0  COMPLEX ROOTS
X1=  .21928946  +  .27027961 I
X2=  .21928946  -  .27027961 I
0  COMPLEX ROOTS
X1=  -.56662276  +  1.73585454 I
X2=  -.56662276  -  1.73585454 I
0  REAL ROOTS
X1=  1.98247564  X2 =  -1.24059294
0  COMPLEX ROOTS
X1=  .21787901  +  .26831203 I
X2=  .21787901  -  .26831203 I
0  COMPLEX ROOTS
X1=  -.58882037  +  1.74837913 I
X2=  -.58882037  -  1.74837913 I
0  REAL ROOTS
X1=  2.00578251  X2 =  -1.21760643
0  COMPLEX ROOTS
X1=  .21650226  +  .26641439 I
X2=  .21650226  -  .26641439 I
0  COMPLEX ROOTS
X1=  -.61059032  +  1.76113077 I
X2=  -.61059032  -  1.76113077 I
0  REAL ROOTS
X1=  2.02862707  X2 =  -1.19509146
0  COMPLEX ROOTS
X1=  .21515806  +  .26458219 I
X2=  .21515806  -  .26458219 I
0  COMPLEX ROOTS
X1=  -.63192587  +  1.77409281 I
X2=  -.63192587  -  1.77409281 I
```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185-2