NASA TECHNICAL NOTE

NASA TN D-3734

NASA TN D-3734

GPO PRICE $ ____
CFSTI PRICE(S) $ 2.50

Hard copy (HC) ____
Microfiche (MF) .75
ff 653 July 65

N67 11957 (ACCESSION NUMBER)	(THRU)
59 (PAGES)	1 (CODE)
(NASA CR OR TMX OR AD NUMBER)	12 (CATEGORY)

FACILITY FORM 602

PSEUDO-SONIC VELOCITY AND PSEUDO-MACH NUMBER CONCEPTS IN TWO-PHASE FLOWS

by Joseph W. Bursik

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1966

NASA TN D-3734

PSEUDO-SONIC VELOCITY AND PSEUDO-MACH NUMBER CONCEPTS IN TWO-PHASE FLOWS

By Joseph W. Bursik

George C. Marshall Space Flight Center
Huntsville, Ala.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $2.50

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

LIST OF TABLES

DEFINITION OF SYMBOLS

Symbol	Definition
t	temperature (° F)
T	temperature (° R)
p_T	pressure $\left(\frac{\mathrm{lb_f}}{\mathrm{ft^2}}\right)$; (denoted as a function of temperature)
p'_T	$\frac{dp_T}{dT}$; also used simply as p'
p''_T	$\frac{d^2p_T}{dT^2}$; also used simply as p''
Ψ	specific Gibbs' function; ie, $\Psi = h - Ts$ $\left(\frac{\mathrm{Btu}}{\mathrm{lbm}}\right)$
h	specific enthalpy $\left(\frac{\mathrm{Btu}}{\mathrm{lbm}}\right)$
s	specific entropy $\left(\frac{\mathrm{Btu}}{\mathrm{lbm\ deg\ R}}\right)$
Ψ_T	two-phase specific Gibbs' function; (denoted as a function of temperature)
Ψ'_T	$\frac{d\Psi_T}{dT}$; also used simply as Ψ'
Ψ''_T	$\frac{d^2\Psi}{dT^2}$; also used simply as Ψ''
v	specific volume ($\mathrm{ft^3}$/lbm)
u	specific internal energy (Btu/lbm)

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
h_{f_T}	values of above functions on saturation liquid curve; all are denoted as temperature functions
s_{f_T}	
v_{f_T}	
u_{f_T}	
h_{fg_T}	change of above functions after complete isothermal phase change; all are denoted as temperature functions
s_{fg_T}	
v_{fg_T}	
u_{fg_T}	
c_v	single-phase constant volume specific heat capacity (Btu/lbm °F)
c_λ	single-phase polytropic specific heat capacity
σ_v	two-phase constant volume specific heat capacity
α	single-phase sonic velocity (ft/sec)
a	two-phase pseudo-sonic velocity (ft/sec)
M	pseudo-Mach number ($\overline{V}/a$)
$\overline{V}$	fluid flow velocity (ft/sec)

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
q	specific heat transfer (Btu/lbm)
A	cross-sectional area of duct (ft^2)
G	mass flow rate per unit cross-sectional area (lbm/sec-ft^2)
W	total mass in system
x	quality of two-phase mixture
L	flow system length (ft)
T_3	absolute triple point temperature (°R) (491.69 for H_2O)
r	ratio of absolute temperature in °R to triple point temperature
$f(T)$	an arbitrary function of temperature formed from the partial integration of the internal energy function derivative
$f'(T)$	$\frac{df(T)}{dT}$
J	conversion factor 778.26 $\frac{ft/lb_f}{Btu}$
g_c	conversion factor 32.17 $\frac{lbm}{lb_f}\ \frac{ft}{sec^2}$
$\overline{a}$	an empirical coefficient (489.286 Btu/lbm for H_2O)
b	an empirical coefficient (489.281 Btu/lbm for H_2O)
λ	polytropic exponent (1.252 for H_2O)
τ	time (seconds)

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
Subscript	
f	denotes liquid phase
g	denotes gas or vapor phase
t	denotes throat conditions
1	denotes a reference state; arbitrarily taken as the duct inlet state in numerical examples
Superscript	
$*$	denotes choke conditions for Fanno or Rayleigh flow

ACKNOWLEDGEMENT

This report was written as a partial result of the activities of the author while a summer faculty fellow at the Marshall Space Flight Center, Huntsville, Alabama. The fellowship program was under the joint auspices of NASA and ASEE acting through the administrative agency of Auburn University.

PSEUDO-SONIC VELOCITY AND PSEUDO-MACH NUMBER CONCEPTS IN TWO-PHASE FLOWS

SUMMARY

In ideal gas theory, the introduction of the Mach number into the analysis of steady, one-dimensional duct flow results in associating choking conditions with a Mach number of unity for several well known flows, such as Rayleigh and Fanno pipe flows and isentropic converging-diverging duct flow. Replacing the ideal gas by a liquid-vapor, two phase mixture in the above named flows complicates the analysis even when equilibrium between the phases is assumed. However, it is possible to introduce a pseudo-sonic velocity concept with a correspondingly defined pseudo-Mach number concept such that these three two-phase flows exhibit choking conditions at a pseudo-Mach number of unity.

It is further shown that a point-by-point, non-tabular analysis of the above two-phase flows is possible when the equations for the Gibbs' function and the vapor pressure curve are known. In the case of the liquid-vapor region of H_2O an empirical equation for the Gibbs' function is derived, while the vapor pressure curve is developed theoretically by the imposition of a transformation property normally encountered in ideal gas theory.

INTRODUCTION

The use of the pseudo-sonic velocity will be delayed until a few thermodynamic relations can first be reviewed and specialized for two-phase usage prior to their incorporation into the definition of the new variable.

For a closed system, the differential of the Gibbs' function defined by

$$\Psi = h - Ts \tag{1}$$

is given by

$$d\Psi = \frac{1}{J} vdp - sdT \ . \tag{2}$$

In this form the pressure and the temperature are the independent variables except when the pressure depends only upon the temperature, as is the case in equilibrium, two-phase thermodynamics. For this case if the pressure dependence upon the temperature is expressed by

$$p = p_T \tag{3}$$

then

$$dp = p'_T dT \tag{4}$$

where p'_T is the slope of the vapor pressure curve at the temperature T. If dp from this equation is substituted into equation (2) the temperature derivative of the Gibbs' function becomes

$$\Psi'_T = \frac{1}{J} vp'_T - s \ . \tag{5}$$

That the right side is indeed a temperature function becomes apparent when the two-phase equations

$$v = v_{f_T} + xv_{fg_T} \tag{6}$$

$$s = s_{f_T} + xv_{fg_T} \tag{7}$$

and the Clapeyron equation

$$p'_T = \frac{Js_{fg_T}}{v_{fg_T}} \tag{8}$$

are substituted into the right side of equation (5) to give

$$\Psi'_T = \frac{1}{J} v_{f_T} p'_T - s_{f_T} \quad . \tag{9}$$

When equation (5) is used to define the entropy in a two-phase region as a function of T and v, the entropy differential becomes

$$ds = \left(\frac{1}{J} vp''_T - \Psi''_T\right) dT + \frac{1}{J} p'_T dv \quad . \tag{10}$$

Hence for a reversible process

$$dq = T\left(\frac{1}{J} vp'' - \Psi''\right) dT + \frac{1}{J} Tp'dv \quad . \tag{11}$$

From this equation it is seen that the coefficient of dT represents a theoretical two-phase specific heat capacity at constant volume. It is designated by the symbol σ_v to distinguish it from the usual single-phase c_v symbol. In general, the chart of two-phase properties representing the T-s plane shows that a curve of constant volume possesses discontinuous slopes at the saturation envelope. This means that the saturation value of σ_v is not the same as the saturation value of c_v. Thus, for any point inside the two-phase region

$$\sigma_v = T\left(\frac{1}{J} vp'' - \Psi''\right) \quad . \tag{12}$$

As a final digression before defining the pseudo-sonic velocity it is recalled that the velocity of propagation of a small disturbance in a non-condensable single phase is given by

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{13}$$

where α is the acoustic or sonic velocity. In terms of the specific volume equation (13) becomes

$$\alpha^2 = -g_c v^2 \left(\frac{\partial p}{\partial v}\right)_s \quad . \tag{14}$$

By direct analogy, the right side of equation (14) will be defined as the pseudo-sonic velocity parameter for two-phase analysis and the symbol "a" will be used for its description ; thus

$$-g_c v^2 \left(\frac{\partial p}{\partial v}\right)_s \equiv a^2 . \tag{15}$$

To evaluate the partial derivative in this relation, equations (4) and (12) are used to transform equation (10) into

$$ds = \frac{\sigma_v}{Tp'} dp + \frac{1}{J} p'dv \quad . \tag{16}$$

From this the desired partial derivative is readily obtained and substituted into equation (15) to give for the square of the pseudo-acoustic velocity

$$a^2 = \frac{g_c T v^2 p'^2}{J\sigma_v} \quad . \tag{17}$$

These relationships are discussed further in Appendix C.

For positive σ_v the right side is likewise positive; hence the pseudo-sonic velocity parameter is real when σ_v is positive, which is the usual case. This may be readily checked by observing the slopes of constant volume curves in the T-s plane. As examples, these slopes are positive for the liquid-vapor regions of water, nitrogen, para and normal hydrogen, etc.

From equations (17) and (12) it is apparent that "a" depends explicitly upon T and v, and possesses additional implicit temperature dependence through the first and second derivatives of the pressure and the second derivative of the Gibbs' function. Hence, knowledge of these functions is required before equation (17) can be used for calculating "a" at any state. This point will be returned to later.

ISENTROPIC TWO-PHASE FLOW THROUGH A HORIZONTAL CONVERGING-DIVERGING DUCT

One of many common equations used to describe the isentropic flow of an ideal gas through a horizontal duct is given by

$$\frac{dA}{A} = (M^2 - 1)\frac{d\overline{V}}{\overline{V}} . \tag{18}$$

In this equation, A is the cross sectional area of the duct, M is the Mach number and $\overline{V}$ is the gas velocity. With the use of the pseudo-sonic velocity an identical equation will be derived for two-phase flow in which a pseudo-Mach number will replace its counterpart of ideal gas flow.

For isentropic two-phase flow of liquid-vapor phases through a horizontal converging diverging duct, the continuity equation, general energy flow equation and the combined first and second laws are, respectively:

$$\frac{dA}{A} = \frac{dv}{v} - \frac{d\overline{V}}{\overline{V}} \tag{19}$$

$$dh = - \frac{\overline{V}d\overline{V}}{Jg_c} \tag{20}$$

$$dh = \frac{vdp}{J} . \tag{21}$$

Equations (20) and (21) can be combined to give

$$dp = - \frac{\overline{V}d\overline{V}}{g_c v} \tag{22}$$

Substitution of equation (22) and ds = 0 into equation (16) yields

$$dv = \frac{J\sigma_v}{g_c T p'^2} \frac{\overline{V}}{v} d\overline{V} . \tag{23}$$

When this is used to eliminate dv in equation (19) the result is

$$\frac{dA}{A} = \left(\frac{J\sigma_v}{g_c T p'^2} \cdot \frac{\bar{V}^2}{v^2} - 1 \right) \frac{d\bar{V}}{\bar{V}} \quad . \tag{24}$$

This result is then transformed by use of pseudo-sonic relation of equation (17) into

$$\frac{dA}{A} = \left[\left(\frac{\bar{V}}{a} \right)^2 - 1 \right] \frac{d\bar{V}}{\bar{V}} \quad . \tag{25}$$

If a pseudo-Mach number is now defined as

$$M = \frac{\bar{V}}{a} \tag{26}$$

equation (25) becomes

$$\frac{dA}{A} = (M^2 - 1) \frac{d\bar{V}}{\bar{V}}$$

and this is identical in form to the ideal gas counterpart.

Equations (27), (23), (22), (21) and (4) yield information of the way various quantities increase or decrease depending on whether the flow is for $M > 1$ or $M < 1$ and further, whether the flow is for a converging section or for a diverging section. This is summarized in Table I in which the sign dT corresponds to an assumed positive p'_T.

The analogy with ideal gas flow is carried further in recalling that in this flow an improper back pressure on a supersonic nozzle can cause a standing normal shock in the diverging section such that the Mach number ahead of the shock is greater than one and less than one behind the shock. Further the flow is accelerating with falling pressure in the diverging section ahead of the shock and diffusing with rising pressure after the shock. Table I suggests a similar situation in two-phase flow since both $M > 1$ and $M < 1$ flows, for example, apparently can be accommodated in the diverging section, the former characterized by falling pressure and the latter by increasing pressure.

TABLE I. EFFECTS OF AREA CHANGE ON VELOCITY, SPECIFIC VOLUME, PRESSURE, ENTHALPY AND TEMPERATURE IN TWO-PHASE FLOW ABOVE AND BELOW PSEUDO-SONIC VELOCITY

	dA	$d\bar{V}$	dv	dp	dh	dT
M < 1	-	+	+	-	-	-
M < 1	+	-	-	+	+	+
M > 1	-	-	-	+	+	+
M > 1	+	+	+	-	-	-

At the throat $M_t = 1$; therefore,

$$\bar{V}_t^2 = a_t^2 \tag{28}$$

where the t subscript refers to throat conditions. If the mass flow rate per unit cross-sectional area is denoted by G the continuity equation then becomes

$$\bar{V}_t^2 = G_t^2 v_t^2 \quad . \tag{29}$$

Substituting a_t^2 from equation (17) for $\bar{V}_t^2$ in equation (29) gives for the throat specific volume after substituting (12) for σ_v:

$$v_t p_t'' = J\Psi_t'' + \frac{g_c P_t'^2}{G_t^2} \tag{30}$$

If p_t'' is not zero v_t can be eliminated from equations (30) and (5) to give a throat temperature equation in the form

$$\frac{1}{p_t''}\left[\Psi_t'' + \frac{g_c}{J}\left(\frac{p_t'}{G_t}\right)^2\right] = \frac{s + \Psi_t'}{p_t'} \qquad (31)$$

No throat subscript is used for the entropy since isentropic conditions are assumed. In this form it is again apparent that knowledge of the Gibbs' function and the vapor pressure curve are needed to calculate the critical throat temperature. Further discussions of these functions will be postponed at this point.

TWO-PHASE FANNO FLOW

For this adiabatic pipe flow with friction an analogy between two-phase equilibrium flow and ideal gas flow will again be developed; in particular, it will be shown that choking occurs in the two-phase flow at pseudo-Mach one.

In this flow the continuity equation is

$$\bar{V} = Gv \qquad (32)$$

with G now a constant. The energy equation for steady flow is

$$h + \frac{\bar{V}^2}{2Jg_c} = h_0 \qquad (33)$$

where h_0 is a constant. These two equations are then equivalent to

$$dh = -\frac{G^2}{Jg_c}\, vdv \ . \qquad (34)$$

When equations (34) and (4) are used in the general thermodynamic relation

$$ds = \frac{1}{T}\, dh - \frac{1}{JT}\, vdp \qquad (35)$$

to eliminate dh and dp, the result is

$$ds = -\frac{vp'}{JT}\, dT - \frac{G^2}{Jg_c}\,\frac{v}{T}\, dv \ . \tag{36}$$

An expression for the differential temperature change is obtained by equating the right sides of equations (10) and (36) and rearranging the terms to give

$$dT = -\left(\frac{\dfrac{G^2 v}{g_c} + Tp'}{J\sigma_v + vp'}\right) dv \ . \tag{37}$$

Elimination of dT in equation (10) by use of equation (37) together with the use of equations (12), (17) and (26) finally leads to

$$ds = \frac{1}{J}\,\frac{vp'^2}{J\sigma_v + vp'}\,[1 - M^2]\, dv \ . \tag{38}$$

Since Fanno flow requires a positive ds, equation (38) initiates a sign analysis of flows for which M is greater than one and for which M is less than one. Table II is a summary of the sign analysis obtained from equations (38), (37), (34), (32) and (4) where in the use of (4), it is again assumed that p'_T is positive. Once again the analogy with ideal gas Fanno flow is apparent.

TABLE II. EFFECTS OF FRICTIONAL LOSSES ON VELOCITY, SPECIFIC VOLUME, PRESSURE, ENTHALPY AND TEMPERATURE IN TWO-PHASE FLOW ABOVE AND BELOW PSEUDO-SONIC VELOCITY

	dA	$d\bar{V}$	dv	dp	dh	dT
$M < 1$	0	+	+	-	-	-
$M > 1$	0	-	-	+	+	+

Next an integral will be obtained for equation (37) that gives the temperature, specific volume relation at any point in the flow. This is accomplished indirectly as follows. Equations (32) and (33) are combined to eliminate $\overline{V}$, giving

$$h + \frac{G^2}{2\,Jg_c} v^2 = h_0 \tag{39}$$

and h is then eliminated by the use of (1), resulting in

$$\Psi_T + Ts + \frac{G^2}{2Jg_c} v^2 = h_0 \ . \tag{40}$$

Replacing s in equation (40) with equation (5) then reveals v as a quadratic function of T which can be solved explicitly for v, giving at any point in the flow

$$v = \frac{g_c\,Tp'}{G^2}\left[-1 + \sqrt{1 - \frac{2G^2J\,(\Psi - T\Psi' - h_0)}{g_c\,T^2p'^2}}\right] . \tag{41}$$

The right side of this equation is a function of temperature through the dependence of p and Ψ on the temperature. The choice of sign preceding the radical corresponds to the assumption of positive p'_T.

The entropy and enthalpy dependence on the temperature are then obtained by eliminating v in equations (5) and (39) by use of equation (41).

Since M = 1 at the choke point (denoted by the superscript*) equations (32), (26), (17) and (12) can be combined to give

$$v^* = \frac{1}{G^2p''_{T^*}}\left(g_c p'^2_{\ T^*} + JG^2\,\Psi''_{T^*}\right) . \tag{42}$$

Then with equation (41) also applied to the choke point, the specific volume may be eliminated between equations (41) and (42) leaving one equation to be solved for the choke temperature T* and, once again, this solution requires knowledge of p_T and Ψ_T. As before, the discussion of these functions will be deferred.

TWO-PHASE RAYLEIGH FLOW

In this horizontal flow of a two-phase mixture only heating effects are considered. Fluid friction is neglected and equilibrium is again assumed to exist between the phases at all points. Just as in the case of ideal flow, this reversible two-phase flow exhibits choking effects at M = 1 and analogous characteristics in the two-phase regimes above and below M = 1.

In this flow a force balance on an element and the continuity equation given by equation (32) combine to give for the two-phase Rayleigh flow

$$p + \frac{G^2}{g_c} v = c \ . \qquad (43)$$

With the use of equation (4) equation (43) becomes

$$p'_T dT + \frac{G^2}{g_c} dv = 0 \qquad (44)$$

and this together with (12) substituted into (10) gives

$$ds = \left[1 - \frac{J\sigma_v G^2}{g_c T p'^2} \right] \frac{p'}{J} dv \ . \qquad (45)$$

Equations (32), (17) and (26) then transform equation (45) into

$$ds = (1 - M^2) \frac{p'}{J} dv \ . \qquad (46)$$

Equations (46), (44), (32), and (4) are summarized with respect to sign behavior in Table III. Once again positive behavior of p'_T is assumed and the table represents heating of the fluid. For cooling, the signs in the table are reversed.

TABLE III. EFFECTS OF HEATING ON ENTROPY, SPECIFIC VOLUME, VELOCITY, TEMPERATURE, AND PRESSURE IN TWO-PHASE FLOW ABOVE AND BELOW PSEUDO-SONIC VELOCITY

	dA	ds	dv	$d\bar{V}$	dT	dp
M < 1	0	+	+	+	-	-
M > 1	0	+	-	-	+	+

For choked flow the asterisk superscript will again be used to represent properties of the fluid at the state M = 1. At this state equation (26) reduces to

$$\bar{V}^{*2} = a^{*2} \quad . \tag{47}$$

When equations (32) and (17) are substituted respectively for the left and right sides of equation (47) the result, after elimination of σ_v by equation (12), can be arranged to give

$$p'^2_{T^*} + [p_{T^*} - c]\, p''_{T^*} + \frac{JG^2}{g_c}\, \Psi''_{T^*} = 0 \quad . \tag{48}$$

For known functions p_T and Ψ_T equation (48) can be solved for the choke temperature and this then gives the choke pressure and Gibbs' function, p^* and Ψ^*. The choke pressure is then used in equation (43) to give the choke value of the specific volume. The choke entropy follows from equation (5) and the enthalpy from equation (1). Once the choke pressure and specific volume are known an alternate method for calculating the entropy and enthalpy is to calculate the quality from the pressure-specific volume values and then use the quality to calculate the desired entropy and enthalpy. In any event, equation (48) pinpoints the choke point when p and Ψ are known functions of temperature. In the absence of known equations representing these functions the choke point must then be obtained graphically by calculating the Rayleigh states with the aid of the usual tabulations of two-phase data.

NUMERICAL EXAMPLE OF RAYLEIGH FLOW

For the Rayleigh flow a numerical example is presented to illustrate the use of equation (48) together with the pseudo-sonic velocity and pseudo-Mach number concepts when empirical formulations of p and Ψ are employed. In the example the entrance conditions are saturated water at 400°F. From these conditions an iterative procedure is used for the calculation of the Rayleigh states in which successively reduced pressures are introduced into equation (43) yielding successive values of the specific volume and, hence, a series of qualities from which enthalpies and entropies are calculated. These results are summarized in Table IV and shown graphically in Figure 1, where all of the states except the choke point are calculated in this manner. The choke state calculation is initiated with the use of equation (48) and the pseudo-Mach numbers are obtained from (12), (17) and (26). For these calculations empirical expressions for p and ψ are taken from Bursik and Onuf [1, 2]. From Reference 1 the empirical expression for the Gibbs' function for the liquid vapor region of H_2O is obtained as

$$\Psi = \bar{a}\,(r - 1) - b\,r \ln r \tag{49}$$

where

$$\bar{a} = 489.286 \tag{50}$$

$$b = 489.281 \tag{51}$$

and

$$r = \frac{T}{T_3} \quad . \tag{52}$$

For computational purposes $\bar{a}$ and b will be taken as equal, i.e.,

$$\bar{a} \cong b \cong 489.3 \quad . \tag{53}$$

TABLE IV. SUMMARY OF TWO-PHASE RAYLEIGH FLOW EXAMPLE

Temp	Pressure	Specific Vol.	Quality	Enthalpy	Entropy	Velocity	Pseudo-Acoustic Velocity	Pseudo-Mach Number
t(°F)	p lb/in^2	v ft^3/lbm	x	h $\frac{\text{Btu}}{\text{lbm}}$	s $\frac{\text{Btu}}{\text{lbm deg}}$	$\bar{V}$ $\frac{\text{ft}}{\text{sec}}$	a $\frac{\text{ft}}{\text{sec}}$	M
400	247.31	0.01864	0	374.97	0.5664	10	43.52	0.230
380	195.77	0.8484	0.358	655.8	0.9014	455.1	848.9	0.536
370	173.37	1.2088	0.457	732.8	0.9987	648.5	983.9	0.659
342	121.2	2.049	0.553	798.4	1.0978	1099	1099	1.000
320	89.66	2.5562	0.518	753.8	1.0583	1371	1058	1.297
300	67.013	2.9208	0.450	679.1	0.9760	1567	972.0	1.612
280	49.203	3.2074	0.370	591.2	0.8721	1721	863.4	1.993

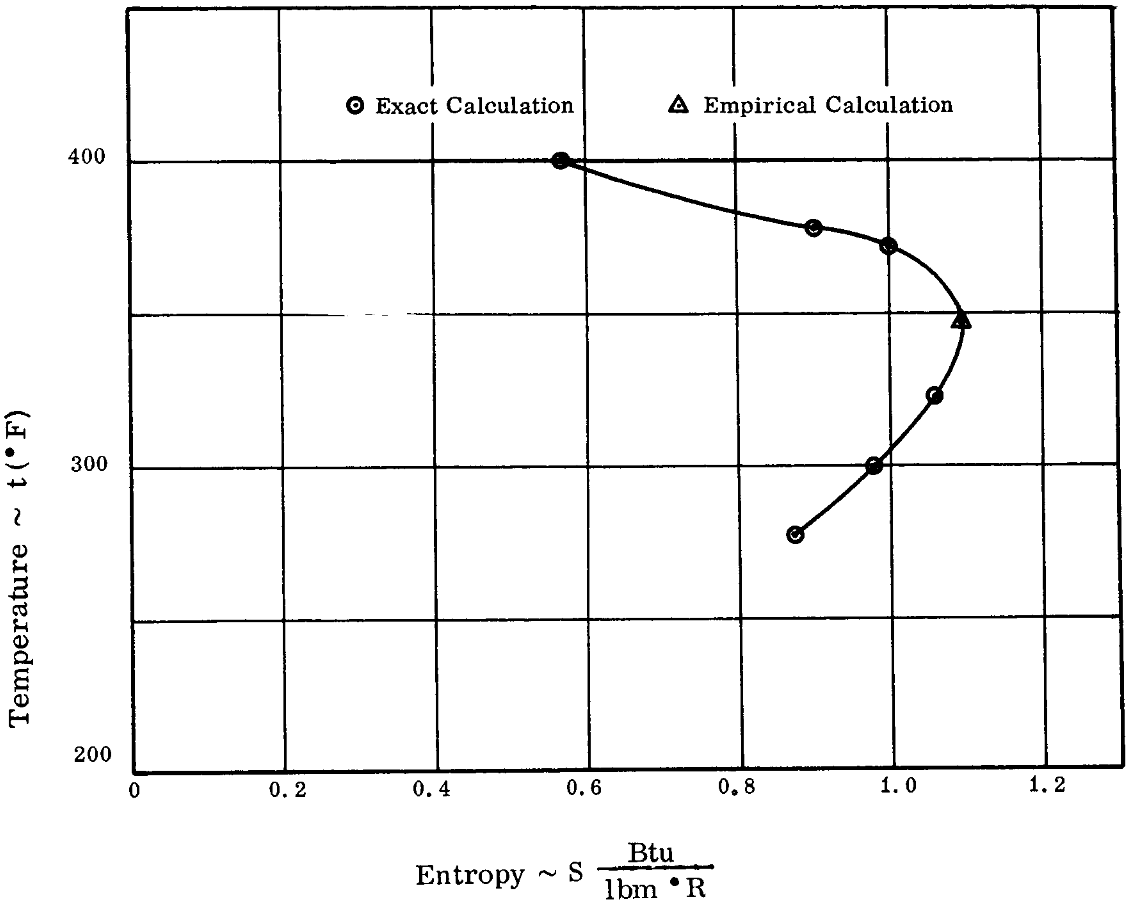

FIGURE 1. DEPENDENCE OF SPECIFIC ENTROPY ON TEMPERATURE IN RAYLEIGH TWO-PHASE FLOW

From equation (49) it follows that

$$\Psi'_T = \frac{1}{T_3} \left[\bar{a} - b\,(1 + \ln r)\right] \tag{54}$$

or

$$\Psi'_T \approx -\frac{b}{T_3}\ln r \tag{55}$$

and

$$\Psi''_T = -\frac{b}{T_3^2}\,\frac{1}{r}\ . \tag{56}$$

It is interesting to note at this point that while $\bar{a}$ and b are very nearly equal, theoretically they must differ. This can be demonstrated by evaluating both equations (54) and (55) at the triple point where r = 1. Thus from equation (54)

$$\Psi'_{T_3} = \frac{\bar{a} - b}{T_3} \tag{57}$$

and from equation (55)

$$\Psi'_{T_3} = 0\ . \tag{58}$$

If equation (5) is now applied to the saturated liquid state of H_2O at the triple point where the Keenan and Keyes [3] steam tables have their entropy datum, the result is that

$$\Psi'_{T_3} = \frac{1}{J}\, v_{f_{T_3}}\ p'_{T_3}\ . \tag{59}$$

By the Clapeyron equation p'_{T_3} is greater than zero; hence Ψ'_{T_3} must likewise be greater than zero. Equation (57) gives a very small positive value for Ψ'_{T_3} which is neglected in equation (58).

Additional discussion of the Gibbs' function is in Appendix A.

The summary of Bursik [2], is included in Appendix B. It shows that if some polytropic family of two-phase curves, defined initially by an unspecified constant λ as

$$pv^{\lambda} = K \tag{60}$$

where K is a constant that depends upon λ, transforms into the T-s plane as a family of parallel curves, then the constant λ must be related to a dimensionless grouping of thermodynamic variables by the following expression:

$$\frac{p'^2_T}{p_T \, p''_T} = \lambda \quad . \tag{61}$$

The degree to which equation (61) represents two-phase data for the liquid-vapor region of H_2O is discussed further in the Appendix. For present purposes equation (61) is taken as a sufficiently accurate description of the H_2O liquid-vapor region when λ is given the value

$$\lambda = 1.252 \quad (H_2O) \quad . \tag{62}$$

This same polytropic theory yields an additional result that will be useful in later numerical examples, namely

$$\frac{p}{p'} = \frac{\lambda - 1}{\lambda} \, T_3 \, (r - r_1) + \frac{p_1 v_{fg1}}{J s_{fg1}} \quad . \tag{63}$$

In this result from the polytropic theory the subscript 1 refers to an arbitrarily selected reference temperature and r_1, p_1, v_{fg1} and s_{fg1} are all then computed at this temperature; hence equation (63) requires p/p' to be a linear function of the temperature or the temperature ratio by virtue of (52). Finally, the polytropic theory yields an equation for the vapor pressure curve in the form

$$T = T_1 + \frac{\lambda}{\lambda - 1} \, \frac{p_1 v_{fg1}}{J s_{fg1}} \left[\left(\frac{p}{p_1} \right)^{\frac{\lambda - 1}{\lambda}} - 1 \right] \quad . \tag{64}$$

When p'' and Ψ'' are eliminated in equation (48) by the substitution of equations (56) and (61) the result is an empirical choke equation

$$\frac{c}{p_{T*}} = \lambda + 1 - \frac{JG^2}{g_c} \, \frac{b}{T_3^2} \, \frac{\lambda}{r_{T*} \, p'^2_{T*}} \quad . \tag{65}$$

At this point equation (65) is an implicit function of the choke temperature. It can be transformed readily into an explicit function of the choke temperature by combining it with equations (63) and (64). However, instead of pursuing this path, an alternative cut-and-dry solution was used by employing steam tables to solve

$$\frac{c}{p_{T*}} = \lambda + 1 - \frac{\lambda G^2 b}{J g_c \, T_3^{\,2}} \; \frac{1}{r_{T*}} \left[\frac{v_{fg_{T*}}}{s_{fg_{T*}}} \right]^2 \tag{66}$$

where p'_{T*} in equation (65) has been replaced by the Clapeyron relation, equation (8).

For the conditions of the example

$$G = 536.5 \text{ lbm/sec ft}^2 \tag{67}$$

$$c = 35780 \text{ lb}_f/\text{ft}^2 \quad . \tag{68}$$

Therefore equation (66) can be reduced to

$$\frac{35780}{p_{T*}} = 2.252 - \frac{14.321}{T*} \left(\frac{v_{fg_{T*}}}{s_{fg_{T*}}} \right)^2 \quad . \tag{69}$$

As already indicated in Table IV the solution of equation (69) yields a choke temperature of 340°F which corresponds to a saturation pressure of 121.2 psia as the choke pressure.

In the same table the pseudo-Mach numbers were calculated from

$$M^2 = \frac{G^2}{g_c} \left[\frac{v}{\lambda p_T} + \frac{b}{J T_3 T} \left(\frac{v_{fg_T}}{s_{fg_T}} \right)^2 \right] \quad . \tag{70}$$

after first calculating v from equation (43). Equation (70) results when equations (61) and (56) are combined with equations (12), (17), (26), (8), (61), (56) and (31). In conclusion, attention is drawn to the tabulations of the pseudo-sonic velocities: a very small value for this quantity at the saturated liquid state is indicated; increasing values then accompany falling temperatures until a maximum pseudo-sonic velocity is reached; and finally, this quantity decreases as the temperature continues to decrease.

Some of the data of Table IV are plotted in Figures 1, 2, and 3.

NUMERICAL EXAMPLE OF NOZZLE FLOW

In this example of reversible two-phase flow through a converging-diverging, horizontal duct, the inlet conditions where a negligible approach velocity is assumed are denoted by the subscript 1 and the throat conditions by a subscript t. A saturated mixture of H_2O is assumed to enter at 400° F and a specific entropy of 1.0468 Btu/lbm deg R. This corresponds to an inlet quality of 0.5.

For this example the velocity of the flow at any section will be calculated in two ways: The first is the result of the substitution of steam table data into the conservation of energy statement for steady flow, namely

$$\bar{V}^2 = 2Jg_c\ (h_1 - h) \quad . \tag{71}$$

The second way will transform the right side of this equation into a function of the temperature ratio by incorporating equation (49) and the constancy of the entropy into it. With the entropy set equal to s_1 in equation (1) and the temperature in the same equation eliminated in terms of r by using equation (52), the enthalpies in equation (71) are replaced by the expressions obtained from (1) applied to the inlet state and to the variable state without the subscript. Finally, equation (49) is substituted into the transformed expression for the velocity to give

$$\bar{V}^2 = 2bJg_c\ [\,(\mu + 1 - \ln r_1)\ r_1 - (\mu + 1 - \ln r)\ r] \tag{72}$$

where μ is defined as the constant

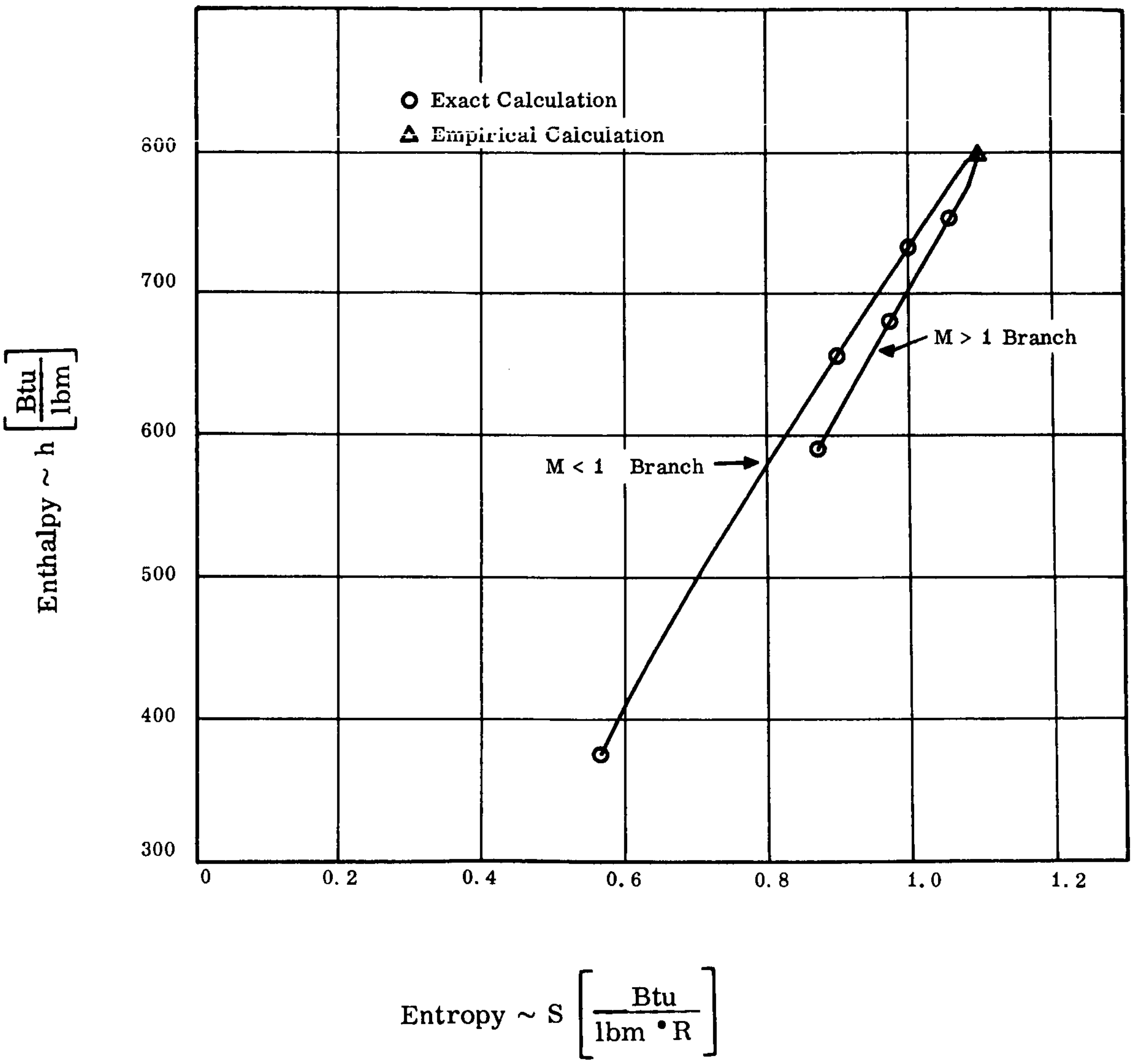

FIGURE 2. SPECIFIC ENTHALPY AS A FUNCTION OF SPECIFIC ENTROPY IN RAYLEIGH TWO-PHASE FLOW

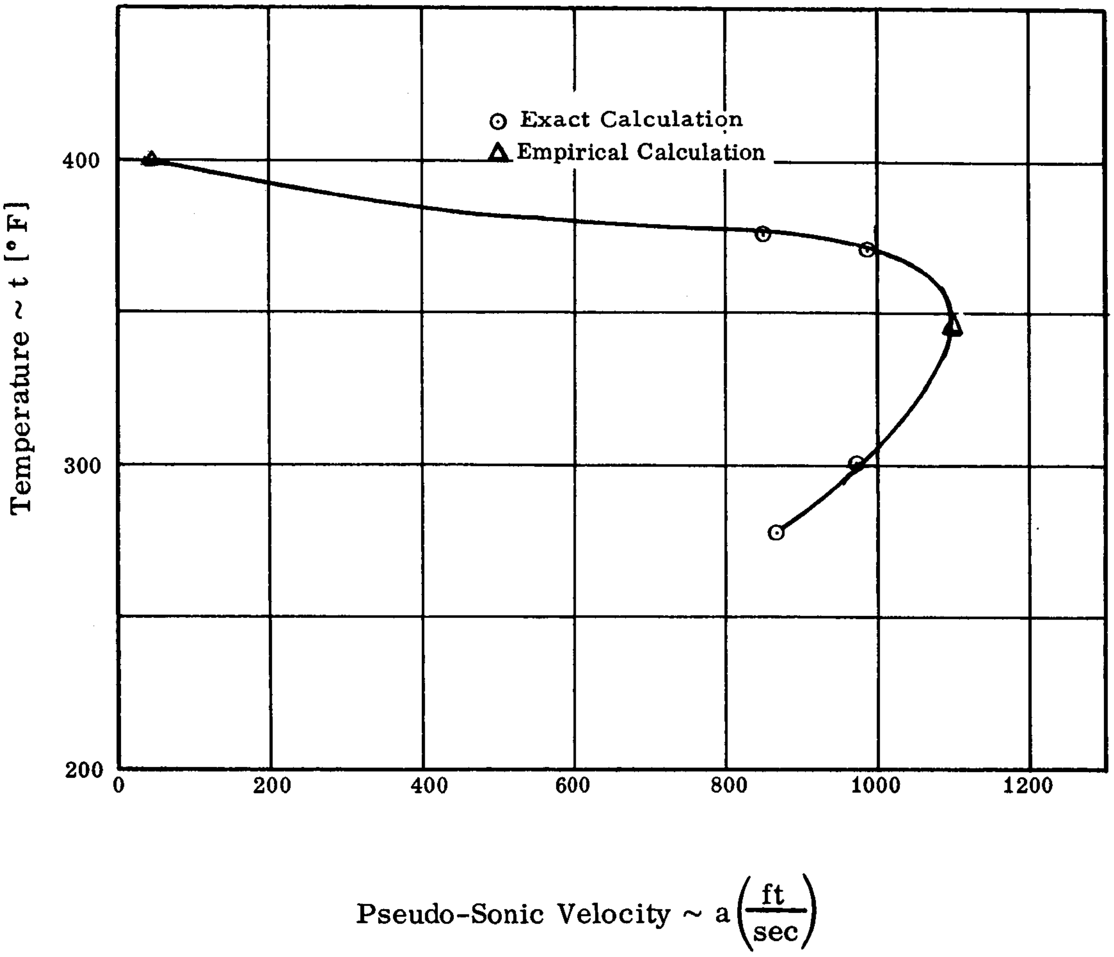

FIGURE 3. DEPENDENCE OF THE PSEUDO-SONIC VELOCITY ON TEMPERATURE IN RAYLEIGH TWO-PHASE FLOW

$$\mu = \frac{T_3 \, s_1}{b} \quad . \tag{73}$$

For the assumptions of the example

$$\mu = \frac{491.69 \,\big|\, 1.048}{489.3 \,\big|\,} = 1.052 \tag{74}$$

$$r_1 = \frac{859.69}{491.69} = 1.748 \tag{75}$$

$$\ln r_1 = 0.5585 \quad . \tag{76}$$

With these, equation (72) reduces to

$$V^2 = 2\ (489.3)\ Jg_c\ [2.611 - (2.052 - \ln r)\ r\,] \quad . \tag{77}$$

Equation (71) and (77) are compared in the first two columns of Table V, except for the critical point of the throat where equation (72) is equated to the square of the pseudo-sonic velocity to locate the critical point. This will become apparent when the pseudo-sonic velocity relation is specialized to the present example.

Substitution of equation (12) into equation (17) gives

$$a^2 = \frac{g_c v^2 p'^2}{vp'' - J\Psi''} \quad . \tag{78}$$

With the entropy held constant at its entrance value s_1, equation (5) is solved for v and the result introduced into equation (78) to give

$$a^2 = \frac{Jg_c\ (s_1 + \Psi')^2}{(s_1 + \Psi')\ \frac{p''}{p'} - \Psi''} \quad . \tag{79}$$

Then p'', Ψ' and Ψ'' are eliminated from equation (79) by equation (55), (56) and (61) to give

TABLE V. SUMMARY OF ISENTROPIC TWO-PHASE NOZZLE FLOW EXAMPLE

t° F	$\bar{V}\ \frac{ft}{sec}$ Eq. (72)	$\bar{V}\ \frac{ft}{sec}$ Eq. (78)	x	$h\ \frac{Btu}{lbm}$	$v\ \frac{ft^3}{lbm}$	$G\ \frac{lbm}{ft^2\ sec}$	A/A_t
400	0	0					
376.18	769						
376		777	0.503	775.9	1.240	626.8	1.079
370		874	0.5037	772.7	1.331	657.1	1.029
360		1015	0.5048	767.4	1.502	675.7	1.001
357.1	1056		0.5051	765.7	1.561	676.4	1.000
351.6	1117						
350		1136	0.5060	762.2	1.700	668.5	1.012
340		1257	0.5066	756.4	1.928	652.1	1.037
328		1387	0.5075	750.0	2.252	615.8	1.098
327.01	1396						
302.43	1638						
278		1858	0.5097	719.0	4.549	408.4	1.656
277.85	1858						
253.26	2064						
252		2075	0.5099	702.0	6.829	303.8	2.226
228.68	2261						
228		2269	0.5093	685.2	0.233	221.7	3.051

$$a^2 = \frac{bJg_c\left(\frac{T_3s_1}{b} - \ln r\right)^2}{\left(\frac{T_3s_1}{b} - \ln r\right)\frac{T_3}{\lambda}\frac{p'}{p} + \frac{1}{r}} \quad . \tag{80}$$

Finally, this equation is transformed into an explicit function of the temperature ratio by eliminating p/p' from it with equation (63). This then results in

$$a^2 = \frac{bJg_c\,(\mu - \ln r)^2}{\frac{1}{\lambda - 1}\,\frac{\mu - \ln r}{r - r_1 + \phi} + \frac{1}{r}} \quad . \tag{81}$$

In this relation the constant μ has already been defined by equation (73) and the additional symbol ϕ represents another constant given by

$$\phi = \frac{\lambda}{\lambda - 1}\,\frac{p_1v_{fg1}}{JT_3s_{fg1}} \quad . \tag{82}$$

A pseudo-Mach number is then formed from equations (72) and (81), giving

$$M^2 = \frac{2\left[(1 + \mu - \ln r_1)\,r_1 - (1 + \mu - \ln r)\,r\right]}{(\mu - \ln r)^2}\left[\frac{\mu - \ln r}{r - r_1 + \phi}\,\frac{1}{\lambda - 1} + \frac{1}{r}\right] \tag{83}$$

Substituting the assumed conditions into equation (82) evaluates the constant as

$$\phi = \frac{1.252}{0.252}\,\Bigg|\,\frac{35613}{778.26}\,\Bigg|\,\frac{1.8447}{491.69}\,\Bigg|\,\frac{}{0.9608} = 0.888 \quad . \tag{84}$$

With all of the constants in equation (83) now known the pseudo-Mach number is set equal to unity for the throat condition and the equation for the throat ratio becomes

$$(1.052 - \ln r_t)^2 = 2\,[2.611 - (2.052 - \ln r_t)\, r_t] \left[3.968 \; \frac{1.052 - \ln r_t}{r_t - 0.86} + \frac{1}{r_t} \right] \tag{85}$$

which yields the solution already indicated in Table V as

$$t_t = 357.1^\circ \text{ F} \quad . \tag{86}$$

Since the inlet temperature was 400° F, this represents a temperature drop of only 43° needed to establish the throat conditions, considerably smaller than that encountered with a diatomic ideal gas expanding from the same temperature.

Some of the results bearing on the accuracy of the empirical relationships used are shown in Figures 4 and 5.

DISCUSSION

The numerical examples used in this paper were not selected for any special purpose other than to illustrate the concepts and investigate the reliability of the empiricism that was also introduced into these examples. Indeed, the empirical relations are necessary before the pseudo-sonic velocity and Mach number can be calculated. The empirical calculations viewed in the context of exact, steam-table calculations are shown graphically in Figures 1, 2, and 5. In each of these graphs the choke point calculation makes use of the empirical equation for Ψ_T and p_T while the other points do not. Apparently the choke points fit in reasonably well with the other points. In Figure 4 a series of points calculated without recourse to the empirical formulations is compared to another empirical series. Again the agreement between the two appears quite satisfactory.

In Bursik and Onuf [1, 2] it is shown that the same appropriate form of the Gibbs' function can be used for the liquid-vapor regions of Freon-12, CO_2, Hg and NH_3, whereas it fails for sulphur and the solid-vapor regions of H_2O and CO_2.

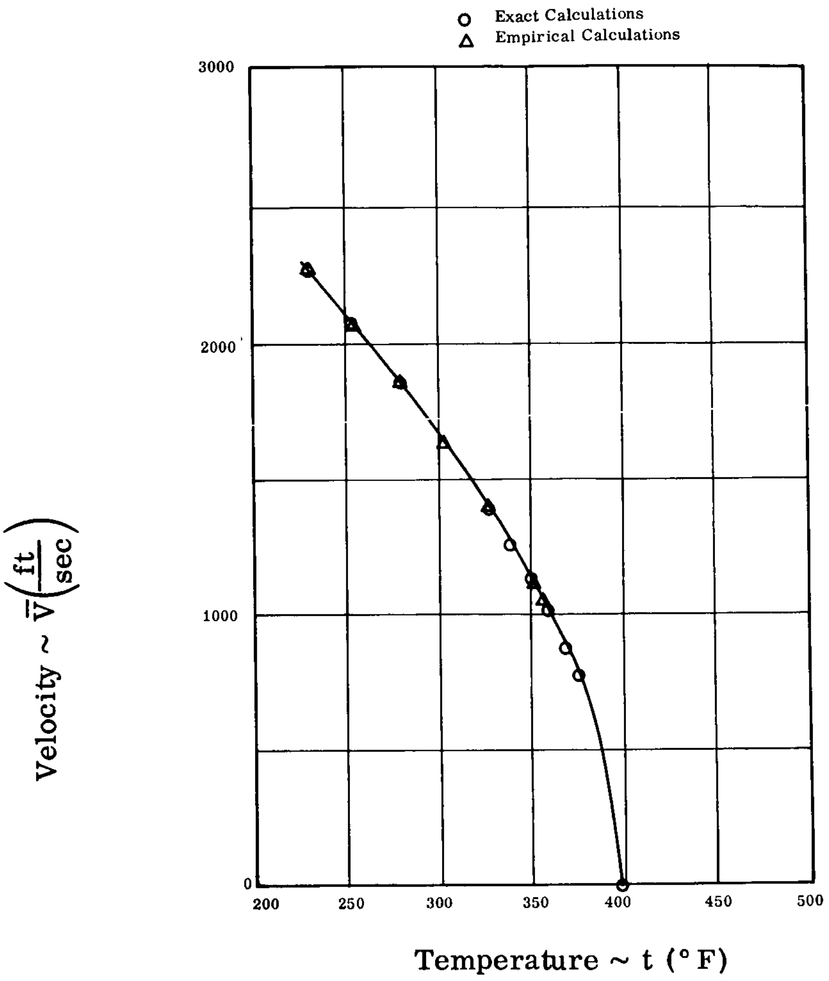

FIGURE 4. DEPENDENCE OF FLOW VELOCITY ON TEMPERATURE IN TWO-PHASE NOZZLE FLOW

In Bursik [2] the polytropic correlation is shown to give results at least as good as the H_2O correlation for the liquid-vapor regions of Freon 12, Hg, NH_3, CO_2, O_2 and sulphur, and for the solid-vapor regions of H_2O and CO_2.

The use of the empirical equations for p and Ψ permits choke point calculations to be made without the use of the quality; hence it is possible to obtain solutions outside the physical two-phase range. In the Rayleigh flow, for example, the pressure obtained after the choke temperature is calculated from equation (66) should be entered into equation (43) to calculate the specific volume at the choke state. This can then be checked against the saturated

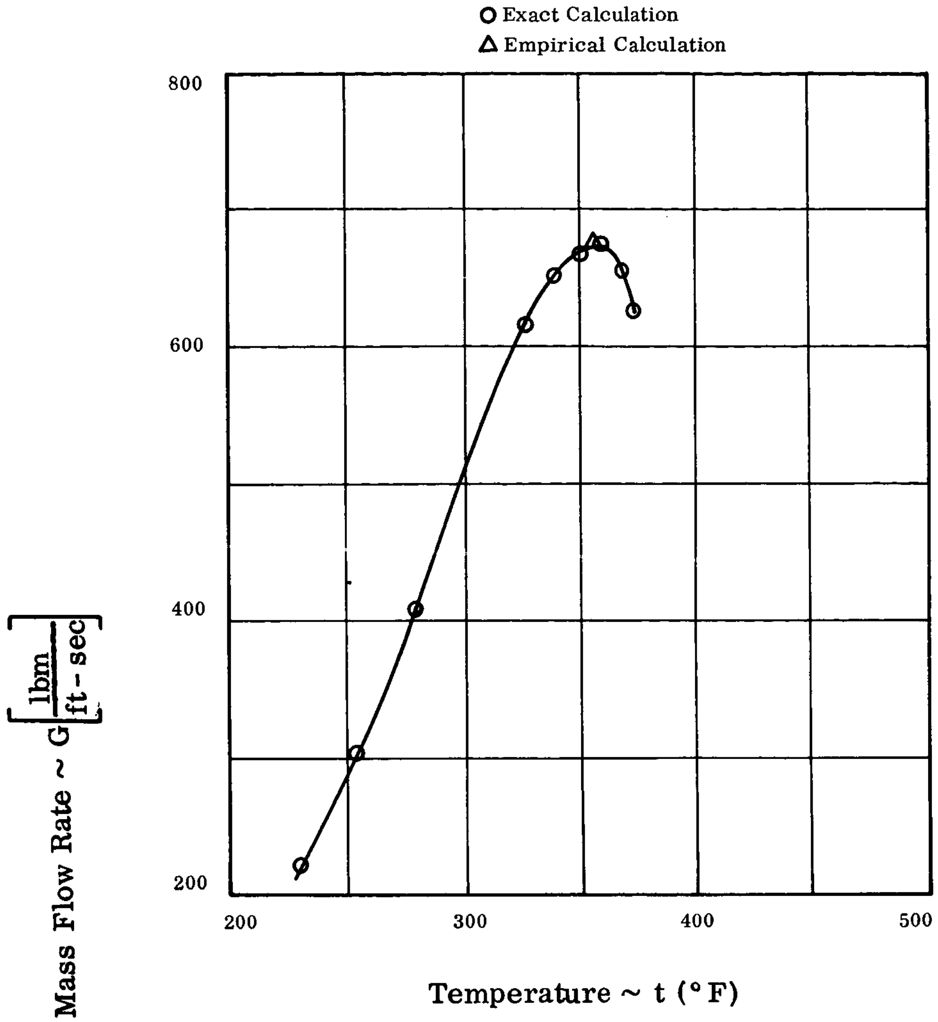

FIGURE 5. EFFECT OF TEMPERATURE ON THE MASS FLOW RATE IN TWO-PHASE NOZZLE FLOW

specific volumes corresponding to the choke temperature (to determine whether a choke solution exists within the two phase region for the assumed inlet conditions).

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, August 12, 1966.

APPENDIX A.

AN EMPIRICAL EQUATION FOR THE GIBBS' FUNCTION

In Bursik and Onuf [1] the empirical approach to an equation representing the Gibbs' function begins with the thermodynamic relation

$$\left(\frac{\partial u}{\partial v}\right)_T = \frac{1}{J}\left[T\left(\frac{\partial p}{\partial T}\right)_v - p\right] \tag{87}$$

which is immediately specialized to two-phase regions by recognizing that the pressure is a function of the temperature. This permits the partial derivative of the pressure to be replaced by p'_T and equation (87) then becomes

$$\left(\frac{\partial u}{\partial v}\right)_T = \frac{1}{J}\,[Tp' - p] \quad . \tag{88}$$

Since the right side of equation (88) is a function of temperature only the equation may be partially integrated to give

$$u = \frac{1}{J}\,(Tp' - p)\ v + f(T) \tag{89}$$

where f(T) is an arbitrary function of temperature resulting from the integration and is related to the Gibbs' function as follows. The two-phase constant volume specific heat capacity is formed from equation (89) by differentiating partially with respect to the temperature. This results in

$$\sigma_v = \frac{1}{J}\,Tp''v + f'(T) \tag{90}$$

where primes again denote differentiation with respect to the temperature. Comparison of equation (90) with equation (12) then provides the connection between f and Ψ in the form:

$$f'(T) = -T\Psi''_T \ . \tag{91}$$

In Bursik and Onuf [1] the main emphasis is on seeking correlations for the arbitrary temperature function, f (T). Then the Gibbs' function falls out as a byproduct through the integration of equation (91).

By virtue of the Clapeyron relation given by equation (8) and the two-phase thermodynamic relation

$$Ts_{fg_T} = u_{fg_T} + \frac{p_T v_{fg_T}}{J} \tag{92}$$

equation (89) becomes

$$f(T) = u - \frac{u_{fg_T}}{v_{fg_T}} v \quad . \tag{93}$$

That the right side of equation (93) is indeed a function of the temperature becomes apparent when u and v are replaced by the two-phase relations

$$u = u_{f_T} + x\, u_{fg_T} \tag{94}$$

and

$$v = v_{f_T} + x\, v_{fg_T} \quad . \tag{95}$$

These substitutions result in

$$f(T) = u_{f_T} - \frac{u_{fg_T}}{v_{fg_T}} v_{f_T} \quad . \tag{96}$$

The right side of equation (96) contains terms that are generally tabulated functions of the temperature. Hence the appropriate tables permit the graphical study of f (T). An alternate form can be used for graphical study by adding and subtracting pv_{f_T} to the right side to introduce the enthalpy function. This procedure gives

$$f(T) = h_{f_T} - \frac{h_{fg_T}}{v_{fg_T}} v_{f_T} \quad . \tag{97}$$

For the liquid-vapor region of H_2O, Table VI shows numerical values of f computed by use of equation (97) in conjunction with the Keenan and Keyes steam tables. When f is plotted as a function of the temperature, as shown in Figure 6, the points very closely approximate a straight line whose equation is

$$f(t) = 0.9951\ t - 31.848 \quad . \tag{98}$$

In this equation the substitution of temperature in Fahrenheit degrees gives f (t) in Btu/lbm. The last column in Table VI enables the comparison of the results obtained from the empirical equation (98) with those obtained from the steam table solution of equation (97).

In terms of the Rankine temperature, equation (98) becomes

$$f(T) = 0.9951\ T - \bar{a} \tag{99}$$

and in terms of the ratio defined by equation (52)

$$f(r) = br - \bar{a} \tag{100}$$

where $\bar{a}$ and b were defined by equations (51) and (52). With these, equation (91) can be transformed to one with r as the independent variable

$$\frac{d\Psi'_r}{dr} + \frac{b}{r} = 0 \tag{101}$$

which has an integral

$$\Psi'_r + b \ln r = c_1 \quad . \tag{102}$$

At the triple point r = 1, therefore, the constant is given by

$$c_1 = \Psi'_1 \tag{103}$$

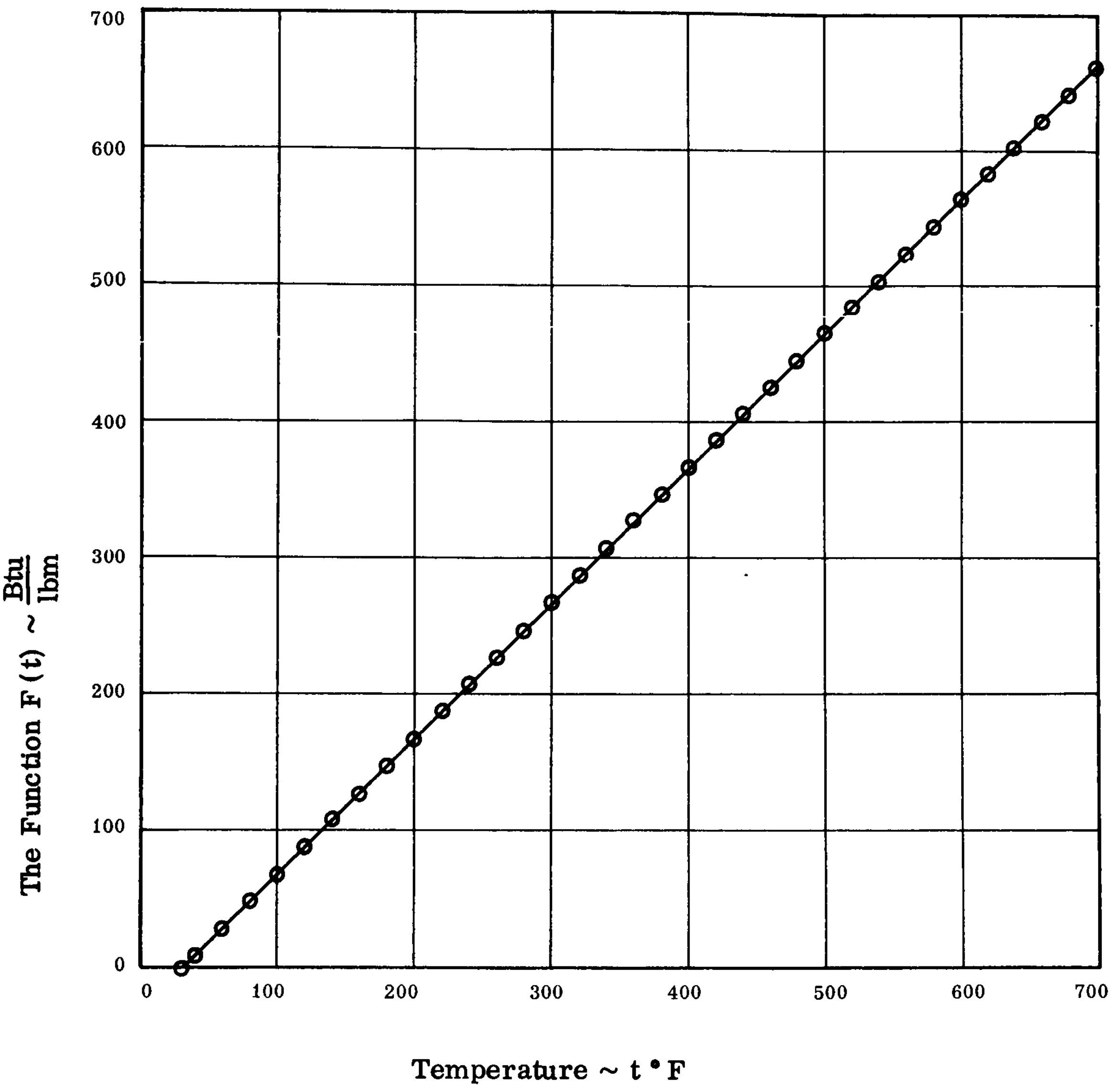

FIGURE 6. THE FUNCTION f (T) FOR H_2O IN THE LIQUID-VAPOR REGION

TABLE VI. THE FUNCTION f(T) FOR H_2O IN THE LIQUID-VAPOR REGION

t(°F)	$f = h_f - \frac{h_{fg}}{v_{fg}} v_f$	f = 0.9951 t - 31.848
32	- 0.005	- 0.005
40	8.000	7.956
60	28.01	27.86
80	48.00	47.76
100	67.70	67.66
120	87.80	87.56
140	107.7	107.5
160	127.6	127.4
180	147.6	147.3
200	167.5	167.2
220	187.4	187.1
240	207.3	207.0
260	227.2	227.0
280	247.2	247.0
300	267.1	267.0
320	287.0	287.0
340	306.9	306.9

TABLE VI. (Concluded)

t(° F)	$f = h_f - \frac{h_{fg}}{v_{fg}} v_f$	$f = 0.9951\ t - 31.848$
360	326.9	326.4
380	346.8	346.3
400	366.7	366.2
420	386.5	386.1
440	406.3	406.0
460	426.1	426.0
480	445.8	445.8
500	465.5	465.7
520	485.2	485.6
540	504.8	505.1
560	524.4	525.1
580	544.0	545.3
600	563.8	565.2
620	583.1	585.1
640	602.1	605.0
660	621.1	624.9
680	640.6	644.8
700	661.3	664.7

and equation (102) becomes

$$\frac{d\Psi_r}{dr} + b \ln r = \Psi'_1 \quad . \tag{104}$$

A second integration gives

$$\Psi_r + b(r \ln r - r) = r\Psi'_1 + c_2 \quad . \tag{105}$$

The constant c_2 is evaluated by noting that at the triple point H_2O the Gibbs' function for the liquid-vapor region is arbitrarily set equal to zero in the Keenan and Keyes steam tables by virtue of the fact that both the entropy and enthalpy of the saturated liquid are set equal to zero at the triple point temperature in these tables. Hence

$$c_2 = -b - \Psi'_1 \quad . \tag{106}$$

With this result, equation (105) becomes

$$\Psi_r = [b + \Psi'_1] \; (r - 1) - b \; r \ln r \quad . \tag{107}$$

It may then be shown that

$$b + \Psi'_1 = \bar{a} \tag{108}$$

in the following manner: h_{fg} is first replaced in equation (97) by Ts_{fg} and then the ratio s_{fg}/v_{fg} is in turn replaced by p'/J by the Clapeyron relation. If the resulting equation is written for the triple point temperature where $h_{f_{T_3}}$ is zero for the Keenan and Keyes tables and then combined with equation (59), the result is

$$\Psi'_{T_3} = -f(T_3) \tag{109}$$

which becomes equation (108) in view of equations (99), (100) and (52). Thus equation (107) is reduced to equation (49) previously used in the numerical examples.

In Bursik and Onuf [1, 2] it is shown that this same approach holds approximately for a number of substances when their liquid-vapor regions are considered.

APPENDIX B.

A POLYTROPIC VAPOR PRESSURE CURVE THEORY

In Bursik [2] a one-assumption theory is developed for the pressure-temperature equation in any two-phase region by constructing an analog with ideal gases. For ideal gases any family of curves defined by assigning a value to the polytropic exponent λ in the equation

$$pv^{\lambda} = c_1 \tag{110}$$

has the property of transforming into the T-s plane such that each member of the family has the same slope at the same temperature as is apparent from the slope expression for any member of the λ family

$$\left.\frac{dT}{ds}\right|_{\lambda} = \frac{T}{c_{\lambda}} \tag{111}$$

where c_{λ} is the well known polytropic specific heat capacity. For a given gas c_{λ} depends on the heat capacity ratio assumed to be constant and on the identity of the family of curves, namely the assigned value of λ. For the constant pressure family of curves c_{λ} becomes c_p and λ becomes 0 while the constant volume family is characterized by $\lambda \rightarrow \infty$ with c_{λ} becoming c_v. In this case as well as in the former, it is apparent that the substitution of c_v into equation (111) requires the constant volume family of curves to have the property that all are parallel in the T-s plane at the same temperature. The same is true for the constant pressure family and for any other family defined by assigning a fixed value to λ.

In the case of two-phase regions the situation is not as simple with respect to the transformation into the T-s plane of polytropic families of curves, again defined by equation (110). Indeed the T-s plane for H_2O as found in the Keenan and Keyes [3] steam tables clearly shows that the curves of the constant volume family are not parallel to one another at the same temperature. The curves of the constant pressure family are parallel in this plane but this is trivial since the pressure is a function of the temperature which in turn is one of the coordinate axes.

If this trivial case is ignored and it is recognized that the polytropic family of constant volume curves has lost the parallelism property under transformation to the T-s plane, the question is whether there exists one family of polytropic curves associated with some one value of λ that retains the parallelism property in the T-s plane.

To analyze the question the assumption is made that for any two-phase region there is one family of polytropic curves that retains the parallelism property under transformation to the T-s plane. When this assumption is imposed on two-phase thermodynamics, the polytropic family is further identified as the family of curves given by

$$vp' = \text{constant} \tag{112}$$

and an expression for the pressure as an explicit function of the temperature is developed as is a procedure for calculating λ. Various other predictions result, all of which can then be tested against tabulated two-phase data.

The development of the polytropic two-phase theory is initiated by rewriting equation (10) as

$$ds = \frac{1}{J}\, d(vp') - \Psi'' dT \tag{113}$$

in which the terms have been regrouped by using the fact that

$$d(vp') = vp'' dT + p'\, dv \quad . \tag{114}$$

After v is eliminated in equation (113) by introducing equation (110) the slope of any polytropic curve in the T-s plane is given by

$$\left.\frac{dT}{ds}\right|_{\lambda} = \frac{1}{\dfrac{c_1^{1/\lambda}}{J}\, \dfrac{d}{dT}\, \dfrac{p'}{p^{1/\lambda}} - \Psi''} \quad . \tag{115}$$

In this equation one assigned value of λ identifies a family of curves and the constant c_1 then identifies a particular member of the family. In general, assigning a fixed number λ does not result in only one value of the slope expression at a given temperature even though p and Ψ are temperature functions only; since, for these conditions, the number c_1 varies from curve to curve of the family. The assumption of the polytropic theory is that from the infinity of numbers that can be assigned to λ, there is one, namely

$$\lambda = \lambda_0 \quad , \tag{116}$$

for which the entire family of curves

$$Pv^{\lambda_0} = c_1 \tag{117}$$

transforms into the T-s plane such that the slope of each member of the family is the same at the same temperature. From equation (115) this can be true only if

$$\frac{d}{dT}\left(\frac{p'}{p^{1/\lambda_0}}\right) = 0 \quad . \tag{118}$$

Then the slope of any member of the family of curves will be given by

$$\left.\frac{dT}{ds}\right|_{\lambda_0} = - \frac{1}{\Psi''} \quad . \tag{119}$$

Since the Gibbs' function depends only on the temperature, equation (119) reflects the basic assumption of the theory in that all members of the family $\lambda = \lambda_0$ now obviously have the same slope at the same temperature. Equally obvious, the polytropic family $\lambda = \lambda_0$ must be simultaneously identified as the $vp' =$ constant family. At this point, the numerical value of λ_0 has not been identified. For simplicity the subscript on λ will be dropped with the understanding that only the polytropic family of curves $\lambda = \lambda_0$ is being discussed. With this convention equation (118) on the one hand integrates into

$$p' = kp^{1/\lambda} \tag{120}$$

and on the other hand differentiates into the following non-dimensional form

$$\frac{p'^2}{pp''} = \lambda \tag{121}$$

where both equations hold for all temperatures in the two-phase region considered; that is, equation (120) represents an integral of equation (121). If p_1 and p'_1 correspond to an arbitrary temperature reference in the two-phase region, say T_1, then the constant k in equation (120) can be written as

$$k = \frac{p_1'}{p_1^{1/\lambda}} \quad . \tag{122}$$

It cannot be evaluated until λ is known. Integration of equation (120) and evaluation of the new constant of integration at the arbitrarily selected T_1 then gives

$$T = T_1 + \frac{1}{k} \frac{\lambda}{\lambda - 1} \left(p^{\frac{\lambda - 1}{\lambda}} - p_1^{\frac{\lambda - 1}{\lambda}} \right) \tag{123}$$

and this reduces to equation (64) when the Clapeyron equation (8) is used to eliminate p_1' in equation (122) and k is then eliminated in equation (123). Hence the temperature and pressure are explicitly related by either equation (123) or (64) and all that remains is to evaluate λ. This evaluation can be performed with the use of equation (63); therefore this equation will be derived next.

From equations (120) and (122) the ratio p/p' is given by

$$\frac{p}{p'} = \frac{p_1}{p_1'} \left(\frac{p}{p_1} \right)^{\frac{\lambda - 1}{\lambda}} \quad . \tag{124}$$

The exponential term is then replaced by equation (64) and p_1' by equation (8), giving

$$\frac{p}{p'} = \frac{\lambda - 1}{\lambda} (T - T_1) + \frac{p_1 v_{fg1}}{J s_{fg1}} \tag{125}$$

which can be reduced to equation (63) by replacing T and T_1 in terms of r and r_1 from equation (52). The reference state corresponding to the subscript 1 is still arbitrary. Replacement of p' in equation (125) by equation (8) yields the form amenable to graphical analysis as

$$\frac{p v_{fg}}{s_{fg}} = J \frac{\lambda - 1}{\lambda} (T - T_1) + \frac{p_1 v_{fg1}}{s_{fg1}} \quad . \tag{126}$$

All of the quantities in the ratio on the left side of the equations are tabulated as a function of T; therefore a plot of the ratio $\frac{p v_{fg}}{s_{fg}}$ as a function of the temperature is predicted as a straight line by equation (126). If the prediction holds, the polytropic exponent can be computed after the slope of the straight line is determined.

A second method of determining λ is obtained from equation (126) by substituting for pv_{fg} from the thermodynamic relation

$$Ts_{fg} = u_{fg} + \frac{1}{J} pv_{fg} \quad . \tag{127}$$

When this is done and the result is rearranged, another linear form is obtained as

$$\frac{u_{fg}}{s_{fg}} = \frac{1}{\lambda} T + \left(1 - \frac{1}{\lambda}\right) T_1 - \frac{p_1 v_{fg1}}{Js_{fg1}} \tag{128}$$

from which λ can be obtained graphically. A final graphical determination of λ can also be made from a log-log plot of equation (120) with p' replaced by equation (8).

Figure 7 shows the raw steam table data plotted in the forms $\frac{u_{fg}}{s_{fg}}$ and $\frac{pv_{fg}}{Js_{fg}}$ as functions of temperature. The encircled points are the raw data points and are shown in relation to the best apparent straight lines drawn through them. The data for the plot appear in Table VII. When the method of least squares was applied to both sets of data, each slope yielded the same value, namely 1.252. In Bursik [2] a further analysis of these data indicates that both sets are curves with inflection points, and an argument is then presented showing that the curves theoretically cannot have these inflection points. This of course cannot be offered in support of the polytropic theory; but on the other hand the data cannot by itself be used to disprove it because of the residual data error associated with the experimental inflection points.

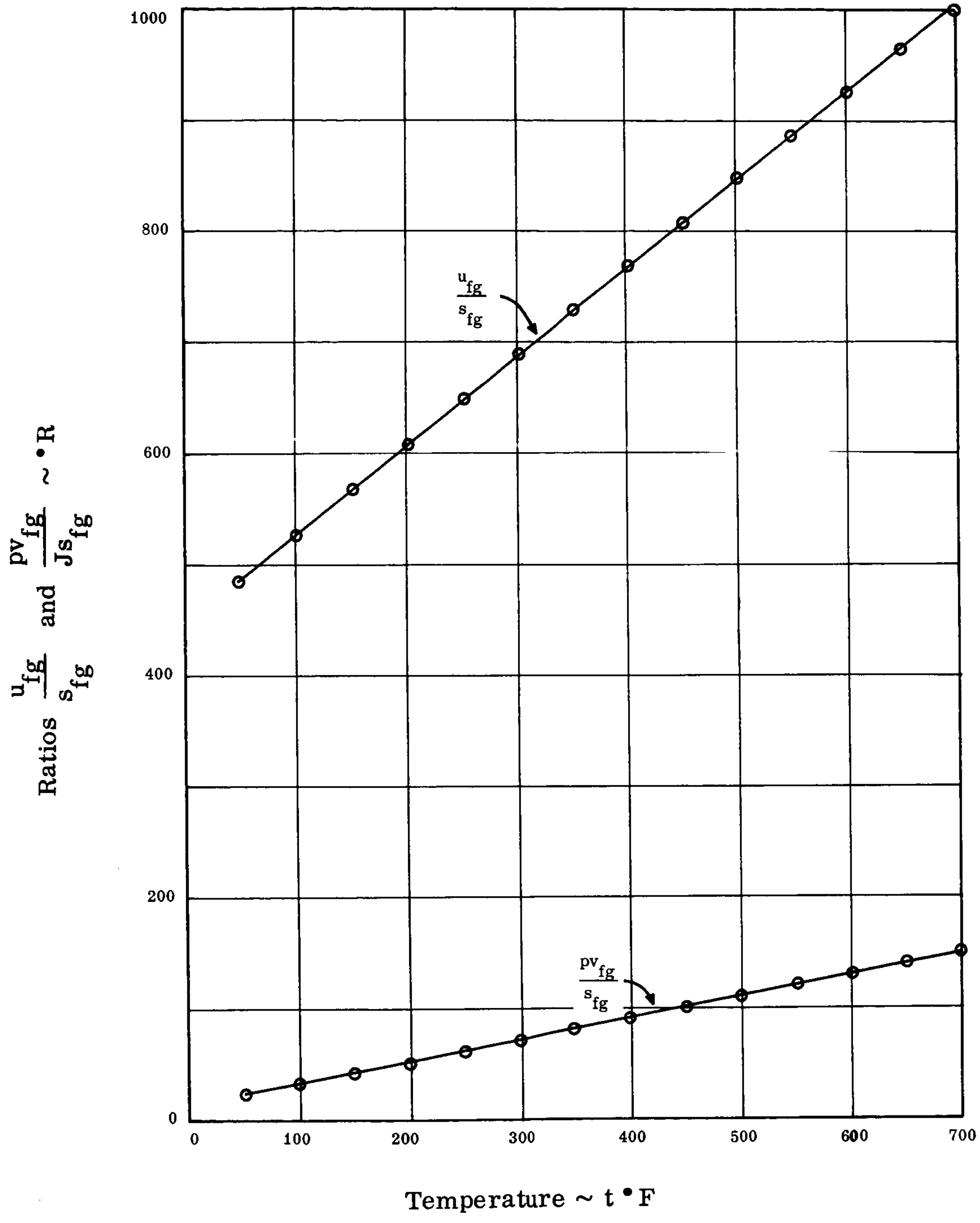

FIGURE 7. THE RATIOS $\frac{u_{fg}}{s_{fg}}$ AND $\frac{pv_{fg}}{Js_{fg}}$ FOR H_2O IN THE LIQUID-VAPOR REGION

TABLE VII. THE RATIOS $\frac{u_{fg}}{s_{fg}}$ AND $\frac{pv_{fg}}{Js_{fg}}$ FOR H_2O IN THE LIQUID-VAPOR REGION

t (°F)	$\frac{pv_{fg}}{Js_{fg}}$ (°R)	$\frac{u_{fg}}{s_{fg}}$ (°R)
50	26.85	482.9
100	33.20	526.5
150	40.38	569.3
200	48.37	611.3
250	57.18	652.5
300	66.75	692.9
350	77.00	732.7
400	87.86	771.9
450	99.19	810.6
500	110.8	849.0
550	122.6	887.2
600	134.1	925.6
650	144.6	965.3
700	151.2	1009

APPENDIX C.

THE EFFECT OF INTER-PHASE HEAT TRANSFER

In single-phase descriptions an infinitesimal disturbance is assumed to propagate through the medium in such a way that condensations and rarefactions take place so rapidly that there is no opportunity for heat to flow from any one element of the fluid to its neighbors. This gives rise to the isentropic pressure-density relation in the right side of equation (13). If the same argument is used in the case of two-phase flow then an immediate time-scale contradiction arises. The constant entropy condition again reflects the thermal isolation of any element from neighboring elements but this condition simultaneously requires the flow of heat between the two phases within the element--hence, the terminology, "pseudo-sonic velocity."

That the isentropic transformation of a saturated two-phase system involves a flow of heat between the phases is illustrated with the use of the model in Figure 8.

In the model it is convenient to think of the closed, saturated, two-phase system as being made up of three closed subsystems. In the initial state shown on the left, these are indicated by "g" for the vapor phase of mass m_g and "f" for the liquid phase which is further subdivided into a liquid system of mass dm and another mass m_f. For the infinitesimally displaced system shown on the right it is assumed that the mass dm of liquid has been transformed to vapor such that now two "g" subsystems and only one "f" subsystem appear. Only first-order terms are considered for the properties in the displaced state. The length dimensions are shown as volumes since they are proportional to one another. An energy description for all of the subsystems in the two states is obtained simply by changing the symbol v for u wherever v appears in the model. Since W is the total mass of the two-phase system the quality is given by

$$x = \frac{m_g}{W} \tag{129}$$

and

$$1 - x = \frac{m_f + dm}{W} \approx \frac{m_f}{W} \quad . \tag{130}$$

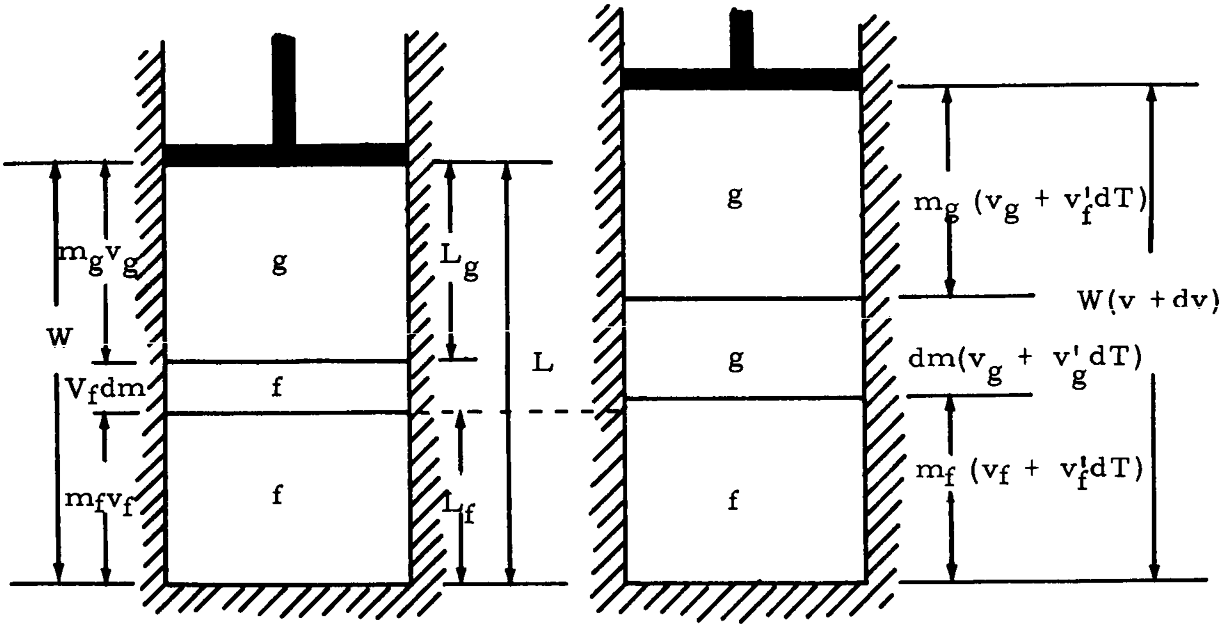

FIGURE 8. MODEL FOR AN ISENTROPIC TRANSFORMATION INVOLVING INTER-PHASE HEAT TRANSFER

To within first order quantities the reversible first law expressions for each of the three closed subsystems become

$$dQ_g = m_g (u_g' + \frac{1}{J} pv_g') \ dT \tag{131}$$

$$dQ_{fg} = (u_{fg} + \frac{1}{J} pv_{fg}) \ dm \tag{132}$$

$$dQ_f = m_f (u_f' + \frac{1}{J} pv_f') \ dT \tag{133}$$

where dQ_g is the heat transferred to the closed vapor system of mass m_g, dQ_{fg} is the heat transferred to the closed system of mass dm that undergoes phase change and dQ_f pertains to the closed liquid system of mass m_f.

If dm is replaced by dm_g then the use of equations (129) and (130) enables the set of equations (131) through (135) to be summed into

$$dQ_g + dq_{fg} + dQ_f = W \left[\left\{ (1 - x)\, u_f{}' + x\, u_g{}' \right\} dT + u_{fg}\, dx + \frac{1}{J}\, p\, dv \right] \qquad (134)$$

The right side of equation (135) is recognized as the heat transfer to the composite two-phase system and is denoted by dQ; that is,

$$dQ = dQ_g + dQ_{fg} + dQ_f \qquad (135)$$

The terms appearing on the right side of equation (135) are then split into external and internal parts since any subsystem may receive heat from the surroundings external to the composite two-phase system as well as from an adjacent subsystem within the composite two-phase system. Thus

$$dQ_g = dQ_{ge} + dQ_{gi} \qquad (136)$$

$$dQ_{fg} = dQ_{fge} + dQ_{fgi} \qquad (137)$$

$$dQ_f = dQ_{fe} + dQ_{fi} \qquad (138)$$

In this manner the formulation approaches that of Prigogine [4]. Here the additional subscripts e and i refer respectively to the external and internal parts.

If the composite system is assumed to be insulated, then each of the three closed subsystems is likewise insulated from the same external surroundings; that is, each of the dQ terms associated with the subscript e is zero and for this case equations (136) through (138) reduce to

$$dQ_g = dQ_{gi} \tag{139}$$

$$dQ_{fg} = dQ_{fgi} \tag{140}$$

$$dQ_f = dQ_{fi} \tag{141}$$

and the right side of equation (134) is zero for this constant entropy process of the composite two-phase system. Substitution of equations (139), (140) and (141) into equation (134) results in

$$dQ_{gi} + dQ_{fgi} + dQ_{fi} = 0 \tag{142}$$

While equation (142) requires the sum of the internal heat transfers to be zero it does not necessarily require the individual terms to be zero. That these terms cannot be individually zero is easily demonstrated by assuming that they are zero and then showing resulting contradictions. For example, setting the right side of equation (140) equal to zero requires from equation (132) that the latent heat of evaporation be likewise equal to zero. Similarly, setting equation (133) equal to zero results in s_f' being zero everywhere on the saturated liquid curve and equation (131) set equal to zero would require s_g' to be zero everywhere on the saturated vapor curve. Since none of these results is true, the conclusion is that an isentropic transformation of a saturated, two-phase system is accompanied by internal heat transfer between the phases of the system. The importance of this inter-phase heat transfer will be investigated for the simple model of Figure 8 by comparing the expression for the pseudo-sonic velocity of equation (15) with an expression for the average velocity of sound to be derived.

When the piston in Figure 8 is withdrawn slightly the disturbance propagates through the vapor with a velocity denoted by a_g. The time τ_g for the disturbance to travel the entire vapor distance L_g is then

$$\tau_g = L_g/a_g \tag{143}$$

but

$$L_g/L = x\, v_g/v \tag{144}$$

therefore

$$\tau_g = \frac{L}{v} \frac{x\,v_g}{a_g} \qquad (145)$$

similarly

$$\tau_f = \frac{L}{v} \frac{(1-x)\,v_f}{a_f} \qquad (146)$$

where τ_f is, of course, the time for the transmitted part of the disturbance to travel through the liquid phase. The total time, τ, taken for the disturbance to travel through both phases is the sum of the two phase times τ_g and τ_f and this is given by

$$\tau = \frac{L}{v} \left[x \frac{v_g}{a_g} + (1-x) \frac{v_f}{a_f} \right] \; . \qquad (147)$$

If the average velocity of propagation, denoted by $\bar{a}$, is taken as the total distance traveled divided by the total time of travel, then

$$\bar{a} = \frac{v}{\dfrac{x\,v_g}{a_g} + \dfrac{(1-x)\,v_f}{a_f}} \; . \qquad (148)$$

This equation will be compared to the pseudo-sonic velocity equation given by a modified form of equation (78) in which Ψ'' is eliminated by equation (56); p'' by (61) and p' by equation (8). This results in the following expression for the pseudo-sonic velocity:

$$a^2 = \frac{g_c}{\dfrac{1}{\lambda\,pv} + \dfrac{b}{JTT_3}\left(\dfrac{v_{fg}}{v\,s_{fg}}\right)^2} \; . \qquad (149)$$

The comparison of these two equations will be done at a fixed temperature of 400° in order to bring out the effect of the quality. At this temperature equation (149) reduces to the computational form

$$a^2 = \frac{32.17\ (10^6)}{\frac{22.428}{v} + \frac{5.4284}{v^2}} \cdot \tag{150}$$

Before equation (148) can be reduced to a computational form, the pair of numbers a_g and a_f in that equation need to be determined. It is assumed that these are obtained from equation (150) by respectively replacing v first by v_g and then by v_f. This, in effect, forces equations (148) and (150) to intersect at $x = 0$ and $x = 1$. The justification for this lies in the fact that at each of these points only one phase is present, hence there is no interphase heat transfer phenomenon.

With this assumption the computational form of equation (148) becomes

$$\bar{a} = \frac{v\ (10^3)}{0.4283 + 0.7832\ x} \tag{151}$$

and this is compared at a temperature of 400° F to equation (150) at eleven uniformly spaced values of the quality in Table VIII. Bearing in mind the empiricism employed in these computations it appears that the maximum deviation between a and $\bar{a}$ occurs at about a quality of 0.5.

As one final note, it emphasized that the $\bar{a}$ of equation (151) corresponds to an infinitesimal expansion of the original saturated vapor and liquid. If the disturbance generated at the piston of Figure 8 is a compression then, in the absence of metastable effects, the saturated liquid is displaced infinitesimally into the subcooled region and the saturated vapor into the superheat region. In this case the saturated sonic velocity corresponding to the single, subcooled phase can be about two orders of magnitude higher than the value computed at the same point from the two-phase side by use of equation (149).

TABLE VIII. COMPARISON OF THE PSEUDO-SONIC VELOCITY WITH AN AVERAGE SONIC VELOCITY AT 400° F

x	a ft/sec	$\bar{a}$ ft/sec
0	43.52	43.52
0.1	363.6	400.9
0.2	583.9	662.6
0.3	758.2	862.6
0.4	905.6	1020.0
0.5	1035	1148.0
0.6	1152	1253.0
0.7	1259	1341
0.8	1357	1417
0.9	1450	1482
1.0	1538	1538

REFERENCES

1. Bursik, J. W., Onuf, B. O.: Final Report. USARTC Grant DA TC 44 177 G, June 1963.

2. Bursik, J. W.: A Thermodynamic Study of Two-Phase Regions. Phd. Theses. Rensselaer Polytechnic Institute, June 1961.

3. Keenan, J. H. and Keyes, F. G.: Thermodynamic Properties of Steam. John Wiley and Sons, Inc., 1936.

4. Prigogine, I.: Introduction to Thermodynamics of Irreversible Processes. Interscience Publishers, 1961.

NASA-Langley, 1966 M339

> *"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."*
>
> —NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546