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ATTITUDE STABILITY OF A SPINNING PASSIVE SATELLITE IN A CIRCULAR ORBIT

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ABSTRACT

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The stability of motion of a satellite orbiting the earth is being investigated. The mathematical model consists of a rigid body with two flexible antennas. A simplified model uses two viscously damped oscillators to simulate the antennas. The motion is described by ten generalized coordinates of which three are used for the motion of the center of the rigid body, three for the angular rotations of the body and four for the displacements of the oscillators.

Almost every investigator working on this problem considers the satellite as rigid and assumes that there is no interaction between the orbital motion and the rotational stability of the body. This is implied in the assumption that the center of mass moves in a given orbit; this assumption is referred to as orbital constraints. Where the body contains moving parts, such as the oscillators, the center of mass shifts with respect to the rigid body. In this case it may be more convenient to work with the center of the rigid body.

For a body moving in a circular orbit it is convenient to introduce a frame of reference orbiting with the body. This investigation is concerned with the possibility that the motion in which the body undergoes no rotation with respect to the orbiting frame of reference is stable. To this end the direct method of Liapounov is used for the case in which the center of the rigid body is constrained to move in a circular orbit as well as for the unconstrained case. The conclusion is that the orbital constraints assumption just defined is not valid. It is approximately valid if the mass of the oscillators is small relative to the mass of the rigid body. Stability criteria have been derived for three equilibrium configurations which are intimately related to the gravity-gradient stabilization case for the rigid body.

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The analysis of the mathematical model using flexible rods to simulate the antennas is more complicated but does not reveal additional information. This work will be reported later.

The case in which the satellite does undergo rotational motion with respect to the orbiting frame of reference is presently being investigated.

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1. Introduction

The purpose of this study is to examine the stability of motion of a rigid body with elastically connected moving parts which undergoes rotational motion with respect to an inertial system, while the center of the rigid body orbits around the earth. The mathematical model consists of a rigid body with principal mass moments of inertia A, B, and C about the principal axes x, y, and z. Along the z axis there are two viscously damped oscillators attached.

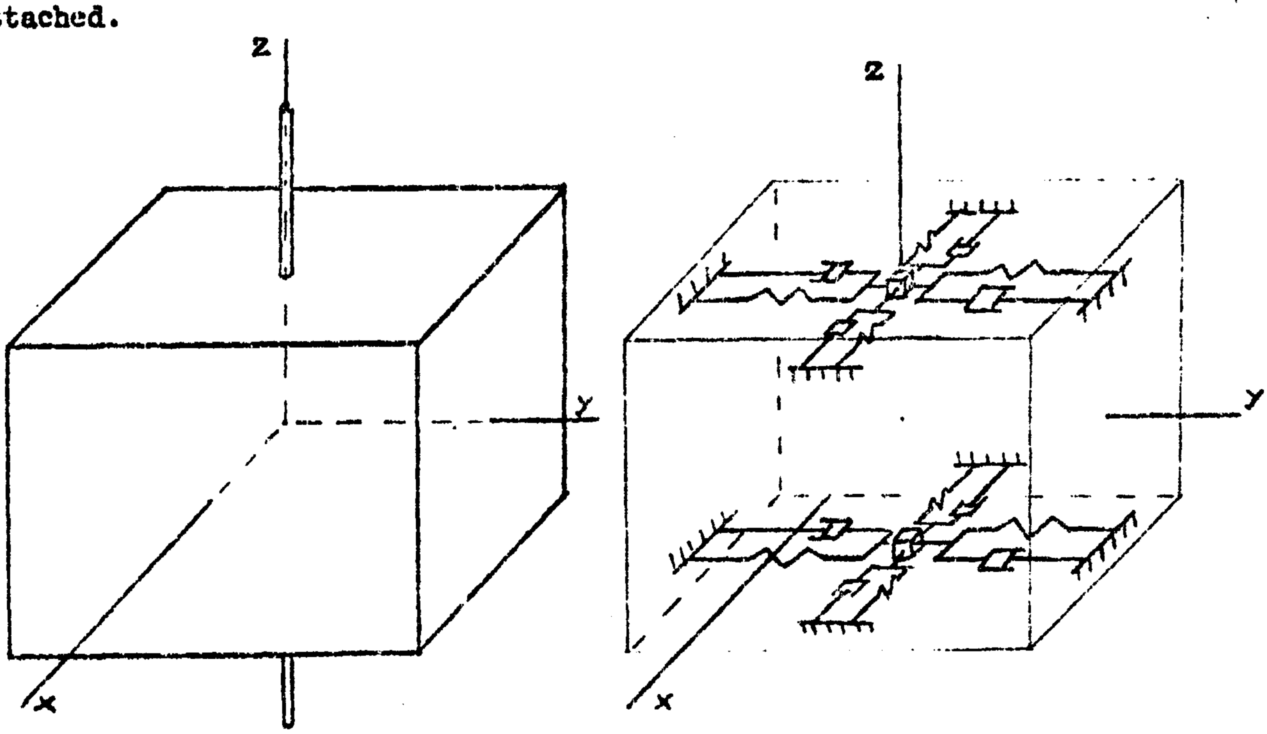

Figure 1

The body represents a man-made satellite where the oscillators are meant to simulate flexible antennas. Under certain circumstances, the more exact analysis obtained by regarding the antennas as flexible rods yields results which do not differ appreciably from the results obtained from the analysis using the oscillators.

Due to functional requirements it is desirable that such a satellite maintain a certain attitude with respect to an inertial system or with respect to a moving system of reference. One such reference system, called orbital system of axes, is that for which the three coordinate axes are aligned with the tangent to the orbit, the radial direction, and the normal to the orbit plane. This, of course, assumes that the orbit is circular.

It is possible to use an active control system to maintain the satellite orientation. We shall be interested, however, in stabilization by passive means.

When one is interested in a satellite that has one axis pointing towards the earth at all times it is possible, in certain circumstances, to use the differential-gravity torque as a passive means of stabilization. This method is known as gravity-gradient stabilization. It is based on the fact that, for a non-spinning body, the gravitational torques tend to align the axis of minimum moment of inertia of the body with the radial direction from the center of force (which is assumed to coincide with the center of mass of the earth) to the center of mass of the satellite. Another passive method of stabilization is known as spin-stabilization. This method is based on the fact that a spinning body tends to maintain the direction of the spin axis fixed in an inertial space if no disturbing torques are present. In this case the differential-gravity torque can be used to impart a steady precession to the body.

A considerable amount of work has been done in the area of satellite stabilization by passive means. The research proposal dated September 14, 1964 contains a survey of the related work. Since that date a number of papers in the same general area have been published. Of these we shall single out the papers by K. H. Wadleigh, A. J. Galloway, and P. N. Mathur*, and P. W. Likins**. The first paper discusses the possibility of passively damping the nutation which might result from any disturbing moments in a spin-stabilized space vehicle. The device consists of a simple damped oscillator. The space vehicle that is considered is not, however, an orbiting satellite and consequently does not include consideration of gravity torques. The second paper, although it does consider the gravitational torques, regards the satellite as a rigid body. This paper, in essence, obtains the same results as those obtained by R. Pringle

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*"Spinning Vehicle Nutation Damper," Journal of Spacecraft, Vol. 1, No. 6, pp. 588-592, Nov.-Dec., 1964.

**"Stability of a Symmetrical Satellite in Attitudes Fixed in an Orbiting Reference Frame," Journal of Astronautical Sciences, Vol. XII, No. 1, pp. 18-24, Spring 1965.

Jr[*] by using a slightly different approach. The interesting part about the paper by Likins is that it uses the same approach as envisioned by the author of this report. The problem that is presently being investigated, however, is considerably more comprehensive due to the fact that the body has elastic components and possesses damping characteristics. In this respect, the present mathematical model is more realistic.

The problem under investigation is formulated by means of methods of analytical mechanics. The variational approach leads to Lagrange's equations of motion. As generalized coordinates for the simplified model which uses oscillators to simulate the flexible antennas, we have three coordinates defining the position of the origin of the x y z system in space, three angular coordinates describing the orientation of the principal axes (x, y, and z) and four linear displacements, two for each oscillator, describing the positions of the oscillators with respect to the axes x, y, and z. This gives a total of ten generalized coordinates. The formulation leads to ten coupled equations of motion which are highly nonlinear.

Since we are interested in the stability of motion, the method of solution that has been employed consists of determining some possible stable configuration from the equations of motion and checking the stability of motion in the neighborhood of this motion. This can be done by means of an <u>infinitesimal analysis</u> which consists of linearizing the equations about a possible stable configuration, known as a dynamic equilibrium or plain equilibrium position, and using a perturbation method. This method, however, can make a positive statement only about the instability of a system and not the stability. Another method of approach is the <u>Liapounov</u> second, or direct, method which can make a statement about the quality of equilibrium without solving the equations of motion. The main difficulty in using this method lies in devising a <u>Liapounov function</u> which is to be used for testing purposes.

Among all possible equilibrium configurations the cases which can be interpreted as the gravity-gradient stabilized case is of particular interest at this point. In almost every previous work on the subject,

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[*]"Bounds of the Librations of a Symmetric Satellite," AIAA Journal, Vol. 2, No. 5, pp. 908-912, May 1964.

it has been assumed that the body is rigid and that orbital constraints are imposed such that the center of mass of the body moves in a circular orbit, R_c = const., at a constant angular velocity Ω_0. This assumption allows a reduction of the number of generalized coordinates by three. When the body contains moving parts, such as the oscillators, the center of mass shifts with respect to the xyz system and it may be more convenient to apply the orbital constraint on the origin O of the xyz system. Analysis was performed with and without the use of orbital constraints and stability criteria were derived for each case. A major conclusion of the present study is that while the orbital constraints just defined are valid for a rigid satellite (for which there is no distinction between the origin O and the center of mass) it is not valid for a satellite containing elastic parts. The assumption is reasonable if the mass of the oscillators is small relative to the total mass of the system. Although this conclusion was reached for a special dynamic equilibrium position and using a mathematical model of a certain configuration, it is felt that the conclusion may be valid under much broader circumstances. Certainly it points out a problem area which should not be overlooked in future investigations.

The gravity-gradient stabilization implies that the body possesses an angular velocity Ω_0 with respect to an inertial space, the angular velocity being equal to the orbital angular velocity. Thus the angular velocity between the body and the orbital system of axes is zero at an equilibrium point. The differential equations describing this motion are autonomous, and solutions by means of the Liapounov method have been accomplished. In the case of spin stabilization in a gravitational field, the angular velocity of the body is different than Ω_0, and the differential equations describing the motion contain periodic coefficients. In this case, a mathematical analysis may present insurmountable problems. Nevertheless an attempt will be made in this direction.

The next stage of the investigation consists of the identification of other dynamic equilibrium configurations and developing stability criteria. To this end, infinitesimal and Liapounov-type solutions will be attempted. Any progress made by using the latter approach will be a contribution toward advancing the current state of the art.

A natural extension of the present investigation should consider elliptical rather than circular orbits. For the case of librating, nonspinning communications satellites one may include the effect of solar radiation pressure. Either of these problems appears worthy of a future research project.

Another problem of interest would be the case in which the satellite contains internal moving parts such that the mass moments of inertia change according to a predetermined pattern. This problem, too, could be the subject of a future research investigation.

2. Coordinate Systems and Coordinate Transformations

As mentioned in the introduction, the motion of the satellite is described by ten generalized coordinates. It will prove convenient to describe the orientation of the body in two different sets of coordinates. In the investigation of the stability about equilibrium positions we shall make use, at different times, of both systems of coordinates. Since the directions of displacements of the oscillators is known with respect to a set of body axes x, y, z, we shall be concerned first with the determination of the orientation of the x, y, z axes in space. To this end let us consider Figure 2 and designate an inertial coordinate system by X, Y, Z.

The projection on the reference plane XY of the radial line, R_c, from the center of force, CF, to the center of the body, O, makes an angle Φ with the reference axis X.

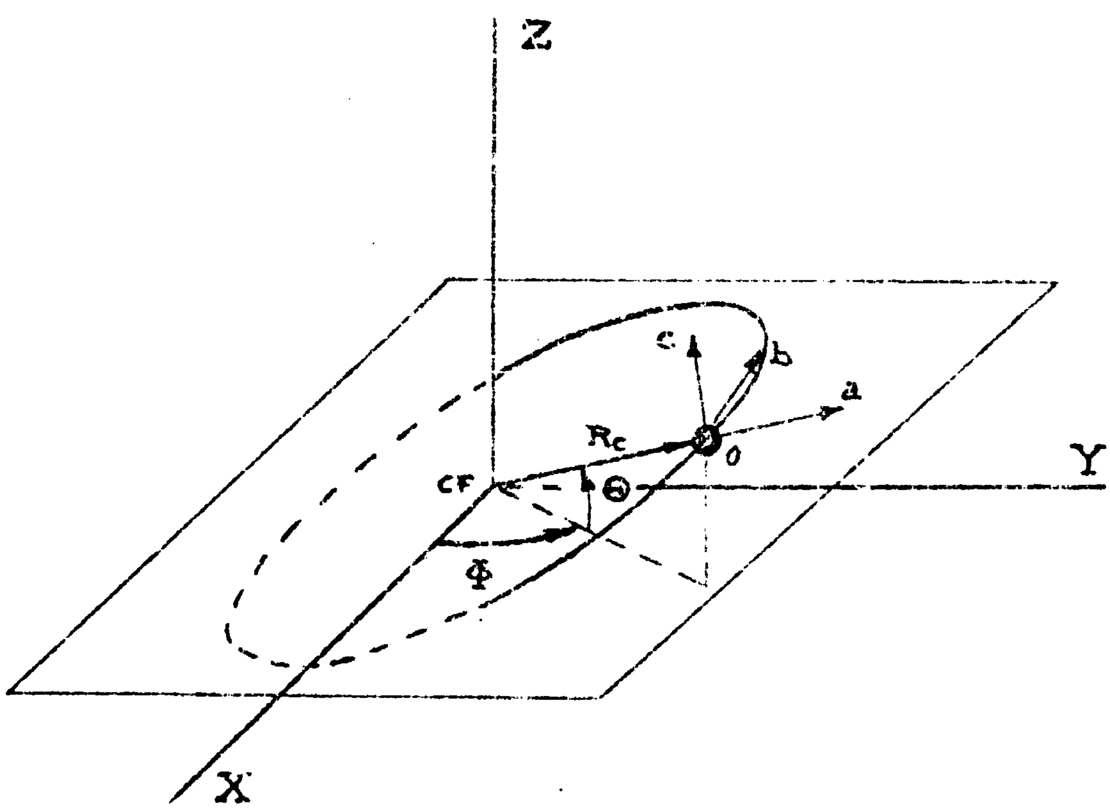
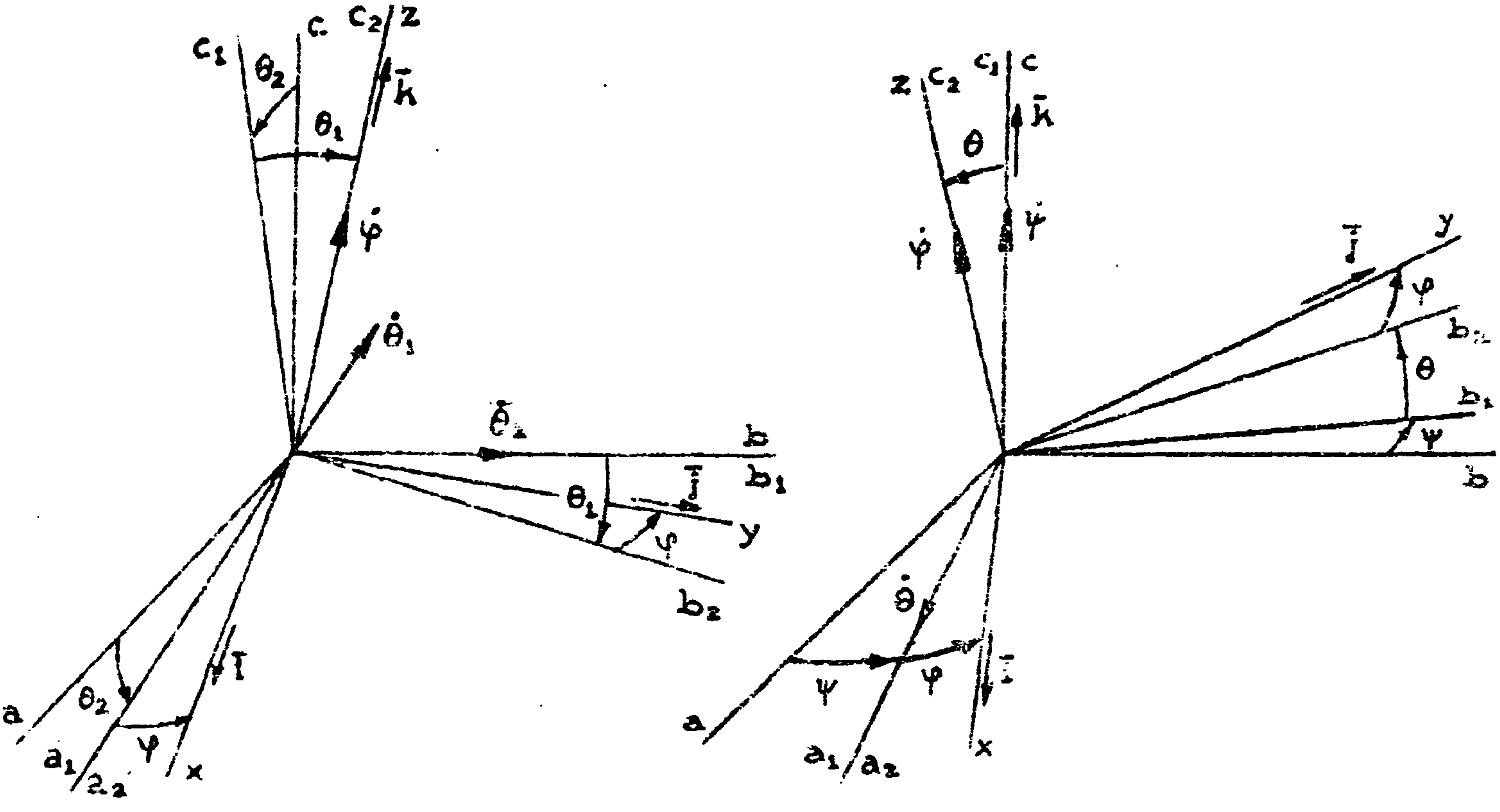


Figure 2

The angle between this projection and the radius vector $\bar{R}_c$ is denoted by θ. The triad a, b, c is such that a and c are in a plane normal to the reference plane XY with a in the direction $\bar{R}_c$ and c normal to $\bar{R}_c$, whereas b is normal to both a and c. The orientation of the body will be measured with respect to this triad in two ways, as shown in Figures 3(a) and 3(b).



(a) (b)

Figure 3

From Figure 3(a), we see that the coordinate transformation relating the coordinates a, b, c and x, y, z can be written in the matrix form

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} l_{xa} & l_{xb} & l_{xc} \\ l_{ya} & l_{yb} & l_{yc} \\ l_{za} & l_{zb} & l_{zc} \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$= \begin{bmatrix} c\theta_2\, c\varphi - s\theta_1\, s\theta_2\, s\varphi & c\theta_1\, s\varphi & -s\theta_2\, c\varphi - c\theta_2\, s\theta_1\, s\varphi \\ -c\theta_2\, s\varphi - s\theta_1\, s\theta_2\, c\varphi & c\theta_1\, c\varphi & s\theta_2\, s\varphi - c\theta_2\, s\theta_1\, c\varphi \\ s\theta_2\, c\theta_1 & s\theta_1 & c\theta_2\, c\theta_1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{2.1}$$

where l_{xa} is the direction cosine between the x axis and the a axis, etc., and

$$c\theta_2 = \cos\theta_2, \qquad s\theta_1 = \sin\theta_1, \text{ etc.}$$

The angular velocity components of the body may be written

$$\begin{aligned} \Omega_x &= \dot{\Phi}\left[(-c\Theta\, s\theta_2 + s\Theta\, c\theta_2)\, c\varphi - (c\Theta\, c\theta_2 + s\Theta\, s\theta_2)\, s\theta_1\, s\varphi\right] \\ &\quad - \dot{\Theta}\, c\theta_1\, s\varphi + \dot{\theta}_2\, c\theta_1\, s\varphi - \dot{\theta}_1\, c\varphi \\ \Omega_y &= \dot{\Phi}\left[(c\Theta\, s\theta_2 - s\Theta\, c\theta_2)\, s\varphi - (c\Theta\, c\theta_2 + s\Theta\, s\theta_2)\, s\theta_1\, c\varphi\right] \\ &\quad - \dot{\Theta}\, c\theta_1\, c\varphi + \dot{\theta}_2\, c\theta_1\, c\varphi + \dot{\theta}_1\, s\varphi \\ \Omega_z &= \dot{\Phi}\,(c\Theta\, c\theta_2 + s\Theta\, s\theta_2)\, c\theta_1 - \dot{\Theta}\, s\theta_1 + \dot{\theta}_2\, s\theta_1 + \dot{\varphi} \end{aligned} \tag{2.2}$$

In a similar way we obtain from Figure 3(b)

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} c\psi\, c\varphi - s\psi\, c\varphi\, s\varphi & s\psi\, c\varphi + c\varphi\, c\theta\, s\varphi & s\theta\, s\varphi \\ -c\psi\, s\varphi - s\psi\, c\theta\, c\varphi & -s\psi\, s\varphi + c\psi\, c\theta\, c\varphi & s\theta\, c\varphi \\ s\psi\, s\theta & -c\psi\, s\theta & c\theta \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{2.3}$$

and the corresponding angular velocity components

$$\begin{aligned} \Omega_x &= \dot{\Phi}\,(s\Theta\, c\psi\, c\varphi + c\Theta\, s\theta\, s\varphi - s\Theta\, s\psi\, c\theta\, s\varphi) - \dot{\Theta}\,(s\psi\, c\varphi \\ &\quad + c\psi\, c\theta\, s\varphi) + \dot{\psi}\, s\theta\, s\varphi + \dot{\theta}\, c\varphi \end{aligned}$$

$$\Omega_y = \dot{\Phi}\,(-s\,\Theta\;c\psi\;s\varphi + c\,\Theta\;s\theta\;c\varphi - s\,\Theta\;s\psi\;c\theta\;c\varphi) + \dot{\Theta}\,(s\psi\;s\varphi \tag{2.4}$$

$$- c\psi\;c\theta\;c\varphi) + \dot{\psi}\;s\theta\;c\varphi - \dot{\theta}\;s\varphi$$

$$\Omega_z = \dot{\Phi}\,(c\,\Theta\;c\theta + s\Theta\;s\psi\;s\theta) + \dot{\Theta}\,c\psi\;\;s\theta + \dot{\psi}\;c\theta + \dot{\varphi}$$

3. Energy Expressions and Rayleigh's Dissipation Function

In order to use Lagrange's formulation of the differential equations of motion we need the kinetic and potential energies of the body as well as Rayleigh's dissipation function. The dissipation function is a quadratic function of velocities which was introduced by Lord Rayleigh to account for damping forces in the Lagrange equations. Figure 4 shows the mathematical model of the body.

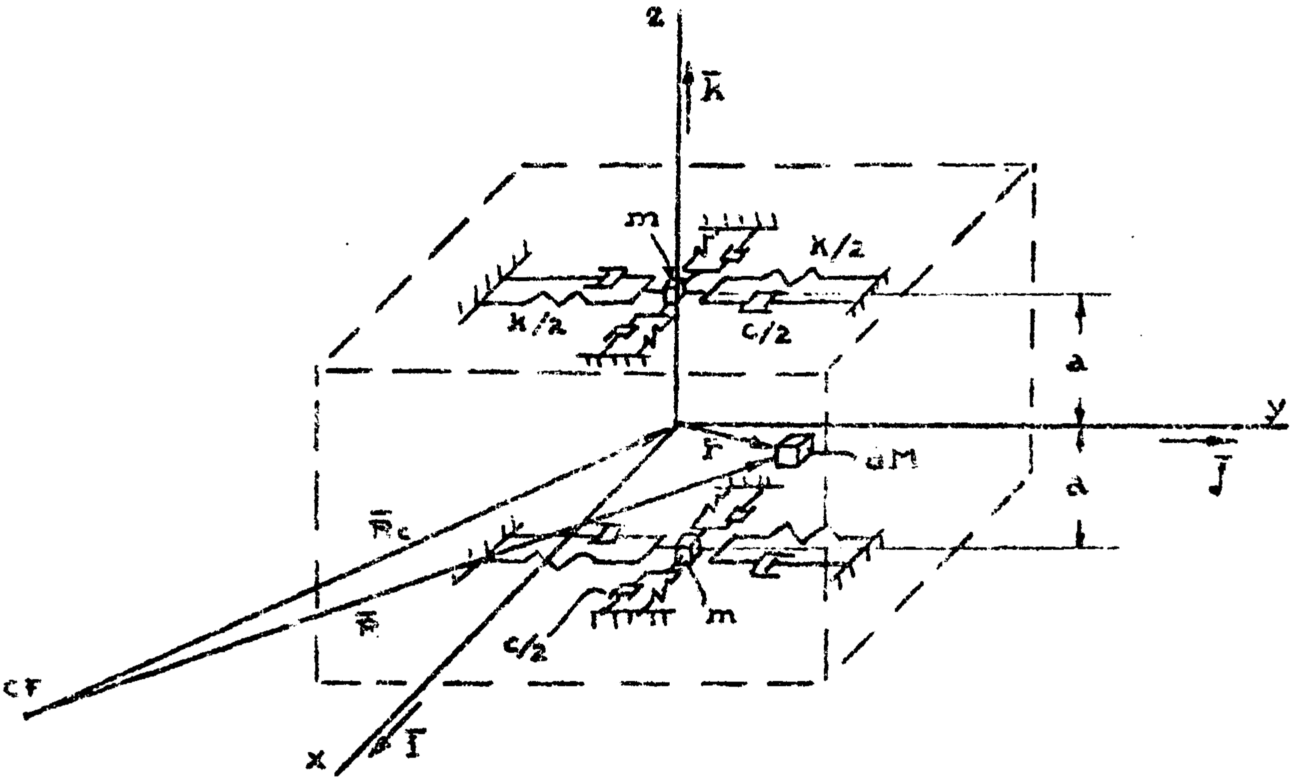


Figure 4

Denoting the velocity vector of an element of the rigid body of mass dM by $\bar{V}$, and denoting the velocity vectors of the two masses by $\bar{v}_1$ and $\bar{v}_2$, we can write the kinetic energy of the entire body in the form

$$T = \frac{1}{2} \iiint (\bar{V} \cdot \bar{V})\, dM + \frac{1}{2}m\,(\bar{v}_1 \cdot \bar{v}_1) + \frac{1}{2}m\,(\bar{v}_2 \cdot \bar{v}_2) \qquad (3.1)$$

where the velocity vectors are given by the expressions

$$\bar{v} = \dot{\bar{R}}_c + \bar{\Omega} \times \bar{r}$$

$$\bar{v}_1 = \dot{\bar{R}}_c + \dot{\bar{r}}_{1\,rel} + \bar{\Omega} \times \bar{r}_1 \qquad (3.2)^*$$

$$\bar{v}_2 = \dot{\bar{R}}_c + \dot{\bar{r}}_{2\,rel} + \bar{\Omega} \times \bar{r}_2$$

where

$$\dot{\bar{R}}_c = \dot{R}_c\,\bar{I}' + R_c\,\dot{\phi}\cos\theta\,\bar{J}' + R_c\,\dot{\theta}\,\bar{K}' \qquad (3.3)$$

is the absolute velocity of the center of mass in terms of components along the abc axes (see Figure 2), and

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\bar{r}_1 = v_1\bar{i} + w_1\bar{j} + a\bar{k} \qquad (3.4)$$

$$r_2 = v_2\bar{i} + w_2\bar{j} - a\bar{k}$$

are the positions of the element of mass, dM, and the two masses, m, relative to the system xyz. In addition, Ω denotes the angular velocity vector of the body with respect to an inertial space and

$$\bar{r}_{1\,rel} = \dot{v}_1\,\bar{i} + \dot{w}_1\,\bar{j}$$

$$(3.5)$$

$$\bar{r}_{2\,rel} = \dot{v}_2\,\bar{i} + \dot{w}_2\,\bar{j}$$

are the velocities of the two masses relative to the body axes xyz. Introducing Eqs. (3.2) through (3.5) into Eq. (3.1) we obtain

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*This assumes that CF is at rest, which is only an approximation.

$$
\begin{aligned}
T = {} & \tfrac{1}{2}(2m + M)(\dot{R}_c^{\,2} + R_c^{\,2}\cos^2\Theta\,\dot{\Phi}^2 + R_c^{\,2}\dot{\Theta}^2) \\
& + \tfrac{1}{2}(A'\Omega_x^2 + B'\Omega_y^2 + C\Omega_z^2) + \tfrac{1}{2}m\,\Big\{\dot{v}_1^{\,2} + \dot{v}_2^{\,2} + \dot{w}_1^{\,2} + \dot{w}_2^{\,2} \\
& + \Omega_x^2(w_1^{\,2} + w_2^{\,2}) + \Omega_y^{\,2}(v_1^{\,2} + v_2^{\,2}) + \Omega_z^2(v_1^{\,2} + v_2^{\,2} + w_1^{\,2} + w_2^{\,2}) \\
& - 2\Omega_x a(\dot{w}_1 - \dot{w}_2) + 2\Omega_y a(\dot{v}_1 - \dot{v}_2) - 2\Omega_z(\dot{v}_1 w_1 - v_1\dot{w}_1 + \dot{v}_2 w_2 - v_2\dot{w}_2) \\
& - 2\Omega_x\Omega_y(v_1 w_1 + v_2 w_2) - 2a\Omega_y\Omega_z(w_1 - w_2) - 2a\Omega_z\Omega_x(v_1 - v_2)\Big\} \\
& + m(\dot{R}_c\, l_{xa} + R_c\cos\Theta\,\dot{\Phi}\, l_{xb} + R_c\,\dot{\Theta}\, l_{xc})\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right] \\
& + m(\dot{R}_c\, l_{ya} + R_c\cos\Theta\,\dot{\Phi}\, l_{yb} + R_c\,\dot{\Theta}\, l_{yc})\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right] \\
& + m(\dot{R}_c\, l_{za} + R_c\cos\Theta\,\dot{\Phi}\, l_{zb} + R_c\,\dot{\Theta}\, l_{zc})\left[\Omega_x(w_1 + w_2) - \Omega_y(v_1 + v_2)\right]
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
A' &= A + 2ma^2 \\
B' &= B + 2ma^2
\end{aligned}
\tag{3.7}
$$

The potential energy consists of two parts, gravitational and elastic potential energy. We shall first evaluate the potential energy due to gravity. For an inverse square central force field the potential energy is given by

$$
V_G = -\iiint \frac{K}{R}\,dM - \frac{K}{R_1}\,m - \frac{K}{R_2}\,m + \text{const} \tag{3.8}
$$

where $K = GM_e$ is the strength of the field which is equal to the product of the universal granitational constant, G, and the mass of the earth, M_e. Of course, this assumes that the earth's density is a function only of the distance from the center of the earth.

The reciprocal of the absolute value of the radius vector $\bar{R}$ can be written in the form

$$R^{-1} = \left[(\bar{R}_c + \bar{r}) \cdot (\bar{R}_c + \bar{r})\right]^{-\frac{1}{2}}$$

$$\cong R_c^{-1} - R_c^{-2} (\bar{i}' \cdot \bar{r}) - R_c^{-3}\left[\frac{r^2}{2} - \frac{3}{2} (\bar{i}' \cdot \bar{r})^2\right] \qquad (3.9)$$

Similar expressions can be written for R_1^{-1} and R_2^{-2} upon noting that $\bar{R}_1 = \bar{R}_c + \bar{r}_1$ and $\bar{R}_2 = \bar{R}_c + \bar{r}_2$ where $\bar{r}_1$ and $\bar{r}_2$ are given by Eqs. (3.4). This allows us to obtain

$$V_G = -\frac{K}{R_c} (2m + M) + \frac{K m}{R_c^2} \left[(v_1 + v_2)\, l_{xa} + (w_1 + w_2)\, l_{ya}\right]$$

$$+ \frac{K}{4 R_c^3} (A' + B' + C) - \frac{3K}{4 R_c^3} \Big[(C + B' - A')\, l_{xa}^2$$

$$+ (C + A' - B')\, l_{ya}^2 + (A' + B' - C)\, l_{za}^2 \Big]$$

$$+ \frac{K m}{2 R_c^3} (v_1^2 + v_2^2 + w_1^2 + w_2^2) - \frac{3 K m}{2 R_c^3} \Big[(v_1^2 + v_2^2)\, l_{xa}^2$$

$$+ (w_1^2 + w_2^2)\, l_{ya}^2 + 2a (v_1 - v_2)\, l_{xa}\, l_{za} + 2a (w_1 - w_2)\, l_{ya}\, l_{za}$$

$$+ 2 (v_1 w_1 + v_2 w_2)\, l_{xa}\, l_{ya}\Big] + \text{const.} \qquad (3.10)$$

Assuming that the springs are linear and that the spring constants in the directions x and y is equal to k we obtain the elastic potential energy

$$V_{EL} = \frac{1}{2} k \ (v_1^2 + v_2^2 + w_1^2 + w_2^2) \qquad (3.11)$$

Rayleigh's dissipation function*, which can be interpreted as one half the rate at which the system dissipates energy, has the form

$$D = \tfrac{1}{2} c (\dot{v}_1^{\,2} + \dot{v}_2^{\,2} + \dot{w}_1^{\,2} + \dot{w}_2^{\,2}) \tag{3.12}$$

*See Goldstein, H., Classical Mechanics, Addison Wesley Press, Inc., 1950, pp. 20-22.

4. The Differential Gravity Torque

Although the equations of motion could be derived directly from the kinetic and potential energies and the dissipation function by means of the Lagrangian formulation, it will prove convenient to write a special form of Lagrange's equations, namely in terms of "quasi-coordinates." This subject will be discussed in detail in the next section. For the moment it suffices to say that we must know the x, y, and z components of torque about the center of mass in order to use the special form of the Lagrange equation. The force vector acting on the differential element of mass d M is

$$d\,\overline{F} = -\frac{K}{R^3}\,\overline{R}\,d\,M \tag{4.1}$$

The torque vector about the center of the body, due to the differential of force $d\,\overline{F}$, is

$$\begin{aligned} d\,\overline{N} &= \overline{r} \times d\,\overline{F} = -\frac{K}{R^3}\,d\,M\,(\overline{r} \times \overline{R}) \\ &= -\frac{K}{R^3}\,d\,M\,(\overline{r} \times \overline{R}_c) \end{aligned} \tag{4.2}$$

in view of the fact that $\overline{R} = \overline{R}_c + \overline{r}$ and $\overline{r} \times \overline{r} = 0$. But, as in the preceding section, we have

$$\begin{aligned} R^{-3} &= \left[(\overline{R}_c + \overline{r}) \cdot (\overline{R}_c + \overline{r})\right]^{-3/2} = R_c^{-3}\left[1 + \frac{2\,\overline{R}_c \cdot \overline{r}}{R_c^{\,2}} + \frac{\overline{r} \cdot \overline{r}}{R_c^{\,2}}\right]^{-3/2} \\ &= R_c^{-3} - 3\,R_c^{-5}\,(\overline{R}_c \cdot \overline{r}) + R_c^{-7}\left[\frac{15}{2}\,(\overline{R}_c \cdot \overline{r}) - \frac{3\,R_c^{\,2} r^2}{2}\right] + \ldots \end{aligned} \tag{4.3}$$

Introducing Eq. (4.3) into (4.2) and integrating over the entire body, and noting that $\iiint \overline{r}\,d\,M = 0$ according to the definition of the center of mass of the rigid body, we obtain

$$\bar{N} = \frac{3K}{R_c^5} \iiint (\bar{r} \times \bar{R}_c)(\bar{R}_c \cdot \bar{r})\, dM - \frac{K}{R_c^3}\, m\left[(\bar{r}_1 \times \bar{R}_c) + (\bar{r}_2 \times \bar{R}_c)\right]$$

$$+ \frac{3\,Km}{R_c^5}\left[(\bar{r}_1 \times \bar{R}_c)(\bar{R}_c \cdot \bar{r}_1) + (\bar{r}_2 \times \bar{R}_c)(\bar{R}_c \cdot \bar{r}_2)\right] + \ldots \qquad (4.4)$$

Performing the above integrations we obtain the components

$$N_x = -\frac{K\,m}{R_c^2}\,(w_1 + w_2)\, l_{za} + \frac{3\,K}{R_c^3}\Big\{ (C - B')\, l_{ya}\, l_{za} +$$

$$+ m\Big[(w_1^2 + w_2^2)\, l_{ya}\, l_{za} + (v_1 w_1 + v_2 w_2)\, l_{xa}\, l_{za}$$

$$- a(v_1 - v_2)\, l_{xa}\, l_{ya} + a(w_1 - w_2)(l_{za}^2 - l_{ya}^2)\Big]\Big\} + \ldots \quad (4.5)$$

$$N_y = \frac{K\,m}{R_c^2}\,(v_1 + v_2)\, l_{za} + \frac{3\,K}{R_c^3}\Big\{(A' - C)\, l_{xa}\, l_{za}$$

$$+ m\Big[- (v_1^2 + v_2^2)\, l_{xa}\, l_{za} + a(w_1 - w_2)\, l_{xa}\, l_{ya} -$$

$$- (v_1 w_1 + v_2 w_2)\, l_{ya}\, l_{za} + a(v_1 - v_2)(l_{xa}^2 - l_{za}^2)\Big]\Big\} + \ldots \quad (4.6)$$

$$N_z = -\frac{K\,m}{R_c^2}\left[(v_1 + v_2)\, l_{ya} - (w_1 + w_2)\, l_{xa}\right] + \frac{3\,K}{R_c^3}\Big\{(B' - A')\, l_{xa}\, l_{ya}$$

$$+ m\Big[(v_1^2 + v_2^2 - w_1^2 - w_2^2)\, l_{xa}\, l_{ya} + a(v_1 - v_2)\, l_{ya}\, l_{za}$$

$$- a\,(w_1 - w_2)\, l_{xa}\, l_{za} + (v_1 w_1 + v_2 w_2)(l_{ya}^2 - l_{xa}^2)\Big]\Big\} + \ldots$$

$$(4.7)$$

Note that terms in R_c^{-4} and smaller have been neglected in Eqs. (4.5) to (4.7).

5. Lagrange's Equations of Motion

We shall dispense with the details of the derivation of the Lagrange's equations of motion, as they are found in any text on classical mechanics.* The differential equations for the generalized coordinates R_c, Φ, and Θ describe the motion of the center of the origin of the xyz system and the generalized coordinates v_1, v_2, v_1, and w_2 describe the motion of the two oscillators relative to the xyz system. These equations have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}_c}\right) - \frac{\partial L}{\partial R_c} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) - \frac{\partial L}{\partial \Phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}}\right) - \frac{\partial L}{\partial \Theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}_1}\right) - \frac{\partial L}{\partial v_1} + \frac{\partial D}{\partial \dot{v}_1} = 0 \tag{5.1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}_2}\right) - \frac{\partial L}{\partial v_2} + \frac{\partial D}{\partial \dot{v}_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}_1}\right) - \frac{\partial L}{\partial w_1} + \frac{\partial D}{\partial \dot{w}_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}_2}\right) - \frac{\partial L}{\partial w_2} + \frac{\partial D}{\partial \dot{w}_2} = 0$$

where L is the Lagrangian which has the expression

$$L = T - V_G - V_{EL}$$

in which the kinetic energy, T, is given by Eq. (3.6), the gravitational potential energy, V_G, by Eq. (3.10), and the elastic potential energy,

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*See Goldstein, Herbert, Classical Mechanics, Addison-Wesley Publishing Co., Reading, Mass., 1959, p. 22.

V_{EL}, by Eq. (3.11). Rayleigh's dissipation function, D, is given by Eq.. (3.12).

One could write another three differential equations by using the Lagrange formulation and the generalized coordinates ψ, θ, and φ (or θ_1, θ_2, and φ) which describe the rotational motion of the body with respect to the triad a b c (see Section 2). It will prove more convenient, however, to use a set of angular motions about the orthogonal body axes x, y, and z. Such coordinates are called quasi-coordinates* and the corresponding equations of motion can be written in form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_x}\right) - \Omega_z \frac{\partial T}{\partial \Omega_y} + \Omega_y \frac{\partial T}{\partial \Omega_z} = N_x$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_y}\right) - \Omega_x \frac{\partial T}{\partial \Omega_z} + \Omega_z \frac{\partial T}{\partial \Omega_x} = N_y \qquad (5.3)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_z}\right) - \Omega_y \frac{\partial T}{\partial \Omega_x} + \Omega_x \frac{\partial T}{\partial \Omega_y} = N_z$$

where Ω_x, Ω_y, and Ω_z are the angular velocity components about the x, y, and z axes as given by Eqs. (2.4) and N_x, N_y, and N_z are the torque components which are given by Eqs. (4.5), (4.6), and (4.7).

The first of Eqs. (5.1) leads to the equation of motion for R_c

$$(2m + M)\left[\ddot{R}_c - R_c(\dot{\psi}^2 \cos^2 \theta + \dot{\theta}^2) + \frac{K}{R_c^2}\right]$$

$$+ m\left\{ \dot{l}_{xa}\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right] + \dot{l}_{ya}\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right]\right.$$

$$\left. + \dot{l}_{za}\left[\Omega_x (w_1 + w_2) - \Omega_y(v_1 + v_2)\right]\right\}$$

$$+ m\left\{ l_{xa}\left[\ddot{v}_1 + \ddot{v}_2 - \dot{\Omega}_z(w_1 + w_2) - \Omega_z(\dot{w}_1 + \dot{w}_2)\right]\right.$$

$$+ l_{ya}\left[\ddot{w}_1 + \ddot{w}_2 + \dot{\Omega}_z(v_1 + v_2) + \Omega_z(\dot{v}_1 + \dot{v}_2)\right]$$

*See Whittaker, E.T.; A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, 1961, p. 41.

$$+ l_{za}\left[\dot{\Omega}_x(w_1 + w_2) + \Omega_x(\dot{w}_1 + \dot{w}_2) - \dot{\Omega}_y(v_1 + v_2) - \Omega_y(\dot{v}_1 + \dot{v}_2)\right]\Big\}$$

$$- m\Big\{\left[\dot{\phi}\cos\theta\; l_{xb} + \dot{\theta}\, l_{xc}\right]\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right]$$

$$+ \left[\dot{\phi}\cos\theta\; l_{yb} + \dot{\theta} l_{yc}\right]\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right]$$

$$+ \left[\dot{\phi}\cos\theta\; l_{zb} + \dot{\theta} l_{zc}\right]\left[\Omega_x(w_1 + w_2) - \Omega_y(v_1 + v_2)\right]$$

$$- \frac{2\,K\,m}{R_c^{\,3}}\left[(v_1 + v_2)\, l_{xa} + (w_1 + w_2)\, l_{ya}\right] = 0 \qquad (5.4)$$

where terms of negative powers of R_c larger than three have been neglected as small and $(\dot{\ })$ implies differentiation with respect to time.

In a similar way we obtain

$$\frac{d}{dt}\Big\{(2m + M)\, R_c^{\,2}\,\dot{\phi}\cos^2\theta + \frac{\partial L}{\partial\Omega_x}\frac{\partial\Omega_x}{\partial\dot{\phi}} + \frac{\partial L}{\partial\Omega_y}\frac{\partial\Omega_y}{\partial\dot{\phi}} + \frac{\partial L}{\partial\Omega_z}\frac{\partial\Omega_z}{\partial\dot{\phi}}$$

$$+ m R_c\cos\theta\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right] l_{xb}$$

$$+ m R_c\cos\theta\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right] l_{yb}$$

$$+ m R_c\cos\theta\left[\Omega_x(w_1 + w_2) - \Omega_y(v_1 + v_2)\right] l_{zb}\Big\} = 0 \qquad (5.5)$$

$$\frac{d}{dt}\Big\{(2m + M)\, R_c^{\,2}\,\dot{\theta} + \frac{\partial L}{\partial\Omega_x}\frac{\partial\Omega_x}{\partial\dot{\theta}} + \frac{\partial L}{\partial\Omega_y}\frac{\partial\Omega_y}{\partial\dot{\theta}} + \frac{\partial L}{\partial\Omega_z}\frac{\partial\Omega_z}{\partial\dot{\theta}}$$

$$+ m R_c\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right] l_{xc} + m R_c\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right] l_{yc}$$

$$+ m R_c\left[\Omega_x(w_1 + w_2) - \Omega_y(v_1 + v_2)\right] l_{zc}\Big\} + (2m + M)\, R_c^{\,2}\,\dot{\phi}^2\sin\theta\cos\theta$$

$$+ m R_c\sin\theta\,\dot{\phi}\, l_{xb}\left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right]$$

$$+ m R_c\sin\theta\,\dot{\phi}\, l_{yb}\left[\dot{w}_1 + \dot{w}_2 + \Omega_z(v_1 + v_2)\right]$$

$$+ m R_c \sin \Theta \, \dot{\Phi} \, l_{zb} \left[\Omega_x (w_1 + w_2) - \Omega_y (v_1 + v_2) \right]$$

$$- \frac{\partial L}{\partial \Omega_x} \frac{\partial \Omega_x}{\partial \Theta} - \frac{\partial L}{\partial \Omega_y} \frac{\partial \Omega_y}{\partial \Theta} - \frac{\partial L}{\partial \Omega_z} \frac{\partial \Omega_z}{\partial \Theta} = 0 \qquad (5.6)$$

where in Eqs. (5.5) and (5.6)

$$\frac{\partial L}{\partial \Omega_x} = \frac{\partial T}{\partial \Omega_x} = \left[A' + m(w_1^2 + w_2^2) \right] \Omega_x - m \left[a(\dot{w}_1 - \dot{w}_2) \right.$$

$$\left. + \Omega_y (v_1 w_1 + v_2 w_2) + a \, \Omega_z (v_1 - v_2) \right]$$

$$+ m \left[\dot{R}_c \, l_{za} + R_c \cos \Theta \, \dot{\Phi} \, l_{zb} + R_c \, \dot{\Theta} \, l_{zc} \right] (w_1 + w_2)$$

$$\frac{\partial L}{\partial \Omega_y} = \frac{\partial T}{\partial \Omega_y} = \left[B' + m(v_1^2 + v_2^2) \right] \Omega_y - m \left[a(\dot{v}_1 - \dot{v}_2) - \Omega_x (v_1 w_1 + v_2 w_2) \right.$$

$$\left. - a \, \Omega_z (w_1 - w_2) \right] - m \left[\dot{R}_c \, l_{za} + R_c \cos \Theta \, \dot{\Phi} \, l_{zb} + R_c \, \dot{\Theta} \, l_{zc} \right] (v_1 + v_2)$$

$$\frac{\partial L}{\partial \Omega_z} = \frac{\partial T}{\partial \Omega_z} = \left[C + m(v_1^2 + v_2^2 + w_1^2 + w_2^2) \right] \Omega_z$$

$$- m \left[\dot{v}_1 w_1 - v_1 \dot{w}_1 + \dot{v}_2 w_2 - v_2 \dot{w}_2 \right.$$

$$\left. + a \, \dot{\Omega}_y (w_1 - w_2) + a \, \Omega_x (v_1 - v_2) \right] + m \left\{ - \left[\dot{R}_c \, l_{xa} \right. \right.$$

$$\left. + R_c \cos \Theta \, \dot{\Phi} \, l_{xb} + R_c \, \dot{\Theta} \, l_{xc} \right] (w_1 + w_2)$$

$$+ \left[\dot{R}_c \, l_{xa} + R_c \cos \Theta \, \dot{\Phi} \, l_{xb} + R_c \, \dot{\Theta} \, l_{yc} \right] (v_1 + v_2)$$

The last four of Eqs. (5.1) yield

$$m \frac{d}{dt} \left\{ \dot{v}_1 + a \Omega_y - w_1 \Omega_z + \left[\dot{R}_c 1_{xa} + R_c \cos \Theta \dot{\Phi} 1_{xb} + R_c \dot{\Theta} 1_{xc} \right] \right\}$$

$$- m \left\{ (\Omega_y^2 + \Omega_z^2) v_1 + \Omega_z \dot{w}_1 - \Omega_x \Omega_y w_1 - a \Omega_z \Omega_x + \left[\dot{R}_c 1_{ya} \right. \right.$$

$$\left. + R_c \cos \Theta \dot{\Phi} 1_{yb} + R_c \dot{\Theta} 1_{yc} \right] \Omega_z - \left[\dot{R}_c 1_{za} + R_c \cos \Theta \dot{\Phi} 1_{zb} \right.$$

$$\left. \left. + R_c \dot{\Theta} 1_{zc} \right] \Omega_y \right\} + m \left\{ \frac{K}{R_c^2} 1_{xa} + \frac{K}{R_c^3} v_1 - \frac{3K}{R_c^3} \left[v_1 1_{xa}^2 \right. \right.$$

$$\left. \left. + a 1_{xa} 1_{za} + w_1 1_{xa} 1_{ya} \right] + \frac{k}{m} v_1 \right\} + c \dot{v}_1 = 0 \tag{5.7}$$

$$m \frac{d}{dt} \left\{ \dot{v}_2 - a \Omega_y - w_2 \Omega_z + \left[\dot{R}_c 1_{xa} + R_c \cos \Theta \dot{\Phi} 1_{xb} + R_c \dot{\Theta} 1_{xc} \right] \right\}$$

$$- m \left\{ (\Omega_y^2 + \Omega_z^2) v_2 + \Omega_z \dot{w}_2 - \Omega_x \Omega_y w_2 + a \Omega_z \Omega_x + \left[\dot{R}_c 1_{ya} \right. \right.$$

$$\left. + R_c \cos \Theta \dot{\Phi} 1_{yb} + R_c \dot{\Theta} 1_{yc} \right] \Omega_z - \left[\dot{R}_c 1_{za} + R_c \cos \Theta \dot{\Phi} 1_{zb} \right.$$

$$\left. \left. + R_c \dot{\Theta} 1_{zc} \right] \Omega_y \right\} + m \left\{ \frac{K}{R_c^2} 1_{xa} + \frac{K}{R_c^3} v_2 - \frac{3K}{R_c^3} \left[v_2 1_{xa}^2 \right. \right.$$

$$\left. \left. - a 1_{xa} 1_{za} + w_2 1_{xa} 1_{ya} \right] + \frac{k}{m} v_2 \right\} + c \dot{v}_2 = 0 \tag{5.8}$$

$$m \frac{d}{dt} \left\{ \dot{w}_1 - a \Omega_x + v_1 \Omega_z + \left[\dot{R}_c 1_{ya} + R_c \cos \Theta \dot{\Phi} 1_{yb} + R_c \dot{\Theta} 1_{yc} \right] \right\}$$

$$- m \left\{ (\Omega_x^2 + \Omega_z^2) w_1 - \Omega_z \dot{v}_1 - \Omega_x \Omega_y v_1 - a \Omega_y \Omega_z - \left[\dot{R}_c 1_{xa} \right. \right.$$

$$\left. + R_c \cos \Theta \dot{\Phi} 1_{xb} + R_c \dot{\Theta} 1_{xc} \right] \Omega_z + \left[\dot{R}_c 1_{za} + R_c \cos \Theta \dot{\Phi} 1_{zb} \right.$$

$$\left. \left. + R_c \dot{\Theta} 1_{zc} \right] \Omega_x \right\} + m \left\{ \frac{K}{R_c^2} 1_{ya} + \frac{K}{R_c^3} w_1 - \frac{3K}{R_c^3} \left[w_1 1_{ya}^2 \right. \right.$$

$$+ a\, l_{ya}\, l_{za} + v_1\, l_{ya}\, l_{xa} \Big] + \frac{k}{m}\, w_1 \Big\} + c\, \dot{w}_1 = 0 \tag{5.9}$$

$$m \frac{d}{dt} \Big\{ \dot{w}_2 + a\, \Omega_x + v_2\, \Omega_z + \Big[\dot{R}_c\, l_{ya} + R_c \cos\theta\, \dot{\phi}\, l_{yb} + R_c\, \dot{\theta}\, l_{yc} \Big] \Big\}$$

$$- m \Big\{ (\Omega_x^2 + \Omega_z^2)\, w_2 - \Omega_z\, \dot{v}_2 - \Omega_x\, \Omega_y\, v_2 + a\, \Omega_y\, \Omega_z - \Big[\dot{R}_c\, l_{xa}$$

$$+ R_c \cos\theta\, \dot{\phi}\, l_{xb} + R_c\, \dot{\theta} \Big] \Omega_z + \Big[\dot{R}_c\, l_{za} + R_c \cos\theta\, \dot{\phi}\, l_{zb}$$

$$+ R_c\, \dot{\theta}\, l_{zc} \Big] \Omega_x \Big\} + m \Big\{ \frac{K}{R_c^2}\, l_{ya} + \frac{K}{R_c^3}\, w_2 - \frac{3K}{R_c^3} \Big[w_2\, l_{ya}^2$$

$$- a\, l_{ya}\, l_{za} + v_2\, l_{ya}\, l_{xa} \Big] + \frac{k}{m} \Big\} + c\, \dot{w}_2 = 0 \tag{5.10}$$

The final three equations of motion, taken from Eqs. (5.3), are the following

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_x} \right) - \Omega_z \frac{\partial T}{\partial \Omega_y} + \Omega_y \frac{\partial T}{\partial \Omega_z} = N_x \tag{5.11}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_y} \right) - \Omega_x \frac{\partial T}{\partial \Omega_z} + \Omega_z \frac{\partial T}{\partial \Omega_x} = N_y \tag{5.12}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \Omega_z} \right) - \Omega_y \frac{\partial T}{\partial \Omega_x} + \Omega_x \frac{\partial T}{\partial \Omega_y} = N_z \tag{5.13}$$

in which $\frac{\partial T}{\partial \Omega_x}$, $\frac{\partial T}{\partial \Omega_y}$, and $\frac{\partial T}{\partial \Omega_y}$ are given by the expressions appearing after Eq. (5.6) and the torque components N_x, N_y, and N_z are given by Eqs. (4.5) through (4.7).

6. Discussion of the Methods of Solution

It is easy to see that the equations of motion, Eqs. (5.4) through (5.13), are coupled and highly nonlinear. The present state of the art does not permit a closed form solution of the equations. A purely numerical solution, by necessity, is of restricted value since one must assign numerical values to various system parameters such as the mass moments of inertia, spring stiffness, damping coefficient, mass of the oscillators, etc. A closed form solution may be possible under restricting assumptions.

The case of spin stabilization in which the effect of the gravity torque is ignored as small in relation to the effect of high spin was solved under a different project*. Other restrictions were that the body be symmetric about the spin axis, $B = A$, and that the angular velocity component about the spin axis is much larger than the components about the transverse axes, $\Omega_z >> \Omega_x$ and $\Omega_z >> \Omega_y$. A closed form solution of the linearized equations led to stability criteria (or rather instability criteria). Specifically it was first concluded that the spin velocity is constant, $\Omega_z = \Omega_s =$ constant, and furthermore, if the mass moments of inertia are such that $C > A + 2\ ma^2$, the motion is unstable regardless of the value of the spin velocity, Ω_s. On the other hand if $C < A + 2\ ma^2$ the stability is determined by the magnitude of the ratio Ω_s/ω_n, where $\omega_n = \sqrt{k/m}$ is the natural frequency of one oscillator. Curves of Ω_s/ω_n versus C/A were plotted for various values of $2\ ma^2/A$ and regions of instability were identified. A numerical solution of the nonlinear equations corroborated the above conclusions. A paper summarizing the above work is being prepared and will be submitted for publication in a national journal.

The route followed in the present work is entirely different. No restrictive assumptions are being made and the problem of stability of motion is considered in its entirety. No explicit closed form solution for the motion in terms of the generalized coordinates is sought. In contrast, however, we are interested in deriving criteria for the stability of motion. The method consists of identifying various

*"Attitude Stability of a Spinning Body of Revolution in a Circular Orbit," NSF, GP-3087. Principal Investigator: Leonard Meirovitch

equilibrium configurations in the neighborhood of which stable motion may take place, and testing the stability of this motion. These positions are stationary points in a 2 n-dimensional space defined by the n generalized coordinates and n generalized velocities, and will be referred to as dynamic equilibrium positions. In order to test stability at a dynamic equilibrium position we must devise a function and prove that the function is positive definite and has a minimum at the equilibrium position. Such a function is known as a Liapounov function and the procedure is called Liapounov's second method (also known as direct method). The above is only a brief description of the method and details will be discussed later.

7. Determination of Dynamic Equilibrium Positions

The Lagrange formulation of the equations of motion of a dynamical, n degree-of-freedom system consists of a set of n second order differential equations in terms of the generalized coordinates q_i, $(i = 1, 2 \ldots, n)$. These equations can be replaced by 2 n first order differential equations by introducing, in addition to the n generalized coordinates, another set of n generalized momenta p_i $(i = 1, 2 \ldots, n)$. The resulting 2 n differential equations are called Hamilton's canonical equations. An alternate approach for the formulation of the dynamical problem is to regard the n generalized velocities as the additional set of generalized coordinates.

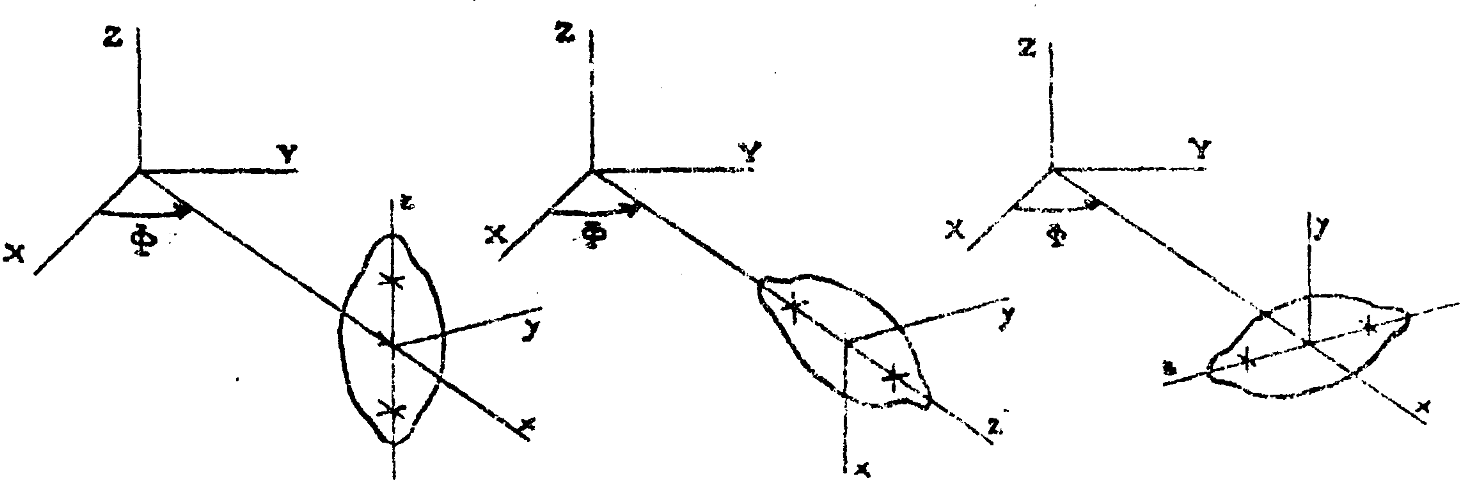
The 2 n coordinates can be thought of as defining a 2 n-dimensional space called a phase space. The points for which the time derivatives of the 2 n coordinates are zero are called stationary or equilibrium points. Physically this means that at such points, the velocities and the accelerations are zero. The stability of motion depends upon what happens to the system if displaced slightly from such an equilibrium position. The motion is considered stable if these displacements remain in this neighborhood and do not tend to increase with time. Mathematically, the stability is established if the system can be characterized by a suitable scalar function which is positive definite in a certain domain and has a relative minimum in the equilibrium position.

An equilibrium point that occurs at a point other than the origin may be translated to the origin by means of a simple, linear transformation. A function meeting these requirements is known as a Liapounov function. In the next section we shall enlarge on this definition of the Liapounov function. In this section we will be concerned with the determination of the equilibrium points.

An inspection of the equations of motion, Eqs. (5.4) through 5.13), reveals that equilibrium positions exist when the generalized displacements and velocities assume the values

$$
\begin{array}{lll}
\dot{R}_c = 0 & R_c = \text{const.} & \\
\dot{\Phi} = \Omega_0 = \text{const.} & \Phi = \text{arbitrary}^* & \\
\dot{\Theta} = 0 & \Theta = 0 & \\
\dot{\theta}_1 = 0 & \theta_1 = i\,\pi/2 & (i = 0, 1, 2, \ldots) \\
\dot{\theta}_2 = 0 & \theta_2 = j\,\pi/2 & (j = 0, 1, 2, \ldots) \\
\dot{\varphi} = 0 & \varphi = k\,\pi/2 & (k = 0, 1, 2, \ldots) \\
\dot{v}_1 = 0 & v_1 = 0 & \\
\dot{v}_2 = 0 & v_2 = 0 & \\
\dot{w}_1 = 0 & w_1 = 0 & \\
\dot{w}_2 = 0 & w_2 = 0 &
\end{array}
\qquad (7.1)
$$

Although mathematically θ_1, θ_2, and φ can assume values which are any integer multiples of $\pi/2$, consideration of the physical system shows that Eqs. (7.1) represent only three distinct equilibrium configurations. These configurations denoted E_1, E_2, and E_3 are shown in Figure 5.



E_1 : $\theta_1 = 0$, $\theta_2 = 0$, $\varphi = 0$

E_2 : $\theta_1 = 0$, $\theta_2 = \pi/2$, $\varphi = 0$

E_3 : $\theta_1 = -\pi/2$, $\theta_2 = 0$, $\varphi = 0$

Figure 5

There may be equilibrium positions other than the ones given by Eqs. (7-1). This possibility has not been fully explored yet. The

*Examining Eq. (5.4) we conclude that the Φ may be arbitrary because Φ is an ignorable or cyclic coordinate which provides immediately an integral of the motion. This subject will be discussed further in a subsequent section.

present report discusses the equilibrium positions shown in Figure 5.

We note, however, that the angular velocities $\dot{\theta}_2$ and $\dot{\varphi}$ are colinear in the equilibrium position E_3. The analysis of this case would involve indeterminate expressions, as will be shown later. To avoid this difficulty we shall use the rotational coordinates ψ, θ, and φ in analyzing this case. On the other hand the coordinates ψ, θ, and φ would lead to the same problem in the analysis of configuration E_1. For the case E_2 either set of angular coordinates can be used. We shall use the coordinates θ_1, θ_2, and φ for the cases E_1 and E_2 and the coordinates ψ, θ, and φ for case E_3.

Noting that $\dot{\psi} = \Omega_o = \text{const}$, where Ω_o is the orbital velocity, and that the body, in the equilibrium configuration, does not have any angular velocity about the axes x, y, and z, we conclude that cases E_1, E_2, and E_3 correspond to the gravity-gradient stabilization case in which the body has an angular velocity Ω_o with respect to an inertial space but no angular velocity with respect to an orbiting frame of reference. The case in which the body, in the equilibrium configuration, does have an angular velocity with respect to the orbiting frame of reference is presently under investigation and will not be discussed here.

8. The Case of Zero Motion Relative to the Orbiting Frame of Reference. Liapounov Stability Analysis

The three dynamical equilibrium configurations of the preceding section can all be identified as cases of zero motion relative to the orbiting reference system x, y, and z. We shall examine the stability in these configurations in a Liapounov sense. The direct, or second method of Liapounov allows one to make a statement as to the stability of the perturbed motion in the neighborhood of an equilibrium position without having to solve for the explicit form of the perturbed motion. This method is very useful when the solution of the differential equations is not possible and a stability statement suffices. This is often the case in dynamical problems described by nonlinear differential equations. Following is a brief description of the method.

We shall be concerned with the solution of a set of autonomous differential equations*.

$$\dot{x}_i(t) = f_i\,(x_1, x_2, \ldots x_{2n}), \quad (i = 1, 2, \ldots 2n) \tag{8.1}$$

in the neighborhood of the origin.**

The Liapounov theorem of stability states that: If there exists a differentiable function $V_L\,(x_1, x_2, \ldots x_{2n})$ known as a Liapounov function, that satisfies the conditions

$$1)\quad V_L(x_1, x_2, \ldots x_{2n}) \geq 0 \tag{8.2}$$

in a definable region surrounding the origin where the equal sign applies only at the origin (i.e. V_L is positive definite in the neighborhood of the origin with a relative minimum at the origin)

$$2)\quad \frac{d\,V_L}{d\,t} = \sum_{i=1}^{2n} \frac{\partial V_L}{\partial x_i}\,\dot{x}_i = \sum_{i=1}^{2n} \frac{\partial V_L}{\partial x_i}\, f_i(x_1, x_2, \ldots x_{2n}) \leq 0 \tag{8.3}$$

where $\frac{d\,V_L}{dt}$ is taken along an integral curve $x_i(t)$, $(i = 1, 2, \ldots 2n)$

*The functions f_i do not contain the time t explicitly.

**As mentioned previously, a simple transformation of coordinates can translate the origin of the system to any point in the phase space.

then the equilibrium point $x_i = 0$ $(i = 1, 2, \ldots 2n)$ is stable. The equilibrium will be asymptotically stable if $\frac{\partial V_L}{\partial t} \leq 0$ at all points in the neighborhood of the equilibrium and the equal sign applies only at the origin.

The main problem in a Liapounov analysis is to devise the Liapounov function. In order to obtain a clue we shall make a small digression into the area of classical mechanics.

The motion of an n-degree-of-freedom system can be described by n generalized coordinates q_i, $(i = 1, 2, \ldots, n)$. When the Lagrangian L does not depend explicitly on one of these coordinates, say q_s, then one can immediately obtain one integral of the differential equations which, in effect, reduces the number of degrees of freedom by one. The integral in this case is the generalized momentum $\pi_s = \frac{\partial L}{\partial \dot{q}_s}$.

The coordinate q_s is called ignorable and the procedure for reducing the degree of freedom of the system is called the Routh process for the ignoration of coordinates*.

Let us consider a dynamical system for which the Lagrangian is a function of m generalized displacements and n generalized velocities, with $m < n$. In other words the system possesses $s = n - m$ ignorable (also called cyclic) coordinates q_i, $(i = m+1, \ldots n)$. This requires the assumption that the dissipation function D, which is a quadratic function of the generalized velocities, does not depend on the generalized velocities associated with the ignorable coordinates. We shall assume that this is the case so that the system can be described by the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0, \quad (i = 1, 2, \ldots, m) \tag{8.4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = 0 \quad , \quad (s = m+1, \ldots, n) \tag{8.5}$$

The n-m equations for the ignorable coordinates, (Eqs. (8.5), can be integrated immediately to obtain

*See Whittaker, E. T., Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Doner Publications, New York, 1944, p. 54.

$$\frac{\partial L}{\partial \dot{q}_s} = \alpha_s = \text{const.}, \quad (s = m+1, \ldots, n) \tag{8.6}$$

where

α_s, $(s = m+1, m+2, \ldots, n)$, are integral invariants of the system, the values of which are determined by the initial conditions.

The time rate of change of the Lagrangian can be written as

$$\frac{dL}{dt} = \sum_{i=1}^{m} \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \sum_{s=m+1}^{n} \frac{\partial L}{\partial \dot{q}_s} \frac{d\dot{q}_s}{dt} + \frac{\partial L}{\partial t} \tag{8.7}$$

and if the Lagrangian is not an explicit function of time, $\frac{\partial L}{\partial t} = 0$. Introducing Eqs. (8.4) and (8.5) into Eq. (8.7) we obtain

$$\frac{d}{dt}\left[\sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \sum_{s=m+1}^{n} \alpha_s \dot{q}_s - L\right] = -\sum_{i=1}^{m} \frac{\partial D}{\partial \dot{q}_i} \dot{q}_i \tag{8.8}$$

The expression in parenthesis is defined as the Hamiltonian function H.

Next, define the Routhian of the system by

$$R = \sum_{s=m+1}^{n} \alpha_s \dot{q}_s - L \tag{8.9*}$$

and use Eqs. (8.6) to eliminate the generalized velocities associated with the ignorable coordinates q_s, $(s = m+1, \ldots, n)$, from Eq. (8.9) so that the functional dependence of the Routhian can be written in the form

$$R = R\,(q_i, \dot{q}_i, \alpha_s) \tag{8.10}$$

where the latter form of the Routhian contains only the m non-ignorable coordinates and the n-m constants α_s. The Lagrange equations of motion can be shown to assume the form

*In most texts on classical mechanics, the Routhian is defined as the negative of the one given by Eq. (8.9).

$$-\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right)+\frac{\partial R}{\partial q_i}+\frac{\partial D}{\partial \dot{q}_i}=0, \quad (i=1,2,\ldots,n) \tag{8.11}$$

and there are only m equations instead of n. Of course, the reason for the reduced number of equations lies in the fact that we already have the n-m integrals, α_s, in our possession. Correspondingly, the Hamiltonian assumes the form

$$H = H(q_i, \dot{q}_i, \alpha_s) = R - \sum_{i=1}^{m} \frac{\partial R}{\partial \dot{q}_i}\dot{q}_i \tag{8.12}$$

so that

$$\frac{dH}{dt} = -\sum_{i=1}^{m} \frac{\partial D}{\partial \dot{q}_i}\dot{q}_i = -c\sum_{i=1}^{m} \dot{q}_i^{\,2} < 0 \tag{8.13}$$

The above discussion of classical mechanics leads us to investigate the Hamiltonian, Eq. (8.12) as a possible Liapounov function. Eq. (8.13) shows us that the Eulerian derivative of the Hamiltonian is negative or zero over all configuration space, satisfying the second requirement of the Liapounov stability theorem, Eq. (8.3). If it can be shown that the Hamiltonian is positive definite in the neighborhood of an equilibrium point and has a relative minimum at the equilibrium point, thus satisfying the first requirement of the Liapounov stability theorem, Eq. (8.2), then the motion will be stable in the neighborhood of the equilibrium point.

Next let us define a Hessian matrix in the form of the partitioned matrix

$$\mathcal{H}^E = \left[\begin{array}{c|c} \dfrac{\partial^2 H}{\partial q_i\, \partial q_j} & \dfrac{\partial^2 H}{\partial \dot{q}_i\, \partial q_j} \\ \hline \dfrac{\partial^2 H}{\partial q_i\, \partial \dot{q}_j} & \dfrac{\partial^2 H}{\partial \dot{q}_i\, \partial \dot{q}_j} \end{array}\right]_{\substack{\text{at equilibrium}\\ \text{point E}}}, \quad (i,j = 1, 2, \ldots, m) \tag{8.14}$$

The elements of the Hessian matrix are equal to twice the coefficients of the quadratic terms in a Taylor series expansion of the Hamiltonian about the equilibrium E. It can be shown that, for small q_i and $\dot{q}_i$ in the neighborhood of the equilibrium E, the Hamiltonian function will be positive definite if the Hessian matrix is positive definite. It can also be shown that the existance of an equilibrium point at E guarantees an extremum of the Hamiltonian function at E. (More specifically, the first order partial derivatives of the Hamiltonian with respect to the generalized displacements and generalized velocities are zero for an autonomous system at an equilibrium position.) Further, the extremum at E must be a relative minimum when the Hamiltonian is positive definite in the neighborhood of E.

Due to the fact that the Hessian matrix is evaluated at an equilibrium point, after some slightly elaborate manipulations, it can be shown that the matrix can be written in the simpler form

$$\mathcal{H}^E = \left[\begin{array}{c|c} \mathcal{H}_1^E & 0 \\ \hline 0 & \mathcal{H}_2^E \end{array}\right]_{\text{at } E} \tag{8.15}$$

where

$$\mathcal{H}_1^E = \left[\frac{\partial^2 T}{\partial \dot{q}_i \, \partial \dot{q}_j}\right]_{\text{at } E}, \qquad (i, j = 1, 2, \ldots n) \tag{8.16}$$

$$\mathcal{H}_2^E = \left[-\frac{\partial^2 L}{\partial q_i \, \partial q_j}\right]_{\text{at } E}$$

The matrix is positive definite if both $\mathcal{H}_1^E$ and $\mathcal{H}_2^E$ are positive definite. But $\mathcal{H}_1^E$ is the matrix of the coefficients of the kinetic energy expression and hence, by definition, positive definite. It remains to show that $\mathcal{H}_2^E$ is positive definite.

According to Sylvester's criterion, a matrix is positive definite if all of the principal determinants of the matrix are positive*. Hence the matrix $\mathcal{H}_2^E$ is positive definite if

$$-\left|\frac{\partial^2 L}{\partial q_i \, \partial q_j}\right|_{\text{at } E} > 0 \qquad \begin{array}{l} (i, j = 1, 2, \ldots, r) \\ (r = 1, 2, \ldots, n) \end{array} \tag{8.17}$$

Equations (8.17) represent sufficient conditions for the stability of the motion in the neighborhood of the equilibrium positions in question.

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*See Malkin, I.G., Theory of Stability of Motion (translation from Russian). AEC-tr-3352, Office of Technical Services, Department of Commerce, Washington 25, D.C., 1952 (Russian original), p. 19.

A large number of authors* assume, in their work on the subject, that the center of mass of the system moves in a circular orbit at a constant angular velocity Ω_o and that there is no interaction between the orbital motion and the rotational motion of the body about the center of mass. When the body contains moving parts, such as the oscillators, the center of mass shifts constantly relative to the x y z system. In this case it is more convenient to use as orbital constraints the assumptions that the origin of the x y z system moves in a circular orbit with the angular velocity Ω_o. If that is done we conclude that there is interaction between the orbital motion and the rotational motion of the body about the origin of the x y z system. To show this the problem is solved in two ways: first by using the orbital constraints and second by relaxing these constraints. Furthermore, the stability criteria are not entirely the same for both cases. They become approximately the same if the mass of the oscillators is small relative to the total mass of the system.

*See Likins, Pringle, etc.

9. Stability Criteria for the System with Orbital Constraints

We shall first discuss the case in which the origin of the x, y, z system is constrained to move in a circular orbit*. Mathematically, this can be expressed by the following constraint equations

$$
\begin{aligned}
R_c &= \text{const.}, & \dot{R}_c &= 0 \\
\dot{\phi} &= \Omega_0 = \text{const.}, & \ddot{\phi} &= 0 \qquad (9.1) \\
\theta &= 0 = \text{const.}, & \dot{\theta} &= 0
\end{aligned}
$$

With these assumptions the Lagrangian reduces to

$$
\begin{aligned}
L = \tfrac{1}{2} (A' \Omega_x^2 + B' \Omega_y^2 + C \Omega_z^2) + \tfrac{1}{2} m \Big\{ & \dot{v}_1^2 + \dot{v}_2^2 + \dot{w}_1^2 + \dot{w}_2^2 \\
& + \Omega_x^2 (w_1^2 + w_2^2) + \Omega_y^2 (v_1^2 + v_2^2) + \Omega_z^2 (v_1^2 + v_2^2 + w_1^2 + w_2^2) \\
& - 2 \Omega_x a (\dot{w}_1 - \dot{w}_2) + 2 \Omega_y a (\dot{v}_1 + \dot{v}_2) - 2 \Omega_z (\dot{v}_1 w_1 - v_1 \dot{w}_1 + \dot{v}_2 w_2 - v_2 \dot{w}_2) \\
& - 2 \Omega_x \Omega_y (v_1 w_1 + v_2 w_2) - 2 \Omega_y \Omega_z a (w_1 - w_2) \\
& - 2 \Omega_x \Omega_z a (v_1 - v_2) \Big\} + m \Omega_0 R_c \Big\{ \left[\dot{v}_1 + \dot{v}_2 - \Omega_z (w_1 + w_2) \right] l_{xb} \\
& + \left[\dot{w}_1 + \dot{w}_2 + \Omega_z (v_1 + v_2) \right] l_{yb} + \left[\Omega_x (w_1 + w_2) \right. \\
& \left. - \Omega_y (v_1 + v_2) \right] l_{zb} \Big\} - \frac{K m}{R_c^2} \left[(v_1 + v_2) l_{xa} + (w_1 + w_2) l_{ya} \right] \\
& + \frac{3 K}{4 R_c^3} \left[(C + B' - A') l_{xa}^2 + (A' + C - B') l_{ya}^2 + (B' + A' - C) l_{za}^2 \right] \\
& - \frac{K m}{2 R_c^3} (v_1^2 + v_2^2 + w_1^2 + w_2^2) + \frac{3 K m}{2 R_c^3} \Big[(v_1^2 + v_2^2) l_{xa}^2 \\
& + (w_1^2 + w_2^2) l_{ya}^2 + 2 a (v_1 - v_2) l_{xa} l_{za} + 2 a (w_1 - w_2) l_{ya} l_{za}
\end{aligned}
$$

---

*Note that for a rigid body the origin of the x, y, z axes coincides with the center of mass of the body.

$$+ 2(v_1 w_1 + v_2 w_2)\, l_{xa}\, l_{ya}\Big] - \tfrac{1}{2} k(v_1^2 + v_2^2 + w_1^2 + w_2^2) + \text{const.} \tag{9.2}$$
[*]

and from Eqs. (2.2) we obtain the angular velocity components in terms of the angular coordinates θ_1, θ_2, and φ in the form

$$\begin{aligned} \Omega_x &= \Omega_0 (- s\theta_2\, c\varphi - c\theta_2\, s\theta_1\, s\varphi) + \dot\theta_2\, c\theta_1\, s\varphi - \dot\theta_1\, c\varphi \\ \Omega_y &= \Omega_0 (s\theta_2\, s\varphi - c\theta_2\, s\theta_1\, c\varphi) + \dot\theta_2\, c\theta_1\, c\varphi + \dot\theta_1\, s\varphi \\ \Omega_z &= \Omega_0\, c\theta_2\, c\theta_1 + \dot\theta_2\, s\theta_1 + \dot\varphi \end{aligned} \tag{9.3}$$

and in a similar way for the ψ, θ, φ coordinates we obtain

$$\begin{aligned} \Omega_x &= (\Omega_0 + \dot\psi)\, s\theta\, s\varphi + \dot\theta\, c\varphi \\ \Omega_y &= (\Omega_0 + \dot\psi)\, s\theta\, c\varphi - \dot\theta\, s\varphi \\ \Omega_z &= (\Omega_0 + \dot\psi)\, c\theta + \dot\varphi \end{aligned} \tag{9.4}$$

For the constrained case there are no ignorable coordinates so that, from Eqs. (8.9) and (8.2), the Hamiltonian becomes

$$H = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q_i}\, \dot q_i - L$$

which can be written explicitly for the θ_1, θ_2, φ system in the form

$$\begin{aligned} H = {} & \frac{\partial L}{\partial \dot\theta_1}\, \dot\theta_1 + \frac{\partial L}{\partial \dot\theta_2}\, \dot\theta_2 + \frac{\partial L}{\partial \dot\varphi}\, \dot\varphi + \frac{\partial L}{\partial \dot v_1}\, \dot v_1 + \frac{\partial L}{\partial \dot w_1}\, \dot w_1 \\ & + \frac{\partial L}{\partial \dot v_2}\, \dot v_2 + \frac{\partial L}{\partial \dot w_2}\, \dot w_2 - L \end{aligned} \tag{9.6}$$

---

*The constant includes constant terms from the potential as well as the kinetic energy, as a result of the constraints.

and since

$$\frac{\partial}{\partial \theta_1} = -c\varphi \frac{\partial}{\partial \Omega_x} + s\varphi \frac{\partial}{\partial \Omega_y}$$

$$\frac{\partial}{\partial \theta_2} = c\,\theta_1\, s\varphi \frac{\partial}{\partial \Omega_x} + c\varphi\, c\theta_1 \frac{\partial}{\partial \Omega_y} + s\theta_1 \frac{\partial}{\partial \Omega_z} \tag{9.7}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \Omega_z}$$

Eq. (9.6) becomes

$$\begin{aligned}
H = {} & (-\dot\theta_1\, c\varphi + \dot\theta_2\, c\theta_1\, s\varphi) \frac{\partial L}{\partial \Omega_x} + (\dot\theta_1\, s\varphi + \dot\theta_2\, c\theta_1\, c\varphi) \frac{\partial L}{\partial \Omega_y} \\
& + (\dot\theta_2\, s\theta_1 + \dot\varphi) \frac{\partial L}{\partial \Omega_z} + \frac{\partial L}{\partial \dot v_1} \dot v_1 + \frac{\partial L}{\partial \dot w_1} \dot w_1 + \frac{\partial L}{\partial \dot v_2} \dot v_2 + \frac{\partial L}{\partial \dot w_2} \dot w_2 - L \\
= {} & \frac{\partial L}{\partial \Omega_x} \Omega_x + \frac{\partial L}{\partial \Omega_y} \Omega_y + \frac{\partial L}{\partial \Omega_z} \Omega_z + \frac{\partial L}{\partial \dot v_1} \dot v_1 + \frac{\partial L}{\partial \dot w_1} \dot w_1 \\
& + \frac{\partial L}{\partial \dot v_2} \dot v_2 + \frac{\partial L}{\partial \dot w_2} \dot w_2 - L - \Omega_0 \Big\{ -(s\theta_2\, c\varphi + c\theta_2\, s\theta_1\, s\varphi) \frac{\partial L}{\partial \Omega_x} \\
& + (s\theta_2\, s\varphi - c\theta_2\, s\theta_1\, c\varphi) \frac{\partial L}{\partial \Omega_y} + c\theta_2\, c\theta_1 \frac{\partial L}{\partial \Omega_z} \Big\} = T + V \\
& - \Omega_0 \Big\{ -(s\theta_2\, c\varphi + c\theta_2\, s\theta_1\, s\varphi) \frac{\partial L}{\partial \Omega_x} + (s\theta_2\, c\varphi - c\theta_2\, c\theta_1\, c\varphi) \frac{\partial L}{\partial \Omega_y} \\
& + c\theta_2\, c\theta_1 \frac{\partial L}{\partial \Omega_z} + mR_c \left[\dot v_1 + \dot v_2 - \Omega_z (w_1 + w_2) \right] 1_{xb} + mR_c \left[\dot w_1 + \dot w_2 \right. \\
& \left. + \Omega_z (v_1 + v_2) \right] 1_{yb} + mR_c \left[\Omega_x (w_1 + w_2) - \Omega_z (v_1 + v_2) \right] 1_{zb} \Big\}
\end{aligned} \tag{9.8}$$

We note that, as a result of constraints in the system, the Hamiltonian is not equal to the sum of the potential and kinetic energy as might be expected (see Eq. 10.3). It should be remembered that the constrained system does not exist in reality and has been employed only as a mathematical convenience.

In a similar way we obtain the Hamiltonian for the coordinates ψ, θ, and φ is the form

$$H = T + V - \Omega_0 \Big\{ s\theta\ s\varphi \frac{\partial L}{\partial \Omega_x} + s\theta\ c\varphi \frac{\partial L}{\partial \Omega_y} + c\theta \frac{\partial L}{\partial \Omega_z}$$

$$+ mR_c \left[\dot{v}_1 + \dot{v}_2 - \Omega_z(w_1 + w_2)\right] 1_{xb} + mR_c \left[\dot{w}_1 + \dot{w}_2\right.$$

$$\left. + \Omega_z(v_1 + v_2)\right] 1_{yb} + mR_c \left[\Omega_x(w_1 + w_2) - \Omega_y(v_1 + v_2)\right] 1_{zb} \Big\} \qquad (9.9)$$

Again we conclude that the Hamiltonian is not equal to the sum of the kinetic and potential energies. This is always the case when the kinetic energy expression contains terms which are linear in the generalized velocities.

Letting $q_1, q_2, \ldots, q_7$ be equal to ϑ_2, v_1, v_2, θ_1, w_1, w_2, and φ respectively, we obtain from the second of Eqs (8.16)

$$\mathcal{H}_2^{F1} = \Omega_0^2 \begin{bmatrix} 4(C-A') & -4\ ma & -4\ ma & 0 & 0 & 0 & 0 \\ -4\ ma & m(r^2-3) & 0 & 0 & 0 & 0 & 0 \\ 4\ ma & 0 & m(r^2-3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (C-B') & -ma & ma & 0 \\ 0 & 0 & 0 & -ma & mr^2 & 0 & 0 \\ 0 & 0 & 0 & ma & 0 & mr^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(B'-A') \end{bmatrix} \qquad (9.10)$$

where $r = \omega_h/\Omega_0$

Applying Sylvester's criterion we conclude that the equilibrium E_1 is asymptotically stable if the following inequalities are satisfied

(a) $\quad 4(C - A') > 0$

(b) $\quad 4m(C - A')(r^2 - 3) - 16\ m^2a^2 > 0$

$$\text{(c)} \qquad \left[4m(C - A')(r^2 - 3) - 32\, m^2 a^2\right] m\,(r^2 - 3) > 0$$

$$\text{(d)} \qquad (C - B') > 0 \qquad (9.11)$$

$$\text{(e)} \qquad (C - B')\, mr^2 - m^2 a^2 > 0$$

$$\text{(f)} \qquad \left[(C - B')\, mr^2 - 2\, m^2 a^2\right] m r^2 > 0$$

$$\text{(g)} \qquad 3\,(B' - A') > 0$$

Not all of Eqs. (9.11) are independent. It can be shown that they can be reduced to the following conditions

$$C > B' > A'$$

$$r^2 > \frac{3(C - A') + 8\, ma^2}{(C - A')} \qquad (9.12)$$

$$r^2 > \frac{2\, ma^2}{(C - B')})$$

It can be noted that as the mass of the oscillators reduces to zero the last two of Eqs. (9.12) become meaningless. On the other hand, however, the first of Eqs. (9.12) reduces to

$$C > B > A \qquad (9.13)$$

which implies that stability is obtained if x is the axis of minimum moment of inertia. Examining the position E_1 in Figure 5 we conclude that in this case the axis of maximum moment of inertia is normal to the orbital plane whereas the axis of minimum moment of inertia is aligned with the radial direction of the center of force, which is precisely the stability condition for the gravity-gradient stabilization of a rigid body.

In a similar way, by maintaining the same order of the generalized coordinates in the Hessian matrix, we obtain for the E_2 equilibrium position

$$\mathcal{H}_2^{E_2} = \Omega_0^2 \begin{bmatrix} 4(A'-C) & 4ma & -4ma & 0 & 0 & 0 & 0 \\ 4\ ma & m(r^2+1) & 0 & 0 & 0 & 0 & 0 \\ -4\ ma & 0 & m(r^2+1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3(B'-C) & 3\ ma & -3\ ma & 0 \\ 0 & 0 & 0 & 3\ ma & mr^2 & 0 & 0 \\ 0 & 0 & 0 & -3\ ma & 0 & mr^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (A'-B') \end{bmatrix} \tag{9.14}$$

which leads to the conditions

$$\begin{aligned} & A' > B' > C \\ & r^2 > \frac{6\ ma^2}{(B'-C)} \\ & r^2 > \frac{8\ ma^2 - (A' - C)}{(A' - C)} \end{aligned} \tag{9.15}$$

The rigid body counterpart of the first of Eqs. (9.15) is the case in which x is the axis of maximum moment of inertia and z is the axis of minimum moment of inertia. Noting that Ω_x for this case is equal to $-\Omega_0$ we conclude that this again leads to the gravity-gradient stabilization case.

Finally, for the E_3 equilibrium position, we use as generalized coordinates ψ, v_1, v_2, θ, w_1, w_2, and φ, in that order and obtain

$$\mathcal{H}_2^{E_3} = \Omega_0^2 \begin{bmatrix} 3(C-A') & -3\ ma & 3\ ma & 0 & 0 & 0 & 0 \\ -3\ ma & m(r^2-3) & 0 & 0 & 0 & 0 & 0 \\ 3\ ma & 0 & m(r^2-3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (B'-C) & -ma & ma & 0 \\ 0 & 0 & 0 & -ma & r^2+1 & 0 & 0 \\ 0 & 0 & 0 & ma & 0 & r^2+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4(B'-A) \end{bmatrix} \tag{9.16}$$

for which the stability conditions are

$$B' > C > A'$$

$$r^2 > \frac{3(C - A') + 6\, ma^2}{(C - A')} \tag{9.17}$$

$$r^2 > \frac{2\, ma^2 - (B' - C)}{(B' - C)}$$

The rigid body case can easily be shown to conform to the previous conclusions.

10. The Unconstrained System

In contrast with the constrained case, in the unconstrained case we make no assumptions regarding the motion of the origin of the x, y, z axes. The Hamiltonian for the θ_1, θ_2, φ coordinates is

$$H = \frac{\partial L}{\partial \dot{R}_c}\dot{R}_c + \frac{\partial L}{\partial \dot{\Phi}}\dot{\Phi} + \frac{\partial L}{\partial \dot{\Theta}}\dot{\Theta} + \frac{\partial L}{\partial \dot{\theta}_1}\dot{\theta}_1 + \frac{\partial L}{\partial \dot{\theta}_2}\dot{\theta}_2 + \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi}$$

$$+ \frac{\partial L}{\partial \dot{v}_1}\dot{v}_1 + \frac{\partial L}{\partial \dot{w}_1}\dot{w}_1 + \frac{\partial L}{\partial \dot{v}_2}\dot{v}_2 + \frac{\partial L}{\partial \dot{w}_2}\dot{w}_2 - L \tag{10.1}$$

whereas the Hamiltonian for the ψ, θ, φ coordinates is

$$H = \frac{\partial L}{\partial \dot{R}_c}\dot{R}_c + \frac{\partial L}{\partial \dot{\Phi}}\dot{\Phi} + \frac{\partial L}{\partial \dot{\Theta}}\dot{\Theta} + \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi}$$

$$+ \frac{\partial L}{\partial \dot{v}_1}\dot{v}_1 + \frac{\partial L}{\partial \dot{w}_1}\dot{w}_1 + \frac{\partial L}{\partial \dot{v}_2}\dot{v}_2 + \frac{\partial L}{\partial \dot{w}_2}\dot{w}_2 - L \tag{10.2}$$

Introducing Eqs. (3.6), (3.10), (3.11), and (5.2) into Eq. (10.1) (or Eq. 10.2) and using Eqs. (2.2) (or Eqs. (2.4)) it can be shown that

$$H = T + V \tag{10.3}$$

as expected.

Examining the Lagrangian we notice that it is independent of Φ. It also must be noticed that $\dot{\Phi}$ is not contained in the dissipation function D so that Φ is an ignorable coordinate and the derivations of Section 8 apply here. Hence Eqs. (8.6) leads to

$$\frac{\partial L}{\partial \dot{\Phi}} = \alpha_\Phi = \text{const.} \tag{10.4}$$

which can be solved for $\dot{\Phi}$. Upon introducing the value of $\dot{\Phi}$, thus obtained, into the Lagrangian we reduce the degree of freedom by one.

The procedure of the preceding two sections can be applied to the unconstrained case. For the equilibrium configurations E_1 and E_2 we use the generalized coordinates in the order R_c, Θ, θ_2, v_1, v_2, θ_1, w_1, w_2 , and φ and for configuration E_3 the coordinates R_c, ψ, v_1, v_2, θ, θ, φ, w_1, and w_2 to derive the Hessian matrices $\mathcal{K}_2^{E1}$, $\mathcal{K}_2^{E2}$, and $\mathcal{K}_2^{E3}$. These matrices have the form

$$
E_1 = \Omega_0^2 \begin{bmatrix}
M'\dfrac{M'R_c^2 - C}{M'R_c^2 + C} & 0 & 0 & -m + \dfrac{4mM'R_c^2}{M'R_c^2 + C} & -m + \dfrac{4mM'R_c^2}{M'R_c^2 + C} & 0 & 0 & 0 & 0 \\
0 & M'R_c^2 - A' + C & A'-C & ma & -ma & 0 & 0 & 0 & 0 \\
0 & A'-C & 4(C-A') & -4\,ma & 4\,ma & 0 & 0 & 0 & 0 \\
-m + \dfrac{4mM'R_c^2}{M'R_c^2 + C} & ma & -4\,ma & m(r^2-3) + \dfrac{4m^2R_c^{\,2}}{M'R_c^2 + C} & \dfrac{4m^2R_c^{\,2}}{M'R_c^2 + C} & 0 & 0 & 0 & 0 \\
-m + \dfrac{4mM'R_c^2}{M'R_c^2 + C} & -ma & 4\,ma & \dfrac{4m^2R_c^{\,2}}{M'R_c^2 + C} & m(r^2-3) + \dfrac{4m^2R_c^{\,2}}{M'R_c^2 + C} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C-B' & -ma & ma & 0 \\
0 & 0 & 0 & 0 & 0 & -ma & mr^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & ma & 0 & mr^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3(B'-A')
\end{bmatrix} \qquad (10.5)
$$

$$
E_2 = \Omega_0^2 \begin{bmatrix}
M'\dfrac{M'R_c^2-A'}{M'R_c^2+A'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & M'R_c^2-C+A' & C-A' & -m(R_c+a) & -m(R_c-a) & 0 & 0 & 0 & 0 \\
0 & C-A' & -4(C-A') & 4\,ma & -4\,ma & 0 & 0 & 0 & 0 \\
0 & -m(R_c+a) & 4\,ma & m(r^2+1) & 0 & 0 & 0 & 0 & 0 \\
0 & -m(R_c-a) & -4\,ma & 0 & m(r^2+1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3(C-B') & 3\,ma & -3\,ma & 0 \\
0 & 0 & 0 & 0 & 0 & 3\,ma & m\,r^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3\,ma & 0 & m\,r^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A'-B'
\end{bmatrix} E \qquad (10.5)
$$

$$E_3 = \Omega_0^2 \begin{bmatrix} M'\frac{M'R_c^2-B'}{M'R_c^2+B'} & 0 & \frac{4m^2R_c^2}{M'R_c^2+B'}-m & \frac{4m^2R_c^2}{M'R_c^2+B'}-m & 0 & 0 & 0 & 0 & 0 \\ 0 & 3(C-A') & -3\,ma & 3\,ma & 0 & 0 & 0 & 0 & 0 \\ \frac{4m^2R_c^2}{M'R_c^2+B'}-m & -3\,ma & m(r^2-3)+\frac{4m^2R_c^2}{M'R_c^2+B'} & \frac{4m^2R_c^2}{M'R_c^2+B'} & 0 & 0 & 0 & 0 & 0 \\ \frac{4m^2R_c^2}{M'R_c^2+B'}-m & 3\,ma & \frac{4m^2R_c^2}{M'R_c^2+B'} & m(r^2-3)+\frac{4m^2R_c^2}{M'R_c^2+B'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M'R_c^2-A'+B' & 0 & (A'-B')\left(1-\frac{2M'R_c^2}{M'R_c^2+B'}\right) & mR_c & mR_c \\ 0 & 0 & 0 & 0 & 0 & B'-C & 0 & -ma & ma \\ 0 & 0 & 0 & 0 & (A'-B')\left(1-\frac{2M'R_c^2}{M'R_c^2+B'}\right) & 0 & 4(B'-A') & 0 & 0 \\ 0 & 0 & 0 & 0 & mR_c & -ma & 0 & m(r^2+1) & 0 \\ 0 & 0 & 0 & 0 & mR_c & ma & 0 & 0 & m(r^2+1) \end{bmatrix} \quad (10.7)$$

where $r = \omega_h/\Omega_0$ and $M' = 2m + M$.

Applying Sylvester's criterion to Eq. (10.5) we obtain for the position E_1 the stability conditions

$$\begin{aligned}
&C > B' > A' \\
&r^2 > 3 + 10\,\frac{m}{2m + M} \\
&(C-A')\left[(r^2 - 3) - \frac{5\,m}{2m + M}\right] > 4\,ma^2 \qquad (10.8)\\
&(C-A')(r^2 - 3) > 8\,ma^2 \\
&r^2 > \frac{2\,ma^2}{C - B'},
\end{aligned}$$

Similarly the stability conditions for the equilibrium configuration E_2 are

$$\begin{aligned}
&A' > B' > C \\
&r^2 > \frac{2m}{2m + M} - 1 \\
&r^2 > \frac{6\,ma^2}{(B'-C)} \qquad (10.9)\\
&(A'-C)\left[(r^2 + 1) - \frac{m}{2m + M}\right] > 4\,ma^2 \\
&(A'-C)\,(r^2 + 1) > 8\,ma^2
\end{aligned}$$

and for the equilibrium configuration E_3

$$B' > C > A'$$

$$r^2 > 3 + \frac{10\,m}{2m + M}$$

$$(C - A')\left[(r^2 - 3) - \frac{3m}{2m + M}\right] > 3\,ma^2$$

$$(C - A')\,(r^2 - 3) > 6\,ma^2 \qquad (10.10)$$

$$(B' - C)\left[(r^2 + 1) - \frac{m}{2m + M}\right] > ma^2$$

$$(B' - C)\,(r^2 + 1) > 2\,ma^2$$

The satisfaction of the conditions of Eqs. (10.8), (10.9), or (10.10) ensures that the motion is asymptotically stable in the neighborhood of the equilibrium configurations E_1, E_2, or E_3, respectively.

Comparing the inequalities, Eqs. (9.12) and (10.8), we see that Eqs. (10.8) contain all of the criteria specified in Eqs. (9.12) and, furthermore, they contain two additional conditions. Therefore, it must be concluded that the constrained and unconstrained cases yield different criteria for the stability of motion in the neighborhood of the equilibrium configuration E_1. When the ratio $\frac{m}{2m + M}$ is small, however, we notice that the two additional conditions, the second and third of Eqs. (10.8), are approximately duplicated by the fourth of Eqs. (10.8).. Hence we conclude that, if the mass of the oscillators is small relative to the total mass of the system, the assumptions of the constrained system can be regarded as approximately valid for the purpose of deriving stability criteria for the motion of the unconstrained system about the equilibrium position E_1. Similar conclusions can be drawn in connection with the validity of the constrained model in the analysis of stability of motion about the equilibrium positions E_2 and E_3.

Although the above conclusions as to the validity of the orbital constraints assumptions were based on a specific mathematical model, there are reasons to believe that the conclusions are valid under broader conditions. At any rate, this investigation certainly points out a problem area which should not be overlooked when the orbital constraints assumptions are used in deriving stability criteria.

11. Summary and Conclusions

The stability of motion of a satellite that orbits the earth is under investigation. The mathematical model consists of a rigid body with principal moments of inertia A, B, C about a set of axes x, y, and z which are fixed with respect to the rigid body, and two viscously damped oscillators each consisting of a mass m, a spring k, and a dashpot c as shown in Figure 1. The origin of the x, y, z axes moves in a circular orbit while the body undergoes rotational motion with respect to an inertial space. The motion of the system can be represented by ten generalized coordinates, three for the motion of the origin of the x y z axes, three for the rotational motion of these axes relative to an inertial space, and four for the motion of the two oscillators.

In addition to the inertial axes and the x y z axes which are fixed with respect to the rigid body, a set of axes orbiting with the body is used. The stability of motion in the neighborhood of positions for which the body axes are aligned with the orbiting frame of reference has been investigated and stability criteria have been derived by means of Liapounov's second, or direct, method.

Almost all of the investigators working on this problem assumed no interaction between the orbital motion and the stability of the rotational motion. These assumptions are referred to as orbital constraints. In addition, the above investigators were concerned with rigid bodies. The present investigation deals with rigid bodies containing elastic parts and orbital constraints, if any, are applied to the origin of the x y z system. Due to the elastic motion this origin may be slightly different than the center of mass of the system. Analyses for both cases, with and without orbital constraints, were performed. These analyses reveal that while the orbital constraints assumptions are valid for a rigid body they are not valid for a body with elastic parts. These assumptions are, however, approximately valid if the mass of the elastic part of the satellite is relatively small relative to the total mass of the satellite. The reason for the use of orbital constraints is, of course, that they simplify the analysis.

Stability criteria have been derived for three equilibrium configurations. These criteria are shown to reduce to the gravity-gradient stabilization case if the mass of the oscillators reduces to zero.

The next stage of the investigation deals with the examination of stability of motion when the body has rotational motion relative to the orbiting frame of reference. Successful treatment of this case will require the development of new techniques for establishment of stability in systems that are characterized by differential equations with periodic coefficients.

A discussion of additional problems of satellite stability which are related to the present work and appear worthy of future research programs is included in the introduction.