On the class of admissible nonlinearities for Lur'e's Problem

by

Allan M. Krall 1,2

Let us consider the system \( N \):

\[
X' = AX + \mu S ,
\]

\[
\mu = \phi(\sigma),
\]

\[
\sigma = \langle C, X \rangle,
\]

where \( X = (x_j(t)) \) is a variable \( n \) vector, \( S = (s_j) \) and \( C = (c_j) \) are constant \( n \) vectors, \( A = (a_{ij}) \) is a constant \( n \times n \) matrix, \( \sigma \) is a scalar, \( \langle C, X \rangle = \sum_{j=1}^{n} c_j x_j \), and \( \phi(\sigma) \) is in general a nonlinear function of \( \sigma \). We set our problem in an appropriate \( L^p \) space with \( L^p \) norm. Multiplication by \( A \) is then a bounded transformation with suitable norm. We assume that the norm of \( S \) is 1 in \( L^p \) and that the norm of \( C \) is also 1 in the dual space. We denote all these norms by \( \| \cdot \| \).

For a system such as \( N \), stability means that \( X \) remains bounded for

---

1 McAllister Building, The Pennsylvania State University, University Park, Pennsylvania, 16802

2 This research was supported in part by NASA Grant NGR 39-009-041.
all $t > 0$, and asymptotic stability means that $X$ approaches zero as $t$ approaches infinity. We wish to impose conditions on $\mathcal{O}$ so that the system will be stable or asymptotically stable.

We assume that the linear system $L$:

$$Y' = AY + \mu S,$$

$$\mu = h \sigma,$$

$$\sigma = \langle C, Y \rangle,$$

is stable whenever $k_1 < h < k_2$. Note that this system can be written as $Y' = BY$ where the matrix $B = (b_{ij}) = (a_{ij} + hs_i c_j)$.

**Lemma.** Consider the matrix solution of $Y' = BY, Y(0) = I$ as an operator on $\ell^p$ to $\ell^p$. If all of the characteristic roots of $B$ lie in the left half of the complex plane, then there exist constants $a > 0$ and $b > 0$ such that $\| Y \| < ae^{-bt}$.

If all of the characteristic roots of $B$ lie in the left half of the complex plane or as simple roots on the imaginary axis, then there exists a constant $a > 0$ such that $\| Y \| < a$.

See Bellman [2; p.36].

Let us now approximate the nonlinearity $\mathcal{O}(\sigma)$ by $\alpha \sigma + \Psi(\sigma)$, where $k_1 < \alpha < k_2$.

Note that the linear system $L$ with $h = \alpha$ is asymptotically stable, and that by the lemma, there exist constants $a > 0$ and $b > 0$ such that $Y(t)$, the solution of $L$ with $h = \alpha$ satisfying $Y(0) = I$, satisfies $\| Y(t) \| < ae^{-bt}$. 
Theorem. If \( \varphi(\sigma) \) satisfies
\[
(\alpha - \frac{b}{\sigma}) < \varphi(\sigma)/\sigma < (\alpha + \frac{b}{\sigma})
\]
then \( N \) is asymptotically stable.

If \( \varphi(\sigma) \) satisfies
\[
(\alpha - \frac{b}{\sigma}) \leq \varphi(\sigma)/\sigma \leq (\alpha + \frac{b}{\sigma})
\]
then \( N \) is stable.

Proof. \( N \) is equivalent to
\[
X' = BX + \mu S,
\]
\[
\mu = \psi(\sigma),
\]
\[
\sigma = \langle C, X \rangle.
\]

Thus
\[
X(t) = Y(t)X(0) + \int_0^t Y(t-\tau)\psi(\langle C, X(\tau) \rangle) S \, d\tau.
\]

By the lemma,
\[
\|X(t)\| \leq ae^{-bt} \|X(0)\| + \int_0^t ae^{-b(t-\tau)} |\psi(\langle C, X(\tau) \rangle)| \|S\| \, d\tau.
\]

Now if \( \varphi(\sigma) \) satisfies the hypothesis of the theorem, then
\[
|\psi(\sigma)| \leq \beta |\sigma|, \text{ where } \beta \leq \frac{b}{\sigma}.
\]
Thus
\[
|\psi(\langle C, X(\tau) \rangle)| \leq \beta |\langle C, X(\tau) \rangle|,
\]
\[
|\psi(-C, X(\tau))| \leq \beta \|C\| \|X(\tau)\|.
\]

Since \( \|S\| = 1 \), \( \|C\| = 1 \), multiplying by \( e^{bt} \), we have
By Gronwall's inequality [2; p. 35],
\[ e^{bt} \| X(t) \| \leq a \| X(0) \| + \int_0^t a^\beta e^{b\tau} \| X(\tau) \| d\tau. \]

Remarks. 1. A can have characteristic values in the right half plane. That is, the system with no feedback \( \emptyset \) may be unstable.

2. \( a \) and \( b \) depend upon \( \alpha \). As \( \alpha \) approaches \( k_1 \) or \( k_2 \), \( b/a \) must approach zero. Further, \( \alpha - k_1 \geq b/a \) and \( k_2 - \alpha \geq b/a \).

3. Pliss [6] has shown by example that one cannot expect the interval of stability [\( \alpha - b/a, \alpha + b/a \)] to entirely fill the interval \([k_1, k_2] \). (See also Aizerman and Gantmacher [1].)

4. The optimal choice of \( \alpha \) has not yet been determined.
References


