On the class of admissable nonlinearities
for Lur'e's Problem

by

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Let us consider the system N:

$$X' = AX + \mu S \ ,$$
$$\mu = \phi(\sigma) \ ,$$
$$\sigma = \langle C, X \rangle \ ,$$

where $X = (x_j(t))$ is a variable n vector, $S = (s_j)$ and $C = (c_j)$ are constant n vectors, $A = (a_{ij})$ is a constant $n \times n$ matrix, σ is a scaler, $\langle C, X \rangle = \sum_{j=1}^{n} c_j x_j$, and $\phi(\sigma)$ is in general a nonlinear function of σ. We set our problem in an appropriate ℓ^p space with ℓ^p norm. Multiplication by A is then a bounded transformation with suitable norm. We assume that the norm of S is 1 in ℓ^p and that the norm of C is also 1 in the dual space. We denote all these norms by $\| \cdot \|$.

For a system such as N, stability means that X remains bounded for

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all $t > 0$, and asymptotic stability means that X approaches zero as t approaches infinity. We wish to impose conditions on $\emptyset$ so that the system will be stable or asymptotically stable.

We assume that the linear system L:

$$Y' = AY + \mu S \,,$$

$$\mu = h\,\sigma \,,$$

$$\sigma = \langle C, Y \rangle \,,$$

is stable whenever $k_1 < h < k_2$. Note that this system can be written as $Y' = BY$ where the matrix $B = (b_{ij}) = (a_{ij} + hs_i c_j)$.

__Lemma__. __Consider the matrix solution of__ $Y' = BY$, $Y(o) = I$ __as an operator on__ ℓ^p __to__ ℓ^p. __If all of the characteristic roots of__ B __lie in the left half of the complex plane, then there exist constants__ $a > 0$ __and__ $b > 0$ __such that__ $\| Y \| < ae^{-bt}$.

__If all of the characteristic roots of__ B __lie in the left half of the complex plane or as simple roots on the imaginary axis, then there exists a constant__ $a > 0$ __such that__ $\| Y \| < a$.

See Bellman [2; p.36].

Let us now approximate the nonlinearity $\emptyset(\sigma)$ by $\alpha\sigma + \psi(\sigma)$, where $k_1 < \alpha < k_2$.

Note that the linear system L with $h = \alpha$ is asymptotically stable, and that by the lemma, there exist constants $a > 0$ and $b > 0$ such that $Y(t)$, the solution of L with $h = \alpha$ satisfying $Y(o) = I$, satisfies $\| Y(t) \| < ae^{-bt}$.

<u>Theorem.</u> <u>If $\phi(\sigma)$ satisfies</u>

$$(\alpha - \frac{b}{a}) < \phi(\sigma)/\sigma < (\alpha + \frac{b}{a}) \quad ,$$

<u>then N is asymptotically stable.</u>

<u>If $\phi(\sigma)$ satisfies</u>

$$(\alpha - \frac{b}{a}) \leqq \phi(\sigma)/\sigma \leqq (\alpha + \frac{b}{a}) \quad ,$$

<u>then N is stable.</u>

Proof. N is equivalent to

$$X' = BX + \mu S \ ,$$

$$\mu = \psi(\sigma) \ ,$$

$$\sigma = < C,\ X > .$$

Thus

$$X(t) = Y(t)\ X(o) + \int_0^t Y(t-\tau)\psi(< C,\ X(\tau) >)\ S\ d\tau.$$

By the lemma,

$$\|X(t)\| \leqq ae^{-bt}\ \|X(o)\| + \int_0^t ae^{-b(t-\tau)}|\psi(<C,X(\tau)>)|\ \|S\|d\tau.$$

Now if $\phi(\sigma)$ satisfies the hypothesis of the theorem, then $|\psi(\sigma)| \leqq \beta\ |\sigma|$, where $\beta \leqq \frac{b}{a}$. Thus

$$|\psi(<C,\ X(\tau)>)| \leqq \beta\ |<C,\ X(\tau)>|,$$

$$|\psi(-C,\ X(\tau)>| \leqq \beta\ \|\ C\ \|\ \|\ X(\tau)\|.$$

Since $\|\ S\ \| = 1$, $\|\ C\ \| = 1$, multiplying by e^{bt}, we have

$$e^{bt} \| X(t) \| \leqq a \| X(o) \| + \int_o^t a\beta e^{b\tau} \| X(\tau) \| d\tau.$$

By Gronwall s inequality [2; p. 35],

$$e^{bt} \| X(t) \| \leqq a \| X(o) \| e^{a\beta t} , \text{ and}$$

$\| X(t) \| \leqq a \| X(o) \| e^{(a\beta - b)t}$. If $\beta < \frac{b}{a}$, $\| X(t) \|$ approaches zero exponentially. If $\beta = \frac{b}{a}$, $\| X(t) \|$ is bounded.

Remarks. 1. A can have characteristic values in the right half plane. That is, the system with no feedback ϕ may be unstable.

2. a and b depend upon α. As α approaches k_1 or k_2, b/a must approach zero. Further, $\alpha - k_1 \geqq b/a$ and $k_2 - \alpha \geqq b/a$. $k_2 - k_1 \geqq 2 \frac{b}{a}$.

3. Pliss [6] has shown by example that one cannot expect the interval of stability $[\alpha - b/a , \alpha + b/a]$ to entirely fill the interval $[k_1, k_2]$. (See also Aizerman and Gantmacher [1].)

4. The optimal choice of α has not yet been determined.

References

1. M. A. Aizerman and F. R. Gantmacher, "Absolute stability of regulator systems," Holden-Day, Inc., San Francisco, 1964.

2. Richard Bellman, "Stability theory of differential equations," McGraw-Hill Book Co., Inc., New York, 1953.

3. Joseph LaSalle and Solomon Lefschetz, "Stability by Liapunov's direct method with applications," Academic Press, New York, 1961.

4. Solomon Lefschetz, "Stability of non-linear control systems," Academic Press, New York, 1965.

5. V. M. Popov. "On the absolute stability of non-linear control systems," Auto. i Telemekh. vol. 22, no. 8, 1961.

6. V. A. Pliss, "Certain problems in the theory of stability in the whole," Published by L. G. U. 1958.