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Principal Investigator and Contractor: Prof.F.Cap

Institute of Theoretical Physics, Computer Department

University of Innsbruck, Innsbruck, Austria.

Numerical Computation of Satellite Orbits Using Lie Series. Comparison with other Methods.

by H. Knapp

This report, covering older work, gives a survey on the application and the advantages of the Lie series method in celestial mechanics. It was reprinted on the basis of a request of Dr. Wilson, Applied Mathematics Branch, NASA.

Numerical Computation of Satellite Orbits Using the Lie Series Method. Comparison With Other Methods.

by H. Knapp, Electronic Computer Department, University of Innsbruck, Austria.

Introduction

Using the Lie series theory the formal solution of the astronomical n-body problem in a region where no collisions take place, is easy. It could be demonstrated by a special example (J. Kovalevsky chose this example to test the Lie series method for celestial mechanics) that after the transformation given by W. Groebner*) the Lie-series converge so rapidly that the method in its present form can be successfully employed for calculating the orbits in celestial mechanics. This method of solution is particularly flexible and very general, and good estimates can be given since the theoretical expansions and estimations can be directly applied to general multi-body problems.

Chapter I

Presentation of the problems

§ 1 Preparation

1) Coordinate system: Our calculations are based on the following coordinate system: Let the center of mass of the three celestial bodies be the origin. Due to the vanishingly small mass of the 8th moon of Jupiter, it lies on the connection line Sun - Jupiter. Let the x-axis indicate the direction of the ascending node of Jupiter for the year 195o, let the y-axis be rotated in the direction of Jupiter motion by 90° relative to the x-axis in the Jupiter orbital plane, let the z-axis be directed such that we have an orthogonal

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*) See W. Groebner, Die Lie Reihen und ihre Anwendungen, VEB Deutscher Verlag der Wissenschaften, Berlin 196o, p. 92, Formel (12.3 e).

right-handed system. This coordinate system is then assumed to be an inertial system since only in such a system Newton's law of gravitation holds in the simple form. This may be regarded as fulfilled within the accuracy of calculation required here (up to and inclusive of the 9th significant figure of each step).

2) Designations: For reasons of simplicity we use vectors, thus, e.g.

$\vec{x} = \{x, y, z\}$	is a position vector
$\vec{u} = \{u, v, w\}$	is a velocity vector
$\vec{x}\,\vec{u} = xu + yv + zw$	is the scalar product
$\lvert\vec{x}\rvert = \sqrt{x^2 + y^2 + z^2}$	is the absolute amount
$[\vec{x}.\vec{u}] = \{yw-zv,\ zu-xw,\ xv-yu\}$	is the vector product
$\frac{\partial}{\partial\vec{x}} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$	is the gradient symbol

e.g.:

$$\vec{u}\,\frac{\partial}{\partial\vec{x}} = u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z}$$

Furthermore, we use the following designations:

	Sun	Jupiter	8th moon
position vectors	$\vec{x}_3$	$\vec{x}_2$	$\vec{x}_1$
velocities	$\vec{u}_3$	$\vec{u}_2$	$\vec{u}_1$
masses	M_3	M_2	M_1
mass numbers	m_3	m_2	m_1

f is the gravitational constant and $m_i = fM_i$ holds.

All quantities occurring in our calculations are assumed to be differentiable. The three celestial bodies, the Sun, Jupiter and its eighth satellite are assumed to be replaced by mass points which are subject to gravitation according to Newton's law.

The positions and velocities

$$\vec{x}_i(t_o) = \vec{x}_i^{(o)} \qquad \text{and} \qquad \vec{u}_i(t_o) = \vec{u}_i^{(o)}$$

of the three celestial bodies are given for the initial moment $t = t_o$. The 18 components of the vectors $\vec{x}_i$ and $\vec{u}_i$ $(i = 1, 2, 3)$ are to be determined as functions of time such that the mass points move according to the laws of a three-body problem.

3) Units:

Unit length	1 L = 1 astronomical unit = 1495,04200 . 10^{10} cm
unit time	1 d = 1 mean solar day
unit velocity	1 Ld^{-1}
unit mass	1 μ = mass of the Sun

In these units the gravitational constant f assumes the numerical value:

$$f = 0{,}29591220828559 \cdot 10^{-3}\ \mu^{-1}\ L^3\ d^{-2} \quad *)$$

mass numbers:

$$m_3 = 0{,}295912208 \cdot 10^{-3}\ L^3\ d^{-2}$$

$$m_2 = 0{,}282532864 \cdot 10^{-6}\ L^3\ d^{-2} = m_3 : 1047{,}355$$

$$m_1 = 0 \text{ (vanishingly small as compared to } m_2 \text{ and } m_3\text{)}$$

4) Equations of motion of the mechanical system: According to the general theorems of mechanics we obtain the following system of differential equations for the three-body problem: **)

$$(1.1) \qquad \begin{cases} \dot{\vec{x}}_i = u_i \\ \dot{\vec{u}}_i = - \dfrac{1}{M_i} \dfrac{\partial U}{\partial x_i} \end{cases} \qquad (i = 1, 2, 3)$$

with $\displaystyle U = - \sum_{i<k} f \frac{M_i M_k}{r_{ik}} \qquad r_{ik} = |\vec{x}_i - \vec{x}_k|$

(the dot denotes differentiation with respect to the time t)

Let the operator belonging to the differential equations (1.1) be designated by D;

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*) This and all other numerical values are taken from a paper by J. Kovalevsky. Since we are concerned with the explanation of the method rather than with the values themselves the problem of their accuracy is of minor importance.

**) See W. Groebner, Die Lie Reihen und ihre Anwendungen, p.71 ff.

Since $m_1 = 0$ it has the following form:

$$D = \vec{u}_1 \frac{\partial}{\partial \vec{x}_1} + \vec{u}_2 \frac{\partial}{\partial \vec{x}_2} + \vec{u}_3 \frac{\partial}{\partial \vec{x}_3} + \left[\frac{m_2(\vec{x}_2 - \vec{x}_1)}{r_{12}^3} + \frac{m_3(\vec{x}_3 - \vec{x}_1)}{r_{13}^3} \right] \frac{\partial}{\partial \vec{u}_1} + \tag{1.2}$$
$$+ \frac{m_3(\vec{x}_3 - \vec{x}_2)}{r_{23}^3} \frac{\partial}{\partial \vec{u}_2} + \frac{m_2(\vec{x}_2 - \vec{x}_3)}{r_{23}^3} \frac{\partial}{\partial u_3}$$

5) Known integrals of the system:

Law of conservation of energy

$$E_{kin} + E_{pot} - \frac{1}{2} (M_2 \vec{u}_2{}^2 + M_3 \vec{u}_3{}^2) + U = \text{const, since } D(E_{kin} + E_{pot}) = 0 \tag{1.3}$$

Law of conservation of angular momentum:

$$\vec{P} = M_2 \left[\vec{x}_2 . \vec{u}_2\right] + M_3 \left[\vec{x}_3 . \vec{u}_3\right] = \text{const, since } D\vec{P} = 0 \tag{1.4}$$

Conservation of center of gravity:

$$\vec{x}_S = \frac{1}{m} (m_2 \vec{x}_2 + m_3 \vec{x}_3) \text{ with } m = m_2 + m_3$$

is the position of the center of mass of the three bodies.
Since $D^2 \vec{x}_S = 0$ and owing to the special selection of the coordinate system $\vec{x}_S = 0$ is valid for all times*): the center of gravity rests in the origin of the coordinate system. Hence we have:

$$\vec{x}_S = 0 \text{ and } \vec{u}_S = 0 \qquad \text{or:} \quad \begin{aligned} m_2 \vec{x}_2 + m_3 \vec{x}_3 &= 0 \\ m_2 \vec{u}_2 + m_3 \vec{u}_3 &= 0. \end{aligned} \tag{1.5}$$

The nine components of the vectors $\vec{x}_S$, $\vec{U}_S$, $\vec{P}$ and the constant energy (1.3) are the 1o algebraic integrals of the problem. With these 1o relations between the 18 unknown components of the vectors $\vec{x}_i$ and $\vec{u}_i$ ($i = 1, 2, 3$) the number of unknown functions could be reduced to eight. In our example the conservation laws for energy and angular momentum refer only to the partial problem Sun - Jupiter and permit its complete integration. With the aid of (1.5) however, the six unknown quantities

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*) This choice does not restrict generality. See W. Groebner, Die Lie Reihen und ihre Anwendungen p. 75

can be easily eliminated and the motion can then be described by only two position- and two velocity vectors: $\vec{x}_s$ and $\vec{x}_m$, $\vec{u}_s$ and $\vec{u}_m$.

6) Transformation of variables:

$$(1.6)\qquad \begin{cases} \vec{x}_s = \vec{x}_3 - \vec{x}_2 \\ \vec{x}_m = \vec{x}_1 - \vec{x}_2 \end{cases} \qquad \begin{cases} \vec{u}_s = \vec{u}_3 - \vec{u}_2 \\ \vec{u}_m = \vec{u}_1 - \vec{u}_2 \end{cases}$$

Due to (1.5) this transformation is always reversible:

$$(1.7)\qquad \begin{cases} \vec{x}_1 = -\frac{m_3}{m}\vec{x}_s + \vec{x}_m \\ \vec{x}_2 = -\frac{m_3}{m}\vec{x}_s \\ \vec{x}_3 = \frac{m_2}{m}\vec{x}_s \end{cases} \qquad \begin{cases} \vec{u}_1 = -\frac{m_3}{m}\vec{u}_s + \vec{u}_m \\ \vec{u}_2 = -\frac{m_3}{m}\vec{u}_s \\ \vec{u}_3 = \frac{m_2}{m}\vec{u}_s \end{cases}$$

The converted operator (1.2) has the following form:

$$(1.8)\qquad \begin{aligned} D = {} & \vec{u}_s \frac{\partial}{\partial \vec{x}_s} + \vec{u}_m \frac{\partial}{\partial \vec{x}_m} - \frac{m}{|\vec{x}_s|^3}\vec{x}_s \frac{\partial}{\partial \vec{u}_s} - \frac{m_2}{|\vec{x}_m|^3}\vec{x}_m \frac{\partial}{\partial \vec{u}_m} + \\ & + m_3 \left[\frac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \frac{\vec{x}_s}{|\vec{x}_s|^3} \right] \frac{\partial}{\partial \vec{u}_m} \end{aligned}$$

§ 2 Formulation of the problem

We now have to integrate the system of differential equations

$$(2.1)\qquad \begin{aligned} \dot{\vec{x}}_s &= \vec{u}_s \\ \dot{\vec{x}}_m &= \vec{u}_m \\ \dot{\vec{u}}_s &= -\frac{m}{|\vec{x}_s|^3}\vec{x}_s \\ \dot{\vec{u}}_m &= -\frac{m_2}{\vec{x}_m^{\,3}}\vec{x}_m + m_3 \left[\frac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \frac{\vec{x}_s}{|\vec{x}_s|^3} \right] \end{aligned}$$

which belongs to the operator (1.8) under the initial conditions

$\vec{x}_s(t_c) = \vec{x}_s^{(o)}$, $\vec{x}_m(t_o) = \vec{x}_m^{(o)}$, $\vec{u}_s(t_o) = \vec{u}_s^{(o)}$, and $\vec{u}_m(t_c) = \vec{u}_m^{(o)}$

which are to be calculated from the initial conditions $\vec{x}_i(t_o)$ and $\vec{u}_i(t_o)$ for i = 1, 2, 3 according to the formulas (1.6).

The solution can be easily obtained by Lie series:

If f(t) is an arbitrary function holomorphic in the neighborhood of $t = t_o$ of the twelve sought components of the vectors $\vec{x}_s$, $\vec{x}_m$, $\vec{u}_s$, and $\vec{u}_m$, then the Lie series

$$(2.2) \qquad f(t) = \left[e^{(t-t_o)D} f \right]^{(o)} = \sum_{\nu=o}^{\infty} \frac{(t-t_o)^\nu}{\nu!} \left\{ D^\nu f \right\}^{(o)}$$

holds.

The superscript zero denotes that after application of the operator D instead of the variable components of $\vec{x}_s$, $\vec{x}_m$, $\vec{u}_s$, and $\vec{u}_m$ the components of the constant initial values $\vec{x}_s^{(o)}$, $\vec{x}_m^{(o)}$, $\vec{u}_s^{(o)}$, and $\vec{u}_m^{(o)}$ are to substituted. The trajectories are obtained by writing down this formula for the vectors $\vec{x}_s(t)$ and $\vec{x}_m(t)$ and by analytically continuing the series. In this form, the solutions can, however, not be used for numerical purposes since the series converge too weakly. (This has been distinctly shown by J. Kovalevsky in a comparison with the Cowell method). Hence a transformation is necessary: First, we determine an approximate orbit which is then corrected by a perturbation calculation.

Chapter II

Solution of the problem:

3 Sun - Jupiter as an unperturbed two-body problem

1) Splitting of the operator: We shall now split D into two components:

$$(3.1) \qquad D = D_s + \overline{D} \qquad \text{where}$$

$$(3.2) \qquad D_s = \vec{u}_s \frac{\partial}{\partial \vec{x}_s} - \frac{m}{|\vec{x}_s|^3} \vec{x}_s \frac{\partial}{\partial \vec{u}_s}$$

while the remaining terms of the operator (1.8) are denoted by $\overline{D}$.

2) Calculation of $\vec{x}_s(t)$: The partial operator D_s out of the total operator D will solely act, if in the place of functions depending only on $\vec{x}_s$ and $\vec{u}_s$, but not depending on $\vec{x}_m$ and $\vec{u}_m$, are substituted into the final formula (2.2). Thus, we have, for instance,

$$(3.3) \qquad \vec{x}_s(t) = \left[e^{(t-t_o)D} \vec{x}_s\right]^{(o)} = \left\{e^{(t-t_o)D_s} \vec{x}_s\right\}^{(o)}$$

and the problem visualized by the partial operator D_s can be solved separately. We may say: The variables $\vec{x}_s$ and $\vec{u}_s$ are separated from $\vec{x}_m$ and $\vec{u}_m$ since they do not depend on these. - D_s is, however, the operator of the unperturbed two-body problem Sun - Jupiter. We shall give the solution together with the respective numerical data in Chapter III.

§ 4 Construction of the approximative orbit of the eight satellite of Jupiter

1) Further splitting of the operator: It would be most natural to split up $\overline{D}$ in such a way that its essential part again is the operator of a two-body problem in this case of the fictive two-body problem Jupiter - satellite. Rather voluminous intermediate calculations, which may be a large source of accumulating rounding errors, are required for the determination of the Kepler ellipse as an approximative orbit (particularly in the reversal of Kepler's equation!). In order to avoid these we have decided on calculating with a simpler, although less accurate approximative orbit.

We shall split the operator

$$(4.1) \qquad D = D_s + D_m + \Lambda_m$$

The abbreviations mean

$$(4.2)\quad \begin{cases} D_m = \vec{u}_m \dfrac{\partial}{\partial \vec{x}_m} - c^2 \vec{x}_m \dfrac{\partial}{\partial \vec{u}_m} \\ \Delta_m = \vec{\delta}_m \dfrac{\partial}{\partial u_m} \end{cases}$$

where

$$(4.3)\quad c^2 = \frac{m_2}{|\vec{x}_m^{(o)}|^3}$$

The perturbation function $\vec{\delta}_m$ has the form

$$(4.4)\quad \vec{\delta}_m = \vec{\delta}_{m_I} + \vec{\delta}_{m_{II}}$$

with

$$(4.4')\quad \begin{cases} \vec{\delta}_{m_I} = m_3 \left[\dfrac{\vec{x}_s - \vec{x}_m}{|\vec{x}_s - \vec{x}_m|^3} - \dfrac{\vec{x}_s}{|\vec{x}_s|^3} \right] \\ \vec{\delta}_{m_{II}} = \left[c^2 - \dfrac{m_2}{|\vec{x}_m|^3} \right] \vec{x}_m \end{cases}$$

2) Rough estimation of the order of magnitude of $\vec{\delta}_m$:

(a) if $\vec{x}_s$ and $-\vec{x}_m$, respectively, are substituted in the place of $\vec{a}$ and $\vec{b}$ in the formula

$$(4.5)\quad \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|^3} = (\vec{a} + \vec{b}) \sum_{\nu=0} \binom{-3/2}{\nu} \frac{(2\vec{a}\vec{b} + \vec{b}^2)^\nu}{|\vec{a}|^{2\nu+3}}$$

we obtain for $\vec{\delta}_{m_I}$ an expansion into a series by means of which the order of magnitude can be estimated more easily than by means of the expression (4.4') for $\vec{\delta}_{m_I}$ which contains differences of approximately equal orders:

$$(4.6)\quad \vec{\delta}_{m_I} = - \frac{m_3}{|\vec{x}_s|^3} \left[\vec{x}_m - \frac{3\vec{x}_s \vec{x}_m}{|\vec{x}_s|^2} + \frac{3\vec{x}_s \vec{x}_m}{|\vec{x}_s|^2} \vec{x}_m + \ldots\ldots \right].$$

If we consider the first two terms of the series jointly and observe that

$$\left| \vec{x}_m - \frac{3\vec{x}_s\vec{x}_m}{|\vec{x}_s|^2}\,\vec{x}_s \right| \leq 2|\vec{x}_m| ,$$

$|\vec{x}_s| > 4.95$ L and o.o5 L $< |\vec{x}_m| <$ o.25 L, we will have in the most unfavorable case

$$(4.7) \qquad |\vec{\delta}_{m_I}|_{max} \approx \frac{2m_3}{|\vec{x}_s|^3}\,|\vec{x}_m| < 1.22 \cdot 10^{-6}\ Ld^{-2}.$$

(b) $\vec{\delta}_{m_{II}}$ is less favorable to handle. If we transform $\vec{\delta}_{m_{II}}$ in such a way that the Kepler ellipse relations enter the formula as an approximative orbit we find that

$$(4.8) \qquad |\vec{\delta}_{m_{II}}|_{max} \approx 4.05 \cdot 10^{-6}\ Ld^{-3} . |\Delta t|$$

where $\Delta t = t - t_o$ is the length of the concerned step of calculation. However, we shall not go into these details.

3) <u>Relative orbit of the satellite with respect to Jupiter</u>

We shall first neglect Δ_m in comparison to D_m, since then also the variables $\vec{x}_m$ and $\vec{u}_m$ are separated from $\vec{x}_s$ and $\vec{u}_s$. In this way, the problem represented by the operator D_m may be solved separately. The resulting approximative orbit of couse deviates from the true orbit, owing to (4.7) and (4.8). We should note, however, that extremely unfavorable conditions have been assumed in these estimations; the figures in (4.7) and (4.8) will be smaller in general!

The solution of the systems of differential equations

$$(4.9) \qquad \begin{cases} \dot{\vec{x}}_{ma} = \vec{u}_{ma} \\ \dot{\vec{u}}_{ma} = - c^2\,\vec{x}_{ma} \end{cases}$$

with the operator $D_{ma} = \vec{u}_{ma} \frac{\partial}{\partial \vec{x}_{ma}} - c^2\,\vec{x}_{ma} \frac{\partial}{\partial \vec{u}_{ma}}$ and with the initial values

$$(4.10)\qquad \vec{x}_m^{(o)} = \vec{x}_{ma}^{(o)} \quad \text{and} \quad \vec{u}_m^{(o)} = \vec{u}_{ma}^{(o)}$$

for the moment t_o are obtained in the form of the rather simple approximative orbit (ellipse)

$$(4.11)\qquad \begin{cases} \vec{x}_{ma}(t) = \left[e^{(t-t_o)D_{ma}}\, x_{ma}\right]^{(o)} = \vec{x}_m^{(o)}\cos\left[c(t-t_o)\right] + \vec{u}_m^{(o)}\frac{1}{c}\sin\left[c(t-t_o)\right] \\ \vec{u}_{ma}(t) = \left[e^{(t-t_o)D_{ma}}\, u_{ma}\right]^{(o)} = -\vec{x}_m^{(o)}c\sin\left[c(t-t_o)\right] + \vec{u}_m^{(o)}\cos\left[c(t-t_o)\right] \end{cases}$$

(The additional subscript a is to indicate that these approximative functions, in difference from the sought exact solutions $\vec{x}_m$ and $\vec{u}_m$ of the original three-body problem.)

The connection with time t is evident; the reversal of a Kepler equation is superfluous.

5 Solution of the three-body problem by means of the given approximative orbit; perturbation calculus.

1) Transformation of the solution (2.2): With the new symbol

$$(5.1)\qquad D_1 = D_s + D_m$$

we have

$$(5.2)\qquad f(t) = \sum_{\nu=0}^{\infty} \frac{(t-t_o)^\nu}{\nu!}\left[(D_1 + \Delta_m)f\right]^{(o)}.$$

Expanding $(D_1 + \Delta_m)^\nu$, ordering according to the positions of Δ_m, and applying the exchange theorem to the Lie series, one obtains the formula (siehe W. Groebner; Die Lie-Reihen und ihre Anwendungen p. 92, Formel (12.3e))

$$(5.3)\qquad f(t) = f_a(t) + \sum_{\alpha=0}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!}\left[\Delta_m D^\alpha f(t)\right]_a d\tau,$$

which is very important for the subsequent calculations. This formula expresses how the approximative solution $f_a(t)$ has to be modified in order to yield a solution of the original problem. The expression

$$\left[\Delta_m D^\alpha f(\tau)\right]_a$$

means that $\Delta_m D^\alpha f$ has to be calculated first, and that then the components of $\vec{x}_m$ and $\vec{u}_m$ have to be substituted by the components of the approximative solution $\vec{x}_{ma}(\tau)$ and $\vec{u}_{ma}(\tau)$.

2) Expansion of the essential terms in the series (5.3): We shall now substitute the required special functions $\vec{x}_m(t)$ and $\vec{u}_m(t)$ in the place of the general functions $f(t)$ in formula (5.3). - In the subsequent numerical computation we shall have to break the corresponding series and to confine ourselves to the essential terms. Of course, the accuracy of the result may be increased to any degree if more terms are taken into account. In the present instance, the following approximations may be sufficient:

$$(5.4)\quad \begin{cases} \vec{x}_m(t) = \vec{x}_{ma}(t) + \int_{t_o}^{t} (t-\tau)\,\vec{\delta}_{ma}(\tau)d\tau + \int_{t_o}^{t} \frac{(t-\tau)^3}{3!}\vec{\xi}_{ma}(\tau)d\tau \\ \vec{u}_m(t) = \vec{u}_{ma}(t) + \int_{t_o}^{t} \vec{\delta}_{ma}(\tau)d\tau + \int_{t_o}^{t} \frac{(t-\tau)^2}{2!}\vec{\xi}_{ma}(\tau)d\tau \end{cases}$$

with

$$(5.5)\quad \begin{aligned} \vec{\xi}_{ma} = &- \frac{m_2}{|\vec{x}_{ma}|^3}\left[\vec{\delta}_{ma} - \frac{3(\vec{x}_{ma}\vec{\delta}_{ma})}{|\vec{x}_{ma}|^2}\vec{x}_{ma}\right] \\ &- \frac{m_3}{|\vec{x}_s - \vec{x}_{ma}|^3}\left[\vec{\delta}_{ma} - \frac{3([\vec{x}_s - \vec{x}_{ma}]\vec{\delta}_{ma})}{|\vec{x}_s - \vec{x}_{ma}|^2}(\vec{x}_s - \vec{x}_{ma})\right]. \end{aligned}$$

Naturally, the formulas (5.4) are of use only as long as the time space $|t - t_o|$ is chosen so small that the further terms of the series may be neglected according to the required accuracy. (It is obvious that t may never be outside the region of convergence of the series.)

6 Estimation of the error due to breaking off the series

1) Region of validity of the formulas (5.4): We know from formula (5.4) that it is the solution of the problem (2.1) within a certain region of the t-plane. Within this region, the solution functions constructed by means of formula (5.4) have to satisfy the differential equations (2.1). If $\vec{x}_m(t)$ and $\vec{u}_m(t)$ are calculated from (5.4), one obtains

$$(6.1)\qquad \vec{u}_m(t) = -\frac{m_2}{|\vec{x}_{ma}(t)|^3}\,\vec{x}_{ma}(t) + \vec{\delta}_{ma_I}(t) + \vec{R}(t),$$

where

$$(6.2)\qquad \begin{aligned}\vec{R}(t) &= \sum_{\alpha=0}^{\infty}\int_{t_0}^{t}\frac{(t-\tau)^\alpha}{\alpha!}\Big[\Delta_m\, D^{\alpha+2}\,\vec{x}_m(\tau)\Big]_a\,d\tau = \\ &= \int_{t_0}^{t}(t-\tau)\vec{\xi}_{ma}(\tau)d\tau + \sum_{\alpha=2}^{\infty}\int_{t_0}^{t}\frac{(t-\tau)^\alpha}{\alpha!}\Big[\Delta_m D^{\alpha+2}\vec{x}_m(\tau)\Big]_a\,d\tau\end{aligned}$$

Comparison of (6.1) with (2.1) yields

$$(6.3)\qquad \vec{R}(t) = m_2\left[\frac{\vec{x}_{ma}(t)}{|\vec{x}_{ma}(t)|^3} - \frac{\vec{v}(t)}{|\vec{x}_m(t)|^3}\right] + \vec{\delta}_{m_I}(t) - \vec{\delta}_{ma_I}(t)$$

We shall make use of this in order to determine the order of magnitude of the expression $\vec{R}(t)$. With the abbreviation

$$(6.4)\qquad \vec{x}_m(t) = \vec{x}_{ma}(t) - \vec{\varepsilon}(t)$$

where

$$(6.5)\qquad \vec{\varepsilon}(t) = \int_{t_0}^{t}(t-\tau)\vec{\delta}_{ma}(\tau)d\tau + \int_{t_0}^{t}\left[\int_{t_0}^{\tau}\vec{R}(\xi)d\xi\right]d\tau$$

and with the aid of formula (4.5)*) we obtain

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*) The series converges for $\dfrac{|2\vec{x}_{ma}\vec{\varepsilon} + \vec{\varepsilon}^{\,2}|}{|\vec{x}_{ma}|^2} < 1$, which is certainly fulfilled in a region where formula (5.3) represents the solutions, when $|t - t_0| = |\Delta t|$ is chosen sufficiently small

$$(6.6) \qquad -m_2\left[\frac{\vec{x}_{ma}+\vec{\varepsilon}}{|\vec{x}_{ma}+\vec{\varepsilon}|^3}-\frac{\vec{x}_{ma}}{|\vec{x}_{ma}|^3}\right] = -\frac{m_2}{|\vec{x}_{ma}|^3}\left[\vec{\varepsilon}-\frac{3(\vec{x}_{ma}\vec{\varepsilon})}{|\vec{x}_{ma}|^2}\vec{x}_{ma} + \left(\begin{matrix}\text{terms of higher}\\ \text{order of } |\vec{\varepsilon}|\end{matrix}\right)\right]^{*)}$$

Substitution of (6.4) in (4.6) yields

$$(6.7) \qquad \vec{\delta}_{m_I}-\vec{\delta}_{ma_I} = -\frac{m_3}{|\vec{x}_s|^3}\left[\vec{\varepsilon}-\frac{3(\vec{x}_s\vec{\varepsilon})}{|\vec{x}_s|^2}\vec{x}_s + \left(\begin{matrix}\text{terms of higher}\\ \text{order of } |\vec{\varepsilon}|\end{matrix}\right)\right]$$

so that

$$(6.8) \qquad |\vec{R}(t)|_{max} \approx 2|\vec{\varepsilon}(t)|\left[\frac{m_2}{|\vec{x}_{ma}(t)|^3}+\frac{m_3}{|\vec{x}_s(t)|^3}\right] = 2|\vec{\varepsilon}(t)|K(t)$$

K(t) varies between

$$2.10^{-5}\ d^{-2}\ (\text{for large } |\vec{x}_m|) \text{ and } 2.2\cdot 10^{-3}\ d^{-2}\ (\text{for small } |\vec{x}_m|).$$

By virtue of

$$(6.9) \qquad |\vec{\varepsilon}(t)| < \frac{(t-t_o)^2}{2}\left[|\vec{\delta}_{ma}(t)|_{max}+|\vec{R}(t)|_{max}\right]$$

and with (6.8) we obtain in the most unfavorable case the following estimate for the order of magnitude of $|\vec{R}(t)|$:

$$(6.1o) \qquad |\vec{R}(t)|_{max} \approx \frac{(t-t_o)^2K(t)}{1-(t-t_o)^2K(t)}\,|\delta_{ma}(t)|_{max}.$$

This estimate is critical for $1-(t-t_o)^2K(t) = 0$, which means near the perijove for $|t-t_o| \approx 21$ d

---

*) The terms linear in $|\vec{\varepsilon}|$ are sufficient in estimating the order of magnitude.

near the apojove for $|t - t_o| \approx 220$ d

so that, as it was to be expected, the magnitude of the region of convergence of formula (5.3) depends strongly on the distance between the two celestial bodies. Formula (5.3) is valid in any case for a time space of at least 20 days.

In numerically evaluating the formula it will be desirable to chose the interval rather long. One has to be careful, however, not to come close to the edge of the region of convergence since then the rapid convergence of the series, which is desired in practice, will no longer be given.

2) Residue of the series after the second perturbation integral; choice of proper step length Δt: The comprehensive deliberations which have been made to estimate the expression

$$(6.11) \qquad \vec{R}_\rho(t) = \vec{R}(t) - \int_{t_o}^{t} (t-\tau)\vec{\zeta}_{ma}(\tau)d\tau = \sum_{\alpha=2}^{\infty} \int_{t_o}^{t} \frac{(t-\tau)^\alpha}{\alpha!} \left[\Delta_m D^{\alpha+2}\vec{x}_m(\tau)\right]_a d\tau$$

have shown that the step length needs never be shorter than 0.3 d if the error due to the breaking-off of the series in postulated in one step of calculation to amount to not more than 5.10^{-11} L in the case of $|\vec{x}_m|$ and to not more than 5.10^{-13} Ld^{-1} in the case of $|\vec{u}_m|$. Moreover, one may conclude that the breaking-off error after the second perturbation integral in first approximation amounts to

$$(6.12) \qquad \int_{t_o}^{t} \vec{R}_\rho(\tau)d\tau \approx \frac{t}{4}\vec{R}_\rho(t)$$

in the case of $\vec{u}_m$, and to

$$(6.13) \qquad \int_{t_o}^{t}\left[\int_{t_o}^{\tau} \vec{R}_\zeta(\xi)\,d\xi\right]d\tau \approx \frac{(\Delta t)^2}{20}\vec{R}_\zeta(t)$$

in the case of $\vec{x}_m$. Therefore, these quantities may be calculated at the end of each step*). After this one may determine the step length permissible at the prescribed accuracy.

In practice one will always stay somewhat below the accuracy limit, but will calculate several steps of equal length. Only when approaching this limit one will reduce the step length a little (or increase it if the absolute amounts of the expressions (6.12) and (6.13) have dropped below some certain value). If this is sensibly done by the computer one has nothing to do but to adjust the length of the first step. Obviously, this is of particular significance for calculation of rocket trajectories (when their approximate course is known, and when estimations according to the above pattern can be made only for short sections of the trajectory).

3) Propagation of the breaking-off error in the analytical continu-

---

*) The program-controlled SIE 2oo2 computer at the computing center of Aachen Technical University usually calculates with 1o decimal places only. In this way one can obtain only the order of magnitude of $\vec{R}_j(t)$. However, if the solution series are broken off after the first perturbation integral and if the corresponding calculations are carried out for $\vec{R}(t)$, one will obtain 2 or 3 figures of the components of $\vec{R}(t)$. If in analogy to (6.12) and (6.13) the expressions

$$(6.12') \quad \int_{t_0}^{t} \vec{R}(\tau)d\tau \Rightarrow \frac{\Delta t}{3}\,\vec{R}(t)$$

$$(6.13') \quad \int_{t_0}^{t}\left[\int_{t_0}^{\tau} \vec{R}(\xi)d\xi\right]d\tau = \frac{(\Delta t)^2}{12}\,\vec{R}(t)$$

are formed, and if these quantities are added as corrections to $\vec{x}_m$ and $\vec{u}_m$, respectively, one will obtain improved solutions. A checking calculation, also to ten digits, has shown that after 3o steps the result for $\vec{x}_m$ is exactly the same as that obtained when two perturbation integrals were taken into account. The result for $\vec{u}_m$ differed but insignificantly (rounding errors), but the time required for computation was only half as long! – The same procedure can be made with $\vec{R}(t)$ if the computation covers more than 1o digits.

ation of the solutions: The exact result of the analytical continuation of (5.3) after n steps will be denoted by $f^{[n]}$ throughout this paragraph. The result involving the breaking-off errors (we shall not be concerned with rounding errors) of the previous calculating steps (breaking-off after the second perturbation integral) will be termed $\bar{f}^{[n]}$. For the error quantities

$$(6.14)\quad \begin{cases} p_n = |\vec{x}_m^{[n]} - \bar{\vec{x}}_m^{[n]}| \\ q_n = |\vec{u}_m^{[n]} - \bar{\vec{u}}_m^{[n]}| \end{cases}$$

we obtain the recurrence formulas

$$(6.15)\quad \begin{cases} p_n < (1 + P_n)p_{n-1} + (1 + P_n)|\Delta t|_n\, q_{n-1} + \bar{p}_n \\ q_n < (1 + P_n)7c^2|\Delta t|_n\, p_{n-1} + (1 + P_n)q_{n-1} + \bar{q}_n \end{cases}$$

in which $\bar{p}_n$ denotes the amount of the error in $\vec{x}_m$ at the n-th step, due to breaking-off the series, $\bar{q}_n$ the amount of the breaking-off error in the series for $\vec{u}_m$ after the n-th step.

$$P_n < \frac{3}{2}\left[\frac{m_3}{|\vec{x}_s - \vec{x}_{ma}|^3} + \frac{m_2}{|\vec{x}_{ma}|^3}\right]_{max} |\Delta t|_n^2$$

(i.e. the maximum of this expression in the time interval of the n-th step of calculation).

The solution of the recurrence formulas may be written straightforward, if a good part of the path is computed with the same step length $|\Delta t|$, if the breaking-off errors $\bar{p}_i$ and $\bar{q}_i$ in the formulas (6.15) are replaced by their maximum values $\bar{p}$ and $\bar{q}$, and if P_n is replaced by the maximum P. Thus,

$$(6.16)\quad \begin{cases} p_n < \beta_1 e^{\alpha_1 n} + \beta_2 e^{\alpha_2 n} - p \\ q_n < \gamma_1 e^{\alpha_1 n} + \gamma_2 e^{\alpha_2 n} - q \end{cases}$$

where

$$(6.17)\quad \begin{cases} e^{\alpha_1} = (1 + P)\ (1 + \sqrt{7}\ c|\Delta t|) \\ e^{\alpha_2} = (1 + P)\ (1 - \sqrt{7}\ c|\Delta t|) \end{cases}$$

$$(6.18)\quad \begin{cases} p = \left[\bar{p}P - \bar{q}\ (1+P)|\Delta t|\right] k_1 \\ q = \left[\bar{q}P - \bar{p}\ (1+P)\ 7c^2|\Delta t|\right] k_1 \end{cases} \qquad k_1 = \frac{1}{P^2 - (1+P)^2 7c^2 |\Delta t|^2}$$

β_i and γ_i are the constants of the general solution of the recurrence formulas which make the adaptation to the initial conditions possible. With p^* being the error of the initial data of our calculation in $|\vec{x}_m|$ and q^* the error of the initial data in $|\vec{u}_m|$ we have the relations

$$(6.19)\quad \begin{cases} \beta_1 = k_2\left[(p + p^*)e^{\alpha_2} - a\right] \\ \gamma_1 = k_2\left[(q + q^*)e^{\alpha_2} - b\right] \\ \beta_2 = k_2\left[-(p + p^*)e^{\alpha_1} + a\right] \\ \gamma_2 = k_2\left[-(q + q^*)e^{\alpha_1} + b\right] \end{cases} \qquad \begin{cases} k_2 = -\dfrac{1}{2(1 + P)\sqrt{7}c|\Delta t|} \\ a = (1+P)\left[p^* + |\Delta t|\, q^*\right] + \bar{p} + p \\ b = (1+P)\left[7c^2|\Delta t|\, p^* + q^*\right] + \bar{q} + q \end{cases}$$

§ 7 Calculation of the perturbation integrals

It would be an awful lot of work to evaluate generally the integrals

$$(7.1)\quad \int_{t_o}^{t} \frac{(t - \tau)^{\alpha}}{\alpha!} \left[\Delta_m D^{\alpha} f(\tau)\right]_a d\tau,$$

occurring in (5.3). We rather go another way which yields the integrals in question with sufficient accuracy. We label the wellknown functions

(7.2) $\left[\Delta_m D^\alpha f(\tau)\right]_a = g_\alpha(\tau)$

for the 4 equidistant instants of time*)

(7.3) $t_o, \; t_o+h, \; t_o+2h, \; t_o+3h$

where

(7.4) $h = \frac{\Delta t}{3} = \frac{t - t_o}{3}$,

and with the aid of the differentiating scheme of the table

τ	$g_\alpha(\tau)$	$\Delta g_\alpha(\tau)$	$\Delta^2 g_\alpha(\tau)$	$\Delta^3 g_\alpha(\tau)$
t_o	$g_\alpha(t_o)$	$\Delta g_\alpha(t_o)$		
			$\Delta^2 g_\alpha(t_o)$	
$t_o + h$	$g_\alpha(t_o+h)$	$\Delta g_\alpha(t_o+h)$		$\Delta^3 g_\alpha(t_o)$
			$\Delta^2 g_\alpha(t_o+h)$	
$t_o + 2h$	$g_\alpha(t_o+2h)$	$\Delta g_\alpha(t_o+2h)$		
$t_o + 3h$	$g_\alpha(t_o+3h)$	$\Delta g_\alpha(t_o+3h)$		

we replace the function $g_\alpha(\tau)$ by the Newton interpolation polynomial.

---

*) This is arbitrary! The functions could as well be labeled more finely (in the case of large step lengths this might be necessary; naturally, the integral formula (7.6) would then have to be changed). But since the step length has to be chosen short anyhow in order to keep the breaking-off errors, low, and since it is evident that few but finely graded steps involve just as much work as more steps with a coarser grading, there is no reason to label the functions more finely since the errors due to the chosen interpolation do not reach the amount of the breaking-off errors. This can be demonstrated the most rapidly by calculating forth and back with different step lengths.

The difference $\Delta^{\nu} g_{\alpha}(t_o)$ are defined as

$$(7.5) \quad \Delta^{\nu} g_{\alpha}(t_o) = \Delta^{\nu-1} g_{\alpha}(t_o+h) - \Delta^{\nu-1} g_{\alpha}(t_o)$$

We have then

$$(7.6) \quad \int_{t_o}^{t} \frac{(t-\tau)^{\alpha}}{\alpha!} g_{\alpha}(\tau)d\tau = \frac{(\Delta t)^{\alpha+1}}{(\alpha+1)!} \Big\{ g_{\alpha}(t_o) + \frac{3}{\alpha+2}\Delta g_{\alpha}(t_o) - \frac{3}{2}\frac{\alpha-3}{(\alpha+2)(\alpha+3)}\Delta^2 g_{\alpha}(t_o) + \frac{\alpha^2-2\alpha+3}{(\alpha+2)(\alpha+3)(\alpha+4)}\Delta^3 g_{\alpha}(t_o) \Big\}$$

When calculating back, Δt (and also h) has to be taken negative. The difference $\Delta^{\nu} g_{\alpha}(t_o)$ are calculated from their definition (7.5) also in this case.

Chapter III

Numerical computations:

§ 8 Compilation of the special initial values and of the formulas for the solution of one operation

1) Initial instant: Timing begins from Oct. 29, 1958 - the Julian day 2429200.5 - and continues in days.

2) Relative motion of the sun and Jupiter: tabulation of $\vec{x}_s(t)$: for the instants

$$(8.1) \quad t_{\nu} = t_o + \nu h \qquad (\nu = 0,1,2,3) \qquad ^{*)}$$

the corresponding values of E_{ν} are to be determined by inversion of the Kepler equation

$$(8.2) \quad E_{\nu} - E \sin E_{\nu} = \mu t_{\nu} + M.$$

---

*) The step $\Delta t = 3h$ can be chosen arbitrarily

Numerical values:

$$(8.3)\quad \begin{cases} \varepsilon = 0.0484011000 & \text{(eccentricity)} \\ \mu = 0.001450215293\ d^{-1} & \text{(mean motion)} \\ M = 5.645944315 & \text{(mean anomaly)} \\ t_o = 0 & \text{(calendar day)} \end{cases}$$

The solution of (8.2) with respect to E_ν is most easily achieved by iteration of Newton's approximate formula for solving equations:

$$(8.4)\quad E_{\nu II} = E_{\nu I} - \frac{E_{\nu I} - \varepsilon \sin E_{\nu I} - \mu t_\nu - M}{1 - \varepsilon \cos E_I}$$

where $E_{\nu I}$ *) is a value which approximately satisfies Eq. (8.2), and $E_{\nu II}$ is an improved approximate value. Formula (8.4) has to be iterated until $E_\nu = E_{\nu N}$ satisfies Eq. (8.2) with a given accuracy.

Then, $x_s(t_\nu)$ can be calculated from the resulting values of E_ν:

$$(8.5)\quad x_s(t_\nu) = \begin{Bmatrix} 0.015676901 - 4.186636655 \sin E_\nu - 0.323895551 \cos E_\nu \\ -0.251333487 - 0.323515939 \sin E_\nu + 5.192722630 \cos E_\nu \\ 0 \end{Bmatrix} L$$

3) <u>Initial data for the orbit of the moon:</u> Computation is to be carried out with the mass numbers of page 3

$$(8.6)\quad \begin{cases} m_2 = 0.2825328640 \cdot 10^{-6} L^3 d^{-2} \\ m_3 = 0.2959122080 \cdot 10^{-3} L^3 d^{-2} \end{cases}$$

and with the values for the relative position and the relative velocity of the moon, corresponding to the instant t_o:

---

*) The value of $E_{\nu-1}$ corresponding to the preceding instant $t_{\nu-1}$ is best taken as the initial value of $E_{\nu I}$ (starting from E_{oI} = 5.615994607)

$$(8.7) \quad \vec{x}_m^{(o)} = \begin{Bmatrix} -0.1859213874 \\ 0.0071237637 \\ 0.0775628307 \end{Bmatrix} L \qquad \vec{u}_m^{(o)} = \begin{Bmatrix} 0.0002062301590 \\ 0.0008942872800 \\ -0.0003356104520 \end{Bmatrix} L\, d^{-1}$$

4) Approximate orbit for Jupiter's moon: We first calculate

$$(8.8) \qquad c = \sqrt{\frac{m_2}{|\vec{x}_m^{(o)}|^3}}.$$

Then, the position of the moon on its approximate orbit at the instants (8.1) is found from the formula:

$$(8.9) \quad \vec{x}_{ma}(t_\nu) = \vec{x}_m^{(o)} \cos\left[c(t_\nu - t_o)\right] + \vec{u}_m^{(o)} \frac{1}{c} \sin\left[c(t_\nu - t_o)\right]$$

The velocity of the moon on its approximate orbit must be known only for the end point $t_3 = t_o + \Delta t$ of the interval:

$$(8.10) \quad \vec{u}_{ma}(t_3) = -x_m^{(o)}\, c \sin\left[c(t_3 - t_o)\right] + \vec{u}_m^{(o)} \cos\left[c(t_3 - t_o)\right].$$

5) Computation of the perturbation integrals: Now, the functions $\vec{\delta}_{ma}(t)$ and $\vec{\xi}_{ma}(t)$ must be tabulated for the instants (8.1) from the formulas *).

$$(8.11) \begin{cases} \vec{\delta}_{ma}(t_\nu) = m_3\left[\dfrac{\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)}{|\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)|^3} - \dfrac{x_s(t_\nu)}{|x_s(t_\nu)|^3}\right] + \left[c^2 - \dfrac{m_2}{|\vec{x}_{ma}(t_\nu)|^3}\right]\vec{x}_{ma}(t_\nu) \\ \vec{\xi}_{ma}(t_\nu) = \dfrac{m_2}{|\vec{x}_{ma}(t_\nu)|^3}\left[\vec{\delta}_{ma}(t_\nu) - \dfrac{3(\vec{x}_{ma}(t_\nu)\vec{\delta}_{ma}(t_\nu))}{|\vec{x}_{ma}(t_\nu)|^2}\vec{x}_{ma}(t_\nu)\right] - \end{cases}$$

$$\frac{m_3}{|\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)|^3}\, \vec{\delta}_{ma}(t_\nu) - \frac{3(\left[\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)\right]\vec{\delta}_{ma}(t_\nu))}{|\vec{x}_s(t_\nu) - \vec{x}_{ma}(t_\nu)|^2}(\vec{x}_s(t_\nu) -$$

$$- \vec{x}_{ma}(t_\nu))$$

---

*) The second constituent of $\vec{\xi}_{ma}$ hardly influences the result. The delay of the computer is, however, very small if this part is included in the calculation, since all the quantities appearing in it had already to be prepared for the calculation of $\vec{\delta}_{ma}$.

With the aid of the differences between these tables, obtained from (7.5), we are able to calculate the perturbation integrals:

$$
(8.12)\quad
\begin{aligned}
&\int_{t_o}^{t_o+\Delta t} \vec{\delta}_{ma}(\tau)\,d\tau = (\Delta t)\left\{\vec{\delta}_{ma}(t_o)+\tfrac{3}{2}\Delta\vec{\delta}_{ma}(t_o)+\tfrac{3}{4}\Delta^2\vec{\delta}_{ma}(t_o)+\tfrac{1}{8}\Delta^3\vec{\delta}_{ma}(t_o)\right\}\\
&\int_{t_o}^{t_o+\Delta t} (t_o+\Delta t-\tau)\vec{\delta}_{ma}(\tau)\,d\tau = (\Delta t)^2\left\{\tfrac{1}{2}\vec{\delta}_{ma}(t_o)+\tfrac{1}{2}\Delta\vec{\delta}_{ma}(t_o)+\tfrac{1}{8}\Delta^2\vec{\delta}_{ma}(t_o)+\tfrac{1}{60}\Delta^3\vec{\delta}_{ma}(t_o)\right\}.\\
&\int_{t_o}^{t_o+\Delta t} \frac{(t_o+\Delta t-\tau)^2}{2!}\vec{\xi}_{ma}(\tau)\,d\tau = (\Delta t)^3\left\{\tfrac{1}{6}\vec{\xi}_{ma}(t_o)+\tfrac{1}{8}\Delta\vec{\xi}_{ma}(t_o)+\tfrac{1}{80}\Delta^2\vec{\xi}_{ma}(t_o)+\tfrac{1}{240}\Delta^3\vec{\xi}_{ma}(t_o)\right\}\\
&\int_{t_o}^{t_o+\Delta t} \frac{(t_o+\Delta t-\tau)^3}{3!}\vec{\xi}_{ma}(\tau)\,d\tau = (\Delta t)^4\left\{\tfrac{1}{24}\vec{\xi}_{ma}(t_o)+\tfrac{1}{40}\Delta\vec{\xi}_{ma}(t_o)+\tfrac{1}{840}\Delta^3\vec{\xi}_{ma}(t_o)\right\}\quad *)
\end{aligned}
$$

6) <u>Formulas of solution</u>: The perturbation integrals (8.12) are used to correct the approximate solutions (8.9) and (8.10):

$$
(8.13)\quad
\begin{cases}
\vec{x}_m(t_o+\Delta t)=\vec{x}_{ma}(t_o+\Delta t)+\displaystyle\int_{t_o}^{t_o+\Delta t}(t_o+\Delta t-\tau)\vec{\delta}_{ma}(\tau)\,d\tau+\int_{t_o}^{t_o+\Delta t}\frac{(t_o+\Delta t-\tau)^3}{3!}\vec{\xi}_{ma}(\tau)\,d\tau\\
\vec{u}_m(t_o+\Delta t)=\vec{u}_{ma}(t_o+\Delta t)+\displaystyle\int_{t_o}^{t_o+\Delta t}\vec{\delta}_{ma}(\tau)\,d\tau+\int_{t_o}^{t_o+\Delta t}\frac{(t_o+\Delta t-\tau)^2}{2!}\xi_{ma}(\tau)\,d\tau
\end{cases}
$$

Now, we replace t_o by $t_o+(\Delta t)$ in all the formulas of §8 and proceed to another operation, using the values of (8.13) instead of those of (8.7). Again, Δt can be newly chosen.

7) <u>Precautionary measures taken to avoid unnecessary rounding errors</u>: Since the SIE 2002 computer of the TH Aachen, with which our numerical computations were made, usually calculates with no more than 10 digits, some precautionary measures had to be taken to eliminate rounding errors:

---

*) The contribution of this integral manifests itself only with great steps, but the situation is about the same as in the foregoing footnote.

a) Prior to our computations we reduced the quantity M (and E_o) by a factor of 2π in order to maintain the anomaly $|E| < 1$ for some hundred days. Thus, the 10th digit cannot be lost during the inversion of the Kepler equation;

b) Instead of t_o+3h we always calculated $t_o+\Delta t$ since h is equal to $\frac{\Delta t}{3}$ only within rounding errors so that a noticeable error might appear in the time counting;

c) When calculating solutions from (8.13), we first determined the sum of the perturbation integrals and then added the approximate solution. In this way, the rounding error of the additions enters the result only once.

§ 9 Results

1) Trial computations made so far and experience gathered from them: The following trial calculations were made:

a) The first informative computations with different steps (one step forward and one step backward) have shown that formula (7.6) is sufficiently accurate and that the step constitent with the considerations in §6, 2) is approximately 1d.

b) 100 steps were calculated froward and backward with Δt = 1d. *) This was the most important part of our calculations since they could be compared with other results.

J. Kovalevsky pointed out that his 12-digit computations, carried out by Cowell's method with an IBM 650 computer, took 10 sec. for each operation and that the deviations in the coordinates and velocites, obtained when calculating with (Δt) = 5d 100 days forward and backward (i.e., in 40 operations) were less than 50.10^{-10}L and 100.10^{-10}, respectively (unit not given).

---

*) The relevant section of the table may be seen from the enclosed table of data

We obtained the following results by this method:

1o-digit computation with an SIE 2oo2 computer took 2 sec. for each operation (the printing of four lines of data after each operation, which was necessary for informative purposes but could be omitted later, took 1.6 sec.). When calculating with the step $(\Delta t)=1d$ 1oo days forward and backward (i.e., in 2oo operations), the deviations in the coordinates and velocities were less than $15.1o^{-1o}$ L and $1.2 \cdot 1o^{-11}$ L d^{-1}, respectively. On the basis of this result and with the aid of the (still very rough) estimate it could be shown in § 6, 3) that the errors in the analytic continuation at $\Delta t=1d$ accounted for no more than 5o % of the values indicated, whereas the remaining deviations were due to the rounding errors. The same computation with $\Delta t=2d$ yielded deviations in the coordinates and velocities of less than $28.1o^{-1o}$ L and $4.1o^{-11}Ld^{-1}$, respectively. The remaining test time was used for informative computations with greater steps (3d, 5d, 1od). Here, the break-off errors were already noticeable. As a result of these computations, we came to the conclusion that the expressions $\vec{R}_{\iota}(t)$ and $\vec{R}(t)$ might be used for a correction (cf.§6).

c) Integration was performed from $\Delta t=1d$ (then o.8d, o.6d, o.4d) beyond the nearest distance between Jupiter and the moon, and the time left was used for backward calculation. The values obtained again agreed very well. In order to save time, only two lines of values were printed.

d) The modification mentioned in the footnote p. 15 was calculated. At the same time, the printing commands were distributed more conveniently in order to stop the computer for a shorter time. Calculation and printing took about 2 sec. for one operation so that the printing process was hardly interrupted.

2) <u>Influence of errors:</u> The results can be falsified in four ways:

a) by calculating with an insufficient number of protective places. Rounding errors may cause serious errors unless they are smaller than the break-off errors from the very outset;

b) by using too great steps. If a definite number of terms is used, the required rapid convergence of series can be achieved only if the step Δt is reduced;

c) by successively performing many, sufficiently accurate operations (if Δt is definitely chosen, the excessively strong propagation of the break-off error can be eliminated only by allowing for further terms of (5.3). This means, however, that the break-off error is reduced simultaneously. Reduction of the step alone is not very advantageous since the required number of operations increases simultaneously, cf. (6.16) ff.);

d) by inexact tabulation of the functions appearing in the perturbation integrals, which can be avoided either by a more exact tabulation or by reducing the step.

The rounding errors show a random character, whereas the other three error sources reside in the method; however, they can all be controlled:

in b) by observing the increase of (6.11) and by reducing the step in time;

in c) with the aid of the estimate (6.16) which can be improved since we have always taken the maxima of the absolute values of the quantities involved;

in d) by calculating forward and backward (random sampling) and, if necessary, by reducing the step.

When choosing the step Δt, it is necessary that conflicting requirements be compensated:

Results of given accuracy are to be obtained with the greatest possible step and the least possible number of operations. The modification mentioned in § 6 is very helpful in this respect, since it makes it possible to allow for the essential part of the rests of series without determining the required perturbation integrals. Finally, it should be stressed that we have dealt only with a special example and that our method can also be used for the numerical solution of general many-body problems. The elaboration of our method is still under way, and we hope that we shall soon be able to achieve even better results.

Notes on the table of data

Since the data were originally printed only for the purpose of obtaining information on the efficiency of our method, we expressed the numbers in the way they were stored in the computer. The comma was omitted. The last two figures of each numer are the so-called characteristics of the values represented as floating-point numbers (characteristic = exponent + 50; the point of the computer is put behind the sign). The decimal numer +0.7, for example, corresponds to the floating-point number + 700 000 000 050. Another disadvantage of the tables is that the printed numerical values are not clearly arranged. After each operation the values were printed in the following four-line arrangement (dimensions are given in brackets):

time t [d], step Δt [d], components of $\vec{x}_m(t)$ [L], $|\vec{x}_m(t)|$ [L]
components of $\vec{u}_m(t)$ [L d^{-1}]
components of $\vec{R}_{\varsigma}(t)$ [L d^{-2}], $|\vec{R}_{\varsigma}(t)|$ [L d^{-2}]
components of $\vec{x}_s(t)$ [L], $|\vec{x}_s(t)|$ [L]

The numbers in the third line give information only on the order of magnitude of the expression $\vec{R}_{\varsigma}(t)$ (we calculated only with ten digits and several digits were lost in the course of calculation, especially during the determination of the difference between two approximately equal numbers from formula (6.11)): The **first** two **figures** and the characteristic are valid at most, while the other digits are insignificant.

Table of data

We do not reproduce the full table which covers 24 pages. Anyone who is interested to have a copy should write to the author.

A short summary reads

time [d]	step [d]	$(\vec{x}_m)_x$	$\lvert\vec{x}_m\rvert$
0.00000	+ 01.00000	- 185921387450	+ 201577536050
1.00000	+ 01.00000	- 185711957150	+ 201288442250
⋮	⋮	⋮	⋮
99.00000	+ 01.00000	- 129514535750	+ 158151320350
100.00000	+ 01.00000	- 128523006850	+ 157550010150
99.00000	- 01.00000	- 129514535750	+ 158151320350
⋮	⋮	⋮	⋮
0.00000	- 01.00000	- 185921386050	+ 201577534650