THE DYNAMICAL STATE OF THE INTERSTELLAR GAS AND FIELD

II. TURBULENCE AND ENHANCED DIFFUSION

E. N. Parker

THE UNIVERSITY OF CHICAGO

THE ENRICO FERMI INSTITUTE FOR NUCLEAR STUDIES

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E. N. Parker
Enrico Fermi Institute for Nuclear Studies and
Department of Physics
University of Chicago
Chicago, Illinois 60637

Laboratory for Astrophysics and Space Research

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Abstract

The dynamical instability of the interstellar gas, caused by the galactic cosmic rays and magnetic field, is examined for the purpose of gaining a more detailed understanding of the phenomenon. The decrease of the field energy as the gas goes into clouds is illustrated with a simple example. It is shown, too, that a tissue of field, formed by superposing alternate layers of field at right angles to each other, fails to produce stability. As suggested earlier, it appears that there is no stable confinement of magnetic fields by the weight of the interstellar gas.

It is shown that the instability develops very short wavelengths in the direction perpendicular to the galactic gravitational field $\mathbf{g}$ and magnetic field $\mathbf{B}$. The result is that the interstellar gas must be in a continual state of turbulence and fragmentation, terminating only when the clouds become so dense that self-gravitation becomes dominant. The turbulence and fragmentation enhance ambipolar diffusion to such an extent that the field escapes from the gas rapidly compared to the usual ambipolar rates. It is suggest that this may contribute to the escape of the magnetic fields from the interstellar gas which forms stars.

I. Introduction

It was demonstrated earlier (Parker, 1966b, hereafter referred to as I, and Lerche, 1966) that the cosmic rays and the galactic magnetic field both produce an instability which drives the interstellar gas into clumps. The process is basically a Rayleigh-Taylor instability. The clumps of gas are separated by distances of the general order of 100 pc and are to be identified, we suggest, with the observed individual interstellar clouds.

It was shown in two dimensions that the instability may be expressed in terms of an equivalent self-attraction of the interstellar gas, resembling self-gravitation. The self-attraction produced by the instability is a factor $g^2 / G B^2$ times stronger than self-gravitation, where g is the acceleration of the galactic gravitational field perpendicular to the disk of the galaxy, G is the gravitational constant, and B is the galactic magnetic field density. This factor vanishes on the central plane of the galaxy, because g vanishes there, and rises to a value of 5 or 10 at a distance of 100 pc above the central plane. Hence it was pointed out that the cosmic ray, magnetic field instability is probably the dominant force, rather than self-gravitation, for collecting the interstellar gas into clouds and for initiating the collapse of gas clouds to form stars. But this will be discussed in a later paper.

The purpose of the present paper is to look further into the general properties of the cosmic ray, magnetic field instability. The dynamical behavior of the gas clouds produced by the initial instability is a nonlinear phenomenon. It is not subject, therefore, to a simple comprehensive analytical treatment, as was the initial linear instability of an equilibrium atmosphere. The only path open for theoretical exploration of the problem is the construction of a variety of idealized examples, each illustrating some single facet of the more complex behavior of the overall system. This paper introduces a number of formal examples, worked out in the appendices, for the purpose of instructing ourselves on the general behavior of a gas, field, cosmic ray system confined by gravity. The main text of the paper describes the individual examples and points out the general qualitative conclusions which they suggest.

In I the instability was treated by a linear perturbation analysis of the equilibrium states of a number of simple gas-field systems confined by gravity. The equilibria were all found to be unstable for an interstellar gas in which radiative cooling prevented the temperature from rising rapidly upon compression. The physical basis for the instability is easily understood: Perturbing the magnetic lines of force causes the gas to flow downward along the lines of force, accumulating in the low places along the lines, thereby depressing the low places further. The final energy is then lower than the initial energy. The lower final energy is illustrated by direct calculation in section II for a simple case.

The nature of the forces between elements of gas are examined in three dimensions demonstrating both the similarity and difference from self-gravitation. The similarity (pointed out in I) is that in the plane defined by the vectors $\underset{\sim}{g}$ and $\underset{\sim}{B}$ the force between any two elements of gas is of the same form as their gravitational attraction. The difference is the repulsion in the third dimension; elements of gas do not like to remain side by side. This suggests that the initial linear instability of the gas-field distribution is even stronger in three dimensions than shown by the simple two dimensional treatment given in I. A more detailed linear perturbation analysis is taken up in section IV and the conjecture is confirmed. In the absence of diffusion the most unstable modes are those in which the wave number perpendicular to the gB-plane is infinitely large. The limiting effects of diffusion give an optimum wave number for instability. Altogether the calculations indicate that the interstellar gas exists in a dynamical state of progressive fragmentation until the individual clouds become so dense that self-gravitation takes over. The calculations suggest, too, that the combination of dynamical instability and ambi-polar diffusion leads to rapid motion of the gas across the magnetic lines of force. The effect would appear to be important in freeing the gas from the field during the condensation into stars.

II. Energy Decrease Through Clumping of the Gas

The linear perturbation analysis shows that a cosmic ray and magnetic field system confined in equilibrium by the weight of a thermal gas is unstable. It was suggested in I, on the basis of the linear analysis, that the instability is responsible for the observed clumping of the interstellar gas. The statement of the final clumping was made on the basis of the general physical nature of the instability, rather than from calculation, because, of course, the linear perturbation analysis does not apply when the clumping becomes severe. It is instructive, therefore, to look directly at the energy of the system, which can be calculated when the perturbation of the initial uniform equilibrium state has grown to large amplitude. We carry out the calculation for an initial equilibrium of a flat isothermal atmosphere with constant gravity. Many other geometries can be worked out, as in I, but one is sufficient for the present purposes.

Imagine a horizontal magnetic field of density $B(z)$ extending in the y -direction as shown in Fig. 1. Suppose, for the moment, that the cosmic ray pressure can be neglected. The system is confined by the weight of the thermal gas (of density $\rho(z)$ and uniform thermal velocity u such that the gas pressure is $p(z) = \rho(z)u^2$) in the gravitational field $-g$ in the z -direction; the gravitational acceleration g is taken to be a constant in the present illustration. The base of the system is at $z = 0$, representing the central plane of the galactic disk. The reader may suppose that the mirror system of fields fills the space $z < 0$. For the simple case that the magnetic pressure is proportional to the gas pressure throughout the atmosphere, say

$B^2/8\pi = (V_A^2/2u^2)p$ where $V_A^2 = B^2(0)/4\pi\rho(0)$

is a constant, hydrostatic equilibrium requires that

$$\frac{B^2(z)}{B^2(0)} = \frac{\rho(z)}{\rho(0)} = \exp\left(-\frac{2z}{\Lambda}\right) \tag{1}$$

in $z \geq 0$, where the scale height $\Lambda/2$ is $\left(u^2 + \frac{1}{2} V_A^{\,2}\right) / g$.

The reduction in magnetic and cosmic ray energy as a result of clumping of the gas into slabs is readily demonstrated. It is this reduction which pushes the gas into clumps. Imagine that the magnetic field is held in its initial horizontal configuration by a suitable array of demons while the gas is slowly compressed into vertical slabs of thickenss $2a$, separated by intervals of $2b$, by sliding the gas along the lines of force. The configuration, including the straight lines of force, is illustrated in Fig. 2. Suppose that the slabs of gas occupy the regions $(2n+1)b + 2na < y < (2n+1)b + (2n+2)a$, where n is an integer running from $-\infty$ to $+\infty$. Some small amount of work is done in compressing the interstellar gas*, though this is rather small in most cases because of the low temperature.

Following compression of the gas, hold the gas fixed and permit the magnetic field in the spaces between the slabs of gas to relax into the equilibrium form, $\mathbf{B} = -\nabla\psi$, $\nabla^2\psi = 0$. The boundary conditions

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*The work done per unit length in the x - direction in forming each slab, is readily shown to be $[2p(0)\Lambda b/(s-1)]\,(1+a/b)\,[(1+b/a)^{s-1} - 1]$ for the simple case that the pressure is proportional to the s power of the density during compression. If s = 1, the work done is $2p(0)\Lambda b\,(1+a/b)\ln(1+b/a)$. It is readily shown that the work reduces to $2p(0)\Lambda b$ in the limit as $b/a \longrightarrow 0$ and is a monotonically increasing function of b/a when $s > 0$.

are that $\mathbf{B}$ vanishes at $z = +\infty$, that no lines of force cross $z = 0$, and that the field remains frozen into the gas. This boundary value problem is easily solved (see Appendix I). The field takes up the expanded form sketched in Fig. 2. The total energy of the field, per unit length in the x -direction, contained in $-b < y < +b$ decreases below the initial field energy $\mathcal{E}_0 = B^2(0) \Lambda b / 8\pi$ by the amount $\Delta\mathcal{E}$, plotted in Fig. 3. It is readily shown (see Appendix I) that $\Delta\mathcal{E} / \mathcal{E}_0$ increases monotonically with increasing gap width, from zero to one as b/Λ increases from zero to infinity.

We see, then, that the tendency to clump is driven at least in part by the magnetic energy decrease $\Delta\mathcal{E}$ which the clumping permits. If cosmic rays are present, they expand along with the field and make an additional contribution to $\Delta\mathcal{E}$.

Calculation of the magnetic field in the gap between slabs of gas indicates (see Appendix I) that the net force (magnetic plus gravitational) on the slabs of gas (constrained to the density distribution exp($-2z/\Lambda$)) is upward at large z and downward at small z. This indicates the direction in which the gas moves when the constraints on the gas are removed. The larger the gap width b between sheets, the larger is the mass in each sheet. Hence, as one would expect, a larger gap width means that a larger portion of the sheet descends when released. For any finite gap width the clumping of gas into the observed interstellar clouds in the galactic disk is accompanied by a descent of the gas near the central plane of the disk and by an elevation of the gas at large distance from the central plane of the disk.*

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* A more detailed study is given by Lerche (1967).

Generally speaking, when released, the sheets will redistribute themselves vertically, contributing a further reduction of the total energy. Neglecting the thermal energy of the gas, it follows that $\Delta\mathcal{E}$, given in Fig. 3, is a lower limit on the energy change caused by clumping of the gas. The observed clumped state of the interstellar gas is, then, a lower energy state than a more uniform distribution over the galactic disk.

III. Properties of the Magnetic Field Cosmic Ray Instability

Consider the instability of an atmosphere of thermal gas whose weight confines large-scale magnetic fields and/or cosmic ray gas. In I the problem was considered only in the two dimensions defined by the magnetic field $\underset{\sim}{B}$ and the galactic gravitational acceleration $\underset{\sim}{g}$. We now introduce the third dimension $\underset{\sim}{g} \times \underset{\sim}{B}$.

Suppose that a cold gas confines a horizontal magnetic field in the y-direction leading to the equilibrium described by (1) with $u^2 = 0$, sketched in Fig. 1. Ignoring the stabilizing effects of diffusion and viscosity at large wave numbers, it is shown in Appendix II that the system is unstable for all wave numbers (k_x, k_y, k_z) of the perturbation. In discussing the growth rate of the instability it is convenient to treat the wave number k_y (parallel to the magnetic field) as the basic dependent variable and then inquire what effects k_x and k_z may have for a given k_y. The calculations show that, for a given k_y, the fastest growth occurs either for the vertical wave number k_z equal to zero (in which case the growth rate of the instability is independent of the horizontal wave number k_x in the direction $\underset{\sim}{g} \times \underset{\sim}{B}$) or for the horizontal wave number in the direction $\underset{\sim}{g} \times \underset{\sim}{B}$ very large,

$k_x >> k_y, k_z$. Obviously diffusion of any kind would prohibit $k_x \rightarrow \infty$, so the most unstable mode is $k_z = 0$. In this case the growth rate of the instability is a monotonically increasing function of k_y, limited only by diffusion at very large k_y. Neglecting diffusion and denoting the exponential growth with time by exp(t/τ) one finds that

$$\frac{1}{\tau} \cong \left(\frac{g}{\Lambda}\right)^{1/2} k_y \Lambda \tag{2}$$

for $k_y \Lambda \ll 1$, and

$$\frac{1}{\tau} \cong \left(\frac{g}{\Lambda}\right)^{1/2} \tag{3}$$

in the limit of large $k_y \Lambda$. We recognize $(g/\Lambda)^{1/2}$ as the characteristic free-fall time over a distance comparable to the scale height Λ.

Thus we find again, as with all the cases worked out in I, that the characteristic time of growth of the instability is the free fall time.

It is instructive to digress for a moment, before going on to explore the effects of k_x and k_z when the gas is hot and cosmic rays are present, to the more general question considered in I, viz whether there are any special magnetic configurations which permit a stable equilibrium. We explored a number of configurations in I and found them all to be unstable, from which we suggested that there were no stable configurations. Lerche (1967) has since explored a rotating system, finding that it too is unstable in the free fall time. Introduction of the third dimension permits a still different situation to be treated here, and in view of the importance of the question of general instability, we mention it briefly before returning to the inquiry into the effects of k_x and k_z .

It is well known that systems wherein the field confines the gas are more stable if the field contains a strong shear, the reason being that the tension in the lines of force in each layer of field lies across the lines of force in neighboring layers and therefore tends to stabilize fluting instability in the neighboring layers. We investigate, therefore, in Appendix IV the stability of alternate layers of perpendicular field. A horizontal slab of field in the y -direction with a vertical thickness a is overlaid with a horizontal slab of field in the x -direction of thickness a , which in turn is overlaid with a slab in the y -direction, etc. The equilibrium field density follows from (1) independent of the field direction*. The boundary conditions require that the horizontal wave number k_x must be the same in each slab, and so must k_y and τ . The calculations show that the fastest growing mode is $k_z = 0$ with $k_x = k_y$. The growth rate $1/\tau$ is the <u>same</u> function of k_y as when the magnetic field is only in the y -direction, given by (2) and (3). In terms of the total wave number $k = (k_x^2 + k_y^2)^{1/2}$ the growth rate is slower by $2^{1/2}$ when $k\Lambda << 1$ (since $k_y = k/2^{1/2}$),

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* We ignore possible small-scale instabilities which might occur in the high shear at at the interfaces between slabs. They are of a different nature from the gravitational instability with which we are presently dealing, and so far as we can see, they would not enhance the overall stability of the system.

yielding

$$\frac{1}{\tau} = \left(\frac{g}{\Lambda}\right)^{1/2} \frac{k\Lambda}{2^{1/2}} \tag{4}$$

in place of (2). When $k\Lambda >> 1$, the growth rate is unaffected. Thus again we find that the magnetic configuration has little stabilizing effect. The characteristic growth time of the instability is the free fall time irrespective of the form of the magnetic field.

We return, then, to the investigation of the effects of k_x and k_z on the growth rate of the instability in the simple case of a thermal gas confining a horizontal magnetic field with its lines of force all in the y -direction (see Fig. 1). Consider the effect of warming the gas. The equilibrium is described by (1). The stability is treated at length in Appendix II, and an important modification of the simple cold gas appears. Denote the magnetic pressure $B^2(z)/8\pi$ as α times the gas pressure $\rho(z)u^2$, so that the Alfven speed V_A is $(2\alpha)^{1/2}u$; assume that the pressure in an element of gas changes linearly with the density, $\delta p/p = \gamma\, \delta\rho/\rho$ when perturbed. The temperature* of the interstellar gas is regulated so closely by radiative transfer that in most cases $\gamma \lesssim 1$ (See I). The difference from the cold gas system is that short wavelengths

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* The reader may, if he wishes, include turbulent velocities in the gas pressure, but we have no way of knowing what they might be in the hypothetical initial gas distribution. Presumably turbulence is dissipated and regenerated at a more or less constant rate, suggesting that perhaps $\gamma \cong 1$ for turbulence too.

are now stable, their instability being resisted by the non vanishing gas pressure. If $k_x = 0$, it follows from (II 24) that the upper limit on k_y and k_z for instability is

$$\Lambda^2 (k_y^2 + k_z^2) < \frac{(2+2\alpha-\gamma)^2 - \gamma(\gamma+2\alpha)}{2\alpha\gamma} . \tag{5}$$

The left hand side of the inequality is greater than, or equal to, zero. Hence in order that instability exist at some nonvanishing wave number, it is necessary and sufficient that $2(1+\alpha-\gamma)(1+\alpha) > \alpha\gamma$. Since the right hand side of the inequality (5) appears to be of the order of unity in the galaxy, it follows that the shortest unstable wavelengths are of the same general order of magnitude as the scale height of the gas distribution, some 100 pc for the disk as a whole. The wavelengths for maximum growth rate are also $O(\Lambda)$. All longer wavelengths are unstable, but with a diminishing growth rate as the wavelength becomes large compared to Λ.

It is interesting to reflect for a moment what wavelengths will dominate in the final clumped state of the gas (see I for discussion of cloud formation). As already noted, the maximum growth rate in an atmosphere which initially is unstable i. e. the conditions satisfy (5), is for a wave number k_y of the order of $1/\Lambda$, intermediate between zero and the maximum allowed by (5). One would expect some wavelength in the vicinity of the maximum growth rate to dominate the interstellar gas clouds. Roughly speaking then, the cloud spacing would be of the order of the scale height, 100 pc, which, observationally, is the correct order of magnitude. But there is another point of view that should be noted. Imagine that the interstellar gas is initially a uniform slab of gas in the galactic disk which is stable because the gas pressure dominates the magnetic field and cosmic ray pressure with a γ which is significantly greater than 1. Then suppose that conditions change very slowly in the direction of instability. The first wavelengths to become unstable are very long, $\Lambda k_y << 1$. The growth rate is very slow, but if conditions are changing sufficiently slowly, the long wavelengths will have time to develop before shorter wavelengths become unstable. In this way it is theoretically possible for the scale of the clumping of the gas to be extremely large.

It is our impression that the interstellar gas does not start out in a stable state, gradually going unstable, in the galaxy at the present time. Hence we expect that the scales will be of the order of the scale height Λ, ranging, therefore, over 10 - 10^3 pc. But the theoretical possibility of much larger scales in other galaxies, or at other times in our own galaxy, should be kept in mind.

The next step is to consider the effect of the transverse horizontal wave number k_x. The dispersion relation shows that the growth rate of the instability increases with increasing k_x. When we include k_x, the greatest instability is for $\Lambda^2 k_x^2 >> 1$ (see (II 26)) and requires only that

$$\Lambda^2 k_y^2 < 1 + \frac{(2+2\alpha-\gamma)^2 - \gamma(\gamma+2\alpha)}{2\alpha\gamma} \tag{6}$$

there being no restriction on k_z now.

Comparison with (5) shows that the instability extends to larger wave numbers k_y. In order that instability exist at all it is necessary and sufficient to require only that $1 + \alpha > \gamma$.

The fact that the instability has the highest growth rate for very short horizontal transverse wavelengths $k_x \Lambda >> 1$ is interesting, because, as we shall show in the next section, it may permit rapid diffusion of the thermal gas across the magnetic lines of force, which is important in star formation. But before going into this question, consider the nature of the instability driven by cosmic rays, which have been ignored in the calculations discussed above.

Suppose that the thermal gas is not cold and is threaded by a magnetic field whose pressure is negligible and whose lines of force are horizontal. Then the direction of the lines of force in the horizontal planes is unimportant, but to fix ideas suppose that the field in the y -direction, as in Fig. 1. The function of the field is to tie the cosmic ray gas, of pressure $P(z)$, to the thermal gas, whose pressure is $p = \rho(z) u^2$. Writing $P = \beta \rho(z) u^2$, where β is a numerical constant, the equilibrium state is

$$\frac{P(z)}{P(0)} = \frac{\rho(z)}{\rho(0)} \exp\left(-\frac{2z}{\Lambda}\right)$$

where now $\Lambda = 2(1+\beta) u^2/g$. The perturbation of this atmosphere is worked out in Appendix III. The system is unstable provided only that $1 + \beta > \gamma$. The growth rate $1/\tau$ depends upon the wave number $k = (k_x^2 + k_y^2)^{1/2}$ in the same general way as $1/\tau$ depended upon k_y in the cold gas confining a strong field, described by (2) and (3). Even though the thermal gas may be fairly hot, $1/\tau$ increases monotonically with increasing k, approaching the free fall time $(g/\Lambda)^{1/2}$ in the limit of large k .

This is in contrast to the instability of a hot gas and strong magnetic field, in which large k_y is stable. The hot gas and strong field together are able to suppress the instability at large k_y , even though taken separately they cannot. It is readily seen, then, that when cosmic rays, strong magnetic field, and a relatively hot thermal gas with $\gamma > 1$ are all combined, the magnetic field and the hot gas stabilize the large wave numbers k_y , so that the fastest growth rate occurs for large horizontal transverse wave numbers k_x with $k_y = O(1/\Lambda)$. The characteristic growth time is then the free fall time, of course.

IV. Instability at Large Horizontal Transverse Wave Numbers

The instability driven by magnetic field and cosmic rays proceeds somewhat more rapidly when the horizontal transverse wave number k_x is larger, than when k_x is comparable to the wave number k_y along the field (see (6)). This effect may be important in the evolution of the interstellar gas, so its merits further inquiry.

First of all, why does the instability proceed more rapidly when k_x is large? The answer to this question appears to be that the perturbed lines of force crowd each other less if lines separated by small distances over the horizontal transverse direction x are π out of phase with each other. For then the raised portion of one line can expand into the space left by the sinking portion of the neighboring line.

Another way to understand the greater instability at large k_x is worked out in Appendix V. The force which one element of gas exerts on another when both are suspended in a large-scale horizontal magnetic field B is calculated. The geometry is illustrated in Fig. 4, with the galactic gravitational acceleration g in the negative z-direction. The calculations show that, when the separation y along the lines of force is large compared to the distance $(x^2 + z^2)^{1/2}$ between the lines of force through the two elements, the force is independent of the separation. The force is then attractive if the horizontal separation x of the lines of force through the elements is less than their vertical separation z. The force is repulsive if x exceeds z. The magnitude of the force is g^2/GB^2 times the true gravitational attraction of the two elements of gas if they were separated by $(x^2 + z^2)^{1/2}$.

When the separation y is small compared to the distance $(x^2 + z^2)^{1/2}$ between the lines of force, the force is attractive if $x < 2^{1/2} z$ and repulsive otherwise. The magnitude of the force is again g^2/GB^2 times larger than the true gravitational force between the elements.

The physical basis for the forces of attraction and repulsion is straightforward. Consider the downward deflection of the lines of force supporting the weight of an element of gas of mass m, sketched in Fig. 5. The downward deflection displaces the surrounding lines downward if they lie above or below. This forces upward the lines at the sides. Thus the neighboring lines are not level, some sloping towards the suspended element of gas and others sloping away. Another element of gas suspended on the field will slide downward along the lines as a consequence of the galactic gravitational acceleration g. If downward is away, the effect is repulsion. Hence two elements of gas attract if the lines that thread them lie more above or below than beside each other. The elements repel if the lines are beside each other. We would expect, therefore, in view of the repulsion of elements lying side by side, that the instability of an atmosphere suspended in the field will be greatest when elements of gas lying side by side may move apart. Hence large k_x gives the greatest instability.

Consider the consequences of the instability at large k_x. From the simple linear point of view, the gas tends to be sliced up over very short scales perpendicular to the field. This tendency, to break up into small dimensions across the field, continues even after the gas has fallen into condensed clouds suspended in the field. The calculations show that the instability occurs over so broad a spectrum (see Fig. 6) as to produce motions which have some resemblance to white noise or turbulence, even while the motions are still linear. The effect might also be described as progressive fragmentation, particularly after the gas has formed into clouds. The present simple mathematical treatment does not give a more precise picture than is conveyed by the vague terms "turbulence" and "fragmentation". But it is evident that the interstellar gas clouds must be in a continually complicated dynamical state and have ragged forms.

Now ambipolar diffusion, viscosity, resistivity, etc. can be ignored as a first approximation when dealing with motions over 10 - 100 pc. But when the characteristic scale over one or more dimensions decreases below 10 pc such diffusive effects become important. The effects increase as k_x^2 and must be included when k_x becomes large.

The limitation by diffusion is treated formally in Appendix VI. If the field diffuses through the gas with a diffusion coefficient η (η is just $c^2/4\pi\sigma$ in the hydromagnetic case where σ is the scalar electrical conductivity in esu), it is shown that the wave number for maximum growth rate in the linear state is

$$k_x = O(R_m^{1/4}/\Lambda)$$

where R_m is the magnetic Reynolds number $(\Lambda^2 g/\eta^2)^{1/2}$ for the diffusion coefficient η and the characteristic velocity $(g\Lambda)^{1/2}$. Since $R_m >> 1$, it is evident that k_x is large compared to $1/\Lambda$ and hence large compared to k_y and k_z.

The formal calculations show that the maximum in the growth rate is a very broad one because in the absence of diffusion the growth rate $1/\tau$ is approaching only asymptotically to the finite limiting value $O[(g/\Lambda)^{1/2}]$ as $k_x \longrightarrow \infty$. Hence $1/\tau$ increases only very slowly with k_x when $k_x >> 1/\Lambda$, and hence only a very little diffusive dissipation is needed to limit the increase of $1/\tau$. The result is that over the entire range $1/\Lambda < k_x < R_m^{1/2}/\Lambda$ * the growth rate is very nearly equal to the maximum growth rate $O[(g/\Lambda)^{1/2}]$ at $k_x = R_m^{1/4}/\Lambda$. The instability vanishes only as $k_x \longrightarrow \infty$. The broad maximum is illustrated in Fig. 6. Consequently a very broad distribution of instabilities, centered on $k_x = R_m^{1/4}/\Lambda$ is expected.

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* Obtained by equating the diffusion rate to the limiting instability rate .

Consider the quantitative limitations on k_x imposed by diffusion. The most effective diffusive process in the neutral interstellar gas is ambipolar diffusion, with an effective diffusion coefficient (Schluter and Biermann, 1950; Cowling, 1957)

$$\eta = O\left(\frac{B^2/4\pi}{M N_e N A w_i}\right) = O\left(\frac{kT}{M N_e A w_i}\right)$$

with the very rough approximation that $B^2/8\pi$ is comparable to NkT, where M is the mass of the neutral atom (mainly hydrogen), N is the number of neutral atoms per unit volume, N_e is the number of electrons per unit volume, A is the ion-neutral collision cross section, w_i is the ion thermal velocity, and T is the temperature of the neutral gas. The ion masses are typically 10 - 30 times the hydrogen mass. Osterbrook (1961) gives $A w_i$ 2.5×10^{-9} cm^3/sec under the usual interstellar conditions. The electron density N_e is presumably $O(10^{-4} N)$ so that for $T = 10^2$ °K or $B = 5 \times 10^{-6}$ gauss the diffusion coefficient is of the general order of η = $3 \times 10^{22}/N$ cm^2/sec. Hence for a smeared out average of $N = 3/\text{cm}^3$, we have $\eta = O(10^{23})$ cm^2/sec within a factor of ten. The magnetic Reynolds number for $\Lambda = 100$ pc, $v = 3 \times 10^{-9}$ cm/sec is, then,

$$R_m = O(10^4)$$

within a factor of ten.

The large value of the Reynolds number justifies the neglect of diffusion in the preliminary discussion of the instability. The characteristic diffusion time over a scale of 100 pc is Λ^2/η = (10^{18}) sec = 3×10^{10} years. Over 10 pc it is 3×10^8 yrs. These times are to be compared with the characteristic free fall time $(\Lambda/g)^{1/2} \cong 10^7$ yrs.

Now for Λ = 100 pc and R_m = 10^4 it follows that the instability has a scale of the general order of 100 pc along the magnetic field. The growth rate in terms of k_x is plotted in Fig. 6 for $k_y \Lambda$ = 1 and $k_z \Lambda$ = 0.5, 1, and 2 to illustrate the broad range over k_x, from about $1/\Lambda$ to $R_m^{1/2}/\Lambda$, i.e. from about 1 to 100 pc. The maximum growth rate is in the vicinity of 10 pc, but it is evident that the whole range is effectively unstable. The gas is set into motion with scales across the field as small as 1 pc, leading ultimately to diffusion times of only 10^6 - 10^7 years.

Altogether, the broad spectrum of the instability suggests that most interstellar clouds are made up of turbulence and fragments and the clouds are in the process of further fragmentation down to scales somewhere in the general vicinity of 1 pc. It appears that the disordered dynamical state may cease only when the cloud becomes do dense as to be dominated be self-gravitation.

V. Enhanced Diffusion

The tendency for the gas to be broken into small scales across the magnetic field produces an effect which may possibly lead to a rapid separation of the gas from the field. The possibility is not without interest to the theory of the formation of stars, wherein the observed stellar fields of 1 - 10^3 gauss are far below the 10^6 - 10^9 gauss expected when interstellar gas condenses into a star carrying the interstellar field with it. (See review by Mestel, 1966).

Consider the gas in a clump at the low point along a magnetic line of force, as indicated by the point A in Fig. 7. The point B in Fig. 7 is displaced a distance π / k_x from A so that B lies on the high point of a line of force. The gas density at B is relatively low, just as at A it is relatively high. If the diffusion coefficient is η, the gas diffuses from A to B in a time of the order of $1/\eta k_x^2$. Upon arrival at B the gas slides down the line of force to the low point, thereby decreasing its altitude by a distance of the order of $1/k_y$, or Λ, (assuming that the instability has had time to develop significantly). Hence the average rate of descent V of the gas is of the order of $\eta k_x^2 \Lambda$.

As the breakup of the gas progresses, wave numbers k_x appear throughout the interval ($1/\Lambda$, $R_m^{1/2}/\Lambda$). As the clumping progresses, further break up occurs, as discussed in section IV and Appendix V, so that k_x, and hence V, become very large. In the simplest case, $k_x = R_m^{1/4}/\Lambda$ for the maximum growth rate, the rate of descent is $V = O(R_m^{1/2}\eta/\Lambda)$. The characteristic diffusion velocity of the scale Λ would normally be η/Λ, so the instability has increased the overall diffusion rate across the field by the factor $R_m^{1/2}$. The region of maximum instability is so broad, however, (see Fig. 7) that the instability is effective all the way out to $k_x = O(R_m^{1/2}/\Lambda)$ for which $V = O(R_m \eta/\Lambda)$.

A simple formal example is worked out in Appendix VII to illustrate the physical process of the flow of the gas across the field when $k_x >> k_y$. In the example the magnetic lines of force are held fixed while the gas diffuses steadily across them, as outlined above. The process is diffusion limited under the circumstances which apply to $k_x = O(R_m^{1/4}/\Lambda)$ so that the inertial terms for the motion can be neglected. The solution of the problem is straight forward, leading to the mean vertical velocity of the order of $\eta k_x^2 / \Lambda k_y^2$. Then putting $\Lambda k_y \cong 1$ for the most unstable mode, the result is $V = O(\eta k_x^2 \Lambda)$, as stated from the qualitative considerations above.

In section IV the magnetic Reynolds number was estimated to be of the order 10^4. It follows that the dynamical instability enhances the expected rate at which the gas can move across, and separate from, the field by a factor $10^2 - 10^4$. The magnetic field, which might free itself from a gas cloud of 10 pc dimensions in $10^8 - 10^9$ years by ambipolar diffusion alone, may under the combined effects of ambipolar diffusion and dynamical instability, free itself as rapidly as the instability can grow and the cloud can collapse into clumps, some $10^7 - 10^8$ yrs. The field may be enormously reduced in the final gas cloud, which may help to explain why the gas of which stars are composed has so little field threading it, as compared to the interstellar gas.

VI. Summary and Conclusion

Several aspects of the general magnetic field and cosmic ray instability of the interstellar medium have been illustrated. The general results of the calculations given here and in I are that the interstellar gas collects into clouds in times of the order of free fall time $(\Lambda/g)^{1/2}$ of some $10^7 - 10^8$ yrs., whereas self-gravitation alone would not accomplish the observed clumping into clouds. The theoretically predicted spacing of clouds along the field is of the same general order as Λ, the scale height of the gas distribution.

The instability grows vigorously for all wavelengths across the field from $O(\Lambda/R_m^{1/2})$ to $O(\Lambda)$, which is 1 - 100 pc for typical values of Λ and R_m. The result is continual turbulence and fragmentation of the interstellar clouds, indicating that the interstellar gas clouds are in a vigorous and ragged dynamical state until they become sufficiently compact that self-gravitation becomes an important stabilizing force.

The instability of the gas over small scales perpendicular to the magnetic field leads to rapid ambipolar diffusion of the gas across the field. In this way the gas is able to move rapidly downward across the magnetic field by sliding to the lowest point on each line of force encountered during the rapid diffusion across the lines of force. We suggest this effect may contribute to freeing the field from the gas prior to condensation of the gas into stars.

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Appendix I. The Magnetic Field Between Slabs of Gas

Consider the magnetic field $\mathbf{B} = -\nabla\psi$, $\nabla^2\psi = 0$ in the region $-b < y < +b$, $z \geq 0$ subject to the boundary conditions that $B_z = 0$ on $z = 0$, $\mathbf{B} = 0$ at $z = +\infty$, and $B_y = B(z)$ (given by (1)) at $y = \pm b$. Then, in terms of solutions of $\nabla^2\psi = 0$, write

$$\psi(y, z) = \int_{-\infty}^{+\infty} dk\, f(k) \sinh ky \exp ikz \tag{11}$$

with the understanding that $f(k)$ must be an odd function of k so that B_z will vanish at $z = 0$. The transform $f(k)$ is evaluated from the requirement that $B_y = B(z)$ at $y = \pm b$, yielding

$$f(k) = -\frac{B(0)}{\pi k \cosh kb} \frac{\Lambda}{1 + k^2\Lambda^2} . \tag{12}$$

The resulting integrals for B_y and B_z are readily evaluated by closing the contour around the upper half plane and applying Cauchy's theorem,

$$B_y(y, z) = B(0) \left\{ \frac{\cos y/\Lambda}{\cos b/\Lambda} \exp\left(-\frac{z}{\Lambda}\right) \right. \tag{13}$$

$$\left. + \frac{2\Lambda}{b} \sum_{n=0}^{\infty} \frac{(-1)^n \cos(n+\frac{1}{2})\pi y/b \exp\left[-(n+\frac{1}{2})\pi z/b\right]}{1 - (n+\frac{1}{2})^2\pi^2\Lambda^2/b^2} \right)$$

$$B_z(y,z) = -B(0)\left\{ \frac{\sin y/\Lambda}{\cos b/\Lambda} \exp\left(-\frac{z}{\Lambda}\right)\right. \tag{14}$$

$$\left. + \frac{2\Lambda}{b} \sum_{n=0}^{\infty} \frac{(-1)^n \sin(n+\frac{1}{2})\pi y/b \, \exp\left[-(n+\frac{1}{2})\pi z/b\right]}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2} \right\}.$$

The change in field energy is

$$\Delta\mathcal{E} = \mathcal{E}_0 - \int_{-b}^{+b} dy \int_0^{\infty} dz \, \frac{B_y^2 + B_z^2}{8\pi} \tag{15}$$

per unit length in the x -direction, where

$$\mathcal{E}_0 = \int_{-b}^{+b} dy \int_0^{\infty} dz \, \frac{B^2(z)}{8\pi} = \frac{B^2(0)}{8\pi} \Lambda b \tag{16}$$

is the energy of the initial distribution in $-b < y < +b$, given by (1).

The easiest way to perform the integration is to go back to (11). Then

$$\int_{-b}^{+b} dy \int_0^{\infty} dz \, B_y^2 = \frac{1}{2}\int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dk' f(k')\, k' \cosh k'y \, \exp ik'z$$

$$\times \int_{-\infty}^{+\infty} dk'' f(k'')\, k'' \cosh k''y \, \exp ik''z$$

$$= \pi \int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dk \, k^2 f^2(k) \cosh^2 ky \, .$$

after noting that the integration over z gives $2\pi\,\delta(k'+k'')$ and that $f(-k) = -f(k)$. The integral of B_z^2 gives

$$\int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dz\; B_z^2 = \pi \int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dk\; k^2 f^2(k) \sinh^2 ky\,.$$

The integrands have poles at $k\Lambda = \pm i$ and at $k\,b = i\,(n+\frac{1}{2})\pi$ as a consequence of the denominator of $f^2(k)$. The residues of $\int dy \int dz\,(B_y^{\,2} + B_z^{\,2})$ are

$$\frac{-i\cos 2y/\Lambda}{4\Lambda\cos^2 b/\Lambda}\left(1 + \frac{2y}{\Lambda}\tan\frac{2y}{\Lambda} - \frac{2b}{\Lambda}\tan\frac{b}{\Lambda}\right)$$

and

$$-\frac{2i}{b^2}\left\{\frac{y\sin(2n+1)\pi y/b}{\left[1-(n+\frac{1}{2})\pi^2\Lambda^2/b^2\right]^2}\right.$$

$$-\frac{(2n+1)\,\pi\,(\Lambda^2/b)\cos(2n+1)\pi y/b}{\left[1-(n+\frac{1}{2})\pi^2\Lambda^2/b^2\right]^3}$$

respectively. Then, closing the contour around the upper half of the complex plane and using Cauchy's theorem, the integral over k can be evaluated. The remaing integration over y is then elementary, the final result being

$$\Delta\mathcal{E} = \mathcal{E}_0 \Big\{ 1 + \sec^2\frac{b}{\Lambda} - 2\frac{\Lambda}{b}\tan\frac{b}{\Lambda} \tag{17}$$

$$- \frac{4\Lambda}{\pi b}\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})\left[1-(n+\frac{1}{2})\pi^2\Lambda^2/b^2\right]^2}$$

for the decrease of the field energy below the initial value $\mathcal{E}_0$.

The general behavior of $\Delta\mathcal{E}$ is readily established. Beginning with a narrow gap between the slabs of gas, we have $b/\Lambda << 1$. Expanding in ascending powers of b/Λ yields

$$\Delta\mathcal{E} = \frac{1}{3}\mathcal{E}_0\left(\frac{b}{\Lambda}\right)^2\left[1 + O\left(\frac{b}{\Lambda}\right)\right] \tag{18}$$

so that $\Delta\mathcal{E}/\mathcal{E}_0$ increases as the square of the gap width over which the field is freed.

It is readily apparent that the individual terms, including the summations, have singularities at $b/\Lambda = (n+\frac{1}{2})\pi$. The complete expression remains finite however, as can be demonstrated by writing $b/\Lambda = (n+\frac{1}{2})\pi + \epsilon$ and carrying out the expansion in ascending powers of ϵ . The terms of order $1/\epsilon$ and $1/\epsilon^2$ vanish identically.

Finally, in the limit of very large b/Λ it is possible to do the sum by noting that the major contribution is from n in the vicinity of $b/\pi\Lambda$. We know that $\Delta\mathcal{E}$ is a smoothly varying function of b/Λ , so for simplicity suppose that $b/\Lambda = m\pi$ where

m is a large integer. Then let $n = m + \mu$. We have

$$\Delta \mathcal{E} = \mathcal{E}_0 \left\{ 2 - \frac{4}{m\pi^2} \sum_{-\infty}^{+\infty} \frac{m^2}{(m+\frac{1}{2}+\mu)(2\mu+1)^2 \left[1+(2\mu+1)/4m^2\right]^2} \right. \tag{19}$$

where the sum is now over μ. Examination of the terms in the vicinity of $\mu = 0$ shows that the terms in the sum which are $O(m)$ are (1 + 1/9 + 1/25 + 1/49 + ...), yielding $m\pi^2/4$. The terms $O(1)$ vanish identically, and the terms $O(1/m)$ yield the divergent series ... + 1/4 + 1/4 + 1/4 .... It follows, then, that $\Delta \mathcal{E} = \mathcal{E}_0$ in the limit of large b/Λ. The final field energy is small compared to the initial energy because of the unlimited expansion which has taken place. The asymptotic approach of $\Delta \mathcal{E}$ to $\mathcal{E}_0$ is rather slow with increasing b/Λ, as may be seen from Fig. 3 and from the fact that even with $b/\Lambda = 4\pi$ we have $\Delta \mathcal{E}$ equal only to 0.61 $\mathcal{E}_0$.

Finally, it is of interest to consider what stresses are exerted on the gas, so that we may have some idea whether the gas would be raised or lowered if it were released. The magnetic force in the upward direction is

$$F_B(z) = \frac{1}{4\pi} B_y(b,z) B_z(b,z)$$

$$= \frac{B^2(0)}{4\pi} \left\{ \tan\frac{b}{\Lambda} \exp\left(-\frac{2z}{\Lambda}\right) \right.$$

$$+ \frac{2\Lambda}{b} \sum_{n=0}^{\infty} \frac{\exp\left\{-\left[(n+\frac{1}{2})\pi/b + 1/\Lambda\right]z\right\}}{1 - (n+\frac{1}{2})^2 \pi^2 \Lambda^2 / b^2}$$

dynes/cm^2 on each face of the gas slabs. The total downward force on the gas per cm^2 of face is the difference between the gravitational force and the total pressure gradient in the gas,

$$F_G(z) = (a+b)\, g\, \rho(0) \exp\left(-\frac{2z}{\Lambda}\right)$$

$$\times \left\{1 - \left(\frac{a}{a+b}\right)\left(\frac{\frac{1}{2}V_A^2}{u^2+\frac{1}{2}V_A^2}\right) - \left(\frac{a+b}{a}\right)^{s-1}\left(\frac{u^2}{u^2+\frac{1}{2}V_A^2}\right)\right\}$$

if we suppose that the pressure increased in proportion to the s power of the density during the compression. If the upward magnetic force $F_B(z)$ exceeds the net downward force, $F_G(z)$, the gas will tend to move upward when released. If $F_B(z) < F_G(z)$, the gas will move downward. In either case the energy of the system will decrease further. In the simplest case suppose that $a << b$ as a consequence of the gas being very cold $u^2 << \frac{1}{2} V_A^2$. Then the only pressure is magnetic, and $\Lambda = V_A^2/g$. It follows that

$$F_G = \frac{B^2(0)}{4\pi}\,\frac{b}{\Lambda}\exp\left(-\frac{2z}{\Lambda}\right)$$

and

$$F_B(z) - F_G(z) = \frac{B^2(0)}{4\pi}\left\{\left(\tan\frac{b}{\Lambda} - \frac{b}{\Lambda}\right)\exp\left(-\frac{2z}{\Lambda}\right)\right.$$

$$\left. + \frac{2\Lambda}{b}\sum_{n=0}^{\infty}\frac{\exp\left\{-\left[(n+\frac{1}{2})\pi/b + 1/\Lambda\right]z\right\}}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2}\right.$$

Now consider whether the magnetic or gravitational forces dominate. For $z << b, \Lambda$ the exponential can be put equal to 1 in the sum. Then when $b << \Lambda$ each term in the sum can be expanded in descending powers of $n\Lambda/b$, yielding

$$F_B(z) - F_G(z) \cong -\frac{B^2(0)}{4\pi}\frac{b}{\Lambda}\left[1 + O\left(\frac{b^4}{\Lambda^4}\right)\right].$$

When $b >> \Lambda$ suppose that $b = m\pi\Lambda$ where m is an integer. Then write $n = m + \mu$. The sum can be written

$$-\left\{\left[\sum_{\mu=-m}^{m-1} + \sum_{\mu=m}^{\infty}\right]\frac{m}{(2\mu+1)\left[1 + (2\mu+1)/m\right]}\right.$$

$$= \left\{\frac{1}{2}\sum_{\mu=0}^{m-1}\frac{1}{\left[1 - (2\mu+1)^2/16m^2\right]}\right.$$

$$\left. + m\sum_{\mu=m}^{\infty}\frac{1}{(2\mu+1)\left[1 + (2\mu+1)/4m\right]}\right\}.$$

The individual terms in the first sum are all $O(1)$, so the sum is $O(m)$. The first $2m$ terms in the second sum are $O(1/m)$. Thereafter the individual terms decline as $1/\mu^2$, so that the whole quantity in curly braces is $O(m)$. It follows that

$$F_B(z) - F_G(z) \cong - \frac{B^2(0)}{4\pi} \frac{b}{\Lambda} .$$

Intermediate values such as $b = \pi \Lambda$, also give $F_B < F_G$. Thus for small z the gas moves downward when released, for all values of b/Λ.

The situation is rather different when $z \gg b, \Lambda$. If $b \ll \Lambda$, then the sum is negligible because of the exponential factor in each term,

$$F_B(z) - F_G(z) \cong + \frac{B^2(0)}{4\pi} \frac{b^2}{3\Lambda^2} \exp\left(-\frac{2z}{\Lambda}\right).$$

If $b \gg \Lambda$, put $b = m\pi\Lambda$ where m is a large positive integer. The exponential factors in the series are all larger than the factor $\exp(-2z/\Lambda)$ for $n+\frac{1}{2} < m$. These terms are also positive. The first term in the series dominates all the others in the limit of large z, so that $F_G > F_B$. For intermediate values, such as $b = \pi\Lambda$, the same result is obtained. Hence at large z the material moves upward when released.

At intermediate values of z ($z \cong \Lambda$) it is readily shown that $F_B > F_G$ if $b << \Lambda$, and $F_B < F_G$ if $b >> \Lambda$, as one would expect: When b is large, there is so much mass packed into each sheet that the local field cannot support it without sagging.

Appendix II. Instability in the Absence of Cosmic Rays

A. General Equations: Consider the stability of the equilibrium (1) in the presence of small perturbations $\mathbf{v}$, $\mathbf{b}$, $\delta\rho$, and δp . The linearized equation for conservation of mass is

$$\frac{\partial \delta\rho}{\partial t} + v_z \frac{d\rho}{dz} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0, \tag{111}$$

so that if the pressure perturbation δp is related to the density perturbation $\delta\rho$ by $\delta p / p = \gamma\, \delta\rho/\rho$, then δp is given by

$$\frac{\partial \delta p}{\partial t} + v_z \frac{dp}{dz} + \gamma u^2 \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0 \tag{112}$$

The hydromagnetic equation for the field perturbations are

$$\frac{\partial b_x}{\partial t} = + B \frac{\partial v_x}{\partial y} \tag{113}$$

$$\frac{\partial b_y}{\partial t} = - B\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right) - v_z \frac{dB}{dz} \tag{114}$$

$$\frac{\partial b_z}{\partial t} = + B \frac{\partial v_z}{\partial y} \tag{115}$$

assuming that the lines of force are frozen into the gas. The equations of motion are

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial \delta p}{\partial x} - \frac{B}{4\pi}\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right) \tag{116}$$

$$\rho \frac{\partial v_y}{\partial t} = -\frac{\partial \delta p}{\partial y} + \frac{b_z}{4\pi}\frac{dB}{dz} \tag{117}$$

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial \delta p}{\partial z} + \frac{B}{4\pi}\left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}\right) - \frac{b_y}{4\pi}\frac{dB}{dz} - g\,\delta\rho \tag{118}$$

Eliminate $\underset{\sim}{b}$ first by taking the derivative of the equations of motion with respect to time. The result can be written

$$Q_1 v_x = \gamma \upsilon^2 \frac{\partial^2 v_y}{\partial x \partial y} + \left[\left(\gamma \upsilon^2 + V_A^2\right)\frac{\partial}{\partial z} - g\right]\frac{\partial v_z}{\partial x}\,, \tag{119}$$

$$Q_2 v_y = \gamma \upsilon^2 \frac{\partial^2 v_x}{\partial x \partial y} + \left(\gamma \upsilon^2 \frac{\partial}{\partial z} - g\right)\frac{\partial v_z}{\partial y}\,, \tag{1110}$$

$$Q_3 v_z = D_1 \frac{\partial v_x}{\partial x} + D_2 \frac{\partial v_y}{\partial y}\,, \tag{1111}$$

where

$$D_1 \equiv (\gamma u^2 + V_A^2)\frac{\partial}{\partial z} + \frac{(1-\gamma)u^2 - \frac{1}{2}V_A^2}{u^2 + \frac{1}{2}V_A^2}\, g$$

$$D_2 \equiv \gamma u^2 \frac{\partial}{\partial z} + \frac{(1-\gamma)u^2 + \frac{1}{2}V_A^2}{u^2 + \frac{1}{2}V_A^2}\, g$$

$$Q_1 \equiv \frac{\partial^2}{\partial t^2} - (\gamma u^2 + V_A^2)\frac{\partial^2}{\partial x^2} - V_A^2 \frac{\partial^2}{\partial y^2} \tag{II12}$$

$$Q_2 \equiv \frac{\partial^2}{\partial t^2} - \gamma u^2 \frac{\partial^2}{\partial y^2} \tag{II13}$$

$$Q_3 \equiv \frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial y^2} - (\gamma u^2 + V_A^2)\frac{\partial^2}{\partial z^2} + g\left(\frac{\gamma u^2 + V_A^2}{u^2 + \frac{1}{2}V_A^2}\right)\frac{\partial}{\partial z}\,. \tag{II14}$$

Next operate on (II 9) with Q_2 and on (II 10) with Q_1, using (II 10) and (II 9) to eliminate v_y and v_x, respectively. The result is

$$\left(\varphi_1\varphi_2 - \gamma^2\nu^4 \frac{\partial^4}{\partial x^2 \partial y^2}\right) v_x = \tag{II15}$$

$$\left\{\gamma\nu^2\left(\gamma\nu^2\frac{\partial}{\partial z} - g\right)\frac{\partial^2}{\partial y^2} + \left[\left(\gamma\nu^2 + V_A^2\right)\frac{\partial}{\partial z} - g\right]\varphi_2\right\}\frac{\partial v_z}{\partial x},$$

$$\left(\varphi_1\varphi_2 - \gamma^2\nu^4 \frac{\partial^4}{\partial x^2 \partial y^2}\right) v_y = \tag{II16}$$

$$\left\{\gamma\nu^2\left[\left(\gamma\nu^2 + V_A^2\right)\frac{\partial}{\partial z} - g\right]\frac{\partial^2}{\partial x^2} + \left(\gamma\nu^2\frac{\partial}{\partial z} - g\right)\varphi_1\right\}\frac{\partial v_z}{\partial y}.$$

The final step is to operate on (II 11) with $\varphi_1\varphi_2 - \gamma^2\nu^4\,\partial^4/\partial x^2\partial y^2$ and eliminate v_x and v_y with the aid of (II 15) and (II 16).

$$\left(\varphi_1\varphi_2 - \gamma^2\nu^4 \frac{\partial^4}{\partial x^2 \partial y^2}\right)\varphi_3 v_z =$$

$$\left\{\gamma\nu^2\left(\gamma\nu^2\frac{\partial}{\partial z} - g\right)\frac{\partial^2}{\partial y^2} + \left[\left(\gamma\nu^2 + V_A^2\right)\frac{\partial}{\partial z} - g\right]\varphi_2\right\}D_1\frac{\partial^2 v_z}{\partial x^2} \tag{II17}$$

$$+\left\{\gamma\nu^2\left[\left(\gamma\nu^2 + V_A^2\right)\frac{\partial}{\partial z} - g\right]\frac{\partial^2}{\partial x^2} + \left(\gamma\nu^2\frac{\partial}{\partial z} - g\right)\varphi_1\right\}D_2\frac{\partial^2 v_z}{\partial y^2}.$$

We are interested in solutions of the form

$$v_z = \exp\left(\frac{t}{\tau} + ik_x x + ik_y y + ik_z z + \frac{z}{L}\right) \tag{II 18}$$

with the boundary conditions that $v_z = 0$ $\underbrace{}_{\text{at } z = 0}$ $= 0$ and that the perturbation be bounded, in some suitable sense, as $z \longrightarrow \infty$.

It is easily shown from (II 17) that only if $L = \Lambda = 2(u^2 + \frac{1}{2}V_A^2)/g$ can the modes for $\pm k_z$ be paired to give $v_z = 0$ at $z = 0$. For if $L \neq \Lambda$, then $1/\tau$ has an imaginary part whose sign is determined by the sign of k_z. Such individual modes do not then give $v_z = 0$ at $z = 0$. The analysis is much more complicated, requiring a sum over a continuous spectrum in order to satisfy the boundary condition. For the present purposes it is sufficient to put $L = \Lambda$, noting that for this simple mode the kinetic energy density $\frac{1}{2}\rho v^2$ remains bounded as $z \longrightarrow +\infty$, as does the field energy density, with the particle flux ρv going to zero as $z \longrightarrow \infty$. The boundary condition $v_z = 0$ at $z = 0$ is then satisfied by the pair of modes $\pm k_z$ which result.

Substituting (II 18) into (II 17) yields the dispersion relation, which can be put into the form

$$\left(\frac{1}{\tau^2} + \gamma u^2 k_y^2\right)\left[\frac{1}{\tau^2} + V_A^2 k_y^2 + (\gamma u^2 + V_A^2)\frac{1}{\Lambda^2}\right] - \frac{k_y^2}{\Lambda^2}\left[(2-\gamma)u^2 + V_A^2\right]^2 =$$
$$-k_z^2 u^2\left[\frac{\gamma + 2\alpha}{\tau^2} + 2\alpha\gamma u^2 k_y^2\right] + \frac{u^4 k_x^2}{Q_1}\left\{\frac{1}{\tau^2}\left[\gamma^2 k_y^2 + (\gamma+2\alpha)^2 k_z^2 + \frac{(2-\gamma)^2}{\Lambda^2}\right]\right.$$
$$\left. + 2\gamma u^2 k_y^2\left[\alpha(\gamma + 2\alpha)k_z^2 + I(k_y^2)/\Lambda\right]\right\} \tag{II 19}$$

where we have written $V_A^2 = 2\alpha u^2$ on the left hand side of the equation, and $I(x) \equiv \alpha\gamma\Lambda^2 x + 3\alpha\gamma + 2\gamma - 4\alpha - 2$. The quantity $Q_1 = \frac{1}{\tau^2} + (\gamma u^2 + V_A^2)k_x^2 + V_A^2 k_y^2$ is the propagator for fast mode hydromagnetic waves. It is positive and nonvanishing for the waves with which we are concerned here. We have chosen to write the dispersion relation in the form (II 19) because the left hand side alone is the basic dispersion relation for k_y alone ($k_x = k_z = 0$). The effects of k_x and k_z appear on the right hand side.

In the simplest case $u^2 = k_x^2 = k_z^2 = 0$ the dispersion relation reduces to

$$\frac{1}{\tau^4} + \frac{V_A^2}{\tau^2}\left(k_y^2 + \frac{1}{\Lambda^2}\right) - \frac{k_y^2 V_A^4}{\Lambda^2} = 0 . \tag{II20}$$

The unstable root ($\tau > 0$) is $\tau = \tau_0$ where

$$\frac{1}{\tau_0^2} = \frac{V_A^2}{2}\left\{-\left(k_y^2 + \frac{1}{\Lambda^2}\right) + \left[\left(k_y^2 + \frac{1}{\Lambda^2}\right)^2 + \frac{4k_y^2}{\Lambda^2}\right]^{1/2}\right\} . \tag{II21}$$

It is readily shown that $1/\tau_0^2$ is positive for all $k_y^2 > 0$, and is a monotonically increasing function of k_y^2, with the value $1/\tau_0^2 = k_y^2 V_A^2 = (g/\Lambda)(k_y\Lambda)^2$ for $\Lambda k_y << 1$, and approaching asymptotically to the limit $V_A^2/\Lambda = g/\Lambda$ as Λk_y becomes large. We recognize $(\Lambda/g)^{1/2}$ as the free fall time, which characterizes the time in which the instability grows by a factor of e. The shortest wavelengths are the most unstable, subject only to limitations of diffusion, viscosity, etc. which have been ignored here.

It is evident from inspection of (II 20) and (II 21) that the existence and strength of the instability depends upon the amount by which the last term in (II 20) is less than zero. The larger is $k_y^2 V_A^4/\Lambda^2$ the more rapid the growth rate. Thus, it is evident at once that the first term on the right hand side of (II 19) tends to reduce the instability, because when transposed to the left side, it diminishes the negative last term there. This property of the dispersion relation will be exploited throughout the discussions of the effects of k_x^2, k_z^2, and ω^2 which follow.

Suppose now that k_x, $k_z \neq 0$, but ω remains zero. Then with $2\alpha\omega^2 \equiv V_A^2 \neq 0$ the dispersion relation is

$$\frac{1}{\tau^4} + \frac{V_A^2}{\tau^2}\left(k_y^2 + \frac{1}{\Lambda^2}\right) - \frac{V_A^4 k_y^2}{\Lambda^2} = -\frac{V_A^2 k_z^2}{\tau^2}\left(1 - \frac{k_x^2 V_A^2}{\varphi_d}\right) \qquad \text{(II22)}$$

It is evident that the larger is k_z^2, the less rapidly the instability grows, but the inhibiting effects of k_z^2 may be off set by making k_x large. In the limit of large k_x the right hand side is reduced exactly to zero so that (II 21) is recovered. The vertical wave number k_z then has no inhibiting effect.

Note also that in this cold gas the horizontal wave number k_x perpendicular to the magnetic field has an effect only if k_z^2 is nonvanishing.

So far our study of a cold gas has shown that large k_x and k_y give the most rapid growth of the instability, in a period comparable to the free fall over one scale height. We must not be misled by this simple case, however, to conclude that the instability is a small-scale phenomenon in the galactic field. Introduction of even a very modest temperature drastically alters the situation. Suppose that $k_x^2 = 0$, but k_z^2 and u^2 are nonvanishing. Then (II 19) can be written

$$\frac{1}{\tau^4} + \frac{(\gamma u^2 + V_A^2)}{\tau^2}\left(k_y^2 + k_z^2 + \frac{1}{\Lambda^2}\right) = \frac{2u^4 k_y^2}{\Lambda^2}\left[2\alpha^2 - I\,(k_y^2 + k_z^2)\right]. \tag{II23}$$

Then $1/\tau^2$ is positive only if the right hand side is greater than zero,

$$2\alpha^2 > I\,(k_y^2 + k_z^2). \tag{II24}$$

Note that k_y^2 and k_z^2 contribute equally to the right hand side of the inequality. If the inequality is satisfied at all, it is satisfied as k_y, k_z go to zero. In order that the inequality be satisfied at all, then, we obtain the basic criterion (see paper I) that $(1+\alpha)(1+\alpha-\gamma) > \alpha\gamma/2$ for instability. Increasing k_y and k_z increases the right hand side of the inequality, making it more difficult to satisfy. It is evident that the inequality is not satisfied in the limit of large $k_y^2 + k_z^2$. It will be remembered that in a cold gas, increasing k_y^2 enhanced the instability. We now find, with $u^2 \neq 0$, that both k_y^2 and k_z^2 can, if too large, produce stability. It is evident that if $(1+\alpha)(1+\alpha-\gamma) > \alpha\gamma/2$ is satisfied, then there is instability, but the instability occurs only for wave numbers sufficiently small that

$$\Lambda^2 (k_y^2 + k_z^2) < \left[(2-\gamma+2\alpha)^2 - \gamma(\gamma+2\alpha)\right]\frac{1}{2\alpha\gamma} .$$

For small gas temperature $(v^2 << V_A^2 , \alpha >> 1)$ this is

$$\Lambda^2 (k_y^2 + k_z^2) < 2\alpha/\gamma .$$

Thus, only for very low gas temperature can the horizontal wave length of the instability be small compared to the scale height Λ .

It is evident, also, that the wavelength of the instability will never be very large compared to the scale height Λ , because the right hand side of (II 23) vanishes with vanishing k_y^2 . Altogether, the most unstable wavelengths will be for k_y of the order of $1/\Lambda$.

Now suppose the vertical wave number k_z is zero, but that v^2 and k_x^2 do not vanish. What is the effect of k_x^2 ? When $k_x^2 = 0$, instability occurs if and only if $2\alpha^2 > I(k_y)$. Then recall that, for a given k_y^2 , the more positive the right hand side of (II 19), the larger is the growth rate of the instability. Inspection of the coefficient of k_x^2 shows that all the terms in the curly braces are positive, with the possible exception of I. Instability requires that $2\alpha^2 > I$. Hence I can be negative, as for instance if α is large and $\Lambda^2 k_y^2 < 4/\gamma - 3$. But under most unstable circumstances it would appear that I is positive, or at least not so negative as to cause the curly braces to become negative. Hence, for a given k_y^2, increasing k_x^2 increases the instability rate. This was the same effect as obtained for the cold gas with $k_z \neq 0$.

For the general situation in which neither k_x^2, k_z^2, υ^2 is zero, what is the effect of k_z^2 and k_x^2? The right hand side of (II 19) can be written as

$$- k_z^2 \upsilon^2 \left(\frac{\gamma + 2\alpha}{\tau^2} + 2\alpha\gamma\upsilon^2 k_y^2 \right) \left[1 - \frac{\upsilon^2 k_x^2 (\gamma + 2\alpha)}{Q_1} \right]$$

$$+ \frac{\upsilon^4 k_x^2}{Q_1} \left\{ \frac{1}{\tau^2} \left[\gamma^2 k_y^2 + \frac{(2-\gamma)^2}{\Lambda^2} \right] + \frac{2\gamma\upsilon^2 k_y^2 I(k_y^2)}{\Lambda^2} \right\}$$

The first term represents a decrease of the instability with increasing k_z^2. It is evident too that increasing k_x^2 in the first term offsets the stabilizing effect of k_z^2. The stabilizing effect of k_z^2 vanishes in the limit of large k_x^2. What is more, the second term, represented by the curly braces, is generally positive, as discussed above, so that increasing k_x^2 enhances the instability beyond merely nullifying the stabilizing effects of k_z^2. The right hand side of (II 19), as written above, increases monotonically with k_x^2 (assuming the curly braces to be positive), approaching the asymptotic value

$$\frac{\upsilon^4}{\gamma\upsilon^2 + V_A^2} \left\{ \frac{1}{\tau^2} \left[\gamma^2 k_y^2 + \frac{(2-\gamma)^2}{\Lambda^2} \right] + \frac{2\gamma\upsilon^2 k_y^2 I(k_y^2)}{\Lambda^2} \right\}$$

in the limit of large k_x. The growth rate $1/\tau$ is then entirely independent of

k_z^2 . In the limit of large k_x^2 (II 19) reduces to

$$\frac{1}{\tau^4} + \frac{4\upsilon^2}{\tau^2(\gamma+2\alpha)\Lambda^2}\left[\left(\alpha^2+\alpha\gamma+\gamma-1\right)+\alpha(\alpha+\gamma)\Lambda^2 k_y^2\right]$$

$$- \frac{4\alpha\upsilon^4 k_y^2}{\Lambda^2(\gamma+2\alpha)}\left[\alpha(\gamma+2\alpha) - I(k_y^2)\right] = 0 . \qquad \text{(II25)}$$

The criterion for an unstable root $1/\tau^2 > 0$ is that the quantity in the square brackets in the last term be greater than zero,

$$\alpha(\gamma+2\alpha) > I(k_y^2). \qquad \text{(II26)}$$

This requirement is less stringent than (II 24) obtained for $k_x^2 = 0$, as we would expect from the destabilizing effect of k_x .

Appendix III. Instability with Cosmic Rays and a Weak Field

Suppose that the magnetic field is sufficiently weak that its stresses may be neglected, but there is now a significant cosmic ray pressure. The magnetic field ties the cosmic ray gas to the thermal gas. Consider first the simple degenerate case in which the thermal gas is cold. For the equilibrium state write the cosmic ray gas pressure P as $\beta u^2 \rho$ where β is a dimensionless constant and u is a constant characteristic sound velocity. It follows that

$$\frac{1}{\rho}\frac{d\rho}{dz} = \frac{1}{P}\frac{dP}{dz} = -\frac{g}{\beta u^2} . \tag{III 1}$$

Conservation of mass in the perturbed state is described by (II 1). In place of (II 2) we write (see I)

$$\frac{\partial \delta P}{\partial t} + v_z \frac{dP}{dz} = 0 \tag{III 2}$$

which states that the cosmic ray pressure does not vary along the magnetic lines of force.* The momentum equations are

$$\rho \frac{\partial v_x}{\partial t} = -\frac{\partial \delta P}{\partial x} , \tag{III 3}$$

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*This presupposes that $k_y > (k_z^2 + k_x^2)^{1/2} u / c$, for if k_y were small the suprathermal mode (Parker, 1965) would come into play, and we are not interested in that complication.

$$\rho \frac{\partial v_y}{\partial t} = - \frac{\partial \delta P}{\partial y} \quad , \tag{III 4}$$

$$\rho \frac{\partial v_z}{\partial t} = - \frac{\partial \delta P}{\partial z} - g\, \delta\rho \, . \tag{III 5}$$

Differentiate (III 3) - (III 5) with respect to t and use (III 2) to eliminate δP. Then differentiate (II 1) twice with respect to t and eliminate v_x, v_y, obtaining an equation in $\delta\rho$ and v_z. Use the equation for $\rho\, \partial^2 v_z / \partial t^2$ to eliminate $\delta\rho$, obtaining finally,

$$\frac{\partial^4 v_z}{\partial t^4} + g^2 \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right) = 0 \, . \tag{III 6}$$

Then suppose that v_z is of the form $f(z)\, \exp[t/\tau + i k_x x + i k_y y]$. It follows that

$$f(z) \left[\frac{1}{\tau^2} - g^2 (k_x^2 + k_y^2) \right] = 0 \tag{III 7}$$

so that $f(z)$ is an arbitrary function, subject only to such physical requirements that it be finite, single valued, etc. The instability rate is

$$\frac{1}{\tau^2} = g k \qquad \text{(III 8)}$$

where $k^2 = k_x^2 + k_y^2$. The system is unstable for all k_x and k_y, the rate increasing monotonically with both k_x and k_y.

It is interesting to note that if $g = g(z)$, instead of being constant as assumed in the present problem, all the equations (III 1) - (III 5) still apply. In place of (III 6) one obtains

$$\frac{\partial^4 v_z}{\partial t^4} + \frac{dg}{dz}\frac{\partial^2 v_z}{\partial t^2} + g^2\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) = 0. \qquad \text{(III 9)}$$

Again we find that $f(z)$ is arbitrary, and in place of (III 8)

$$\frac{1}{\tau^2} = gk\left\{\left[1 + \left(\frac{1}{2gk}\frac{dg}{dz}\right)^2\right]^{1/2} - \frac{1}{2gk}\frac{dg}{dz}\right\} \qquad \text{(III 10)}$$

Since $(1/g)\,dg/dz$ is generally a positive quantity near the disk of the galaxy the effect of dg/dz is to slow the instability somewhat. But the instability is not removed by dg/dz for any wave numbers. The growth rate still increases monotonically with k_x and k_y, and for the more unstable modes, the effect of dg/dz becomes small without limit.

Now consider the case that the gas is not cold. This removes the degeneracy in the z -dependence. Then the pressure gradient of the thermal gas $\nabla \delta p$ must be added to the right hand side of (III 3) - (III 5), with δp given by (II 2). The procedure is then to differentiate the momentum equations with respect to t, using (II 1), (II 2), and (III 2) to eliminate $\delta\rho$, δp, and δP, respectively. Note that $g = 2(1+\beta)u^2/\Lambda$.

Then operate on the equation for $\partial^2 v_x / \partial t^2$ with $\partial^2/\partial t^2 - \gamma \upsilon^2 \partial^2/\partial y^2$ and use the equation for $\partial^2 v_y/\partial t^2$ to eliminate v_y. The velocity v_x may be eliminated from the equation for v_y in a similar way, yielding the two equations

$$\left(\frac{\partial^2}{\partial t^2} - \gamma \upsilon^2 \nabla_{xy}^2\right) \begin{matrix} v_x \\ v_y \end{matrix} = \left[\gamma \upsilon^2 \frac{\partial}{\partial z} - \frac{2\upsilon^2}{\Lambda}(1+\beta)\right] \begin{matrix} \partial/\partial x \\ \partial/\partial y \end{matrix} \; v_z$$

Then operate on the equation for $\partial^2 v_z / \partial t^2$ with $\partial^2/\partial t^2 - \gamma \upsilon^2 \nabla_{xy}^2$, obtaining

$$\left[\frac{\partial^2}{\partial t^2} - \gamma \upsilon^2 \left(\frac{\partial^2}{\partial z^2} - \frac{2}{\Lambda}\frac{\partial}{\partial z}\right)\right]\left(\frac{\partial^2}{\partial t^2} - \gamma \upsilon^2 \frac{\partial^2}{\partial z^2}\right) v_z$$

$$= \left[\gamma \upsilon^2 \left(\frac{\partial}{\partial z} - \frac{1}{\Lambda}\right) + \frac{\upsilon^2(2+2\beta-\gamma)}{\Lambda}\right]\left[\gamma \upsilon^2\left(\frac{\partial}{\partial z} - \frac{1}{\Lambda}\right) - \frac{\upsilon^2(2+2\beta-\gamma)}{\Lambda}\right]$$

$$\times \nabla_{xy}^2 v_z$$

Assuming that v_z is of the form (II 18) so that the energy density is finite at both $z = \pm \infty$, the dispersion relation is readily shown to be

$$\frac{1}{\tau^4} + \frac{\gamma \upsilon^2}{\tau^2}\left(k_x^2 + k_y^2 + k_z^2 + \frac{1}{\Lambda^2}\right)$$

$$- \frac{\upsilon^4 (k_x^2 + k_y^2)}{\Lambda^2}\left[(2+2\beta-\gamma)^2 - \gamma^2\right] = 0$$

This is the same basic form as was dealt with in Appendix II. There is an unstable root $1/\tau > 0$ if and only if the quantity in square brackets in the last term is greater than zero. Since $\beta, \gamma > 0$ this is equivalent to $1 + \beta > \gamma$.

Note that $k_x^2 + k_y^2$ here plays the same role as k_y^2 for a cold gas in a horizontal magnetic field without cosmic rays. The growth rate is a monotonically increasing function of $k_x^2 + k_y^2$. The effect of k_z^2 is to slow the growth rate, but this effect is removed in the limit as $k_x^2 + k_y^2$ becomes large. For $k_x^2 + k_y^2 << k_z^2 + 1/\Lambda^2$, the growth rate is

$$\frac{1}{\tau} = \upsilon \left(k_x^2 + k_y^2\right)^{1/2} \left[\frac{(2+2\beta-\gamma)^2 - \gamma^2}{\gamma\left(1 + k_z^2 \Lambda^2\right)}\right]^{1/2}.$$

When $k_x^2 + k_y^2 >> k_z^2 + 1/\Lambda^2$, the growth rate increases to

$$\frac{1}{\tau} = \frac{\upsilon}{\Lambda}\left[\frac{(2+2\beta-\gamma)^2 - \gamma^2}{\gamma}\right]^{1/2}$$

which is independent of k_x, k_y, and k_z.

Appendix IV. Stability of Alternate Perpendicular Layers of Magnetic Field

Consider a cold atmosphere without cosmic rays supported by a horizontal magnetic field

$$\underset{\sim}{B} = \begin{cases} \underset{\sim}{e}_y \, B(z) & \text{in } 2na < z < (2n+1)a \\ \underset{\sim}{e}_x \, B(z) & \text{in } (2n-1)a < z < 2na \end{cases} \tag{IV 1}$$

where $n = 0, 1, 2, \ldots$ Write $B^2 = 4\pi\rho V_A^2$, where V_A is a constant Alfven speed. The field density B is given by (1). Ignore possible rapid small-scale instability at the discontinuities $z = na$ between layers of field.

Conservation of mass is described by (II 1) again. Equations (II 3) - (II 7) with $\delta p = 0$ describe the field and the motion in regions where $\underset{\sim}{B} = e_y \, B(z)$. Where $\underset{\sim}{B} = \underset{\sim}{e}_x \, B(z)$ we have

$$\frac{\partial b_x}{\partial t} = - B\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - v_z \frac{dB}{dz} , \tag{IV 2}$$

$$\frac{\partial b_y}{\partial t} = + B \frac{\partial v_y}{\partial x} , \tag{IV 3}$$

$$\frac{\partial b_z}{\partial t} = + B \frac{\partial v_z}{\partial y} , \tag{IV 4}$$

$$\rho \frac{\partial v_x}{\partial t} = \frac{1}{4\pi} \frac{dB}{dz} b_z \quad , \tag{IV 5}$$

$$\rho \frac{\partial v_y}{\partial t} = \frac{B}{4\pi} \left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} \right) \quad , \tag{IV 6}$$

$$\rho \frac{\partial v_z}{\partial t} = -\frac{B}{4\pi} \left(\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} \right) - \frac{b_x}{4\pi} \frac{dB}{dz} - g\, \delta\rho \, . \tag{IV 7}$$

The boundary conditions at $z = na$ are that v_z is continuous and the magnetic pressure is continuous. Hence $|b_x|$, $|b_y|$ on one side of $z = na$ must equal $|b_y|$, $|b_x|$ respectively on the other side. Since k_x, k_y, and τ must be the same in each layer in order that these boundary conditions be satisfied, it is evident that k_z must also be the same because τ is a monotonically decreasing function of k_z. That is to say, k_z is a single valued function of τ.

It is known from I and from Appendix II that the individual layers of field are most unstable when $k_z = 0$ in (II 18). The growth rate $1/\tau$ is then independent of the component of the horizontal wave number which is perpendicular to the equilibrium field $B(z)$, so the problem is greatly simplified. The growth rate $1/\tau$ is then a monotonically increasing function of the wave number parallel to $B(z)$. The fastest growing mode is that for which $k_x = k_y \equiv k/2^{1/2}$

throughout. It is evident from (II 20) that the growth rate is

$$\frac{1}{\tau^2} = \frac{k^2 V_A^2}{4}\left\{-\left(1+\frac{2}{k^2\Lambda^2}\right)+\left[\left(1+\frac{2}{k^2\Lambda^2}\right)^2+\frac{8}{k^2\Lambda^2}\right]^{1/2}\right\}$$

for the mode with maximum growth rate.

Appendix V. Force Between Elements of Gas Suspended On a Horizontal Magnetic Field

Consider the force exerted on a small element of gas suspended on a large-scale magnetic field when a second small element is suspended at an arbitrary position in the same field. Use the geometry shown in Fig. 1, with a gravitational field g in the negative z -direction and the large scale magnetic field B in the positive y -direction. It is sufficient for the present purposes to suppose that the large-scale magnetic field is pervaded by a tenuous plasma which is an excellent conductor of electricity but whose pressure and weight are so slight as to have no sensible dynamical effects on the magnetic field. Hence B may be taken as a uniform magnetic field with infinite scale height. Each of the two small elements of gas suspended in B will be taken to be sufficiently light compared to their small dimensions that the distortion $\Delta \underset{\sim}{B}$ of the large-scale field $\underset{\sim}{e}_y B$ is small everywhere in the field.

Now the calculation may proceed directly from the hydromagnetic equations. But it is much easier, and formally equivalent, in this particular case, to deal directly with the currents (induced by the weight of the gas). The current system is diagrammed in Fig. 8. The current I_1 , shown by the heavy segment of length $2a$ across the middle, represents the current through the element of gas. The Lorentz force $2a\, I_1\, B/c$ of this current supports the weight $m_1 g$ of the gas, so that $I_1 = m_1 c g / 2aB$. The current must stream away from the ends of the element of gas but there can be no Lorentz force exerted on the tenuous plasma filling the space. Hence the current must stream away along the magnetic lines of force, as sketched in Fig. 8. In the present case we suppose that the current flows along the lines of force all the way to $y = \pm \infty$.

It is evident from Maxwell's equations that $\nabla \times \mathbf{B} = 0$ everywhere except at the location of the currents. Hence the field has the same configuration outside the currents as in a vacuum, and the field may be computed from the currents by the same method as in a vacuum. The magnetic field of a current $\mathbf{I}(\mathbf{r}')$ at a point $\mathbf{r}'$ is given by the Biot-Savart law

$$\Delta \mathbf{B}(\mathbf{r}) = \frac{1}{c} \int \frac{ds\, \mathbf{I}(\mathbf{r}) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

where ds is an element of length along the current $\mathbf{I}$. Place the origin of the coordinates at the center of the element of gas, as shown in Fig. 8. Then it is readily shown, after going through the algebra, that the field $\Delta \mathbf{B}(\mathbf{r})$ produced by the current system shown in Fig. 8 is

$$\Delta B_x(\mathbf{r}) = + \frac{I_1 y z}{c} \left(\frac{1}{b_1^2 s_1} - \frac{1}{b_2^2 s_2} \right) \tag{V 1}$$

$$\cong + \frac{4 I_1 a}{c} \frac{x y z}{(x^2+z^2) r} \left(\frac{1}{x^2+z^2} + \frac{1}{2r^2} \right) , \tag{V 2}$$

$$\Delta B_y(\underset{\sim}{r}) = -\frac{2I_1 a}{c}\,\frac{z}{r^3} \qquad \text{(V 3)}$$

$$\Delta B_z(\underset{\sim}{r}) = -\frac{I_1 y}{c}\left(\frac{x-a}{b_1^2 s_1} - \frac{x+a}{b_2^2 s_2}\right) + \frac{2Iay}{cr^3} \qquad \text{(V 4)}$$

$$\cong \frac{2I_1 ay}{cr^3}\left[1 + \frac{z^2+y^2}{z^2+x^2} - \frac{2x^2r^2}{(x^2+z^2)^2}\right] \qquad \text{(V 5)}$$

where

$$r^2 = x^2+y^2+z^2$$

$$b_{1,2} = \left[(x \mp a)^2 + z^2\right]^{1/2}$$

$$s_{1,2} = \left[(x \mp a)^2 + y^2 + z^2\right]^{1/2}.$$

The terms $(2I_1\, a/c)\, z/r^3$ and $(2I_1\, a/c)\, y/r^3$ are the contribution of the current segment $2aI_1$. The other terms are from the currents flowing along the lines of force. The first line of each formula for the components of $\Delta \underset{\sim}{B}$ is exact for the idealized current system shown in Fig. 8. The approximate forms are for either or both x or z large compared to a. When $x << a$, we have $\Delta B_x \cong 0$. The formula for ΔB_y is unchanged, and

$$\Delta B_z \cong \frac{2 I_1 a y}{c} \left[\frac{1}{(a^2+z^2)(a^2+y^2+z^2)^{1/2}} + \frac{1}{r^3} \right] \qquad \text{(V 6)}$$

Now consider what effect the perturbation $\Delta \mathbf{B}$ has on the equilibrium of another small element of gas of mass m_2 suspended elsewhere on the field. The weight of m_2 is supported in equilibrium by the large-scale field $B \mathbf{e}_y$. The perturbation field $\Delta \mathbf{B}$ produced by m_1 introduces an addition force. To calculate the force on m_2 caused by $\Delta \mathbf{B}$ note that the currents flowing to and from the ends of m_2 will flow along the perturbed lines of force, so there is no Lorentz force exerted on them by $\Delta \mathbf{B}$. The only force is $\mathbf{F} = 2a\, \mathbf{I}_2 \times \Delta \mathbf{B} / c$, where $\mathbf{I}_2 = \mathbf{e}_x\, m_2 c g / 2aB$ is the supporting current through m_2. Obviously $F_x = 0$ and

$$F_x = -m_2 g \frac{\Delta B_z}{B},$$

$$F_z = + m_2 g \frac{\Delta B_y}{B}.$$

The vertical component of the force, F_z, is of little present interest. The elements of gas are threaded by the field and hence constrained to slide along the lines of force. The current I_2 adjusts itself so that the Lorentz force in the total horizontal field $B + \Delta B_y$ supports the weight of the gas.

The force F_y causes the element of gas m_2 to slide along the magnetic lines of force, either toward or away from m_1. To illustrate the general form of F_y consider the two cases that m_2, at (x, y, z), is on a line of force passing near m_1, as compared to the distance along the lines of force between m_1 and m_2, or m_2 is on a line remote from the lines through m_1. In the first instance, $y >> x, z$, and

$$\Delta B_z \cong \frac{2Ia}{c} \frac{(z^2-x^2)}{(z^2+x^2)^2}$$

so that

$$F_y = - m_1 m_2 \frac{8^2}{B^2} \frac{z^2-x^2}{(z^2+x^2)^2} \tag{V 7}$$

The force is thus attractive if $z^2 > x^2$ and repulsive if $z^2 < x^2$. The force is undiminished by the distance y between m_1 and m_2. The force falls off inversely as the square of the separation $(z^2+x^2)^{1/2}$ of the magnetic lines of force on which the elements slide.

In the second instance $a^2, y^2 << x^2, z^2$ we have

$$\Delta B_z = \frac{2I_1 a y}{c} \frac{(2z^2-x^2)}{(z^2+x^2)^{5/2}}$$

so that

$$F_y = - m_1 m_2 \frac{g^2}{B^2} \frac{y(2z^2 - x^2)}{(z^2 + x^2)^{5/2}} . \qquad \text{(V 8)}$$

The force is attractive if $z^2 > x^2/2$ and repulsive if $z^2 < x^2/2$. The force is directly proportional to y. If m_2 were threaded on a line of force directly above the line through m_1, then $x = 0$ and the force is

$$F_y = - m_1 m_2 \frac{2g^2}{B^2} \frac{y}{z^3} . \qquad \text{(V 9)}$$

The gravitational force between m_1 and m_2 is $G m_1 m_2 y/z^3$ along the lines of force, indicating that F_y is larger by the factor $2g^2/GB^2$ than self gravitation.

It is evident from (V 8) that the force in $x^2 > z^2 > x^2/2$ is attractive when y is small, and it is evident from (V 9) that the force in $x^2 > z^2 > x^2/2$ is repulsive when y is large. We would expect, then, that in passing from small to large y along a line in $x^2 > z^2 > x^2/2$, that the force F_y must pass through zero. It is readily shown, by equating the right hand side of (V 5) to zero, that the force F_y vanishes at

$$y^2 = (x^2 + z^2)\left(\frac{2z^2 - x^2}{x^2 - z^2}\right).$$

The equilibrium position at this value of y is unstable, of course.

Finally, if the two elements of gas are exactly on the same lines of force, then $x = z = 0$ and it follows from (V 6) that

$$F_y = - m_1 m_2 \frac{g^2}{B^2} \frac{1}{a^2} \left(1 + \frac{a^2}{y^2} \right).$$

The force is attractive for all values of y, approaching the limit $- m_1 m_2 \, g^2 / B^2 a^2$ as y becomes large compared to a.

Altogether it is evident that elements of gas will tend to group in vertical columns, with neighboring columns moving apart. It is evident too that each individual element will tend to shear apart as a consequence of the mutual repulsion of its opposite ends.

Appendix VI. Instability in the Presence of Diffusion

The simplest nontrivial case of the limiting effect of diffusion on the horizontal transverse wave number k_x is a cold gas suspended in a horizontal magnetic field, given by (1) with $\upsilon = 0$. In the absence of dissipation this case gives a monotonic increase of the growth rate with increasing k_x if $k_z \neq 0$ (see (II 22)). Hence it permits the demonstration of the limiting of k_x by diffusive dissipation.

Suppose, then, that the system is dissipative because of resistivity in the medium. The equations of motion are unaffected, so that (II 1) and (II 6) - (II 8) are applicable, with $\delta p = 0$ in the present case that the gas is cold. The equations (II 3) - (II 5) for the magnetic field must be modified by replacing $\partial/\partial t$ by $\partial/\partial T \equiv \partial/\partial t - \eta \nabla^2$ where η is the effective diffusion coefficient. In the simplest case $\eta = c^2/4\pi\sigma$ where σ is the electrical conductivity. The operator $\partial/\partial \tau$ can be approximated as $\partial/\partial t - \eta\, \partial^2/\partial x^2$ in the present instance, since we shall be interested only in the case that the wave number k_x is large compared to k_y, k_z, and $1/\Lambda$.

The motion of the lines of force through the fluid as a consequence of ambipolar diffusion can be treated approximately with the same equations, if we identify η with the ambipolar diffusion coefficient (see Section IV) instead of with $c^2/4\pi\sigma$.

Differentiate (II 6) and (II 7) with respect to T and use the modified (II 3) to (II 5) to eliminate the field, obtaining

$$\left(\frac{\partial^2}{\partial T \partial t} - V_A^2 \nabla_{xy}^2\right) v_x = \left(V_A^2 \frac{\partial}{\partial z} - g\right) \frac{\partial v_z}{\partial x} , \tag{VI 1}$$

$$\frac{\partial^2 v_y}{\partial T \partial t} = - g \frac{\partial v_z}{\partial y} . \tag{VI 2}$$

Then operate on (II 8) with $\partial / \partial T$ so that the magnetic field components can be eliminated, and with $\partial / \partial t$ so that $\delta\rho$ can be eliminated through the use of (II 1). The result can be written

$$\left(\frac{\partial^2}{\partial T \partial t} - V_A^2 \nabla_{yz}^2\right) \frac{\partial v_z}{\partial t} = \left(V_A^2 \frac{\partial}{\partial z} - g\right) \frac{\partial^2 v_x}{\partial x \, \partial t} \tag{VI 3}$$

$$+ g \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial t}\right) \frac{\partial v_x}{\partial x} + g \frac{\partial^2 v_y}{\partial y \, \partial T}$$

$$+ g \left(\frac{\partial}{\partial T} - 3 \frac{\partial}{\partial t}\right) \frac{\partial v_z}{\partial z} + \frac{2 g^2}{V_A^2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial T}\right) v_z .$$

Then operate on (VI 3) with $\partial / \partial t$ so that v_y can be eliminated, and with $\partial^2 / \partial T \partial t - V_A^2 \nabla_{xy}^2$ so that v_x can be eliminated. The resulting equation for v_z can be written

$$\left(\frac{\partial^2}{\partial T \partial t} - V_A^2 \nabla_{xy}^2\right) \times$$

$$\left\{ \left[\left(\frac{\partial^2}{\partial T \partial t} - V_A^2 \nabla_{yz}^2\right)\frac{\partial}{\partial t} + g\left(3\frac{\partial}{\partial t} - \frac{\partial}{\partial T}\right)\frac{\partial}{\partial z} + \frac{2g^2}{V_A^2}\left(\frac{\partial}{\partial T} - \frac{\partial}{\partial t}\right)\right]\frac{\partial v_z}{\partial t} \right.$$

$$\left. + g^2 \frac{\partial^2 v_z}{\partial y^2} \right\} = \tag{VI 4}$$

$$\left[\left(V_A^2 \frac{\partial}{\partial z} - g\right)\frac{\partial}{\partial t} + g\left(\frac{\partial}{\partial T} - \frac{\partial}{\partial t}\right)\right]\left(V_A^2 \frac{\partial}{\partial z} - g\right)\frac{\partial}{\partial t}\frac{\partial^2 v_z}{\partial x^2} .$$

Suppose that v_z is of the form (II 18). Then the dispersion relation can be written

$$\frac{1}{\tau^4} + \frac{\eta k_x^2}{\tau^3} + \frac{V_A^2}{\tau^2}\left[k_y^2 + \frac{1}{\Lambda^2} + k_z^2\left(1 - \frac{k_x^2 V_A^2}{P}\right)\right]$$

$$+ \frac{\eta k_x^2 V_A^2}{\tau}\left[\frac{1}{\Lambda^2} - \frac{i k_z}{\Lambda}\left(1 - \frac{k_x^2 V_A^2}{P}\right)\right] - \frac{V_A^4 k_y^2}{\Lambda^2} = 0 \tag{VI 5}$$

where Λ is the characteristic scale height V_A^2/g and

$$P = \frac{1}{\tau}\left(\frac{1}{\tau} + \eta k_x^2\right) + V_A^2\left(k_x^2 + k_y^2\right) \tag{VI 6}$$

is the propagator for damped Alfven waves. For the cases in which we shall be interested

$\mathrm{Re}\, P > 0$.

Consider what mode is most unstable for a given k_y and k_z. We know from the discussion in Appendix II that large values of k_z suppress the instability and that the most unstable k_y is of the order of $1/\Lambda$ if the thermal motions are included in the calculation. The discussion showed also that increasing k_x offset the stabilizing influence of k_z, with the growth rate given by

$$\frac{1}{\tau} \cong \frac{1}{\tau_0}\left[1 - \frac{k_z^2\left(1 + 1/V_A^2 k_y^2 \tau_0^2\right)}{2k_x^2 \; S^{1/2}}\right] \tag{VI 7}$$

when k_x is large. Here $1/\tau_0$ is the growth rate

$$\frac{1}{\tau_0^2} = \frac{V_A^2}{2\Lambda^2}\left[S^{1/2} - \left(1 + k_y^2\Lambda^2\right)\right] \tag{VI 8}$$

for $k_z = 0$ and

$$S = 1 + 6k_y^2\Lambda^2 + k_y^4\Lambda^4. \tag{VI 9}$$

The important point is that, without diffusion η, the growth rate increases asymptotically to $1/\tau_0$ as $k_x \longrightarrow \infty$. It is evident that inclusion of diffusion must slow the instability at large k_x, indicating that there is an intermediate value of k_x for which the growth rate is a maximum.

The case of interest is for a relatively small diffusion coefficient $\eta \ll V_A\Lambda,\ V_A^2\tau$. Then diffusion does not have much effect until k_x is very large compared to k_y, k_z. But when

$k_x >> k_y, k_z$, the growth rate is increasing only very slowly with k_x, as given by (VI 7), so the damping effect of diffusion becomes important in determining the sign of $d\tau/dk_x$ even when $\eta k_x^2 << 1/\tau$. The maximum in $1/\tau$ occurs in (VI 5) when the terms in ηk_x^2 are still small. Replacing τ by τ_0 in all terms involving η or $1/k_x^2$ it is readily shown that

$$\frac{1}{\tau} = \frac{1}{\tau_0}\left\{1 - \frac{\Lambda^2}{2S^{1/2}V_A^2}\left[\frac{\eta k_x^2}{\tau_0}\left(1+V_A^2\tau_0^2/\Lambda^2\right) + \frac{k_z^2}{k_x^2}\left(V_A^2k_y^2 + 1/\tau_0^2\right)\right]\right\}. \tag{VI 10}$$

The maximum growth rate occurs for the value of k_x given by

$$k_x^4 = \frac{k_z^2}{\eta\tau_0}\left(\frac{1+\tau_0^2V_A^2k_y^2}{1+\tau_0^2V_A^2/\Lambda^2}\right). \tag{VI 11}$$

Hence the most unstable wave number k_x is large compared to k_z because η is small. In terms of the effective magnetic Reynolds number $R_m = 1/k_z^2\eta\tau$, it is evident that $k_x = O(k_z R_m^{1/4})$.

The characteristic diffusion rate $1/t$ is then,

$$\begin{aligned}\frac{1}{t} &\equiv k_x^2\eta \\ &= O\left(R_m^{1/2}/\tau_0\right) \\ &= O\left(k_z^2\eta/R_m^{1/2}\right).\end{aligned} \tag{VI 12}$$

It is faster by $R_m^{1/2}$ than diffusion over the scale Λ or $1/k_z$. This is an enormous enhancement of the diffusion of gas across the field lines. The enhancement is of the same order as the enhancement in the vicinity of a neutral point (see, for instance, Parker, 1963), which is also by the factor $R_m^{1/2}$.

The next step in the development would be to include the thermal motions υ of the gas. It is readily shown that the general result is the same as for $\upsilon = 0$ without goint through the very tedious calculation from the complete set of equations. In the absence of diffusion the dispersion relation is just (II 19). The most unstable k_y is $O(1/\Lambda)$. The vertical wave number k_z tends to decrease the instability rate but it is readily shown that increasing k_x restores the growth rate to the value for $k_z = 0$. This much is the same as when $\upsilon = 0$. So put $k_z = 0$, since there is no need to re-investigate its effect. The remaining dispersion equation can be written, up to terms $O(k_y^2/k_x^2)$, as

$$\frac{1}{\tau^4} + \frac{1}{l^2\tau^2} - \frac{1}{m^4} = -\frac{k_y^2}{n^4 k_x^2} \tag{VI 13}$$

where the coefficients $1/l^2$, $1/m^2$, and $1/n^2$ are all positive quantities, defined by

$$\frac{1}{l^2} = (\gamma\upsilon^2 + V_A^2)\left(k_y^2 + \frac{1}{\Lambda^2}\right) + \frac{\upsilon^4\left[\gamma k_y^2 + (2-\gamma)/\Lambda^2\right]}{\gamma\upsilon^2 + V_A^2} , \tag{VI 14}$$

$$\frac{1}{m^4} = \frac{k_y^2}{\Lambda^2}\left[(2-\gamma)\upsilon^2 + V_A^2\right] + \frac{2\gamma\upsilon^6 k_y^2 I(k_y^2)}{(\gamma\upsilon^2 + V_A^2)\Lambda^2} \tag{VI 15}$$

$$\frac{1}{n^4} = \frac{\upsilon^4 (k_y^2 V_A^2 + 1/\tau^2)}{k_y^2(\gamma\upsilon^2 + V_A^2)}\left\{\frac{1}{\tau^2}\left[\gamma^2 k_y^2 + \frac{(2-\gamma)^2}{\Lambda^2}\right] + \frac{2\gamma\upsilon^2 k_y^2 I(k_y)}{\Lambda^2}\right\}. \tag{VI 16}$$

The right hand side of the equation is small, $O(k_y^2/k_x^2)$ so it is sufficient to replace $1/\tau^2$ in n^4 by $1/\tau_1^2$, where $1/\tau_1^2$ is the root

$$\frac{1}{\tau_1^2} = \frac{1}{2}\left[-\frac{1}{l^2} + \left(\frac{1}{l^4} + \frac{4}{m^4}\right)^{1/2}\right] \tag{VI 17}$$

in the limit as $k_x \longrightarrow 0$. It follows, then, that

$$\frac{1}{\tau} = \frac{1}{\tau_1}\left[1 - \frac{k_y^2\tau_1^2}{2k_x^2 n^4 (1/l^4 + 4/m^4)^{1/2}}\right]. \tag{VI 18}$$

Comparing (VI 18) with (VI 7) it may be seen that $1/\tau$ again approaches asymptotically to the limiting value as $k_x \longrightarrow \infty$. The rate of approach is the same as before, so that when diffusion is introduced, it is evident that the wavenumber for maximum growth rate is

$$k_x^4 = O\left(\frac{k_y^2}{\eta\tau_1}\right) = O\left(\frac{1}{\Lambda^2\eta\tau_1}\right) \tag{VI 19}$$

as in the previous case. The diffusion is enhanced by the square root of the magnetic Reynolds number again.

Appendix VII. Enhanced Diffusion as a Consequence of Instability

Consider the motion of a gas threaded by the fixed lines of force of the static magnetic field

$$\underset{\sim}{\mathbf{B}} = B\left[\underset{\sim}{\mathbf{e}}_y + \underset{\sim}{\mathbf{e}}_z\, \epsilon(x,y)\right] \tag{VII 1}$$

where z is the vertical dimension and $B\underset{\sim}{\mathbf{e}}_y$ represents a uniform horizontal field. The overall field configuration is sketched in Fig. 7. The gas is constrained to move along the magnetic lines of force except for diffusion across the lines of force, represented by the coefficient η cm^2/sec. In the present case the gas density ρ and pressure p are assumed to vary much more rapidly with horizontal distance x across the field than with y or z, so that diffusion is significant only in the x-direction. Assuming that the gas is isothermal and its motion is steady and limited by diffusion, inertia may be neglected and the motion across the field is given by

$$\rho v_x \cong -\eta \frac{\partial \rho}{\partial x} \tag{VII 2}$$

to a first approximation*. Motion along the lines of force is principally in the

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* On purely formal grounds one may start with, say, the hydromagnetic equations $d\underset{\sim}{\mathbf{v}}/dt = -\nabla p + \rho \underset{\sim}{\mathbf{g}} + (\nabla\times\underset{\sim}{\mathbf{B}})\times\underset{\sim}{\mathbf{B}}/4\pi$, $\partial\underset{\sim}{\mathbf{B}}/\partial t = \nabla\times(\underset{\sim}{\mathbf{v}}\times\underset{\sim}{\mathbf{B}}) - q\nabla\times\nabla\times\underset{\sim}{\mathbf{B}}$. Write $\underset{\sim}{\mathbf{B}} = \nabla\times\underset{\sim}{\mathbf{A}}$ and integrate the second equation, obtaining $q\nabla\times\underset{\sim}{\mathbf{B}} = \underset{\sim}{\mathbf{v}}\times\underset{\sim}{\mathbf{B}} - \partial\underset{\sim}{\mathbf{A}}/\partial t + \nabla\psi$ where ψ is a scalar function of position such that $-c\nabla\psi\times\underset{\sim}{\mathbf{B}}/B^2$ represents externally impressed motion of the fluid. Eliminate $\nabla\times\underset{\sim}{\mathbf{B}}$ from the momentum equation, writing the result in the form $\underset{\sim}{\mathbf{v}}_\perp = (\underset{\sim}{\mathbf{g}} - d\underset{\sim}{\mathbf{v}}/dt - \nabla p/\rho)q/V_A^2 + (4\pi/B^2)\underset{\sim}{\mathbf{B}}\times(\partial\underset{\sim}{\mathbf{A}}/\partial t - \nabla\psi)$, where V_A is the local Alfven speed, $B/(4\pi\rho)^{1/2}$. The last term, involving $\partial\underset{\sim}{\mathbf{A}}/\partial t - \nabla\psi$ represents the transverse motion of the lines of force,

so that the motion of the fluid across the lines of force is just $(g - \rho\, dv/dt - \nabla p/\rho)\, q / V_A^2$. Then if $p = \rho u^2$ ($u^2 = \underline{\text{constant}}$) and if $\partial p/\partial x$ is large compared to the gravitational and inertial terms as a consequence of rapid variation of ρ over x, the result is just (VII 2) with $\eta = u^2 q / V_A^2 \cong q$

A similar derivation of (VII 2) can be given for ambipolar diffusion.

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y -direction, assuming that $\epsilon < 1$, so that, neglecting inertia, the momentum balance is

$$0 \cong -\frac{\partial p}{\partial y} - g\rho\, \epsilon(x, y) \qquad \text{(VII 3)}$$

where $-\epsilon g$ is the component of gravity in the direction along the lines of force. Put $p = \rho u^2$. The motion in the z -direction is

$$v_z = v_y\, \epsilon(x, y). \qquad \text{(VII 4)}$$

Finally, if we assume that the system is uniform in the z -direction $(k_x, k_y/\epsilon >> 1/\Lambda)$, conservation of matter requires that

$$\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y \cong 0 \qquad \text{(VII 5)}$$

It is evident that (VII 3) determines ρ in terms of the given function ϵ. To carry out the indicated integration note that the magnetic lines of force in any plane x = constant are given by the one parameter family of curves

$$z - z_0 = \int_{y_0(x)}^{y} dy' \, \epsilon(x, y') \qquad \text{(VII 6)}$$

where z_0 is the lowest point along the particular line of force, occurring at $y = y_0(x)$. Equation (VII 3) states that the gas is in hydrostatic equilibrium along the lines of force. Hence upon integration of (VII 3) the result is

$$\rho(x, y) = \rho(x, y_0) \exp\left[-\frac{g}{U^2}(z - z_0)\right] \qquad \text{(VII 7)}$$

where $z - z_0$ is given by (VII 6) and $\rho(x, y_0)$ represents the density at the lowest point along the line of force through (x, y).

The next step is to calculate V_x. This is accomplished using (VII 2). Then use (VII 2) to eliminate v_y from (VII 5), writing the result as

$$\frac{\partial}{\partial y} \rho v_y = \eta \frac{\partial^2 \rho}{\partial x^2} \, . \qquad \text{(VII 8)}$$

Since ρ is a known function of ϵ, this equation may be integrated over y to give V_y, with V_z following immediately from (VII 4).

For the simple case that the field variations are sinusoidal, of the form

$$\epsilon(x, y) = \xi \sin(k_x x + k_y y), \qquad \text{(VII 9)}$$

where ξ is a dimensionless number, it is readily shown that

$$z - z_0(x) = \frac{\xi}{k_y}\left[1 - \cos\left(k_x x + k_y y\right)\right], \tag{VII 10}$$

so that

$$\rho(x,y) = \rho_0 \exp\left\{-\frac{g\,\xi}{\upsilon^2 k_y}\left[1 - \cos\left(k_x x + k_y y\right)\right]\right\}, \tag{VII 11}$$

$$v_x = \frac{\eta g \xi k_x}{\upsilon^2 k_y} \sin\left(k_x x + k_y y\right). \tag{VII 12}$$

Since now $\partial/\partial y = (k_y / k_x)\, \partial/\partial x$, it follows that (VII 8) may be integrated to give

$$\frac{k_y}{k_x} \rho v_y = \eta \frac{\partial \rho}{\partial x}$$
$$= -\rho v_x$$

so that

$$v_y = -\frac{\eta g \xi k_x^2}{\upsilon^2 k_y^2} \sin\left(k_x x + k_y y\right), \tag{VII 13}$$

$$v_z = -\frac{\eta g \xi^2 k_x^2}{\upsilon^2 k_y^2} \sin^2\left(k_x x + k_y y\right). \tag{VII 14}$$

The interesting quantity is v_z, representing the rate at which the gas moves downward across the field. The rate of motion is $O(\eta g \xi^2 k_x^2 / u^2 k_y^2)$ to be compared with the characteristic rate of motion, if k_x were not large compared to $1/\Lambda$. In that case it is gravity rather than pressure which pushes the gas across the field. The gravitational force ρg leads to a downward drift $v_z' = -\eta g / u^2 = -\eta / \Lambda$ in terms of the characteristic scale height $\Lambda = u^2 / g$. It follows at once that v_z, given by (VII 14) is larger than the usual diffusion velocity v_z' by the factor $\xi^2 k_x^2 / k_y^2$. Since ξ in the actual case is of the order of unity, the enhancement is large.

The instability which leads to large k_x causes the gas to drift out of the magnetic field at a rate k_x^2 / k_y^2 times faster than under gravity alone.

References

Cowling, T. G. 1957, Magnetohydrodynamics (Interscience Pub., New York).

Lerche, I. 1967a, Astrophys. J. (in press)

1967b, ibid (submitted for publication)

Mestel, L. 1965, Quart. J. Roy. Astron. Soc. **6**, 161, 265.

Osterbrock, D. E. 1961, Astrophys. J. **134**, 270.

Parker, E. N. 1963, Astrophys. J. Suppl. **8**, 177, No. 77.

1965, Astrophys. J. **142**, 1086.

1966, ibid **145**, 811.

Schlüter, A. and Biermann, L. 1950, Zeit. f. Naturforsch. **52**, 237.

Figure Captions

Fig. 1 Sketch of the geometry of the simple gas, field, gravity configuration employed in the present discussion of the instability of the gas, field, and cosmic ray system.

Fig. 2 Sketch of the slabs of gas (cross hatched) formed by compressing the gas along the magnetic field. The magnetic lines of force are indicated before their release by the horizontal lines, and afterward by the lines which bow upward between the slabs. The x -direction is perpendicular to the yz -plane of the paper.

Fig. 3 Plot of the energy decrease $\Delta\mathcal{E}$ of the magnetic field between two slabs of gas, in units of the energy $\mathcal{E}_0$ in the field before release. The initial field configuration of the field is given by (1) and is represented by the straight lines of force in Fig. 2. The final field is described by (I 3) and (I 4), represented by the bowed lines of force in Fig. 2.

Fig. 4 Sketch of the coordinate system used in Appendix V to treat the force exerted between two elements of gas of mass $\mathfrak{M}_1$ and $\mathfrak{M}_2$ suspended in the gravitational field g on the lines of force of a horizontal magnetic field.

Fig. 5 Sketch of the displaced lines of force in the neighborhood of a small element of gas $\mathfrak{M}$ suspended in the field. The lines of force above and below $\mathfrak{M}$ are displaced downward along with the lines threading $\mathfrak{M}$. Consequently the lines to the side are displaced upward.

Fig. 6 A plot of the growth rate $1/\tau$ of the instability in units of $1/\tau_0$ as a function of k_x for $k_y \Lambda = 1$, $k_z \Lambda = 0.25$, 0.5, 1.0, 2.0 and $\eta \tau_0 / \Lambda^2 = 3 \times 10^{-5}$, illustrating the broad spectrum of the instability.

Fig. 7 Sketch of two lines of force separated by the small distance π / k_x in the x-direction. The low point on the line at $x = 0$ is represented by A and the adjacent high point of the line at $x = \pi / k_x$ is represented by B.

Fig. 8 Sketch of the currents associated with a small element of gas of length $2a$ and weight mg lying across a magnetic field in the y-direction.

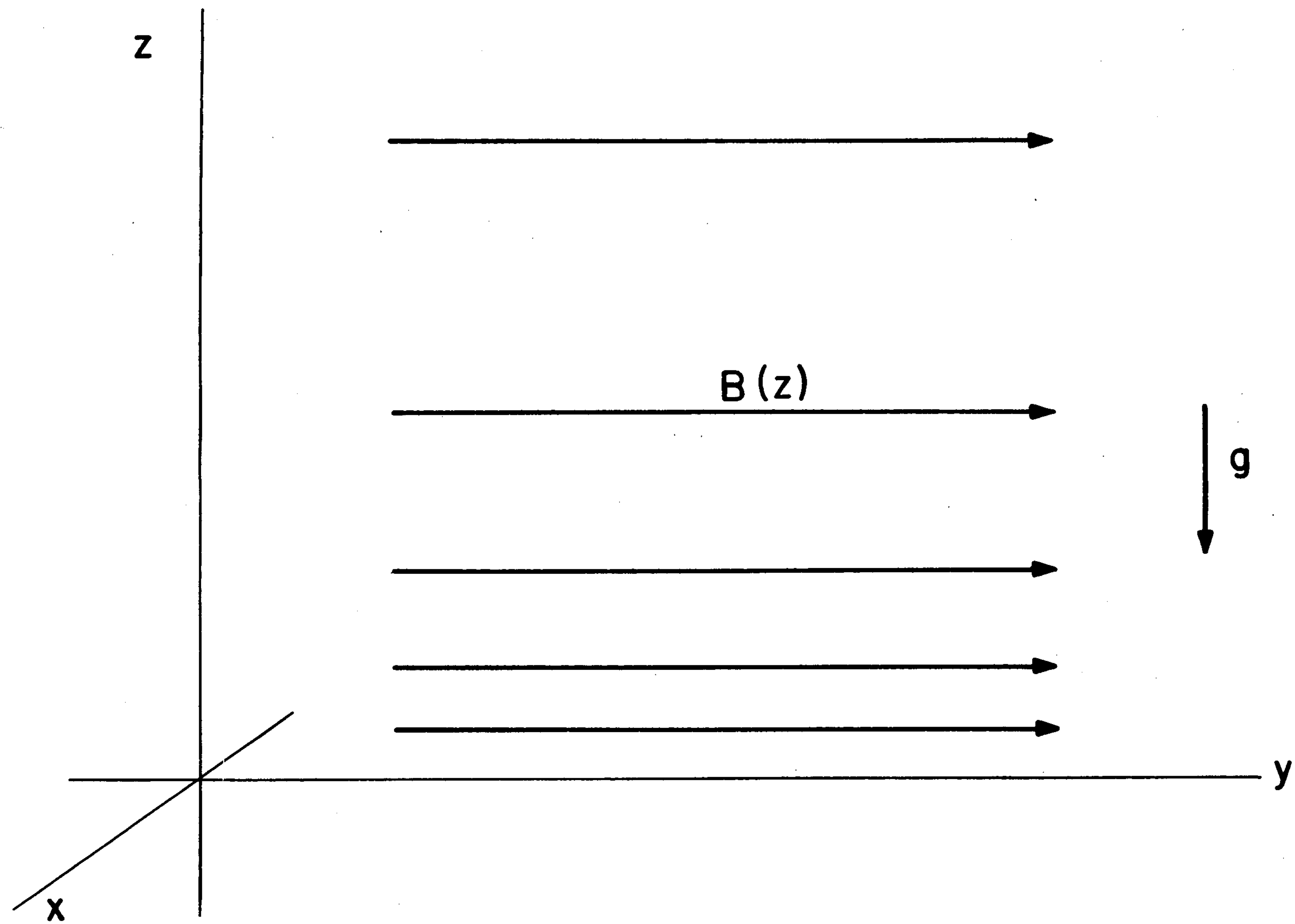

Fig. 1

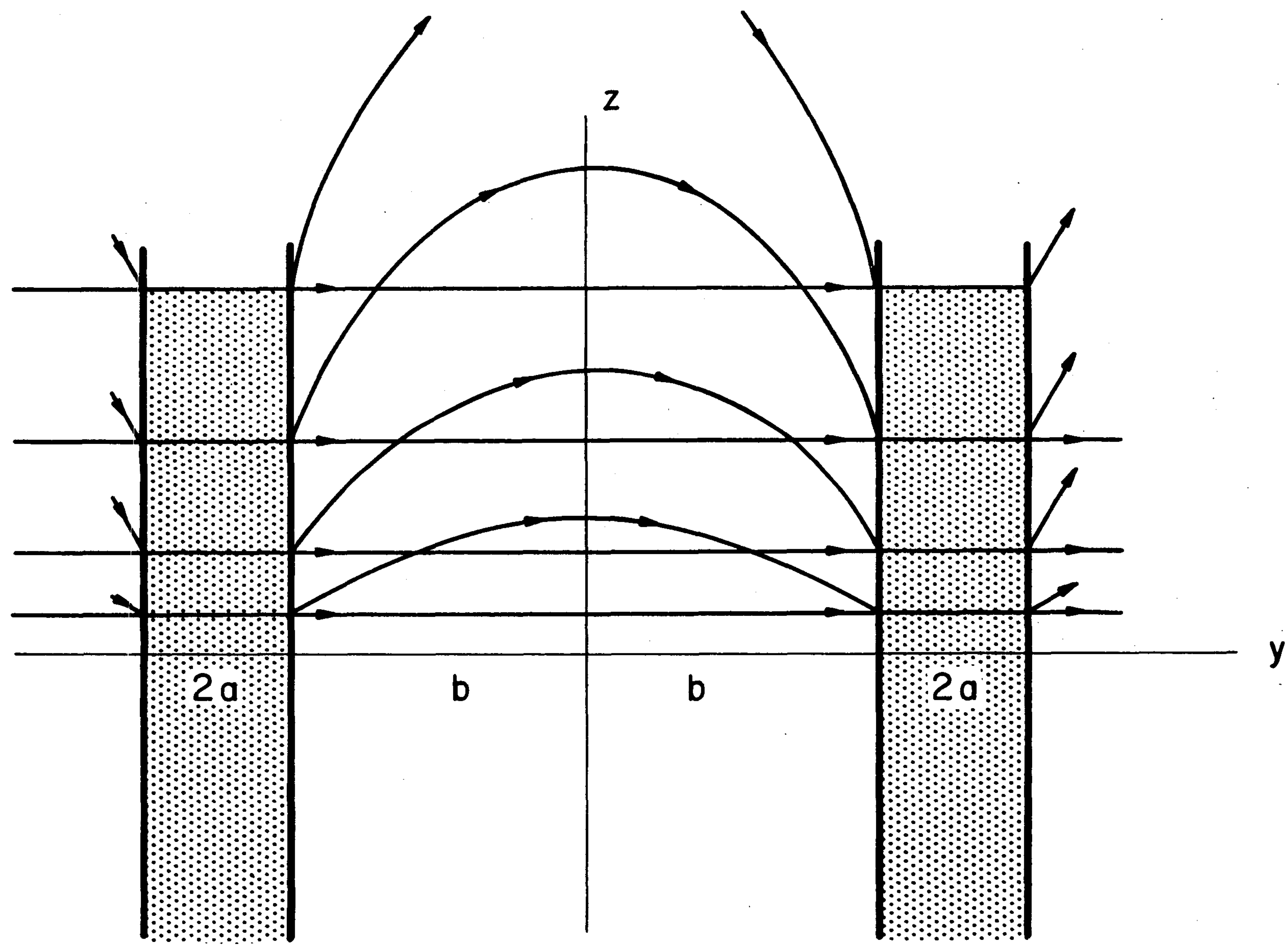

Fig. 2

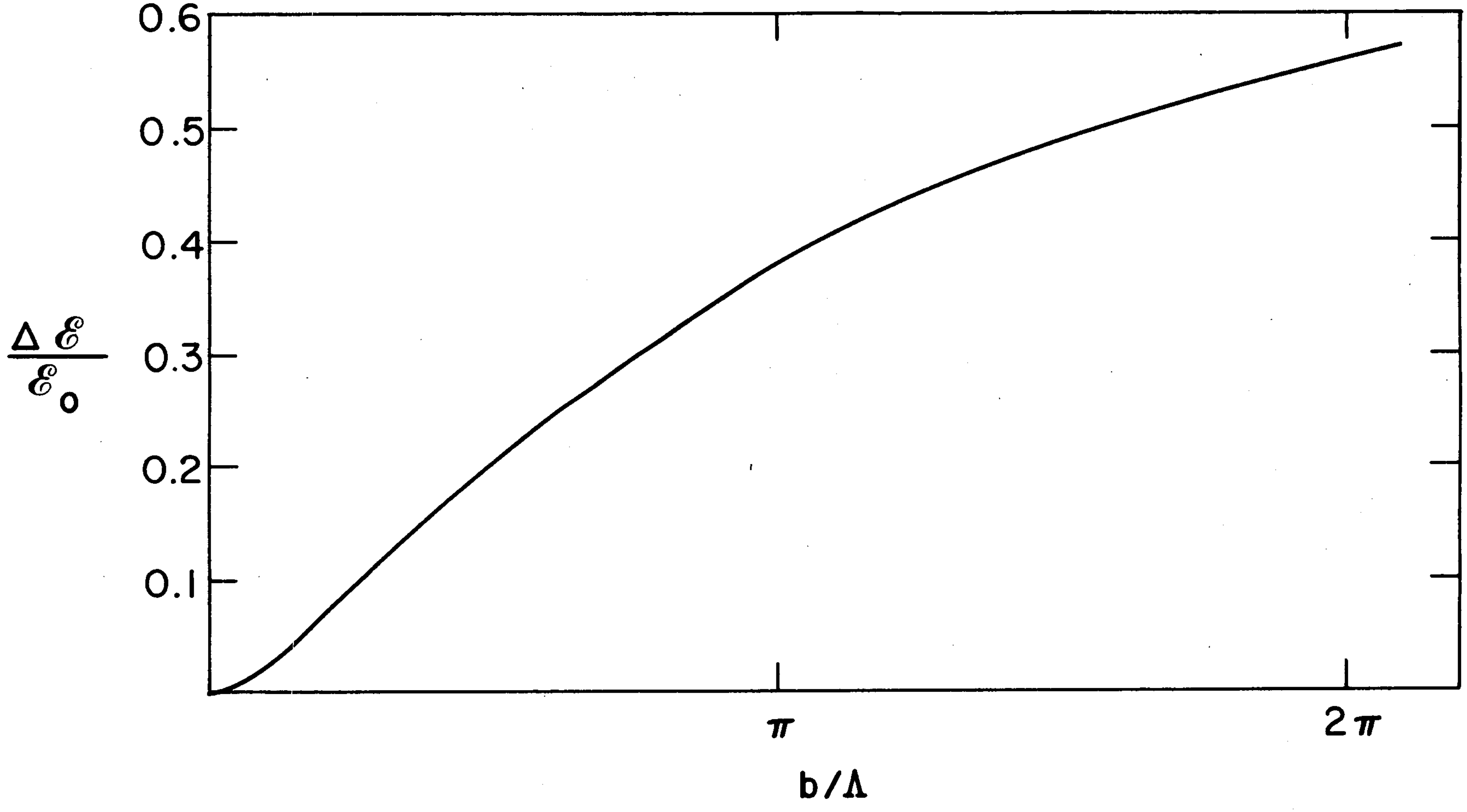
0.6
0.5
0.4
0.3
0.2
0.1
$\frac{\Delta \mathcal{E}}{\mathcal{E}_0}$
π
2π
b/Λ

Fig. 3

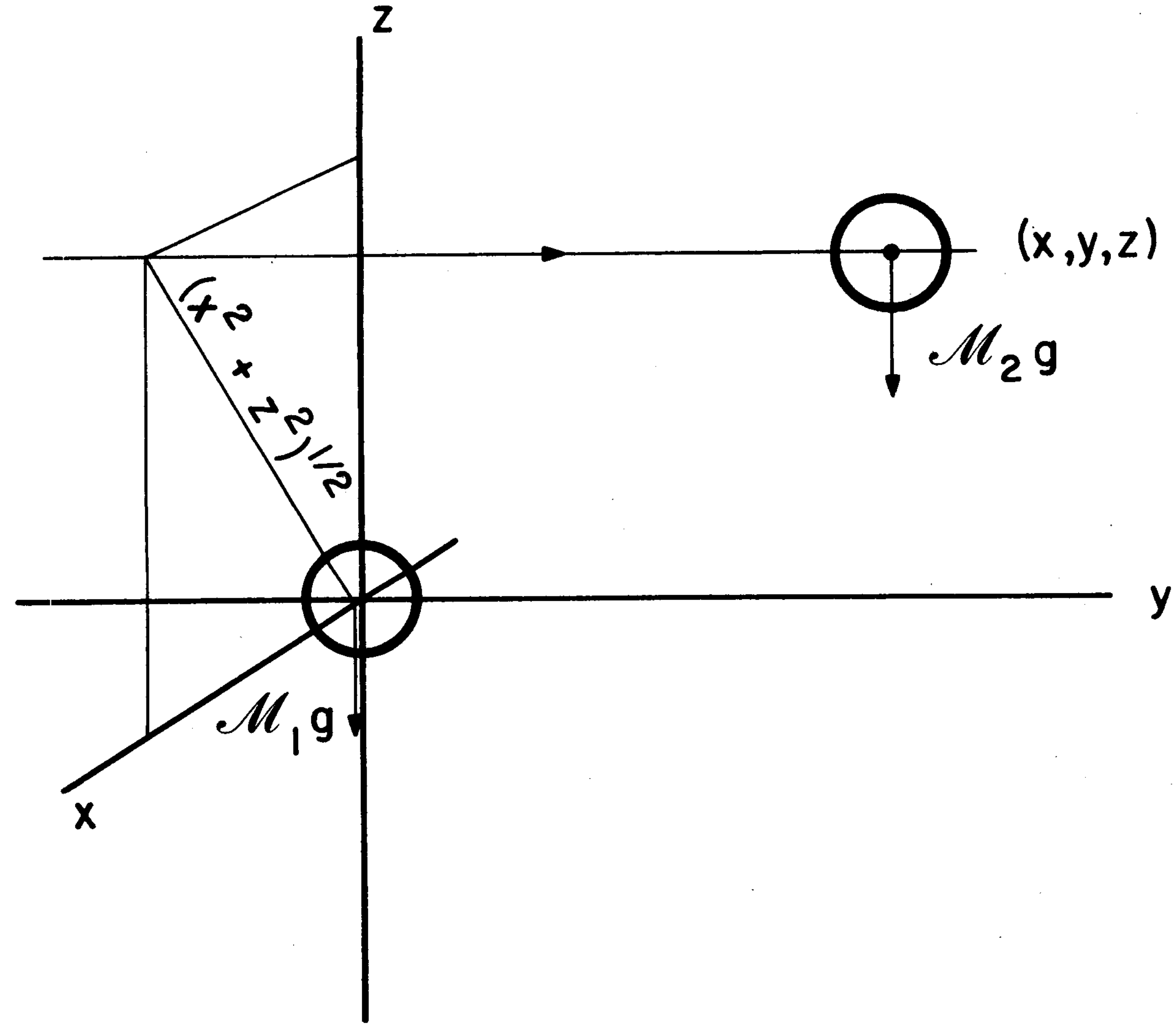

Fig. 4

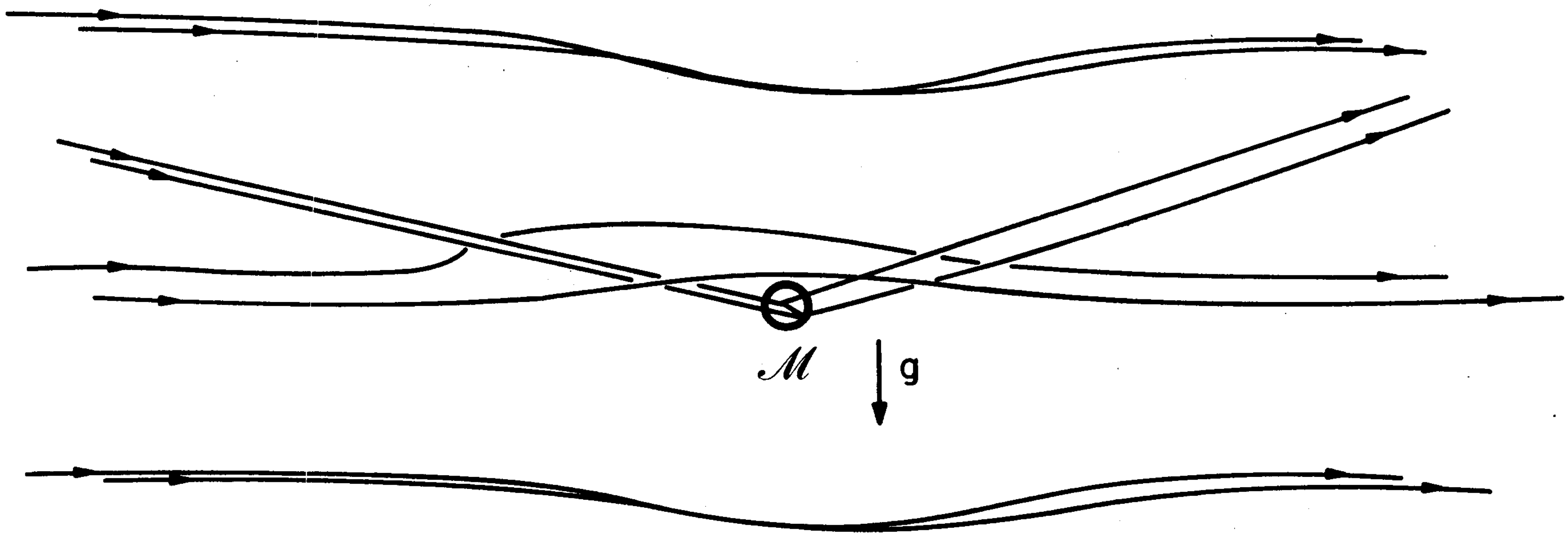

Fig. 5

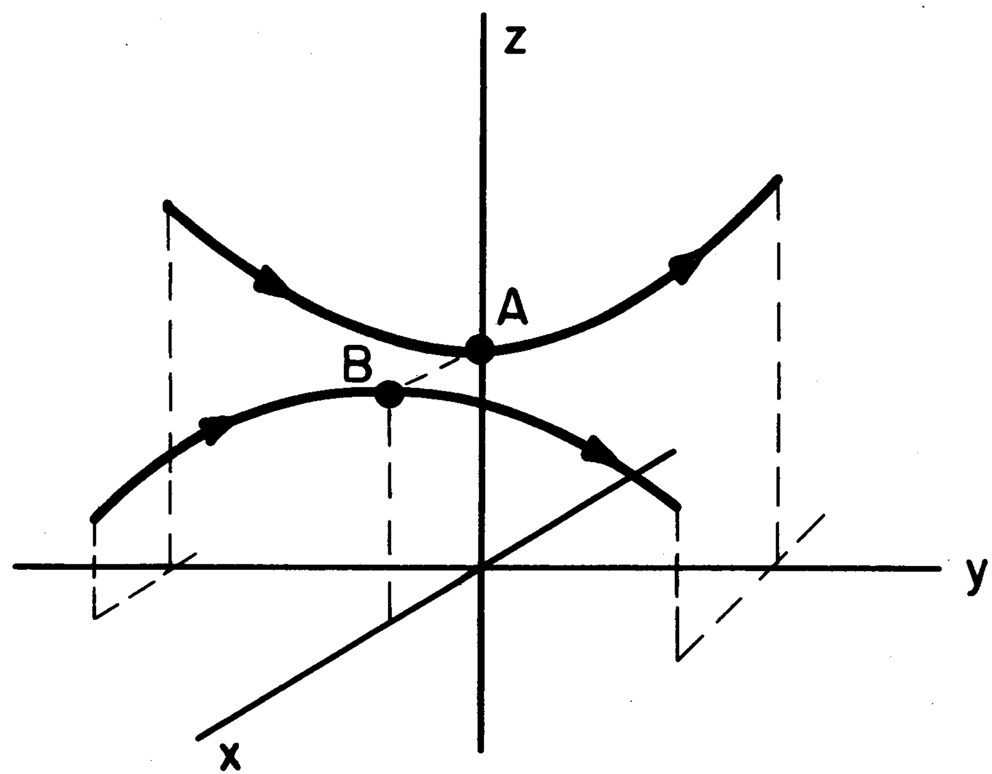

Fig. 6

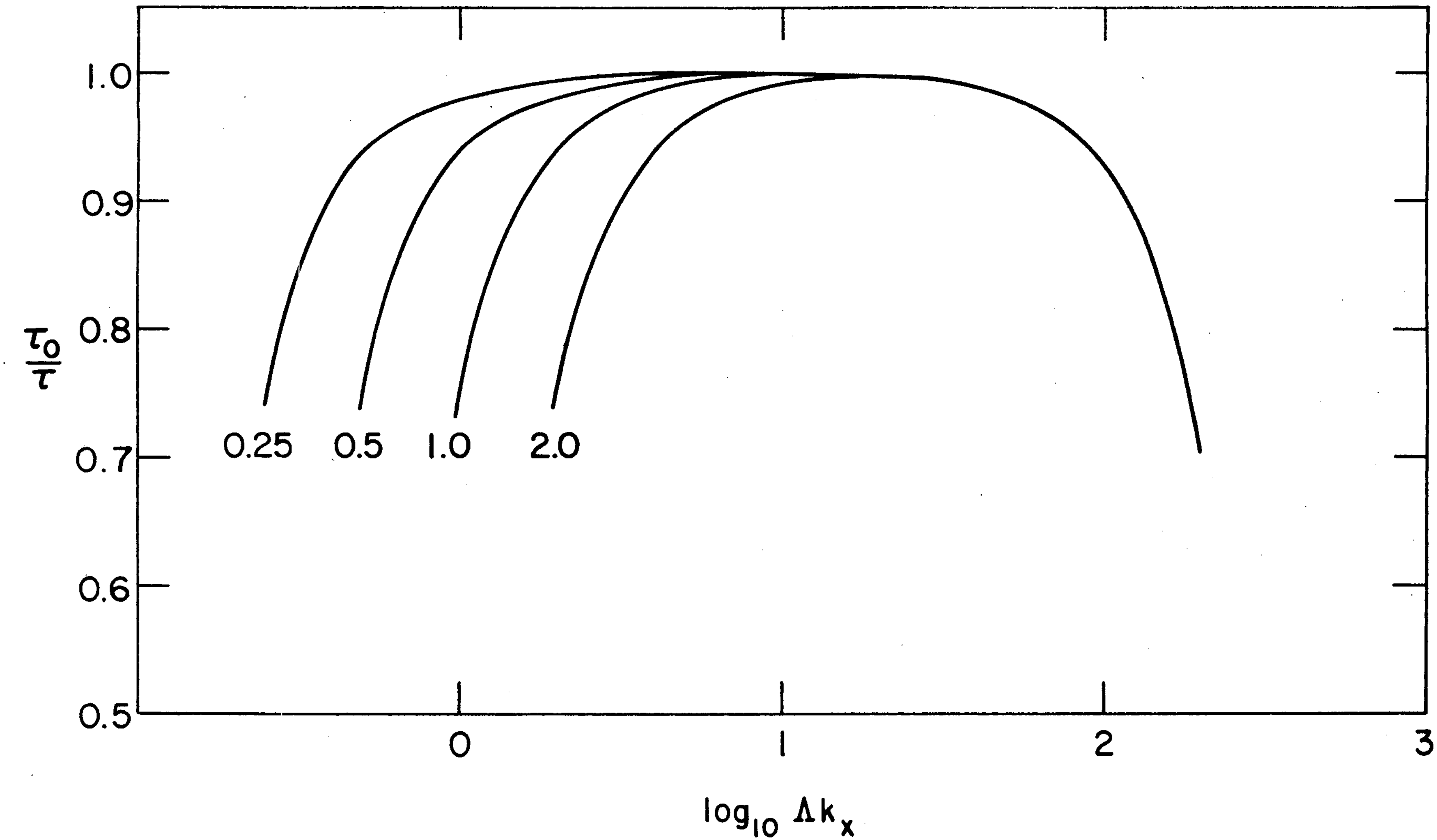

Fig. 7

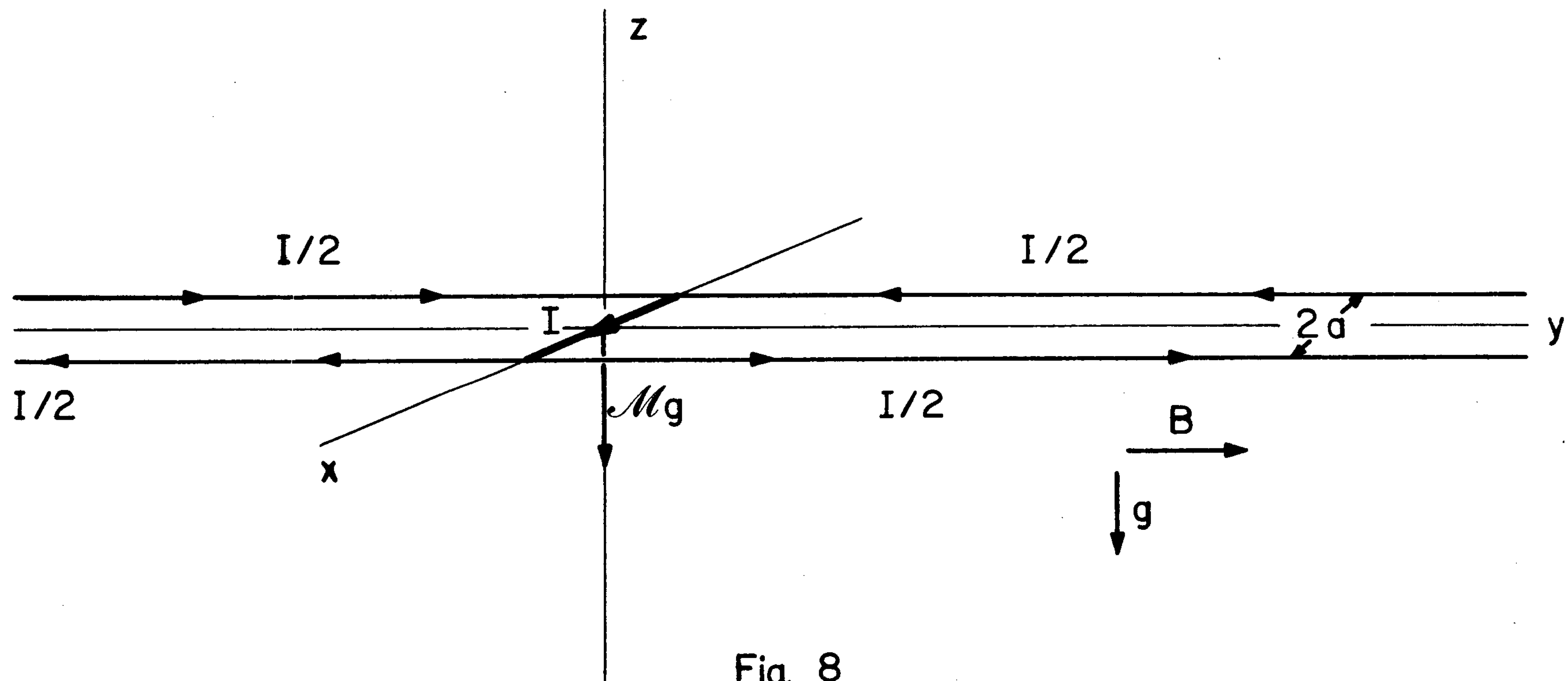

Fig. 8