TECHNICAL REPORT 400-145

DETERMINATION OF HIDDEN EDGES IN POLYHEDRAL FIGURES: CONVEX CASE

Philippe Loutrel

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LABORATORY FOR ELECTROSCIENCE RESEARCH

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ABSTRACT

This report is concerned with the problem of determining the hidden lines of computer-drawn, convex polyhedra. This is the so-called "hidden-line problem," limited here to convex polyhedra. The solution presented should be considered as a step toward the eventual solution of the more general problem of non-convex polyhedra. Results based on actual computer tests are included.

LIST OF SYMBOLS

S	Any convex polyhedral surface
V_i	Vertex number i
F_m	Face number m
V_iV_j	Edge between vertex V_i and vertex V_j
$E(x_o,y_o,z_o)$	Vantage point
M	Total number of vertices
N	Total number of faces
L	Total number of edges
π	Picture plane
(O,xyz)	Original rectangular system of coordinates
(E,XYZ)	System (O,xyz) translated to E
$(O,x'y')$	System of coordinates in π
u_j	Vector supported by EV_j
v_j	Vector $\overrightarrow{EV}_j$
u	Unit vector from E, normal to π
W_i	Vertex number i in the picture plane
λ_i,μ_i,ν_i	Direction cosines
D	Distance from E to π
$v_o(x_o,y_o,z_o)$	Direction vector when E is at infinity
n_{ijk}	Vector normal to the face containing V_i,V_j,V_k
Δ_{ijk}	Triple product

DETERMINATION OF HIDDEN EDGES IN POLYHEDRAL FIGURES: CONVEX CASE

I. INTRODUCTION

With the recent interest in applying digital computers to the solution of graphical problems, a new problem has appeared and received the name of "hidden-line problem." In everyday life, the fact that light can not pass through opaque matter solves the hidden-line problem immediately: the back lines of opaque objects are hidden. In the case of a computer representation of the same object, however, there is no opaque matter to stop the light. The question as to which line should be seen and which should be hidden must be answered mathematically. To date, no practical solution to this problem in its general form has been found.

This report proposes a solution to the hidden-line problem in the restricted case of convex polyhedra. For this case, different solutions exist;[3,6] however, it is believed that the approach taken in this report will lend itself to a generalization to the non-convex case. Given a mathematical description of the polyhedron and a vantage point, the algorithm to be described allows a computer (1) to determine which edges can be seen and which are hidden, and (2) to calculate a projective view of the object as seen by an observer located at the vantage point.

II. FORMAT FOR PRESENTATION OF POLYHEDRAL DATA

A convenient way of describing a polyhedron* is in terms of its vertices and faces:

1) Vertices

A subscript 'i' is attached to each vertex, e.g., V_i (See Fig. 1). Although 'i' may be chosen arbitrarily, for convenience of notations it is assumed that $i = 1,2,\ldots,M$, where M is the total number of vertices. Each vertex V_i is given by three coordinates: $V_i(x_i,y_i,z_i)$ with respect to a three-dimensional system of axes (See Section IV).

2) Faces

Each face of a polyhedron is a polygon[7]. A fixed direction of travel is chosen on this polygon. Starting from any vertex, and traveling around back to the original vertex, a listing of the vertices successively encountered uniquely describes the polygon. For instance, $F_m = p,k,i,j,t,p$ denotes the face whose successive vertices are V_p,V_k,V_i,V_j,V_t. The index 'm' varies from 1 to N, where N is the total number of faces (See Fig. 1).

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* "A polyhedron is a finite set of polygons arranged in space in such a way that every side of each polygon belongs to just one other polygon, with the restriction that no subset has the same property".[5]

Since a convex polyhedron is a two-sided surface, it is possible to define an "inside" and "outside".[2] Using this property, the direction of travel around the faces is chosen such that each face is described in a clockwise direction for an observer outside the polyhedron. With the knowledge of both vertex and face data, it is now possible to determine the edges. From the definition of a polyhedron, an edge is defined as the intersection of two faces. The necessary and sufficient condition for the edge V_iV_j to exist is, therefore, that i and j be adjacent subscripts in the string defining a face. Once i and j are found, say in $F_m = p,k,i,j,t,p$, there must be one and only one other face such that $F_n = a,q,r,j,i,a$.

In the computer representation every edge V_iV_j of the polyhedron is characterized by a block of four adjacent cells in an array, containing respectively:

1) Vertex number: i
2) Vertex number: j
3) Face number: m
4) Face number: n

Eventually, a fifth cell is added to each block. This cell contains the code for visibility or invisibility of the edge. The total number of edges is denoted by L. Since the polyhedron is convex, there exists a fixed relationship between the number of vertices, edges and faces. This relation is known as Euler's equality and is written as: (See Fig. 1)

$$M - L + N = 2$$

Upon determination of L, a test is performed to check whether L satisfies Euler's equality. If it does not, an error is indicated.

III. PROPERTIES OF CONVEX POLYHEDRA

The visibility test to be described in Section VI is based on the following property:

> To an external observer, the faces of a convex polyhedron are either completely visible or completely invisible.

To demonstrate this property, it is first assumed that a partially visible face, F, exists and it is then shown that this yields a contradiction. First, two points, P and Q, are chosen on the face F such that: 1) P is visible, Q is invisible; and 2) Q is not located on an edge and P is not on the boundary between the visible and invisible regions (See Fig. 2). Since the polyhedron is convex, the line EQ intersects the surface S of the polyhedron in two points and two points only: Q and Q'. It is understood that these two points are each on a face of the polyhedron. Since Q is invisible, Q' must lie between E and Q. Consider now Q_1 between Q and Q'. Q_1 is inside S by definition of the convexity. Hence, it follows that all points of PQ_1 are inside S. Since P is not on a visible-invisible boundary (condition 2), there is a point P_1 on PQ such that P_1 is visible. Also since S is a closed surface and P_1 is visible, all points of EP_1 are outside S. This would include the intersection R of EP_1 with PQ_1. But it has already been established that if R is on PQ_1, it is inside S. Since this is a contradiction, it is concluded that no face of a convex polyhedron can be partially visible.

It should be noted that this proof is valid for the isometric case as well (vantage point at infinite distance).

An immediate consequence of the above property is that no edge of a convex polyhedron can be partially visible: it is either visible or invisible.

IV. SYSTEMS OF COORDINATES

Three rectangular systems of coordinates are used: $(0,xyz)$, (E,XYZ), and $(0,x'y')$ (See Fig. 3).

A. E at a Finite Distance

1) System $(0,xyz)$

This is a fixed system of axes with origin at 0. The coordinates of the vertices $V_i(x_i,y_i,z_i)$ and those of the vantage point $E(x_o,y_o,z_o)$ are given in this system.

2) System (E,XYZ)

This variable system of coordinates is simply the system $(0,xyz)$ translated to the vantage point. The transformation is expressed by:

$$\left| \begin{array}{l} X = x - x_o \\ Y = y - y_o \\ Z = z - z_o \end{array} \right.$$

The test for face visibility is performed in this basis.

3) System $(0,x'y')$

This two-dimensional basis represents the <u>picture plane</u>, i.e. the plane of the actual drawing. This picture plane is chosen perpendicular to the direction OE, through the origin 0. To preserve the sensation of vertical and horizontal for an observer located at the vantage point, the axis Ox' is chosen to lie in the xy-plane. The only exception is when E is located on the z-axis, in which case the picture plane simply becomes the original xy-plane.

At this point, a perspective drawing is defined as the set of points W, formed by vectors v from E to points of S piercing the picture plane.* If u is a unit vector from E, normal to the picture plane, a vector w is written as (See Fig. 4).

$$w = v \frac{D}{u \cdot v}$$

Once all $W_i(X_i, Y_i, Z_i)$ are computed, it is desired to find their two-dimensional coordinates $W_i(x_i', y_i')$ in the picture plane π. In its general form, the transformation can be written as:

$$\begin{bmatrix} x_i' \\ y_i' \\ z_i' \end{bmatrix} = \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix} \begin{bmatrix} X_i + x_o \\ Y_i + y_o \\ Z_i + z_o \end{bmatrix}$$

where λ_i, μ_i, ν_i are the direction cosines of the new basis. In addition, the following relations are applicable:

---

* Notation: Lower case letter denote vectors, e.g.: $\vec{EW} = w$

$$\left|\begin{array}{l} \lambda_i^2 + \mu_i^2 + \nu_i^2 = 1 \qquad\qquad i = 1,2,3. \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \\ \mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \\ \nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \\ z_i' = 0 \end{array}\right.$$

Since Π was chosen to be perpendicular to OE, it follows that:

$$\lambda_3 = \frac{x_o}{D} \qquad \mu_3 = \frac{y_o}{D} \qquad \nu_3 = \frac{z_o}{D}$$

$$D = \sqrt{x_o^2 + y_o^2 + z_o^2} \qquad d = \sqrt{x_o^2 + y_o^2}$$

Since Ox' lies in the xy plane,

$$\nu_1 = 0$$

After solving for $\lambda_1, \mu_1, \lambda_2, \mu_2, \nu_2$, the matrix of the transformation is:

$$\begin{bmatrix} x_i' \\ \\ y_i' \end{bmatrix} = \begin{bmatrix} -\dfrac{y_o}{d} & \dfrac{x_o}{d} & 0 \\ \\ -\dfrac{x_o z_o}{dD} & -\dfrac{y_o z_o}{dD} & \dfrac{x_o^2 + y_o^2}{dD} \end{bmatrix} \begin{bmatrix} X_i + x_o \\ Y_i + y_o \\ Z_i + z_o \end{bmatrix}$$

except for the case: $x_o = 0$, $y_o = 0$, for which

$$\left|\begin{array}{l} x_i' = x_i \\ y_i' = y_i \end{array}\right.$$

B. E at a Infinity

1) System $(0,xyz)$

This is the same system as above (A. 1). The direction of the vantage point is given by a vector $v_o(x_o, y_o, z_o)$.

2) System $(0,x'y')$

In contrast to the case of the vantage point at a finite distance, in this case the W_i can not be computed using $w = \frac{D}{u.v}$ because of the infinite values of D, w, v. In general, the condition:

$$z_i' = 0$$

will not be satisfied.

Since the interest is in an isometric drawing of the object as seen from a direction perpendicular to the picture plane, the values of z_i' are irrelevant to the problem. Consequently, the same matrix will be used for the coordinate transformation.

$$\begin{bmatrix} x_i' \\ \\ y_i' \end{bmatrix} = \begin{bmatrix} -\frac{y_o}{d} & \frac{x_o}{d} & 0 \\ \\ -\frac{x_o y_o}{dD} & -\frac{y_o z_o}{dD} & \frac{x_o^2 + y_o^2}{dD} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$

V. DETERMINATION OF HIDDEN FACES

A. Vantage Point at Infinity

The position of the vantage point E at infinity is given by the direction vector v_o. Since a typical face $F_n = i,j,k,t,p,i$ is uniquely defined by three vertices only, the notation F_{ijk} will be used for this face. To test the visibility of F_{ijk}, a normal vector to this face is needed (See Fig. 5). The vector is chosen positively oriented toward the inside of the polyhedron and denoted: n_{ijk}. Then a vector u_j is defined such that: 1) u_j is a finite length vector supported by EV_j, and 2) u_j is positively oriented from E to V_j. Since E is at infinity, all vectors u_j are one and the same:

$$u_j = - v_o$$

To check whether F_{ijk} is facing or looking away from the vantage point, the following dot product is computed (See Figure 5):

$$\Delta_{ijk} = u_j \cdot n_{ijk} = - v_o \cdot n_{ijk}$$

According to the sign of Δ_{ijk} it is concluded that:

$\Delta_{ijk} > 0$	F_{ijk} Visible
$\Delta_{ijk} \leq 0$	F_{ijk} Invisible

The information contained in the input data can now be fully exploited to determine n_{ijk} and Δ_{ijk}. First the inward normal vector n_{ijk} is written as the cross product

$$n_{ijk} = (v_k - v_j) x (v_i - v_j) = v_{kj} x v_{ij}$$

Then Δ_{ijk} can be expressed as the triple product

$$\Delta_{ijk} = n_{ijk} \cdot u_j = -(v_{kj}, v_{ij}, v_o)$$

which can be written in the form

$$\Delta_{ijk} = \begin{vmatrix} x_k - x_j & x_i - x_j & -x_o \\ y_k - y_j & y_i - y_j & -y_o \\ z_k - z_j & z_i - z_j & -z_o \end{vmatrix}$$

It can be seen that the test for face visibility reduces simply to the evaluation of a three-by-three determinant.

B. Vantage Point at a Finite Distance

A greater symmetry is achieved in Δ_{ijk} when E is at a finite distance and E is chosen as the origin of new coordinates. Under this condition, $v_j = \vec{EV}_j$ satisfies the requirements on u_j and one may take $u_j = v_j$ (See Fig. 6). The expression for Δ_{ijk} becomes:

$$\Delta_{ijk} = n_{ijk} \cdot u_j = n_{ijk} \cdot v_j$$

$$\Delta_{ijk} = (v_{kj}, v_{ij}, v_j)$$

$$\Delta_{ijk} = (v_{kj}, v_i, v_j) - (V_{kj}, v_j, v_j)$$

$$\Delta_{ijk} = (v_k, v_i, v_j) - (v_j, v_i, v_j) - (v_{kj}, v_j, v_j)$$

$$\Delta_{ijk} = (v_k, v_i, v_j)$$

since $v_j \times v_i \cdot v_i = 0$. By rearranging:

$$\Delta_{ijk} = (v_i, v_j, v_k)$$

and this can be written as:

$$\Delta_{ijk} = \begin{vmatrix} x_{io} & x_{jo} & x_{ko} \\ y_{io} & y_{jo} & y_{ko} \\ z_{io} & z_{jo} & z_{ko} \end{vmatrix} \qquad \text{with} \qquad \left| \begin{array}{l} x_{io} = x_i - x_o \\ y_{io} = y_i - y_o \\ z_{io} = z_i - z_o \end{array} \right.$$

C. <u>Determination of Hidden Edges</u>

Once each of the N faces has been classified either as visible or invisible, attention can be focussed on the edges. An edge V_iV_j appears in two faces, F_{ijk} and F_{tji}. Since the polyhedron is convex, it is sufficient for V_iV_j to be visible that either of these faces be visible. Conversely, a necessary and sufficient condition for V_iV_j to be invisible is that both these faces be invisible. These results are summarized in the following table:

F_{ijk}	Visible	F_{tji}	Visible	V_iV_j	Visible
F_{ijk}	Invisible	F_{tji}	Visible	V_iV_j	Visible
F_{ijk}	Visible	F_{tji}	Invisible	V_iV_j	Visible
F_{ijk}	Invisible	F_{tji}	Invisible	V_iV_j	Invisible

A formal notation for edge visibility is:*

---

* 'V' denotes the logical 'OR'

$$V_iV_j \text{ Visible } = (\Delta_{ijk} > 0) \vee (\Delta_{tji} > 0)$$

Once the visibility or invisibility of each edge has been determined, it is a simple matter for a computer program to plot the perspective projection of the object, using solid lines for visible edges and dashed lines for invisible edges.[8]

A FORTRAN program for the foregoing algorithm was written and tested. Some of the results obtained are shown in Figs. 7,8,9, and 10. The illustrations are reproductions of the actual computer output as plotted on a CALCOMP plotter.

VI. CONCLUSION

The main objective in developing the foregoing algorithm was to lay a foundation for an algorithm that would hold also for the case of non-convex polyhedra. At this stage of the work, it is too early to tell whether this objective has been achieved. However, the simplicity of the algorithm for handling convex polyhedra is considered encouraging for meeting the ultimate objective. Even in its limited present form, the algorithm should be of interest to persons active in the computer graphics field.

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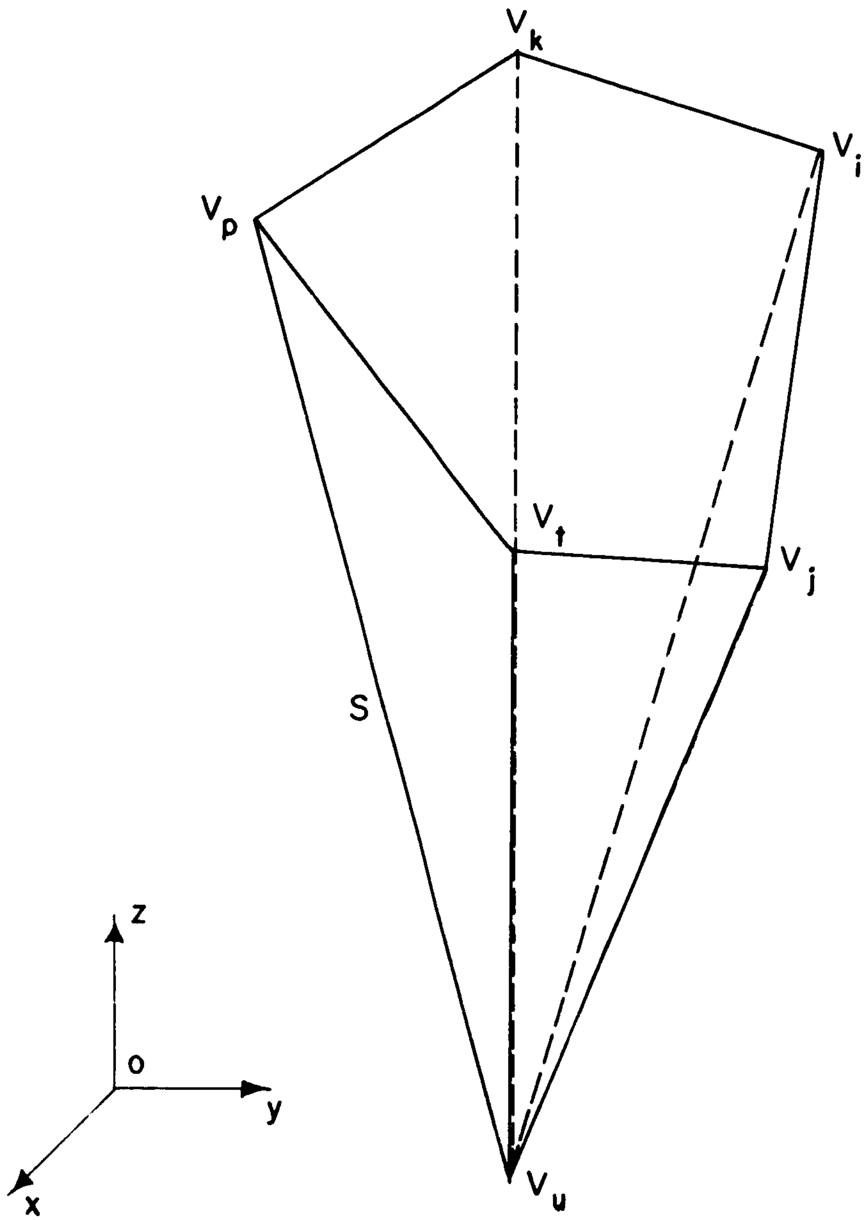

$$F_m = p, k, i, j, t, p$$

$$M - L + N = 6 - 10 + 6 = 2 \quad \left| \begin{array}{l} M = 6 \text{ VERTICES} \\ N = 6 \text{ FACES} \\ L = 10 \text{ EDGES} \end{array} \right.$$

FIG. 1

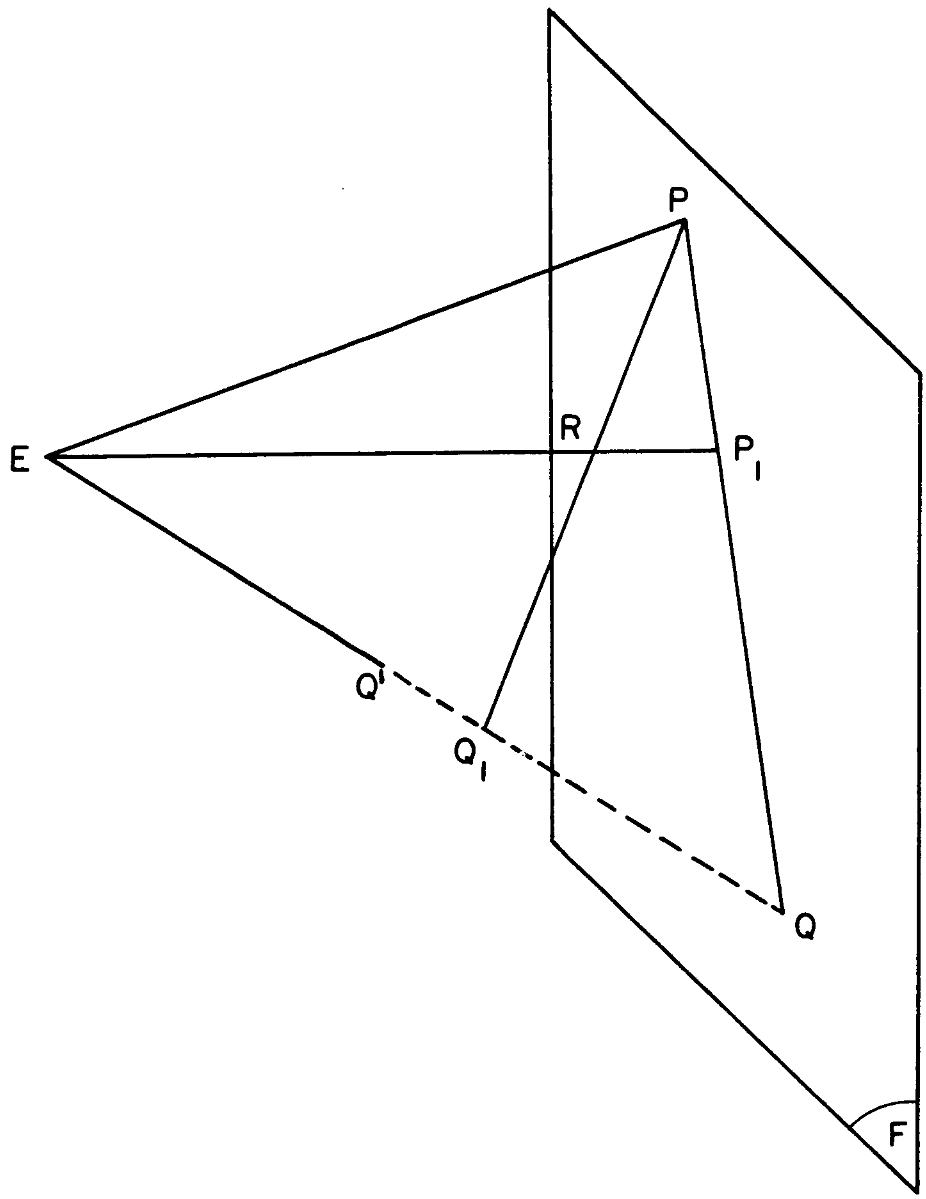

FIG. 2
FACE OF CONVEX POLYHEDRON

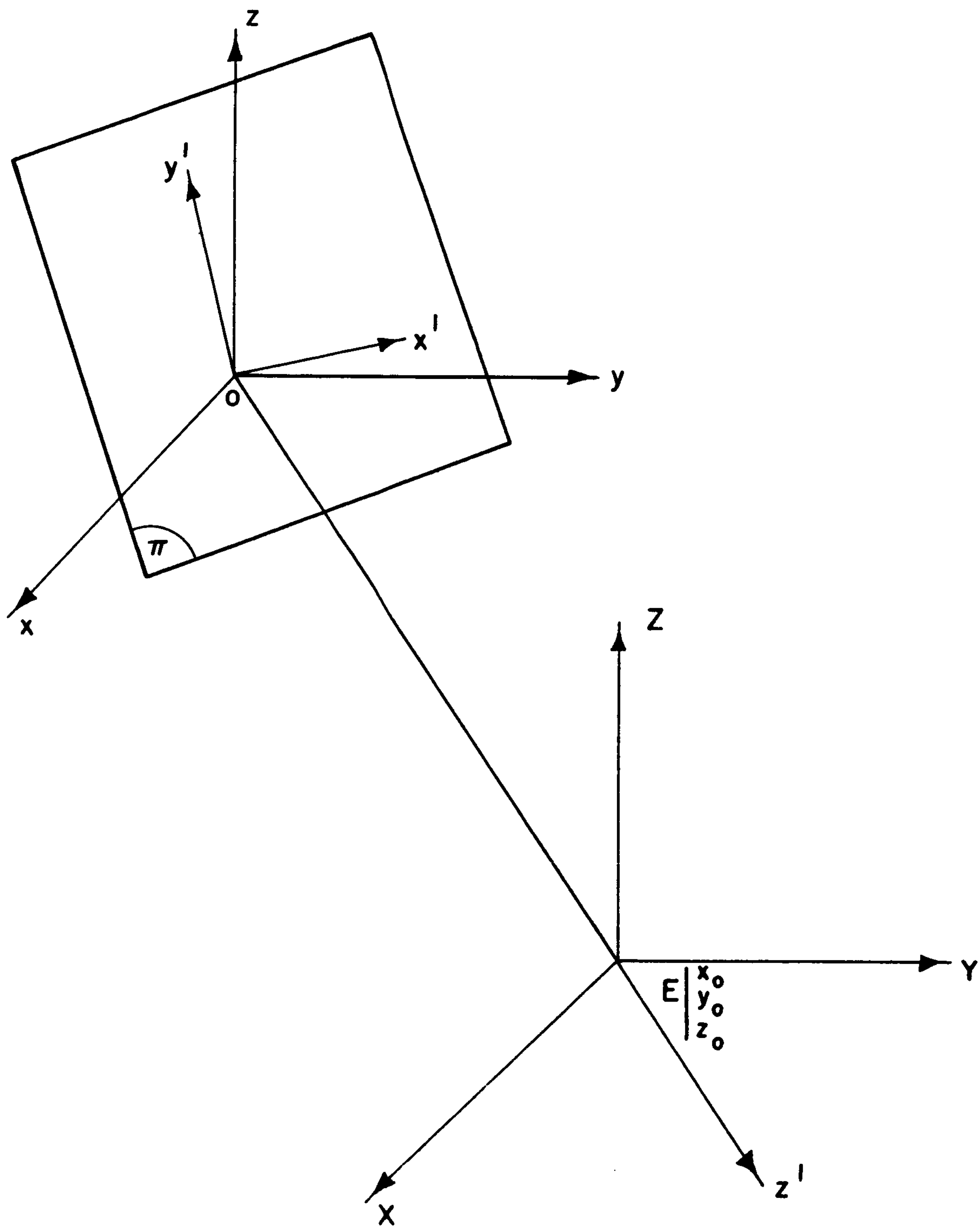

FIG. 3

THE THREE SYSTEMS OF COORDINATES

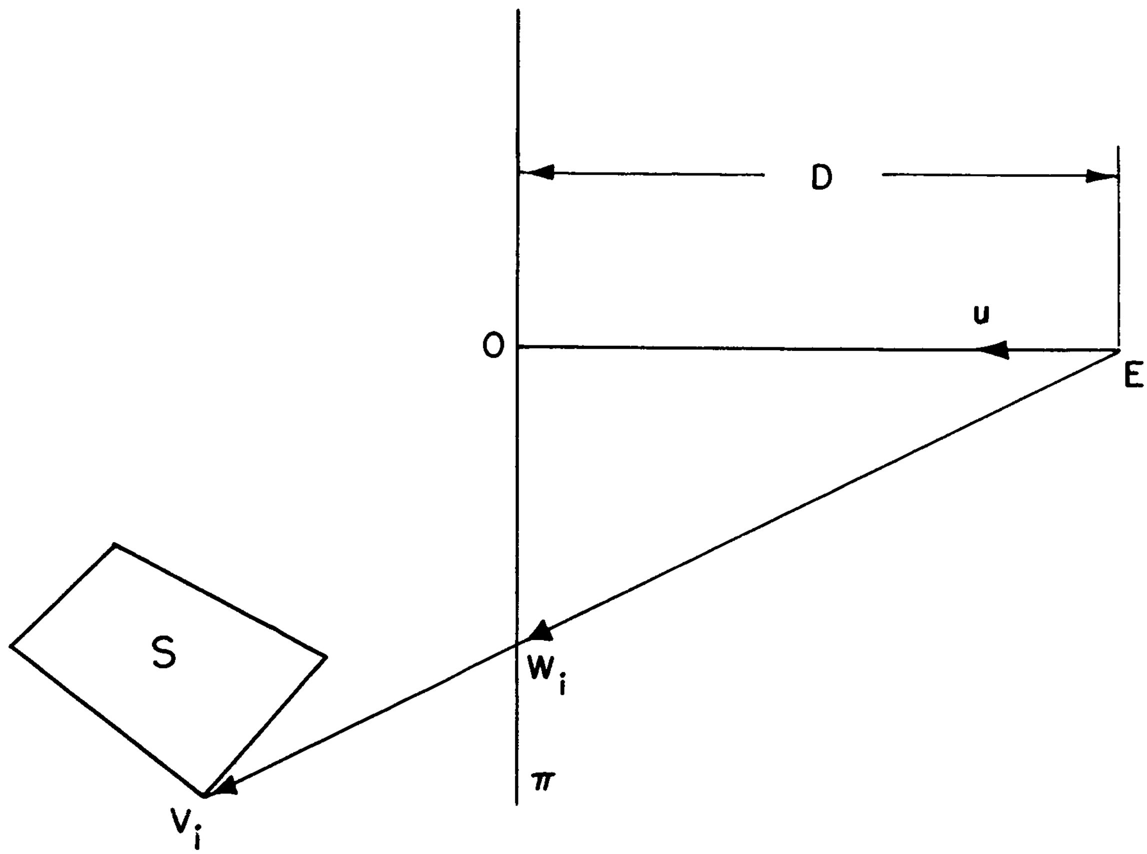

FIG. 4
PROJECTIVE DRAWING

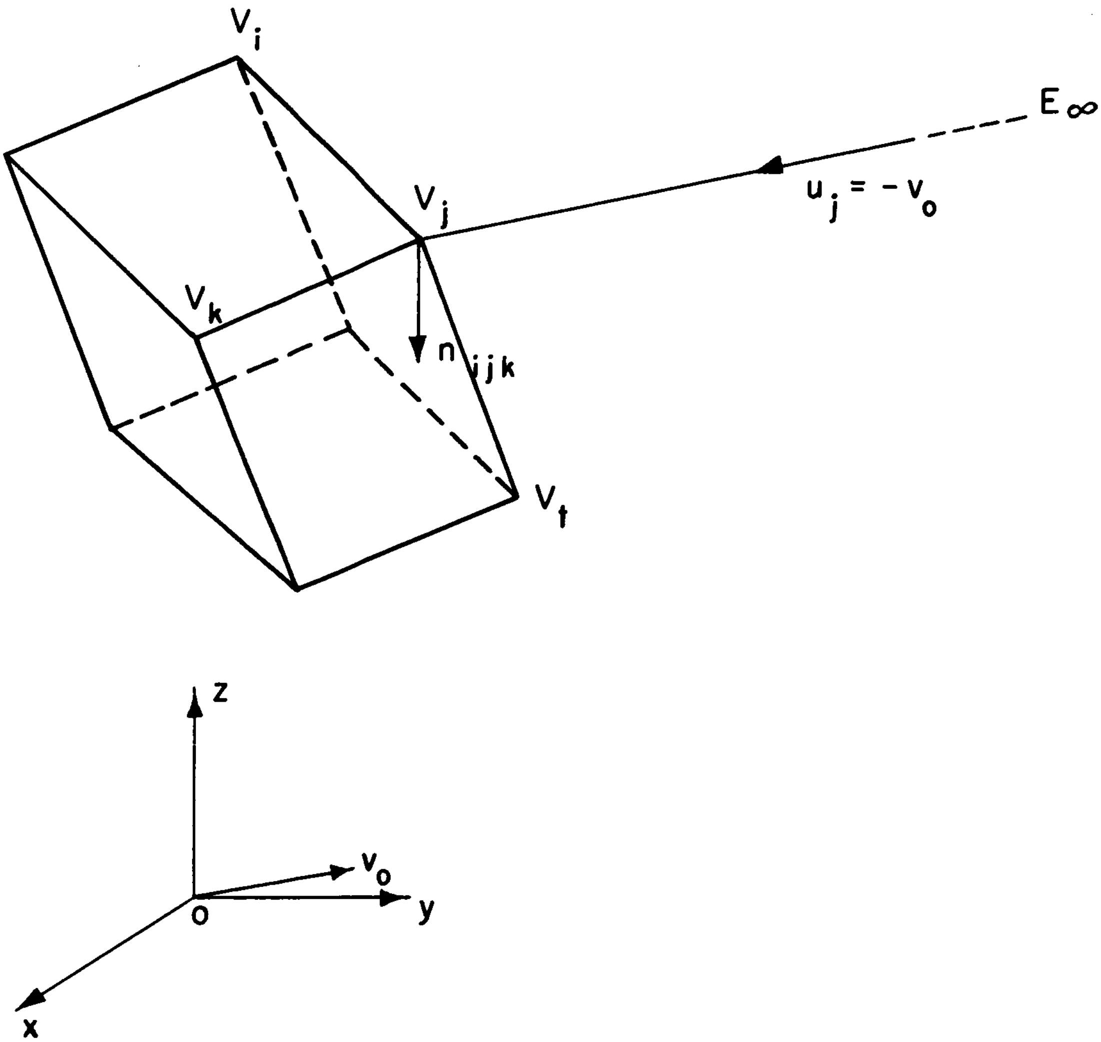

FIG. 5

VANTAGE POINT AT INFINITY

$\Delta_{ijk} = -v_o \cdot n_{ijk}$

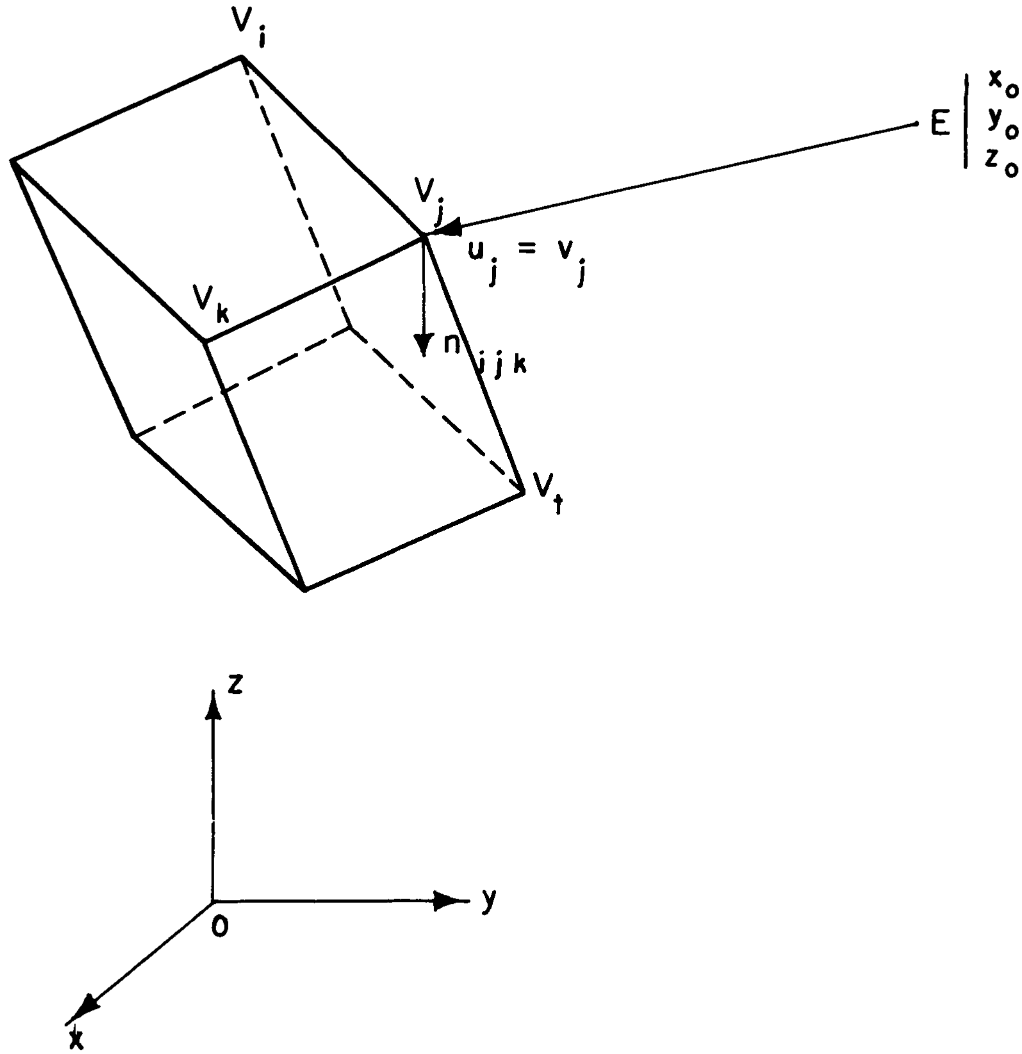

FIG. 6

VANTAGE POINT AT FINITE DISTANCE

$$\Delta_{ijk} = u_j \cdot n_{ijk}$$

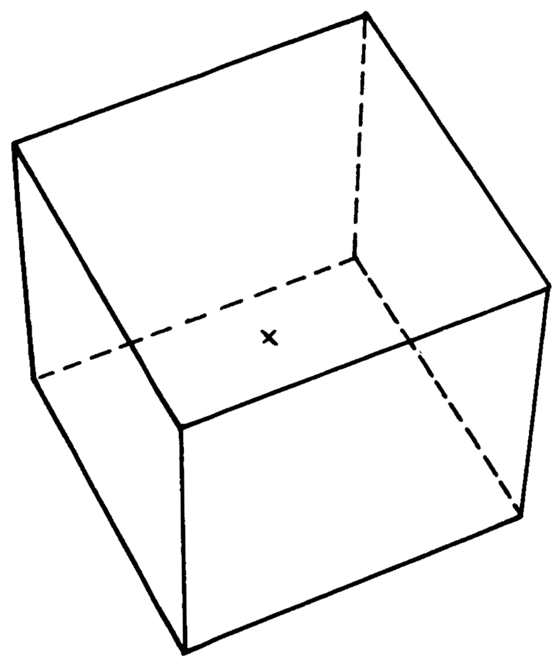

FIG.7

CUBE SEEN FROM E | 1000
2000
3000

NOTE: FIGURES 7, 8, 9, 10 WERE DRAWN BY A CALCOMP PLOTTER

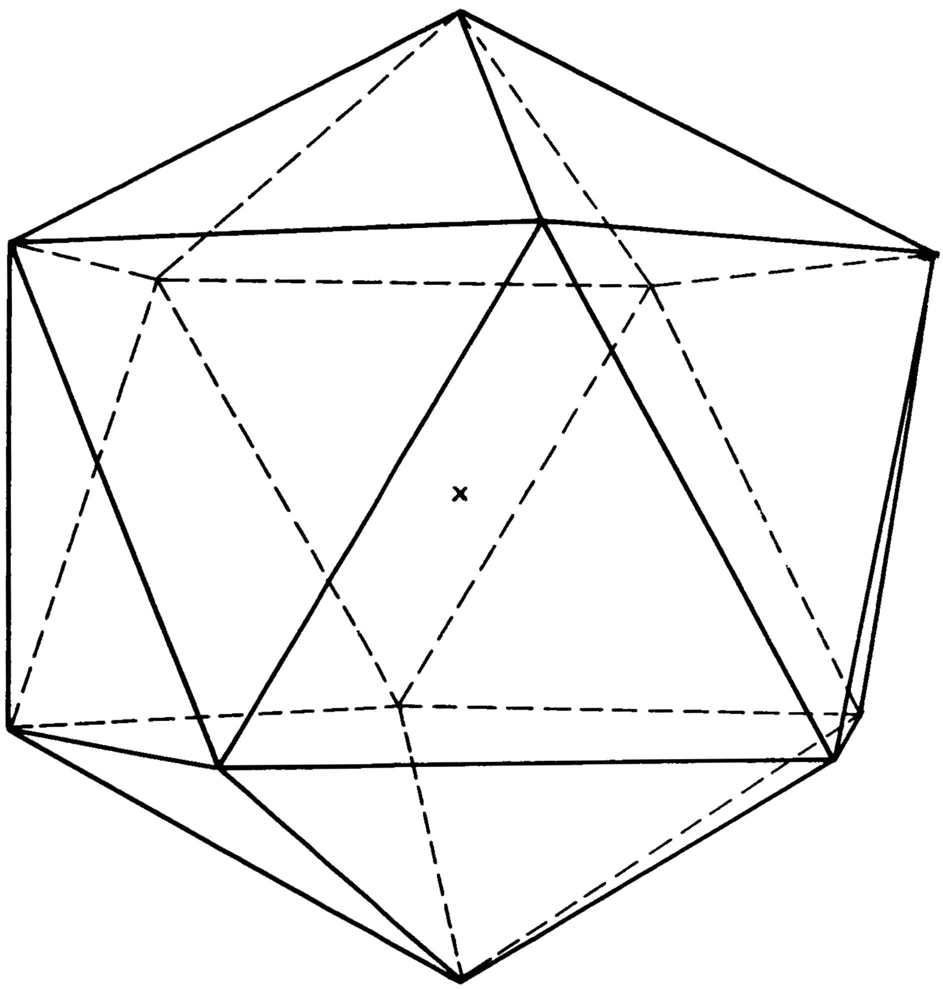

FIG. 8

ICOSAHEDRON SEEN FROM E | 5000
5000
0

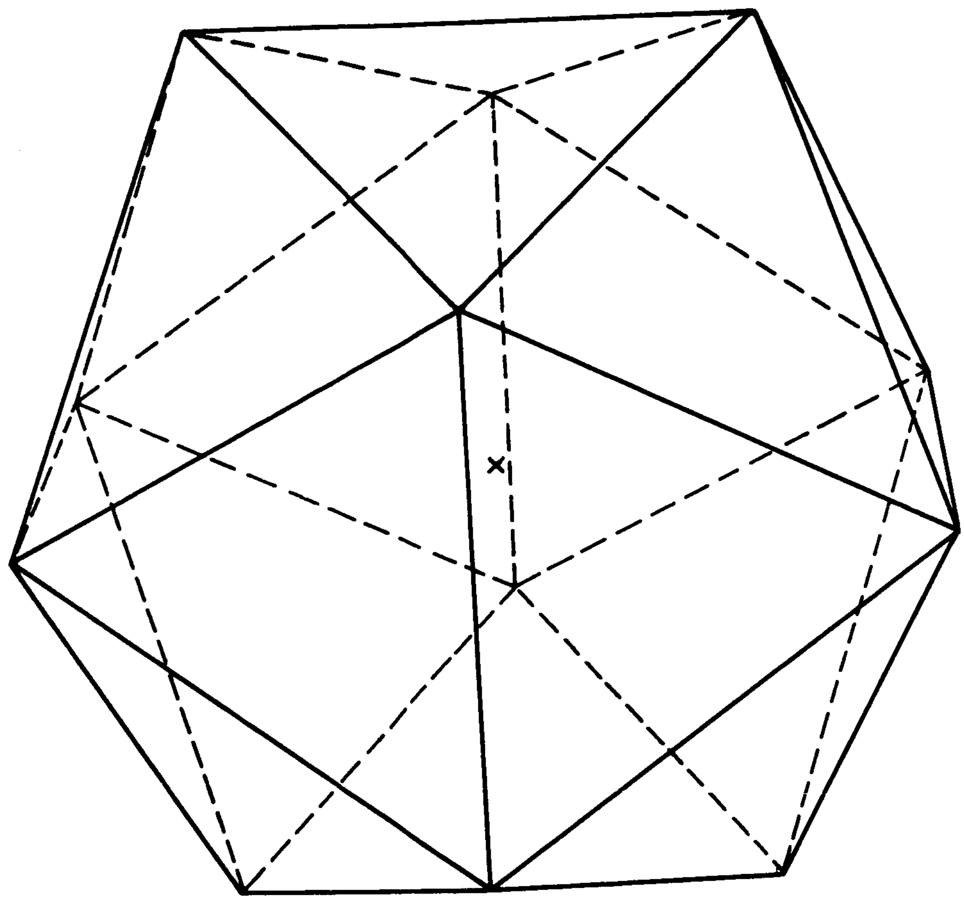

FIG. 9

ICOSAHEDRON SEEN FROM $E\left|\begin{matrix}5000\\6000\\7000\end{matrix}\right.$

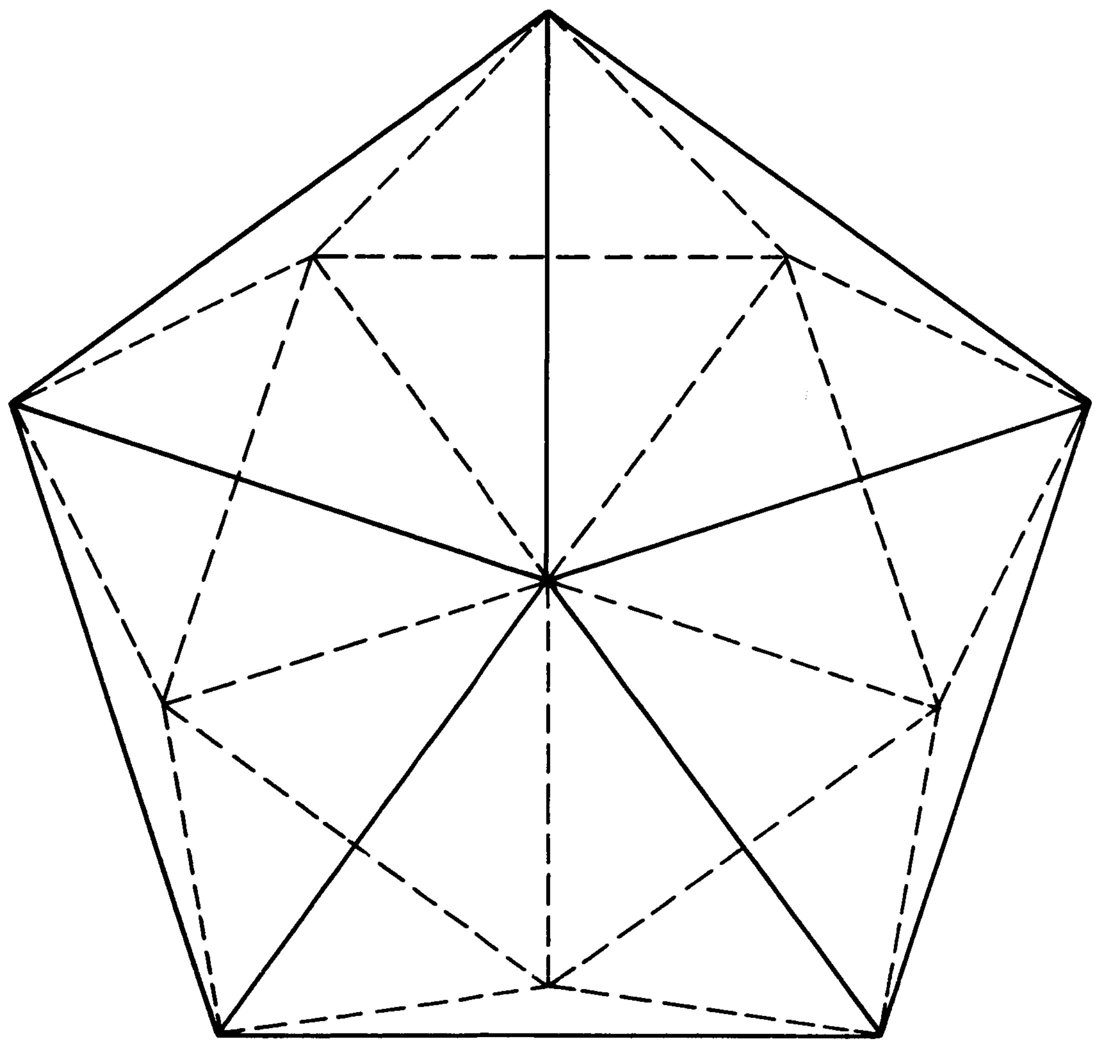

FIG. 10

ICOSAHEDRON SEEN FROM E $\left|\begin{matrix}0\\0\\3000\end{matrix}\right.$