APPLICATION OF HASSE DIAGRAMS FOR COUNTING TOPOLOGIES ON FINITE SETS

by

Richard W. Piotrowski

TECHNICAL REPORT NO. 14

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION

DEPARTMENT OF MATHEMATICS

KENT STATE UNIVERSITY

KENT, OHIO

September 1966

PIOTROWSKI, RICHARD WALTER, M.A., September, 1966, MATHEMATICS

APPLICATION OF HASSE DIAGRAMS FOR COUNTING TOPOLOGIES ON FINITE SETS (36 pp.)

Director of Thesis: T. N. Bhargava

Let A_n be a set consisting of n elements. The main problem considered in this thesis is that of finding the number of elements in the class of all topologies on A_n, i.e. finding the cardinality of the set $\mathcal{T} = \{T : T \text{ is a topology on } A_n\}$. However, the related problems of counting certain kinds of relations and digraphs on A_n are also investigated. By means of digraph topology, Bhargava and Ahlborn have shown that these problems are indeed closely related to each other. Furthermore, Chatterji has shown that these problems can be reduced to those of counting the number of partial orders on A_n and on partitions of A_n. Making use of these results we have obtained the exact number of homeomorphic and non-homeomorphic topologies for $n \leq 5$ and partial results for n=6. This method,which relies on the fact that all partial orders on a set can be represented by Hasse diagrams,also enables us to determine the number of T_o-topologies on A_n. As a consequence, we have also obtained the number of homeomorphic and non-homeomorphic T_o-topologies for $n \leq 5$.

For sake of completeness we also present all known results on counting topologies and related problems, as well as upper and lower bounds for the number of topologies on A_n.

Lastly it is shown that the number of subsets in the $k+1^{st}$, $0<k<n-1$, largest topology for A_n is given by $(2^k+1)2^{n-k-1}$.

We remark here that even some reasonably approximate general results for any n are very hard to get, and the problem remains an open one except for some important contributions made by Chatterji. For higher values of n, i.e. $n>6$, even the use of brute force for counting becomes almost impossible.

Acknowledgments

This thesis could not have been written without the guidance, understanding, and patience of Dr. T. N. Bhargava. To him I extend my deep appreciation.

I owe much to Dr. S. D. Chatterji for his valuable counsel and suggestions, and for letting me use his unpublished paper "On Counting Topologies" as a basis for much of this thesis. I also wish to thank Mr. V. K. Rohatgi for a critical reading of the manuscript.

Finally, I wish to express my thanks for the financial support which was provided by the National Aeronautics and Space Administration through NASA Research Grant No. NsG-568.

TABLE OF CONTENTS

INTRODUCTION AND SUMMARY

The principal objective of this thesis is to present a useful and interesting method, developed by Chatterji [4], of counting topologies on finite sets by means of Hasse diagrams. The problem of counting topologies originally appeared as an exercise in the classical text Set Topology, by Vaidyanathaswamy [11]. This exercise asked: Enumerate all the topologies which can be imposed on a set of four points, and find an expression for t(n) the number of topologies which can be imposed on a set of n points. However, as yet, no general solution seems to have been found. Even for the case $n=4$ the problem becomes very tedious; and for values of n greater than six the task becomes virtually impossible. A few papers have been written on this and related problems, the most notable being that of Chatterji [4], and the most recent being that of Krishnamurthy [8], in which he obtains values for t(n), $n\leq 4$, and a weak upper bound for t(n). It may well be that there does not exist a general expression for t(n), and the best that one can hope for is to obtain sharp bounds.

The purposes of this study are: (a) to reduce the counting of topologies on a finite set A to the counting of reflexive and transitive relations on the set A, which can be systematically enumerated by means of Hasse diagrams, and to illustrate counting (of these relations) by means of Hasse diagrams, (b) to obtain explicitly values of t(n), up to n=5, and partial results for n=6, and finally (c) to discuss a related problem of determining a bound for t(n), n arbitrary but finite.

This thesis is presented in three chapters.

The first consists of definitions, notations, and known results which we use throughout the following chapters.

The second chapter comprises more or less the main part of this thesis. In there we present, following Chatterji [4], various steps needed to reformulate the problem of rather unstructured enumeration of topologies to an equivalent task of systematically counting certain partial orderings by means of Hasse diagrams. This is accomplished by first defining a family $\mathfrak{F}$ of functions from a set A to its power set 2^A. Each function f in $\mathfrak{F}$ is then shown to be a Kuratowski closure operator and hence defining a unique topology on A. We then consider the family $\mathcal{R}$ of reflexive and transitive relations on the set A, and show that there is one-to-one correspondence between the elements of $\mathfrak{F}$ and the elements of $\mathcal{R}$. Finally we show that a further reduction is possible, viz. we can restrict our consideration to the counting of certain partial orderings induced by elements of $\mathcal{R}$. When this stage is reached we have completely reformulated the problem in terms of partial orderings.

The above equivalent formulation of the problem, combined with the known result that each partially ordered set can be represented by a unique Hasse diagram, Szász [9], enables us to use the Hasse diagrams to obtain a more structured and systematic counting procedure than previously available. We find that we need to construct only those Hasse diagrams which represent distinct (non-order isomorphic) partial orderings. Then applying usual combinatorial techniques we determine the total number of partial orderings under consideration. Lastly we develope several formulae which eliminate the need to construct all of the Hasse diagrams. Finally, explicit results obtained for t(n), $n \leq 5$, are derived.

In the third chapter we consider some related counting problems, and present some known results concerning other types of relations. For sake of completeness, results obtained by Chatterji [4], on bounds for t(n) are also given. Finally an interesting approach which also gives information concerning upper bounds for t(n), by determining the maximum number $k < 2^n$ of elements permissible in a family $\mathcal{T}$ of subsets of a set A such that $\mathcal{T}$ forms a topology for A, is presented.

CHAPTER I

PRELIMINARIES

In this chapter we present many definitions, notations and relevant results, a few of which may be only indirectly related to our work, in order to make this thesis completely self contained.

1.1 Definitions and Notations

For the sake of clarity and uniformity definitions concerning the theory of relations and sets are taken from Halmos [6], except for a few which are not in Halmos and as such are taken from Szász [9]; definitions in point set topology are from Kelly [7]; and those in graph theory are from Berge [1], and Bhargava and Ahlborn [2].

Since this thesis is concerned with counting problems related to finite spaces we restrict our definitions to a finite set, consisting of n elements, denoted by $A_n=\{x_1,x_2,...,x_n\}$ and referred to simply as A whenever n is fixed and no ambiguity results. Similarly we use "A" for "set A" whenever no confusion is possible. Throughout, the symbols x,y,z are used to denote arbitrary (not necessarily distinct) elements of A.

Definition 1.1.1. The Cartesian product of a set A with itself, denoted by $A\times A$, is the set of all ordered pairs (x,y), where $x,y \in A$.

<u>Definition 1.1.2</u>. A <u>relation</u> <u>R</u> on a set A is a collection of ordered pairs (x,y) belonging to the Cartesian product $A \times A$. We note that R may be empty. We use the notation xRy to mean that $(x,y) \in R$. Finally when we say "a relation R" we mean a relation R on a set A has been defined.

<u>Definition 1.1.3</u>. A relation R is said to be <u>reflexive</u> if xRx, for all $x \in A$, <u>symmetric</u> if xRy implies yRx for all $x,y \in A$, and <u>transitive</u> if xRy and yRz imply xRz for all $x,y,z \in A$.

<u>Definition 1.1.4</u>. A relation R is said to be an <u>equivalence relation</u> on A if it is simultaneously reflexive, symmetric and transitive.

<u>Definition 1.1.5</u>. Let R be an equivalence relation on A. An <u>equivalence</u> <u>class</u> of A determined by $x \in A$ is the set of all $y \in A$ such that xRy. The family $\mathcal{C}$ of all such subsets is called the <u>set</u> <u>of</u> <u>equivalence</u> <u>classes</u> of A determined by R.

<u>Definition 1.1.6</u>. A <u>partition</u> $\mathcal{P}$ of a set A is a disjoint collection of non-empty sets whose union is A. The elements of $\mathcal{P}$ are called <u>classes</u> of the partition.

<u>Definition 1.1.7</u>. A relation R is said to be <u>antisymmetric</u> if for all $x,y \in A$, the simultaneous validity of xRy and yRx implies that $x=y$.

<u>Definition 1.1.8</u>. A relation R on a set A is said to be a <u>partial</u> <u>ordering</u> if it is simultaneously reflexive, anti-

symmetric, and transitive, and we say that R <u>partially orders</u> the set A.

If a relation R partially orders a set A we write $x \le y$ to mean xRy, and $x < y$ if xRy and $x \ne y$, for all $x, y \in A$; and we say that the set A is partially ordered by the relation "$\le$", or simply a partially ordered set A.

<u>Definition 1.1.9</u>. An element x of a partially ordered set A is said to be a <u>minimal</u> (<u>maximal</u>) element if there is no $y \in A$ such that $y < x$ ($x < y$). An element x of A is called the <u>least</u> (<u>greatest</u>) element of A if $x \le y$ ($y \le x$), for all $y \in A$.

<u>Definition 1.1.10</u>. Let x and y be arbitrary elements of a partially ordered set A. Elements x and y are said to be <u>comparable</u> if $x \le y$ or $y \le x$; <u>incomparable</u> if neither $x \le y$ nor $y \le x$.

<u>Definition 1.1.11</u>. A subset C of a partially ordered set A is said to be a <u>chain</u> if any two elements of C are comparable with respect to the partial ordering on A; and a <u>chain to x</u>, if x is the greatest element in C. If a chain C has k elements we say that the <u>length</u> of C is k-1.

<u>Definition 1.1.12</u>. Let A be a partially ordered set. An element $x \in A$ is said to <u>cover</u> another element $y \in A$ if $y < x$ and there does not exist an element $z \in A$ such that $y < z < x$.

<u>Definition 1.1.13</u>. A <u>Hasse diagram</u> or simply (<u>H-diagram</u>) of a finite non-empty partially ordered set A is a graph

whose vertices are distinct elements of A and furthermore two elements x,y are joined by a line segment if either x covers y or y covers x. If x covers y the vertex x is said to be <u>higher</u> than y.

<u>Definition 1.1.14</u>. Let the sets A and A* be partially ordered with respect to the relations R and R* respectively. A single-valued mapping θ of A into A* is called an <u>order preserving mapping</u> of A into A* if for all $x,y \epsilon A$, xRy implies $\theta(x)\ R^*\ \theta(y)$. If θ^{-1} is also an order preserving mapping then θ is called an <u>order isomorphism</u>. Furthermore, if there is an order isomorphism of A onto A*, then A is said to be <u>order isomorphic</u> to A*.

<u>Definition 1.1.15</u>. Let A be a finite set and $\mathfrak{T}$ be a family of distinct subsets of A. Then $\mathfrak{T}$ is a <u>topology</u> for A if the following axioms are satisfied:

T-1. A and ϕ belong to $\mathfrak{T}$, where ϕ denotes the null set,

T-2. The union of any arbitrary collection of members of $\mathfrak{T}$ is a member of $\mathfrak{T}$, and

T-3. The intersection of any finite number of members of $\mathfrak{T}$ is a member of $\mathfrak{T}$.

The pair $(A,\mathfrak{T})$ is called a <u>topological space</u>. The members of $\mathfrak{T}$ are called <u>open sets</u>, and the complements of members of $\mathfrak{T}$ are called <u>closed sets</u>.

<u>Definition 1.1.16</u>. The <u>closure</u> of a subset B of a topological space $(A,\mathfrak{T})$ is the intersection of the members of

the family of all closed sets containing B. The closure of a set B is the smallest closed set containing B.

Definition 1.1.17. A closure operator "c" on a set A is an operator which assigns to each subset B of A a subset B^c of A such that the following four postulates (called the Kuratowski closure axioms) are satisfied:

K-1. $\phi^c = \phi$,

K-2. For each subset B of A, $B \subseteq B^c$,

K-3. For each subset B of A, $B^{cc} = B^c$, and

K-4. For each pair of subsets B and C of A, $(B \cup C)^c = B^c \cup C^c$.

Definition 1.1.18. A topological space $(A, \mathfrak{T})$ is said to be a T_0-space if for any two distinct points x and y of A there exists an open set containing one of these elements but not the other; a T_1-space if all subsets of A consisting of single elements are closed.

Definition 1.1.19. A function f on a topological space $(A, \mathfrak{T})$ into a topological space $(A^*, \mathfrak{T}^*)$ is continuous if the inverse of each open set in $(A^*, \mathfrak{T}^*)$ is an open set in $(A, \mathfrak{T})$.

Definition 1.1.20. A homeomorphism is a continuous-one-to one mapping f of a topological space $(A, \mathfrak{T})$ onto a topological space $(A^*, \mathfrak{T}^*)$ such that its inverse image f^{-1} is also continuous. If there exists a homeomorphism f on a topological space $(A, \mathfrak{T})$ to a topological space $(A^*, \mathfrak{T}^*)$ the two spaces are said to be homeomorphic.

Definition 1.1.21. A directed graph (or simply digraph) consists of a set A, and a subset E of $A \times A$, $\phi \subseteq E \subseteq A \times A$, and is denoted by $\Gamma(A,E)$. If $B \subseteq A$, $\Gamma(B, E \cap B \times B)$ is said to be a subdigraph of digraph $\Gamma(A,E)$. We note that a digraph is also a relation. An ordered pair $(x,y) \varepsilon E$ is said to be a directed edge from x to y.

Definition 1.1.22. The power set of a finite non-empty set A, denoted by 2^A, is the set of all subsets of A.

Definition 1.1.23. A subdigraph $\Gamma(B, E \cap B \times B)$ of the digraph $\Gamma(A,E)$ is said to define an open set $B \varepsilon 2^A$, where 2^A is the power set, if for every pair of points (x,y) such that $x \varepsilon (A \sim B)$ and $y \varepsilon B$, $(x,y) \not\varepsilon E$. A subdigraph $\Gamma(B, E \cap B \times B)$ of the digraph $\Gamma(A,E)$ is said to define a closed set $B \varepsilon 2^A$ if for every pair of points (x,y), $x \varepsilon B$ and $y \varepsilon (A \sim B)$ imply that $x,y) \not\varepsilon E$.

1.2 Some Relevant Results

In this section we present some well known results which are rather basic to our study in this thesis. These results are stated, without proofs, in the form of theorems for easy accessibility. The proper references are cited immediately following each theorem, along with the page number on which the proof appears in the original work.

Again we restrict our consideration to finite sets, although many of the theorems hold for arbitrary sets.

<u>Theorem 1.2.1</u>. If R is an equivalence relation on A, then the set of equivalence classes is a partition of A that induces the relation R; and if $\mathcal{P}$ is a partition of A, then the induced relation is an equivalence relation whose set of equivalence classes is exactly $\mathcal{P}$ (Halmos [6] p. 28).

<u>Theorem 1.2.2</u>. Order isomorphism is an equivalence relation on the family of partial orderings of a set A (Szász [9]p. 17).

<u>Theorem 1.2.3</u>. Any finite partially ordered set has minimal and maximal members (Birkhoff [3]p. 8).

<u>Theorem 1.2.4</u>. With chains, the notions minimal and least (maximal and greatest) are identical. Hence any finite subset of A has a first (=least) and a last element (Birkhoff [3]p. 9).

<u>Theorem 1.2.5</u>. Every non-void partially ordered set can be represented by a Hasse diagram . Two partially ordered sets can be represented by the same Hasse diagram if and only if they are order isomorphic (Szász [9] pp. 18, 19).

<u>Theorem 1.2.6</u>. There is a one-to-one correspondence between the partial orderings of a set A and the T_0-topologies on A (Birkhoff [3]p. 14).

<u>Theorem 1.2.7</u>. Let "c" be closure operator on A. Let $\mathcal{F}$ be the family of all subsets B of A for which B^c=B, and let $\mathcal{T}$ be the family of complements of members of $\mathcal{F}$, then $\mathcal{T}$ is topology for A and B^c is the closure of B for each subset B

of A (Kelly [7]p. 43).

Theorem 1.2.8. Each digraph $\Gamma(A,E)$ determines a unique topological space $(A,\mathcal{T}_E)$ where $\mathcal{T}_E = \{B: B\epsilon 2^A, B \text{ open}\}$; and the topology has the property of completely additive closure (Bhargava and Ahlborn [2]p. 2).

CHAPTER II

TOPOLOGIES IN TERMS OF RELATIONS

In this chapter we begin by showing the stages in the evolution of the problem from that of counting topologies on a finite non-empty set A to the equivalent problem of counting the number of ways that the set A (or certain families of subsets of A) can be partially ordered. We then proceed to show how Hasse diagrams are employed to facilitate the counting of these partial orderings, and present some formulae that eliminate the need to construct many of the Hasse diagrams. Finally we give explicit results up to n=5 and partial results for n=6.

2.1 Reformulation of the problem

Let $\mathfrak{F}$ be a family of functions from the finite non-empty set A to the power set 2^A, such that each member f of $\mathfrak{F}$ has the following properties:

F-1. For all $x \in A$, $x \in f(x)$, and

F-2. For all $x,y \in A$, $y \in f(x)$ imply $f(y) \subseteq f(x)$.

Furthermore we define $f(\phi)=\phi$ and for any subset B of A we define $f(B)=\bigcup_{x \in B} f(x)$, which we write as $\bigcup\{f(x): x \in B\}$. Thus $f(x)=f(\{x\})$, $f(A)=A$, and also $y \in f(x)$ and $x \in f(y)$ implies $f(x)=f(y)$. For any $B \subseteq A$ we write $f(f(B))=f^2(B)$.

The family $\mathfrak{F}$ of functions satisfying the above conditions is certainly not empty. For, let A be any finite non-empty set and let f be a function from A to 2^A such that for all $x \in A$, $f(x)=\{x\}$. Then f meets the requirements of the definition.

Before proceeding to the theorem 2.1.2 which shows that all functions $f \in \mathcal{F}$ are closure operators on A, we present an example which gives an indication of the usefulness of these functions.

Example 2.1.1. Let A= {a,b,c,d,e} and let f be a function from A to 2^A such that:

i. f(a)={a,b,c,d,e}, iii. f(c)={c,d,e}, v. f(e)={e},

ii. f(b)={b,e}, iv. f(d)={d},

Conditions F-1 and F-2 are clearly satisfied. Proceeding further we obtain:

vi. f({d,e})={d,e}, ix. f({c,d,e})={c,d,e},

vii. f({b,e})={b,e}, x. f({b,c,d,e})={b,c,d,e}

viii. f({b,d,e})={b,d,e},

These, of course, do not list all the possibilities, but are sufficient to illustrate the method. Except for i, ii, and iii we have listed only those subsets B of A for which f(B)=B. We recall that $f(\phi)=\phi, f(A)=A$. Upon taking complements, denoted by $\tilde{f}$, of the last seven subsets we get:

iv'. $\tilde{f}$({d})={a,b,c,e}, viii'. $\tilde{f}$({b,d,e})={a,c},

v'. $\tilde{f}$({e})={a,b,c,d}, ix'. $\tilde{f}$({c,d,e})={a,b},

vi'. $\tilde{f}$({d,e})={a,b,c}, x'. $\tilde{f}$({b,c,d,e})={a}

vii'. $\tilde{f}$({b,e})={a,c,d},

It can be easily shown that the family of subsets iv' through x' along with ϕ and A forms a topology for A. Thus the function f is a closure operator on A.

Lemma 2.1.1. Let "c" be a closure operator on a finite non-void empty set A, and let B and C be subsets of A. Then $B \supseteq C$ implies $B^c \supseteq C^c$.

Proof. For any subsets B, C of A, such that $B \supseteq C$ we have:

$B = C \cup (B \sim C)$, so that

$B^c = (C \cup (B \sim C))^c$

$B^c = C^c \cup (B \sim C)^c$, by K-4, definition 1.1.17, and

$B^c \supseteq C^c$

Theorem 2.1.2. Let $\mathcal{F}$ be the family of all functions from an arbitrary finite non-empty set A into 2^A, satisfying conditions F-1 and F-2. Then each $f \varepsilon \mathcal{F}$ is a closure operator on A. Hence each f determines a unique topology for A.

Proof. We show first that any $f \varepsilon \mathcal{F}$ satisfies postulates K-1 through K-4 of definition 1.1.17.

K-1'. $f(\phi) = \phi$

K-2'. We show that for any $B \subseteq A$, $B \subseteq f(B)$. By definition $f(B) = \bigcup\{f(x) : x \varepsilon B\}$, and the result now follows by F-1.

K-3'. Next we show that $f^2(B) = f(B)$, for any subset B of A. Since f is a function from A to 2^A, $f(B) \subseteq A$, for any $B \subseteq A$. By K-2', $f(B) \subseteq f^2(B)$.

Conversely, we note

$f^2(B) = f(f(B)) = \bigcup\{f(y) : y \varepsilon f(B)\} = \bigcup\{f(y) : y \varepsilon \bigcup\{f(x) : x \varepsilon B\}\}$.

For any $z \varepsilon f^2(B)$ there exists at least one $y \varepsilon f(B)$ such that $z \varepsilon f(y)$. Similarly $y \varepsilon f(B)$ implies that there exists an $x \varepsilon B$

such that $y \varepsilon f(x)$. By F-2 we have $f(y) \subseteq f(x)$. Hence $z \varepsilon f(x)$. Since this is true for all $z \varepsilon f^2(B)$, we have $f^2(B) \subseteq \bigcup \{f(x): x \varepsilon B\} = f(B)$. This concludes the proof that $f^2(B)=f(B)$.

K-4'. Finally if B and C are arbitrary subsets of A, then it can easily be checked that $f(B \cup C)=f(B) \cup f(C)$.

Now we show that each closure operator "c" on A satisfies F-1 and F-2.

F-1'. Let $\{x\} \subseteq A$, then by K-2 $\{x\} \subseteq \{x\}^c$, and it follows that $x \varepsilon \{x\}^c$.

F-2'. Let $x,y \varepsilon A$ with $y \varepsilon \{x\}^c$, then $\{y\} \subseteq \{x\}^c$. It follows by Lemma 1.2.1 that $\{y\}^c \subseteq \{x\}^{cc}$, and by K-3 that $\{y\}^c \subseteq \{x\}^c$.

Thus each $f \varepsilon \mathfrak{F}$ is a closure operator on A and hence by theorem 1.2.7 determines a unique topology for A.

Now we give a formulation in terms of relations. Let $\mathfrak{R}$ be a family of relations on A such that a relation $R \varepsilon \mathfrak{R}$ if and only if there is a corresponding function $f \varepsilon \mathfrak{F}$ such that for all $x,y \varepsilon A$, xRy if and only if $y \varepsilon f(x)$.

<u>Theorem 2.1.3</u>. Let $\mathfrak{F}$ be the family of all functions from a finite non-void set A into 2^A, satisfying conditions F-1 and F-2, and let $\mathfrak{R}$ be the family of relations defined above. Then for each $R \varepsilon \mathfrak{R}$ there is a corresponding unique $f \varepsilon \mathfrak{F}$, such that the conditions F-1 and F-2 on f are equivalent to the following conditions on R:

R-1. For all $x \varepsilon A$, xRx (reflexive), and

R-2. For all $x,y,z \varepsilon A$, xRy and yRz imply xRz (transitive).

Proof. We first show that conditions F-1 and F-2 on f satisfy R-1 and R-2 on R.

F-1'. For all $x \in A$, $x \in f(x)$ by definition of R.

F-2'. For $x,y,z \in A$, let $y \in f(x)$ and $z \in f(y)$.

By F-2, $f(y) \subseteq f(x)$, and $f(z) \subseteq f(y)$. Hence $f(z) \subseteq f(x)$.

By F-1, $z \in f(z)$. It follows that $z \in f(x)$ or xRz.

Now we show that conditions R-1 and R-2 on R satisfy F-1 and F-2 of f.

R-1'. For all $x \in A$ we have xRx and it follows that $x \in f(x)$.

R-2'. Let $x,y \in A$ such that $y \in f(x)$, then xRy. For any $z \in f(y)$ we have yRz, and by R-2 xRz, which imply $z \in f(x)$.

The two preceeding theorems establish a one-to-one correspondence between the reflexive and transitive relations on the set A and the topologies on A. We can summarize these results in the following manner:

T-1,T-2,T-3 $\Leftrightarrow$ K-1,K-2,K-3,K-4 $\Leftrightarrow$ F-1,F-2 $\Leftrightarrow$ R-1,R-2.

We can simplify the problem still further to the counting of certain partial ordering relations by means of theorem 2.1.5.

Lemma 2.1.4. Let $\mathfrak{F}$ be a family of functions from A to 2^A, and let "$\sim$" be a relation on A such that for every $x,y \in A$, $x \sim y$ if and only if $f(x)=f(y)$, where "=" means set equality and $f \in \mathfrak{F}$. Then "$\sim$" is an equivalence relation on A.

Proof. i) Since "=" is reflexive, $f(x)=f(x)$ for all x. Hence $x \sim x$. ii) For $x,y \in A$, let $x \sim y$. Then by definition $f(x)=f(y)$;

and since "=" is symmetric we have $f(y)=f(x)$. Hence $y \sim x$.

iii) For $x,y,z \varepsilon A$ let $x \sim y$, $y \sim z$, it follows that $f(x)=f(y)$ and $f(y)=f(z)$. By the transitive property of "=" we obtain $f(x)=f(z)$. Hence $x \sim z$.

Thus the relation "$\sim$" has the properties of reflexivity, symmetry and transitivity and is therefore an equivalence relation on A.

By theorem 1.2.1 the relation "$\sim$" in the above lemma induces a partition $\mathcal{C}$ of A. We denote the equivalence class of A determined by x with respect to "$\sim$" by $\mathcal{C}(x)$, i.e. $\mathcal{C}(x)=\{y: x \sim y\}$.

Remark 2.1.1. Since in the above lemma $f(x)=f(y)$ is equivalent to saying that x and y are symmetrically related we can restate the definition of "$\sim$" by saying $x \sim y$ if and only if xRy and yRx.

Theorem 2.1.5. Let $\mathcal{R}$ be the family of all reflexive and transitive relations on a finite non-empty set A. Then each $R \varepsilon \mathcal{R}$ is either a partial ordering on A, or induces a unique partial ordering on family $\mathcal{C}$ of disjoint subsets of A whose union is A.

Proof. For each reflexive transitive relation $R \varepsilon \mathcal{R}$ we have two possibilities:

i. If R is also antisymmetric then by definition R is a partial ordering on A, and

ii. If R is not anti-symmetric then there exists at least one pair of elements $x,y \varepsilon A$, such that xRy and yRx.

By theorem 2.1.3 there is a unique function f from A into 2^A satisfying F-1 and F-2, with $y \in f(x)$ and $x \in f(y)$. Hence $f(x)=f(y)$. By lemma 2.1.4 for each such function f there exists a unique equivalence relation "$\sim$" on A such that for $x,y \in A$, $x \sim y$ if and only if $f(x)=f(y)$. For each $x \in A$, we denote the equivalence class of A induced by "$\sim$" as $\mathcal{C}(x)$.

Now consider the relation R' on the family $\mathcal{C}$ of equivalence classes of A, induced by "$\sim$", such that $\mathcal{C}(x)R'\mathcal{C}(y)$ if and only if there exists an $a \in \mathcal{C}(x)$ and $b \in \mathcal{C}(y)$ such that aRb. We will show that R' is a partial ordering on $\mathcal{C}$.

i. R' is reflexive. For each $a \in A$, aRa by definition, and it follows that $\mathcal{C}(x)R'\mathcal{C}(x)$, where $a \in \mathcal{C}(x)$.

ii. Let $\mathcal{C}(x), \mathcal{C}(y), \mathcal{C}(z) \in \mathcal{C}$, and let $\mathcal{C}(x)R'\mathcal{C}(y)$ and $\mathcal{C}(y)R'\mathcal{C}(z)$. It follows that there exist a,b, and c belonging to $\mathcal{C}(x)$, $\mathcal{C}(y)$ and $\mathcal{C}(z)$ respectively such that aRb and bRc. By the transitivity of R we have aRc. Hence $\mathcal{C}(x)R'\mathcal{C}(z)$.

iii. To show that R' is antisymmetric, consider two distinct elements $\mathcal{C}(x), \mathcal{C}(y) \in \mathcal{C}$. If $\mathcal{C}(x)R'\mathcal{C}(y)$ and $\mathcal{C}(y)R'\mathcal{C}(x)$ then there exist $a,a' \in \mathcal{C}(x)$ and $b,b' \in \mathcal{C}(y)$ such that aRb and b'Ra'. By the remark 2.1.1 a'Ra and bRb'. The transitive property of R implies a'Rb and a'Rb', from this it follows $f(a)=f(b')$ or $\mathcal{C}(a')=\mathcal{C}(b')$, a contradiction. Hence R' is antisymmetric.

Thus R' is a partial ordering on $\mathcal{C}$, and the theorem is proved.

We have essentially reduced the problem of determining t(n), the number of topologies on a finite set, to counting the number of partial orders on A, and on disjoint families of subsets of A whose union is A. We write $t_o(n)$ for the number of partial orders on A, where A is a finite non-empty set consisting of n elements.

We also note here that by theorem 1.2.6 every partial ordering relation on the set A represents a T_o-topology. Hence $t_o(n)$ is also the number of T_o-topologies on A. Thus all the topologies on a finite non-empty set A can be determined by counting only the T_o-topologies on A and on partitions of A. Let $t_1(n)$ be the number of T_1-topologies on A.

Lemma 2.1.6. For any finite non-empty set A, $t_1(n)=1$, and this topology $\mathcal{T}$ is the discrete topology (i.e. every subset B of A is in $\mathcal{T}$).

Proof. Let $\mathcal{T}$ be any T_1-topological space on A. Then by definition 1.1.18 for all $x \in A, \{x\}^c = \{x\}$. By theorem 2.1.2 there exists an $f \in \mathfrak{F}$, such that $f(\{x\}) = f(x) = \{x\}$. It follows that for every $B \subseteq A$, $f(B)=B$, which in turn implies that for all $B \subseteq A$, $\tilde{f}(b) \in \mathcal{T}$. Thus for every $B \subseteq A$, $B \in \mathcal{T}$. Certainly there is only one discrete topology on each set A. Hence f is unique.

Lemma 2.1.7. Let R' be a partial ordering on a disjoint family $\mathcal{C}$ of sets whose union is a finite non-empty set A. Then there is a corresponding reflexive, transitive relation R on A.

Proof. For all $x \in A$ let $\mathcal{C}(x)$ be the member of $\mathcal{C}$ to which x belongs. By theorem 1.2.1 there exists an equivalence relation, say "$\sim$" on $\mathcal{C}$ such that the members of $\mathcal{C}$ are precisely the equivalence classes induced by "$\sim$".

Now we define a relation R on A by saying that for all $x,y \in A$, xRy if and only if $\mathcal{C}(x)\ R'\ \mathcal{C}(y)$. Using the equivalence relation "$\sim$" we can easily show that R is a reflexive, transitive relation on A.

For every partition of a finite non-void set A consisting of n elements the family $\mathcal{C}$ of equivalence classes has at most n members. Using the two preceeding results we now say that: $t(n)= \sum_{k=1}^{n} \Pi_{n,k}\ t_o(k)$, where $\Pi_{n,k}$ is the number of partitions of A into k non-empty disjoint subsets.

With this we conclude the reformulation of the problem. Now it remains to be shown how we propose to count the partial orderings by means of Hasse diagrams.

2.2 Hasse Diagrams

In this section we give a detailed description of the so called Hasse diagrams (H-diagrams), and show how they are used to represent the partial orderings (T_o-topologies) on a finite non-empty set A_n. We also show how we obtain the open sets of the topologies determined by the Hasse diagrams. For convenience we repeat the remark that if a relation R partially orders a set A we write $x \leq y$ to mean xRy, and $x<y$ if xRy and $x \neq y$.

Let d be a map on set A_n, or simply A (partially ordered by the relation "$\leq$") into the set of non-negative integers such that for all $x \varepsilon A$, $d(x)$ is length of the maximal chain to x, i.e. $d(x)=\max\{k \mid \exists\ x_0<x_1< \dots <x_{k-1}<x\}$. We adopt the convention that $d(x)=0$ if and only if there is no $y (y \neq x)$ in A such that $y \leq x$, that is, x is unrelated to any other element in A_n. Clearly, $0 \leq k \leq n-1$.

Lemma 2.2.1. If x and y are distinct elements of a finite non-empty partially ordered set A, and $d(x)=d(y)$, then x and y are not comparable.

Proof. Let $x,y \varepsilon A$, $x \neq y$ with $d(x)=d(y)$, and assume that x and y are comparable. It follows that either $x<y$ or $y<x$; let us suppose that $x<y$. If $d(x) = k$, $0 \leq k \leq n-2$, then $d(y)=\max\{(k+\ell) \mid \exists\ x_0<x_1<\dots <x_{k-1}<x<y_{k+1}<\dots<y_{k+\ell-1}<y\}$, $1 \leq \ell \leq (n-k-1)$. Since $k+\ell>k$ we cannot have $d(y)=k$. Hence we obtain a contradiction of the hypothesis that $d(x)=d(y)$, and x and y are not comparable.

Lemma 2.2.2. Let A be a finite non-void set partially ordered by the relation "R". Then there exists an $a \varepsilon A$ such that $d(a)=k$, where $k=\max_{x \varepsilon A} d(x)$.

Proof. By theorem 1.2.3 every partially ordered set has at least one maximal element. Let M be the set of maximal elements of A. For each $x \varepsilon M$ let $d(x)=i$, $0 \leq i \leq n-1$, and let $I=\bigcup_{x \varepsilon M}\{i: d(x)=i\}$. Clearly I is a finite partially ordered set and furthermore for each pair $i,j \varepsilon I$, either $i<j$ or $j<i$.

Thus I is a chain and it follows by theorem 1.2.4 that I has a greatest element, say, k; i.e. $k=\max_{x\varepsilon A} d(x)$. Therefore there exists an element $a\varepsilon A$ such that $d(a)=k$, where $k=\max_{x\varepsilon A} d(x)$

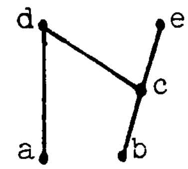
Remark 2.2.1. It follows from the definition of "d" that if $L=\max d(x)$ there is a chain $C=\{x_0, x_1,\ldots,x_L\}$ consisting of L+1 elements and an $x_L\varepsilon C$ such that $d(x_L)=L$. It is also clear that for each $x_i\varepsilon C$, $d(x_i)=i$, for if this was not the case we would have a contradiction that $L=\max_{x\varepsilon A} d(x)$.

Theorem 2.2.3. Let A be a finite non-void set partially ordered by the relation "$\leq$". Then the function d induces a partition of A into k+1 classes S_j such that $S_j=\{x:d(x)=j\}$, $j=0,1,2,\ldots,k$, where $k=\max_{x\varepsilon A} d(x)$, $0\leq k\leq n-1$.

Proof. Let "$\sim$" be a relation on A such that for all $x,y\varepsilon A$, $x\sim y$ if and only if $d(x)=d(y)$. Now "$\sim$" is an equivalence relation on A. The proof of this is exactly the same as that of lemma 2.1.4. with f replaced by d. By theorem 1.2.1, it follows that "$\sim$" induces a partition of A into a family of equivalence classes denoted by $S(x)=\{y: x\sim y\}=\{y: d(x)=d(y)\}$. By the remark 2.2.1 $k=\max_{x\varepsilon A} d(x)$ implies that there is a chain C of k+1 elements such that for each $x_i\varepsilon C$, $d(x_i)=i$, $i=0,1,2,\ldots,k$. It is then clear that d induces a partition of A into k+1 classes such that $S_j=S(x_j)=\{x:d(x)=j\}$.

With the help of these results we proceed to form the Hasse diagrams. Let A be a non-void set partially ordered by the relation "$\leq$", and let $k=\max_{x\varepsilon A} d(x)$. Then by theorem 2.2.3 there exists a partition of A into k+1 subsets denoted by $S_j = \{x: d(x)=j\}$, $j=0,1,\ldots,k$. We place the elements in each S_j, $j=0,1,\ldots k$, into rows, such that only the elements of S_j are in the j^{th} row. Arrange rows so that the j^{th} row is above the i^{th} row if and only if j is greater than i. Let $x,y \varepsilon A$, be elements of the $i^{th}(S_i)$ and $j^{th}(S_j)$ rows respectively and let $j>i$. We join x and y (by a line segment) if and only if y covers x. Two points in the same row are not to be joined since by lemma 2.2.1 they are not comparable. Furthermore each point of the i^{th} row must be joined to some point of the $i-1^{st}$ row. A point x of the i^{th} row may be joined with a point y of the j^{th} row $(j>i)$ only if there is no point in an intermediate row which is joined to both of them, i.e. if and only if y covers x. We now give a graphical illustration of the notion of Hasse diagrams.

<u>Example 2.2.1</u>. Let $A = \{a,b,c,d,e\}$ be partially ordered by the relation "$<$" in the following manner, $a<d$, $b<c$, $b<d$, $b<e$, $c<e$, $c<d$ with the remaining pairs of elements being incomparable. Since there is no element $x \varepsilon A$ such that $x<a$ we have $d(a)=0$, similarly $d(b)=0$, the only element which precedes c is b, so $d(c)=1$. By the transitive property of partially ordered sets we have $b<c<d$ as the maximal chain to d, hence $d(d)=2$, and likewise $d(e)=2$. We have $S_o = \{a,b\}$, $S_1=\{c\}$ and $S_2=\{d,e\}$. Placing the elements of S_o in the 0^{th} row, S_1 in the 1^{st} row and S_2 in the 2^{nd} row, and connecting the rows in the prescribed manner we obtain the Hasse diagram below:



In order to obtain open sets from H-diagrams we note that by definition 1.1.2 each partial ordering relation on a set A is a subset of A×A. Hence each partial ordering relation denoted by "$\leq$" can be represented by a digraph $\Gamma(A,\leq)$. It follows from the definition of an open set of a digraph that a set $B\varepsilon 2^A$ is open if for every pair of points (x,y) such that $x\varepsilon(A\sim B)$ and $y\varepsilon B$ then $(x,y)\not\varepsilon\leq$. In terms of relations a set $B\varepsilon 2^A$ is open if for every $x\varepsilon(A\sim B)$ and $y\varepsilon B$ $x\not\leq y$, that is, no element in B covers an element in $(A\sim B)$. Thus in the preceding example the sets which satisfy this condition are

A,ϕ,{a},{b},{a,b},{b,c},{a,b,c},{b,c,e},{a,b,c,d}, and {a,b,c,e}. It is easily shown that this family of subsets of A does indeed form a topology for A.

Thus in this section we have characterized Hasse diagrams and shown how a topology is determined by each H-diagram. In the next section we present a procedure which makes the process of counting more systematic than if we tried to count topologies without resorting to the reformulation given in section 2.1.

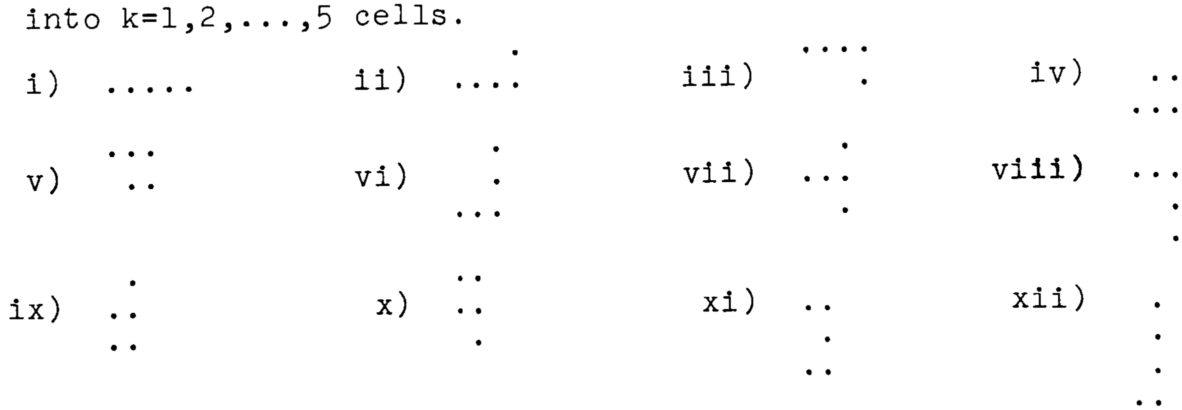
2.3 Counting Procedure

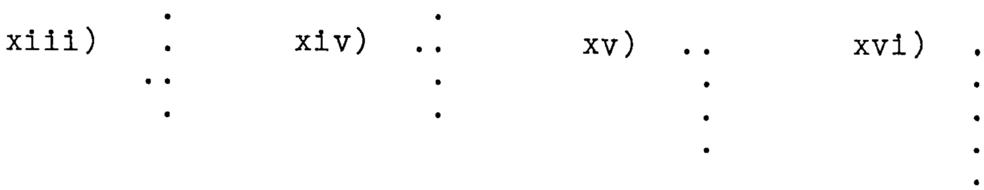
It is not necessary to construct every Hasse diagram which represents a partial ordering on a finite non-void set A partially ordered by the relation "$\leq$", for as theorem

1.2.5 states, two partially ordered sets can be represented by the same Hasse diagram if and only if they are order isomorphic. Hence we need construct only those H-diagrams which represent distinct (non-order isomorphic) partial orderings on A. Then using elementary combinatorial techniques we count the order isomorphic partial orderings represented by each distinct Hasse diagram.

The first step is to obtain the number of partitions of the set A into k+1, $k=0,1,2,\ldots,n-1$, subsets S_j, $j=0,1,\ldots,k$, where $k=\max_{x\in A} d(x)$. In combinatorial terminology the problem reduces to finding the number of ways in which n indistinguishable objects can be placed into k+1 cells such that none of the cells is empty. There are $\binom{n-1}{k}$ ways in which this can be done (see Feller [5] p. 37). It follows that the total number of such partitions of A is $\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$. We now give an example to show how a set of five elements can be partitioned in the desired manner.

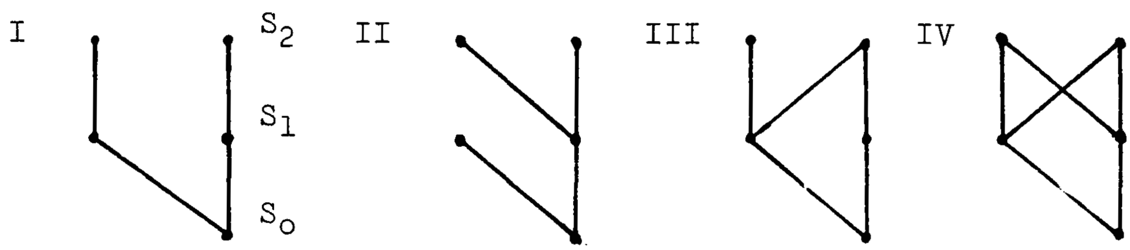
<u>Example 2.3.1</u>. For any set of five indistinguishable elements there are $2^4=16$ distinct partitions of the set into $k=1,2,\ldots,5$ cells.





The next step is to determine the number of distinct H-diagrams which can be obtained from each of the 2^{n-1} partitions. There is no known formulation which directly gives the total number of H-diagrams. However, we have some formulae presented in the next section which give us a partial solution to the problem. First we give (using partition (x) of example 2.3.1) some examples which indicate the counting procedure when no general formula is available.

Example 2.3.2.



These are the only distinct H-diagrams: any other H-diagram would be order isomorphic to one of the above H-diagrams. Consider the following example.

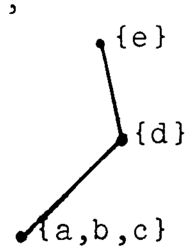
Example 2.3.3. We give a diagram III' which at first appears to be different than III of the above example but is actually order isomorphic to it. For sake of clarity and without loss of generality we label the points with elements of the two sets A={a,b,c,d,e} and A'={a',b',c',d',e'} as follows:

To show that the two sets A and A' are order isomorphic define a function θ from A onto A' as follows: $\theta(a)=a'$, $\theta(b)=b'$, $\theta(c)=d'$, $\theta(d)=c'$ and $\theta(e)=e'$. This is clearly a one-to-one mapping of A onto A' which preserves order. Hence III and III' are order isomorphic.

The final step is to determine how many order isomorphic partial orderings of the set A are represented by each distinct H-diagram. We use the H-diagrams of example 2.3.2 to demonstrate how this is done. For figure I of example 2.3.2 there are $\binom{5}{2}$ ways of choosing the elements in S_2, $\binom{3}{2}$ ways of choosing the 2 elements in S_1 and only 1 way of putting the remaining element in S_0 and 2 ways of connecting the elements in S_2 with the elements in S_1. Hence there are $\binom{5}{2} \cdot \binom{3}{2} \cdot 2=60$ order isomorphic partial orders represented by I. Similarly there are also 60 partial orderings represented by each of figures II, III, and 30 represented by IV. Thus there are 210 T_0-topologies obtained from just one partition of A.

Using the method described above we are theoretically able to determine all the partial orderings of a given set A. If R is a reflexive, transitive relation on a set A which is not a partial ordering of A then the following example illustrates how we count the T_0-topologies induced by R on a family of subsets of A.

Example 2.3.4. Let A= {a,b,c,d,e} and let R be a relation such that aRb, bRa, bRc, cRb, bRe, aRc, cRa, aRd, cRd, cRe. R is not a partial ordering on A, since the pairs (a,b), (a,c), and (b,c) are symmetrically related by R. Using theorem 2.1.5 it follows that R induces a partial ordering R' on the family $\mathcal{C}$ = {{a,b,c}, {d}, {e}} such that {a,b,c,} R'{d},{a,b,c,} R'{e} and {d}R'{e}. The H-diagram of this partial ordering is ,



We see that this H-diagram represents twenty order isomorphic reflexive and transitive relations on A. In the actual counting of the relations which are not partial orderings of a set A we do not explicitly use the method above but rely on the formula $f(n)= \sum_{k=1}^{n-1} \Pi_{n,k} f_o(k)+f_o(n)$.

2.4 Some counting formulae

In this section we present a number of formulae which reduce the task of constructing all the H-diagrams representing the partial orderings of a set. For convenience we introduce some additional notation. Let $P_k(n)$ be the number of partial orderings on a set A with $k = \max_{x \in A} d(x)$.

Then $f_o(n)=\sum_{k=1}^{n-1} P_k(n)$. The first two formulae in this following theorem were developed by Chatterji [4].

Theorem 2.4.1. Let A be a finite non-empty partially ordered set. Then,

P-0. $P_o(n)=1$

P-1. $P_1(n)=\sum_{k=1}^{n} \binom{n}{r}(2^{n-r}-1)^r$, and

P-2. $P_{n-1}(n)=n!$

Proof. P-0. If $\max_{x\varepsilon A} d(x)=0$, then for all $x\varepsilon A$, $d(x)=0$. It follows that $P_o(n)=1$.

P-1. For k=1 we have a partition of A into two disjoint non-empty subsets S_o and S_1 with $A=S_o \cup S_1$. There are $\binom{n}{r}$ ways of choosing the r elements in S_1. For each of these we have the two alternatives of connecting or not connecting it to each of n-r elements in S_o; i.e. 2^{n-r} choices. Each of the elements in S_1 must be connected to at least one of the elements in S_o so for each element in S_1 we have $2^{n-r}-1$ choices. Thus for each $r(1\leq r\leq n-1)$ we get $(2^{n-r}-1)^r$ different H-diagrams (partial orderings). Summing from r=1 to r=n-1 we arrive at the desired result.

P-2. Since $n-1=\max_{x\varepsilon A} d(x)$ we have a partition of A into n subsets, denoted by S_j, j=0,1,2,...,n-1. This can be done in only one distinct way. The n elements can be placed in the n S_j's in n! ways. Hence $P_{n-1}(n)=n!$.

2.5 Counting Results

In this section we present the results which we obtained using the method described in the preceeding sections. We summarize our findings in the two tables below. The first table contains the number of T_0-topologies found by using the formulae in section 2.4, or by actual construction of Hasse diagrams when no formula was available. The number of topologies which are not T_0 was found by summing the first n-1 terms of the formula $f(n)= \sum_{k=1}^{n} \Pi_{n,k} f_0(k)$. In the second chart we present only the number of homeomorphic topologies, that is, the topologies which are associated with the non-order isomorphic partial orderings. We present complete results for n=5 and only partial results for n=6.

Table 2.5.1

n=	1	2	3	4	5	6
$f_0(n)=$	1	3	19	219	4411	
$\sum_{k=1}^{n-1} \Pi_{n,k} f_0(k)=$	0	1	10	136	2749	82,173
$f(n)=$	1	4	29	355	7160	

Table 2.5.2

n=	1	2	3	4	5
$f^*_0(n)=$	1	2	5	17	63
$\sum_{k=1}^{n-1} \Pi_{n,k} f^*_0(k)=$	0	1	4	17	76
$f^*(n)=$	1	3	9	34	139

In table 2.5.2 $f^*(n)$ is the number of non-homeomorphic topologies, and $f^*_0(n)$ the number of non-homeomorphic T_0-topologies on a finite set A_n. Using computer techniques Krishnamurthy [9] obtained the values 1,4, 29 and 355 respectively. However he does not distinguish between the number of T_0 and non-T_0-topologies, nor does he obtain values for the number of non-homeomorphic topologies given in Table 2.5.2.

CHAPTER III

RELATED PROBLEMS

In this chapter we consider some problems related to the counting of topologies on finite sets. The first section contains information concerning the number of certain types of relations on a set, and also bounds found by Chatterji [4] for t(n).

In the second section we consider the problem of determining the maximum number $k<2^n$ of subsets permissible in a family $\mathfrak{T}$ of subsets of a finite set A such that $\mathfrak{T}$ forms a topology for A.

Section 3.1 Bounds for t(n)

First we present some known results concerning the number of various types of relations on a finite set A.

Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$ respectively be the families of reflexive, transitive, symmetric, anti-symmetric relations on a finite set A, and $\mathcal{R}$ be the family of all relations on A.

We present in the following table some known results (see Chatterji [4]) on the number of various types of relations on a set A. For any subset R of $\mathcal{R}$ let $\mu(R)$ be the number of elements in R.

Table 3.1

R	$\mu(R)$	R	$\mu(R)$
$\mathcal{R}_1$	$2^{n(n-1)}$	$\mathcal{R}_1 \cap \mathcal{R}_2$	$f(n)$
$\mathcal{R}_2$	$\frac{n(n+1)}{2}$	$\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_4$	$f_0(n)$
$\mathcal{R}_3$	$2^{\frac{n(n+1)}{2}}$	$\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$	B_n

B_n is an exponential number; for further details we refer to Rota [10].

In his paper Chatterji obtains bounds for t(n). A lower bound for $P_1(n)$ is obtained as a lower bound for t(n), and as an upper bound we use the number of reflexive relations on A. Thus,

$$2^{\frac{n^2}{4}} < t(n) < 2^{n(n-1)}$$

The upper bound is the same as the one obtained by Krishnamurthy [8].

3.2 Another approach

As we have shown, by lemma 2.1.6, for each non-empty finite set A consisting of n elements there is a topology for A consisting of all the 2^n subsets of A; this is the largest topology possible on A. An interesting question is what is the next largest number of subsets of A which form a topology for A? The largest topology on A corresponds to the case when each of the singleton sets is closed. The next largest corresponds to the case when one of the singleton sets is not closed. Using theorems 2.1.2 and 2.1.3 the last statement is equivalent to saying that there is a reflexive and transitive relation R on A such that there exists a pair of elements say x_n and x_{n-1} of A with $x_n R x_{n-1}$ and for all other $x_i, x_j \varepsilon A$, $i \neq j$, $x_i \not R x_j$. The relation R is evidently a partial ordering relation and hence can be represented by a Hasse diagram of the following form:

$$\cdot x_1 \;\; \cdot x_2, \ldots, \cdot x_{n-2} \quad \begin{array}{c} x_n \\ \big/ \\ x_{n-1} \end{array}$$

Now a subset B of A is open if no element in B covers an element in $(A \sim B)$. The singleton **sets** $\{x_1\}$ through $\{x_{n-1}\}$ in the above H-diagram obviously satisfy this condition, and $\{x_n, x_{n-1}\}$ is also open. Since the union of any arbitrary number of open sets is open, we obtain 2^{n-1} open sets from the first n-1 singleton sets. The union of each of the 2^{n-2} open sets, obtained by arbitrary unions of the first n-2 singleton sets, with the set $\{x_n, x_{n-1}\}$ is also open. Now any other set cannot be open since it would contain x_n and not x_{n-1} and hence would not satisfy the necessary condition for openness. Adding 2^{n-1} and 2^{n-2} we get $3 \cdot 2^{n-2}$ open sets in the second largest topology on A. To get the number of elements in the third largest topology on A we would have x_n covering two elements in the above H-diagram, and proceeding in the manner above we obtain $5 \cdot 2^{n-3}$ open sets.

In general if an element say x_n of A covers the k elements x_{n-k} to x_{n-1} with each of the remaining n-k-1 elements being unrelated to any other element, we have 2^{n-1} open sets formed by taking arbitrary unions of the singleton sets $\{x_1\}$ through $\{x_{n-1}\}$. The sets formed by taking the union of the set $\{x_{n-k}, x_{n-k+1},\ldots,x_n\}$ with each of the 2^{n-k-1} sets formed by taking unions of the singleton sets $\{x_1\}, \{x_2\},\ldots\{x_{n-k-1}\}$ will also be open. Adding 2^{n-1} and 2^{n-k-1} we obtain $(2^k+1)\ 2^{n-k-1}$ open sets. Any other subset B of A will not be open, since it will contain x_n but will not contain at least one of the elements $x_{n-k}, x_{n-k+1},\ldots,x_{n-1}$. This will be the number of sets in the $k+1^{st}$ largest topology on A. Thus we have proved the following theorem.

Theorem 3.2.1. Let A be a finite non-void set consisting of n elements. Then the number of elements in the $(k+1)^{st}$, $k=0,1,\ldots,n-1$, largest topology for A is $(2^k+1)2^{n-k-1}$.

In closing we wish to remark that there is more than one such topology for each k. In fact there are $n\cdot\binom{n-1}{k}$ such families.

REFERENCES

[1] Berge, C., Theory of Graphs, Translated by Alison Doig, John Wiley and Sons, Inc., N.Y., 1962.

[2] Bhargava, T. N. and Ahlborn, T. J., On Digraph Topology, submitted for publication.

[3] Birkhoff, G., Lattice Theory, American Mathematical Society, New York City, 1940.

[4] Chatterji, S. D., On Counting Topologies, Technical Report No. 4, NASA Grant No. NsG-568, Kent, Ohio, 1964.

[5] Feller, W., An Introduction to Probability Theory and its Applications, Second Edition, Wiley, New York, 1957.

[6] Halmos, P., Naive Set Theory, Van Nostrand, Princeton, N.J., 1960.

[7] Kelley, J. L., General Topology, Van Nostrand, Princeton, N.J., 1955.

[8] Krishnamurthy, V., "On the Number of Topologies on a Finite Set", The American Mathematical Monthly, Buffalo, N. Y., V. 73, 1966.

[9] Szász, G., Introduction to Lattice Theory, Academic Press, New York, 1963.

[10] Rota, G., "The Number of Partitions of a Set", The American Mathematical Monthly, Buffalo, N.Y., V. 69, 1962.

[11] Vaidyanathaswany, R., Set Topology, Chelsea Publishing Co., New York, 1960.