PRESSURE STABILITY OF PROLATE SPHEROIDS

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SUMMARY

A theoretical investigation of prolate spheroids was conducted using the Donnell type equation for doubly curved shells. The results cover the range from a sphere to a cylinder, and show how the simplified form of Mushtari's relation is approached asymptotically from above within that range.

Theory and experiment are shown to be in reasonable agreement for some of the test data available. However, the simplified theoretical approach underestimated the strengths of shells with large axis ratios.

SYMBOLS

a, b	semimajor, semiminor axes of ellipse forming section of prolate spheroid, in.
D	plate stiffness, $Et^3/[12(1-\nu^2)]$, in-lb.
E	Young's modulus, psi.
E_s	secant modulus, psi.
E_f	tangent modulus, psi.
k	buckling coefficient
L	length of shell, in.
N	membrane loading, lb/in.
N_x, N_y	membrane loading along x and y axes at point of shell, lb/in.
p	hydrostatic pressure acting externally on shell, psi.
R	radius at shell equator, chosen equal to b, in.
r	N_x/N_y
R_x, R_y	radii in x and y directions at point of shell, in.
t	thickness of shell wall, in.
α	R_x/R
β	λ_x/λ_y
η	plasticity reduction factor
λ_x, λ_y	buckle half wavelengths in x and y directions
ν	Poisson's ratio
σ_x, σ_y	N_x/t, N_y/t, psi.

INTRODUCTION

Prolate spheroids are receiving attention as a possible candidate for externally pressurized shells because of the internal geometry superiority over spheres, in which a large proportion of the volume is of doubtful utility. Furthermore, prolate spheroids are capable of carrying larger external pressures than cylinders of the same diameter. This double advantage may offer weight savings in certain cases. However, the theory of instability of prolate spheroids is not yet applicable to weight minimization because of the difficulty of obtaining agreement of theory with experiment.

The double curvature shell Donnell equation of Ref. (1) was used to examine the nature of buckling of a prolate spheroid under external pressure in an attempt to understand the behavior as a function of the ellipse axis ratio, a/b, and to seek better correlation with theory than currently exists. In principle the shell Donnell equation is applicable to instability problems only if buckling begins in a region of nearly constant curvatures in two locally orthogonal directions. Despite this limitation, it was considered useful to explore the range from a sphere to a cylinder to gain the insight it offered. Consequently, no attempt was made to assess the accuracy of the method. As correlation with experiment indicates, however, it appears to be useful in the form depicted.

ELASTIC SHELL THEORY

A. Basic Equations

The basic axisymmetric shell Donnell equation is (Ref. 1)

$$D\nabla^8 w + t\nabla^4\left[\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right] + \frac{Et}{R^2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{\alpha}\frac{\partial^2}{\partial y^2}\right]^2 (w) = 0 \qquad (1)$$

Assume

$$w = w_o \sin(\pi x/\lambda_x)\sin(\pi y/\lambda_y) \qquad (2)$$

then

$$t\left[\frac{\sigma_x}{(\lambda_x)^2} + \frac{\sigma_y}{(\lambda_y)^2}\right] = D\pi^2\left[\frac{1}{(\lambda_x)^2} + \frac{1}{(\lambda_y)^2}\right]^2 + \frac{Et}{\pi^2 R^2}\left[\frac{\frac{1}{(\lambda_x)^2} + \frac{1}{\alpha}\frac{1}{(\lambda_y)^2}}{\frac{1}{(\lambda_x)^2} + \frac{1}{(\lambda_y)^2}}\right]^2 \qquad (3)$$

If

$$N = N_x = t\sigma_x = pR/2, \quad N_x/R_x + N_y/R_y = p, \quad R_x = \alpha R_y = \alpha R \qquad (4)$$

then

$$N_y = N(2 - \frac{1}{\alpha}) = t\sigma_y \qquad (5)$$

Consequently, if $\lambda_x = \beta\lambda_y$, then

$$\frac{N}{(\lambda_x)^2}\left[1 + (2 - \frac{1}{\alpha})\beta^2\right] = \frac{D\pi^2}{(\lambda_x)^4}(1+\beta^2)^2 + \frac{Et}{\pi^2 R^2}\left[\frac{1+\beta^2/\alpha}{1+\beta^2}\right]^2 \qquad (6)$$

If

$$N = t\sigma_x = k\frac{\pi^2 E t^3}{12(1-\nu^2)L^2} = k\pi^2 D/L^2 \qquad (7)$$

$$Z^2 = (L^4/R^2 t^2)(1-\nu^2) \qquad (8)$$

then the general relation for the buckling coefficient is obtained

$$k = \left[1 + (2 - \frac{1}{\alpha})\,\beta^2\right]^{-1} \left\{ \left(\frac{L}{\lambda_x^2}\right)^2 (1+\beta^2)^2 + \frac{12}{\pi^4} Z^2 \left(\frac{\lambda_x}{L}\right)^2 \left[\frac{1+\beta^2/\alpha}{1+\beta^2}\right]^2 \right\} \tag{9}$$

It is important to note that k is the buckling coefficient for the stress, σ_x, that would be parallel to the axis of a cylinder instead of the circumferential stress, σ_y, for which 2k would be the proper cylinder buckling coefficient. In general, $k_y = (2-1/\alpha)k$.

B. Limiting Cases

Case 1: Cylinder ($\alpha=\infty$)

In the moderate length range $\beta^2 >> 1$ and

$$2k = \beta^2 + \frac{12}{\pi^4} Z^2/\beta^6 \tag{10}$$

which leads to the well known result for minimum k (Ref. 2)

$$2k = (3^{1/4} + 3^{-3/4})(12Z^2/\pi^4)^{1/4} = 1.039Z^{1/2} \tag{11}$$

Case 2: Sphere ($\alpha = \beta = 1$)

$$2k = (L/\lambda_x)^2 + \frac{12}{\pi^4} Z^2 (\lambda_x/L)^2 \tag{12}$$

From the minimization process,

$$k = 2(12Z^2/\pi^4)^{1/2} = [4(3)^{1/2}/\pi^2]\,Z \tag{13}$$

resulting in the classical relation for a sphere (Refs. 1, 3)

$$\sigma_x = \sigma_y = pR/2t = [3(1-\nu^2)]^{-1/2}\,Et/R \tag{14}$$

Case 3: Mushtari's Prolate Spheroid (Ref. 4)

Mushtari's relation for a prolate spheroid may be obtained from Eq. (9) on the assumptions that $\beta^2 >> 1$

$$k = \frac{\alpha}{2\alpha - 1}\left\{\left(\frac{L\beta}{\lambda_x}\right)^2 + \frac{12}{\pi^4}\left(\frac{Z}{\alpha}\right)^2\left(\frac{\lambda_x}{L\beta}\right)^2\right\} \tag{15}$$

the minimum value of which is

$$k = 4\,(3)^{1/2}\,(2\alpha - 1)^{-1}\,Z/\pi^2 \tag{16}$$

which leads to Mushtari's relation for stress

$$pR/2t = \sigma = \left[3(1-\nu^2)\right]^{-1/2}\,(Et/R)(2\alpha-1)^{-1} \tag{17}$$

Case 4: General Prolate Spheroid

The buckling stress for a general prolate spheroid should lie between a sphere ($\alpha = \beta = 1$) and a cylinder (α, $\beta >> 1$). As a consequence, the results of Eqs. (16) and (17) are actually fortuitous in the case of the sphere and wrong in the case of the cylinder. This is demonstrated in Fig. 1 which contains k - Z curves for various values of α plotted from Eq. (9) to obtain the lower envelopes of the families of curves for given α values. The curves follow the cylinder curve at low Z, pass through a transition from the cylinder curve to Mushtari's limit at moderate Z, and converge from above to Mushtari's limit for large Z.

In this process, the lower envelope is obtained with $L/\lambda_x = 1$ in all cases of $\alpha > 1$. For the sphere ($\alpha = 1$), β does not figure in locating the limiting curve.

5837

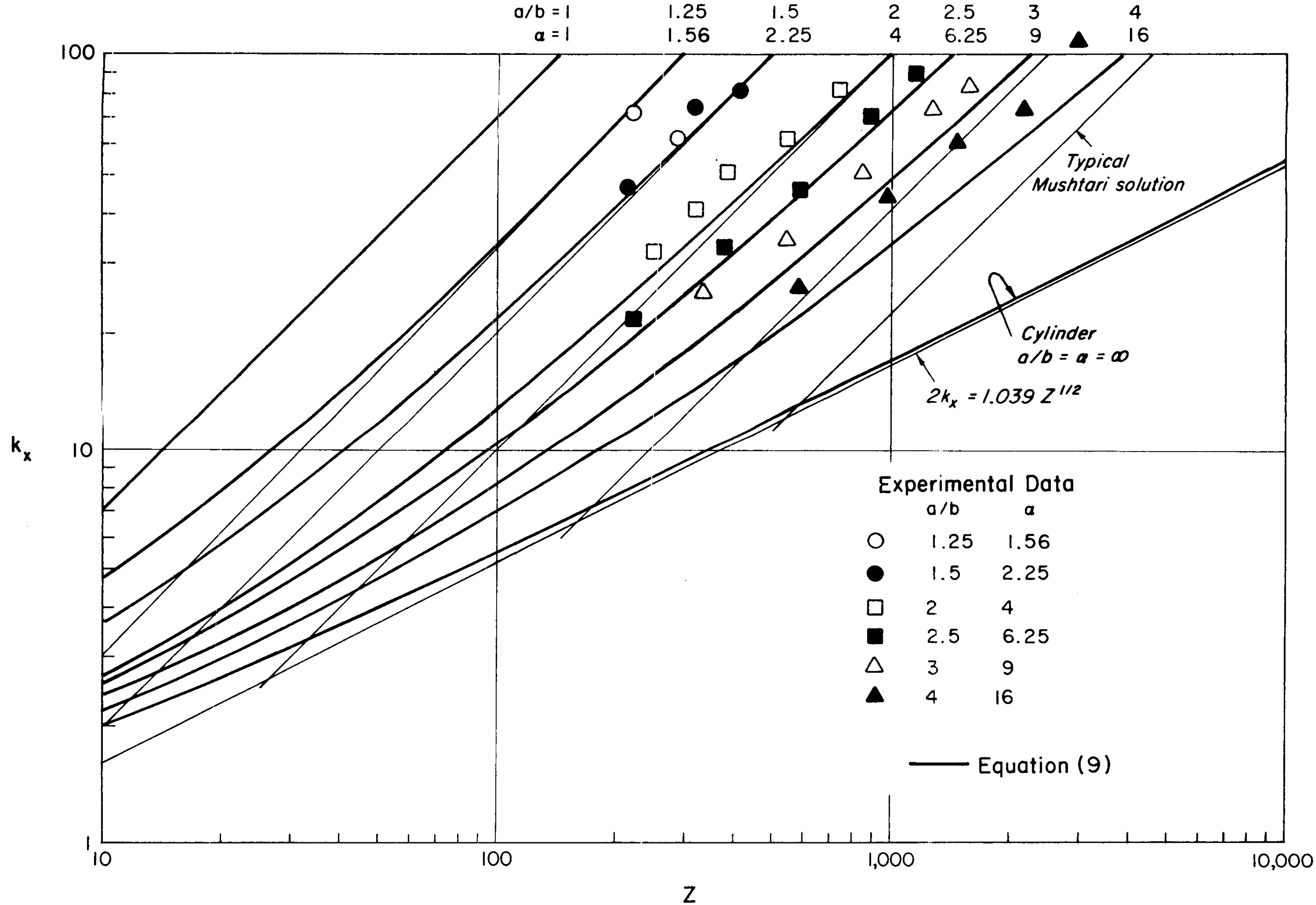

Figure 1. Comparison of Theory with Data of Healey on Plastic Spheroids

COMPARISON WITH EXPERIMENTAL DATA

Experiments were performed on externally pressurized isotropic prolate spheroid shells by Healey (Refs. 5 and 6) and by Nickell (Ref. 7). The data of Ref. 5, plotted in Fig. 1, show generally good agreement with theory at a/b near 1, but the experimental results approach 30 percent above Eq. (9) for slender spheroids. A similar situation is seen in Fig. 2a. This obviously requires more study, although the agreement is not bad considering the approximate nature of Eq. (9). The experimental values of k were obtained using Eqs. (7) and (8) together with the reported test pressures.

In regard to Nickell's data, the shells were half-length spheroids with transverse bulkheads at the minor axes. The assumption of simple support appeared reasonable since no rigid rotation restraint was provided to the shell by the bulkheads. Buckling occurred approximately midway between the nose and the bulkhead. The average thickness values were used to obtain empirical curves in Ref. 7 which also contained tables of measured thickness that made it possible to identify a range of minimum wall thicknesses for each spheroid. The k-Z results for both types of thickness data are shown in Fig. 2b. In view of the fact that $120 < R/t < 240$, the scatter is small.

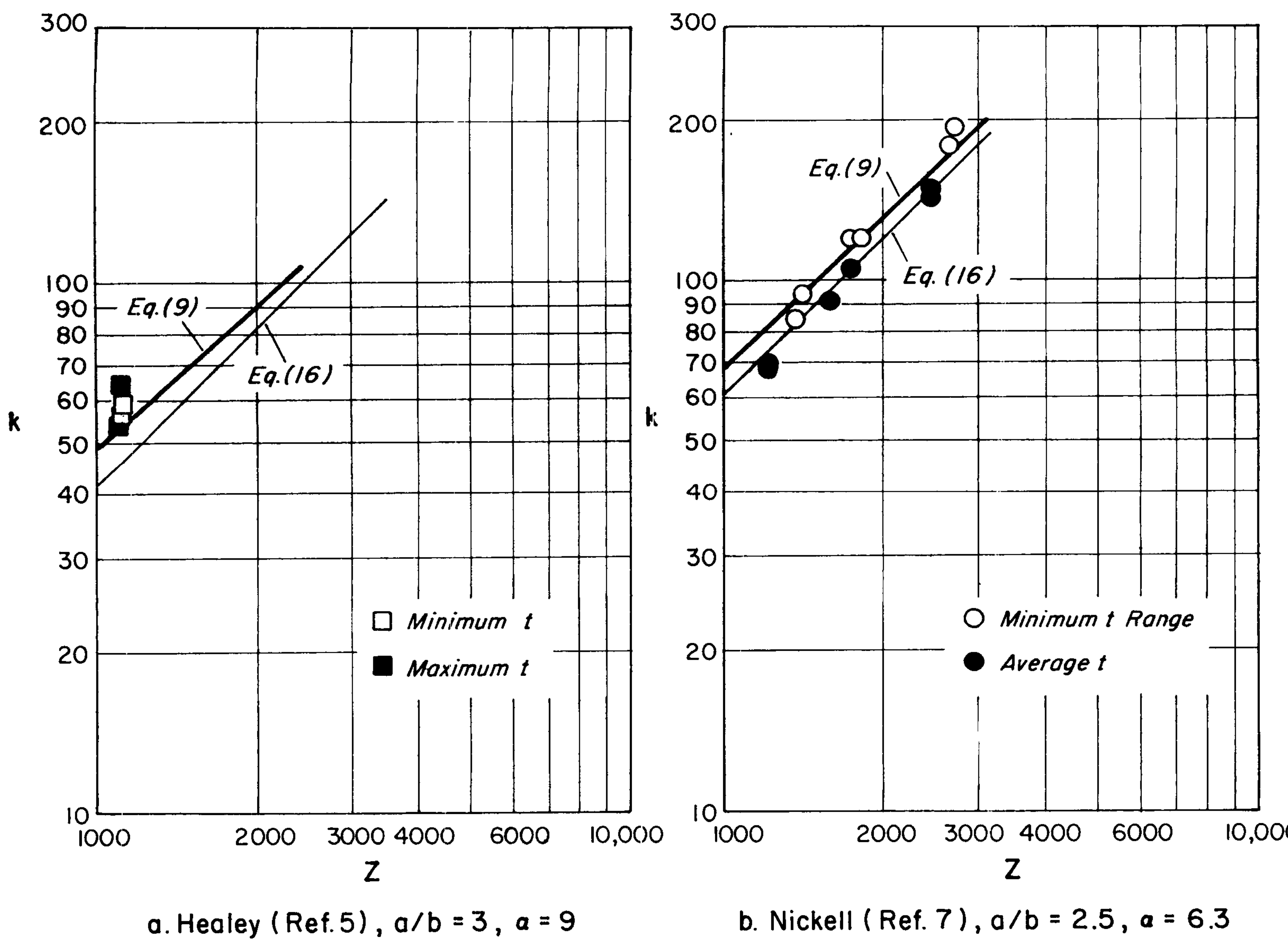

Figure 2. Comparison of Theory with Data in Metal Cylinders

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