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TAPPED DELAY LINE SIMULATION OF
RANDOMLY TIME-VARIANT CHANNELS

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TAPPED DELAY LINE SIMULATION OF RANDOM TIME-VARIANT CHANNELS

ERRATA

<u>Position</u>	<u>Correction</u>
Errors on pages 2-2 Line 6 from bottom	Dispersive also is dispersive
2-4 Line 6 from bottom	$\rho(r, \Delta)$ should be $\rho(r, \delta)$
3-3 Equation (3-9)	$\phi_H(r, l, \delta) = q(\delta)$ should be $\phi_H(r, 0, \delta) = q(\delta)$
4-1 Equation (4-3)	$d = \frac{\partial^2}{\partial t^2} \phi(r, t) \quad \text{should be}$ $d = \frac{\partial^2}{\partial t^2} \phi(r, t) \Big _{t = t_0}$
4-2 Equation (5-6)	$= BR_0 \int_{-BT_{\text{max}}}^{BT_{\text{max}}} \frac{\sin \pi (u-x)}{(u-x)} \frac{\sin \pi (u-x)}{(u-x)} du$ should be $= BR_0 \int_{-BT_{\text{max}}}^{BT_{\text{max}}} \frac{\sin \pi (u-x)}{\pi^2 (u-x)^2} \frac{\sin \pi (u-x)}{(u-x)} du$
5-6 Line 5 from bottom delete Line 5	Should read: "This evaluation has been done for both equations with the help of a digital computer."
5-7 Line 8 from top	$f = 0.5$ should be $f = 0.05$
6-1 Line 3 from bottom	Time-variant should be time-variant
8-2 Column Number 0 Row 6 from top	.000000 should be .900000

1. Representation of Time-Variant Channels

In this report, a theory for representation of randomly time-variant channels is outlined and applied to the development of a tapped delay line method of simulating such channels. The simulation method is due to Stein¹ and much of the notation of this report coincides with that of reference 1. It will be assumed that the channel under consideration can be represented as a time-variant linear system.

We will represent the transmitted signal $x'(t)$ as

$$x'(t) = \operatorname{Re}\{x(t)e^{j2\pi f_0 t}\} \quad (1-1)$$

where $x(t)$ is the complex envelope of $x'(t)$. It will be assumed that $x'(t)$ is a bandlimited signal with spectrum existing only in the range

$$(f_0 - \frac{B}{2}, f_0 + \frac{B}{2}).$$

The channel, being a linear time-variant system, is described by its time-variant impulse response $h(\tau, t)$, which can be written as

$$h(\tau, t) = \operatorname{Re}\{\beta(\tau, t)e^{j2\pi f_0 \tau}\} \quad (1-2)$$

$h(\tau, t)$ is defined as the response of the channel at time t to an impulse applied to the channel τ seconds earlier (i.e., at time $t-\tau$). $\beta(\tau, t)$, the complex envelope of $h(\tau, t)$, will be called the equivalent low-pass impulse response of the channel.

If the signal $x'(t)$ is applied to the input of the channel, then the response of the channel will be $y'(t)$, where $y'(t)$ is given by:

$$y'(t) = \int_{-\infty}^{\infty} h(t-\tau, t) x'(\tau) d\tau \quad (1-3)$$

Now $y'(t)$ can be written in complex envelope form

$$y'(t) = \text{Re}\{y(t)e^{j2\pi f_0 t}\} \quad (1-4)$$

It can be shown that when the bandwidth B is small with respect to the frequency f_0 , $y(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} \beta(t-\tau, t) x(\tau) d\tau \quad (1-5)$$

If we consider the spectra of $x(t)$ and $y(t)$, given by

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \quad (1-6)$$

and

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft} dt \quad (1-7)$$

and define the channel low-pass transfer function as

$$H(f, t) = \int_{-\infty}^{\infty} \beta(\tau, t)e^{-j2\pi f\tau} d\tau \quad (1-8)$$

then we have

$$y(t) = \int_{-\infty}^{\infty} X(f) H(f, t)e^{+j2\pi ft} df \quad (1-9)$$

Note that $H(f, t)$ is the amplitude of the equivalent low-pass channel's response at time t to a sinusoidal excitation at frequency f . This can be shown by noting that if $x(\tau)$ in (1-5) is $e^{j2\pi f\tau}$, then $y(t)$ becomes

$$y(t) = H(f, t)e^{j2\pi ft}$$

It will be assumed that $\beta(\tau, t)$ is a sample function from a complex, zero-mean stationary Gaussian random process. Therefore, the statistics of $\beta(\tau, t)$ are completely described by its autocorrelation function with respect to each of its two variables. The correlation function corresponding to $\beta(\tau, t)$ is given by

$$R_{\beta}(\tau, t, \mu, \delta) = \overline{\beta(\tau, t) \beta^*(\tau + \mu, t + \delta)} \quad (1-10)$$

We will assume that the channel to be modeled obeys probability laws which are independent of the variable t (i.e., $\beta(\tau, t)$ is wide-sense stationary in the variable t). That is, only the time difference and not the actual time of evaluation is pertinent in the evaluation of R_{β} . Thus we can write

$$R_{\beta}(\tau, t, \mu, \delta) = R_{\beta}(\tau, \mu, \delta)$$

We can define a correlation function for $H(f, t)$ in a manner similar to (1-10).

$$R_H(f, t, \Omega, \delta) = \overline{H(f, t) H^*(f + \Omega, t + \delta)} \quad (1-11)$$

R_{β} can be related to R_H as follows:

$$\begin{aligned} R_{\beta}(\tau, \mu, \delta) &= \overline{\beta(\tau, t) \beta^*(\tau + \mu, t + \delta)} \\ &= \overline{\int_{-\infty}^{\infty} H(f, t) e^{j2\pi f \tau} df \int_{-\infty}^{\infty} H^*(f_1, t + \delta) e^{-j2\pi f_1(\tau + \mu)} df_1} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{H(f, t) H^*(f_1, t + \delta)} e^{j2\pi \tau(f - f_1)} e^{-j2\pi f_1 \mu} df df_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_H(f, t, f_1 - f, \delta) e^{j2\pi \tau(f - f_1)} e^{-j2\pi f_1 \mu} df df_1 \end{aligned} \quad (1-12)$$

We note that since R_{β} is independent of t , then R_H must also be independent of t . Thus we have

$$R_{\beta}(\tau, \mu, \delta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_H(f, f_1 - f, \delta) e^{j2\pi \tau(f_1 - f)} e^{-j2\pi f_1 \mu} df df_1$$

Let us now assume that the statistical properties of the channel being considered are such that R_H depends only on the frequency difference between the two frequency variables of interest and not the actual values of the frequency variables (i.e., $H(f, t)$ is wide-sense stationary in the variable f).

This means that $R_H(f, f_1 - f, \delta)$ is only a function of $f_1 - f$ and not of f explicitly.

Thus

$$R_H(f, f_1 - f, \delta) = R_H(f_1 - f, \delta)$$

Therefore R_B can be written

$$R_B(\tau, \mu, \delta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_H(f_1 - f, \delta) e^{-j2\pi\tau(f_1 - f)} e^{-j2\pi f_1 \mu} df df_1 \quad (1-13)$$

Letting $\Omega = f_1 - f$, $d\Omega = -df$

$$\begin{aligned} R_B(\tau, \mu, \delta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_H(\Omega, \delta) e^{-j2\pi\Omega\tau} e^{-j2\pi f_1 \mu} d\Omega df_1 \\ &= \int_{-\infty}^{\infty} e^{-j2\pi f_1 \mu} df_1 \int_{-\infty}^{\infty} R_H(\Omega, \delta) e^{-j2\pi\Omega\tau} d\Omega \end{aligned}$$

Thus

$$R_B(\tau, \mu, \delta) = \delta(\mu) \rho(\tau, \delta) \quad (1-14)$$

$$\text{where } \rho(\tau, \delta) = \int_{-\infty}^{\infty} R_H(\Omega, \delta) e^{-j2\pi\Omega\tau} d\Omega \quad (1-15)$$

Physically we may envision this result as indicating that the channel scatterer or path which introduces delay τ in the transmitted signal has an associated transmission gain which is uncorrelated with that of any different delay.²

II. Tapped Delay Line Simulation of Time-Variant Channels

In this section a basis for tapped delay line simulation of the equivalent low-pass channel is developed. We recall that $x'(t)$, the transmitted signal, was defined to be a bandlimited signal whose frequency components exist only in the range $(f_0 - \frac{B}{2}, f_0 + \frac{B}{2})$. This means that $x(t)$ (see Eq. (1-1)) is also a bandlimited signal existing in the range $(-\frac{B}{2}, +\frac{B}{2})$. Thus we can write

$$x(t) = \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \frac{\sin \pi B\left(t - \frac{m}{B}\right)}{\pi B\left(t - \frac{m}{B}\right)} \quad (2-1)$$

Therefore

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} \beta(t-\tau, t) x(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \beta(t-\tau, t) \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \frac{\sin \pi B\left(\tau - \frac{m}{B}\right)}{\pi B\left(\tau - \frac{m}{B}\right)} d\tau \\ y(t) &= \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \int_{-\infty}^{\infty} \beta(t-\tau, t) \frac{\sin \pi B\left(\tau - \frac{m}{B}\right)}{\pi B\left(\tau - \frac{m}{B}\right)} d\tau \end{aligned}$$

letting

$$u = \tau - \frac{m}{B}, \quad d\tau = du$$

$$\begin{aligned} y(t) &= \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \int_{-\infty}^{\infty} \beta\left(t-u-\frac{m}{B}, t\right) \frac{\sin \pi B u}{\pi B u} du \\ y(t) &= \frac{1}{B} \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \hat{\beta}\left(t - \frac{m}{B}, t\right) \end{aligned} \quad (2-2)$$

where

$$\hat{\beta}(z, t) \equiv \int_{-\infty}^{\infty} \beta(z-u, t) \frac{\sin \pi B u}{\pi u} du = \int_{-\infty}^{\infty} \beta(u, t) \frac{\sin \pi B(z-u)}{\pi(z-u)} du \quad (2-3)$$

Now note that $\hat{\beta}$ is the convolution of a $\frac{\sin \pi x}{x}$ function and $\beta(\tau, t)$. Therefore $\hat{\beta}$ is a bandlimited function and we can use the sampling representation for it

$$\hat{\beta}(z, t) = \sum_{k=-\infty}^{\infty} \hat{\beta}\left(\frac{k}{B}, t\right) \frac{\sin \pi B(z - \frac{k}{B})}{\pi B(z - \frac{k}{B})} \quad (2-4)$$

Therefore

$$\begin{aligned} y(t) &= \frac{1}{B} \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \hat{\beta}\left(t - \frac{m}{B}, t\right) \\ &= \frac{1}{B} \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \sum_{k=-\infty}^{\infty} \hat{\beta}\left(\frac{k}{B}, t\right) \frac{\sin \pi B(t - \frac{m}{B} - \frac{k}{B})}{\pi B(t - \frac{m}{B} - \frac{k}{B})} \\ &= \frac{1}{B} \sum_{k=-\infty}^{\infty} \hat{\beta}\left(\frac{k}{B}, t\right) \sum_{m=-\infty}^{\infty} x\left(\frac{m}{B}\right) \frac{\sin \pi B(t - \frac{m}{B} - \frac{k}{B})}{\pi B(t - \frac{m}{B} - \frac{k}{B})} \\ y(t) &= \frac{1}{B} \sum_{k=-\infty}^{\infty} \hat{\beta}\left(\frac{k}{B}, t\right) x\left(t - \frac{k}{B}\right) \end{aligned} \quad (2-5)$$

Now since we desire to associate $\beta(\tau, t)$ with a physical dispersive channel, we expect that $\beta(\tau, t)$ should be zero for some value of τ greater than $\tau_{\max}$, say. Therefore $\hat{\beta}(\tau, t)$ should also be essentially time-limited in that there should be some quantity T_m such that the energy

$\int_{|\tau| > T_m} |\hat{\beta}(\tau, t)|^2 d\tau$ should be negligible compared to the energy

$\int_{|\tau| < T_m} |\hat{\beta}(\tau, t)|^2 d\tau$. Therefore, the following approximation is valid*

* An actual channel, being physically realizable, will have some average delay T_0 such that $\beta(\tau, t)$ will be nonzero in the range $(T_0 - T_m, T_0 + T_m)$, $T_0 \geq T_m$. Since the value of T_0 has no effect on the shape of the received waveforms, it has been assumed to be zero for convenience.

$$y(t) \approx \frac{1}{B} \sum_{m=-[BT_m]}^{[BT_m]} \hat{\beta}\left(\frac{m}{B}, t\right) x\left(t - \frac{m}{B}\right) \quad (2-6)$$

where $[w]$ is the largest integer in w .

Now if we define $b_m(t) \equiv \hat{\beta}\left(\frac{m}{B}, t\right)$ and let $\Delta = \frac{1}{B}$ we can use the following block diagram for a representation of the computing of $y(t)$ from $x(t)$

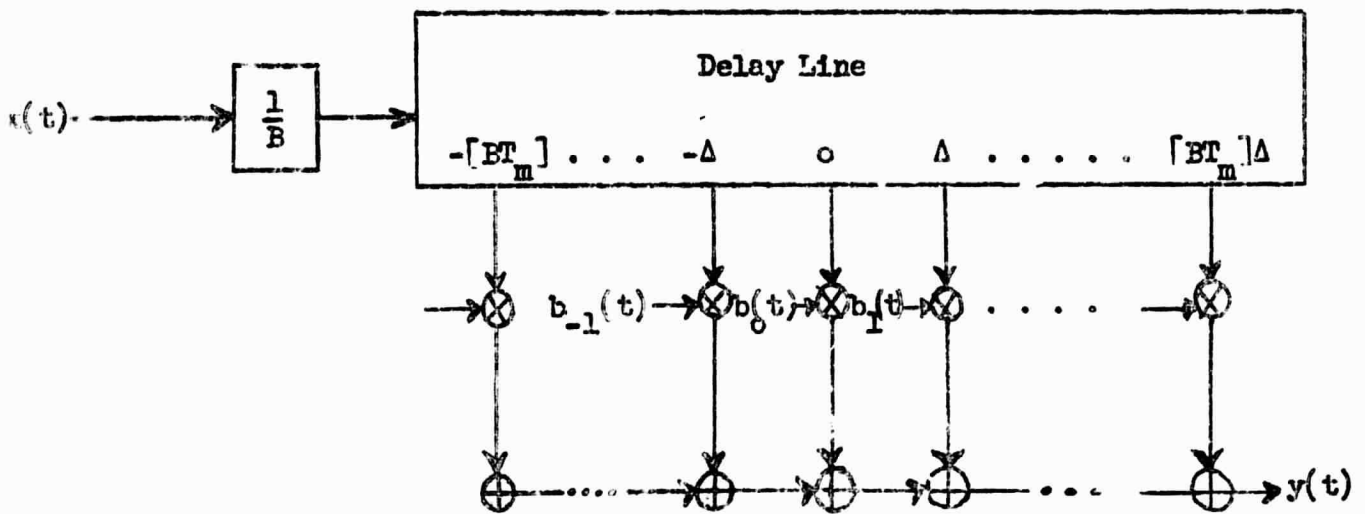


Figure 1

This diagram forms the basis for the tapped delay line simulation of the time-variant channel.

As was stated previously, the statistical nature of $\beta(\tau, t)$ is completely specified by its correlation function. Now in order to accurately simulate the channel with a tapped delay line model, it is necessary that the $b_m(t)$'s possess the statistics which will produce the proper statistical characterization of $\beta(\tau, t)$. The statistical properties of the $b_m(t)$'s will now be determined from the statistics of $\beta(\tau, t)$.

We first note that the $b_m(t)$'s are linear functionals defined on $\beta(\tau, t)$. Thus, they are stationary Gaussian random processes. It therefore suffices to compute their correlation functions to complete their statistical description. Recall that the gain function for the m th tap at time t was found to be (see Eq. (2-6)).

$$b_m(t) = \beta(m\Delta, t) = \int_{-\infty}^{\infty} \beta(u, t) \frac{\sin \pi B(m\Delta - u)}{\pi(m\Delta - u)} du$$

Let us now define the correlation function associated with these tap gains as follows

$$\begin{aligned} \rho_{mn}(\delta) &= b_m(t) b_n^*(t + \delta) = \beta(m\Delta, t) \beta^*(n\Delta, t + \delta) \\ &= \int_{-\infty}^{\infty} \beta(u, t) \frac{\sin \pi B(m\Delta - u)}{\pi(m\Delta - u)} du \int_{-\infty}^{\infty} \beta^*(z, t + \delta) \frac{\sin \pi B(n\Delta - z)}{\pi(n\Delta - z)} dz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta(u, t) \beta^*(z, t + \delta) \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - z)}{\pi(m\Delta - u) \pi(n\Delta - z)} dz du \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\beta}(u, z - u, \delta) \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - z)}{\pi(m\Delta - u) \pi(n\Delta - z)} dz du \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(z - u) \rho(u, \delta) \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - z)}{\pi(m\Delta - u) \pi(n\Delta - z)} dz du \\ \rho_{mn}(\delta) &= \int_{-\infty}^{\infty} \rho(u, \delta) \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - u)}{\pi^2(m\Delta - u)(n\Delta - u)} du \quad (2-7) \end{aligned}$$

Now for values of m and n which are not near to each other, the product of the $\frac{\sin x}{x}$ functions is small and the value of $\rho_{mn}(\delta)$ should be small. For values of m and n which are near to each other (say $|m - n| = 1$ or 2), the orthogonality of the $\frac{\sin x}{x}$ functions will cause the value of $\rho_{mn}(\delta)$ to be small provided $\rho(\tau, \Delta)$ is reasonably constant over a range of width 5Δ to 10Δ near $\tau = m\Delta$ or $n\Delta$. We can expect that $\rho(\tau, \delta)$ will be reasonably constant over regions of width 5Δ to 10Δ if the total multipath delay spread of the channel is much larger than Δ . This approximation is not as valid for taps near the ends of the delay line as for taps near the center; however, the strengths of the tap gain functions at the taps near the ends will normally

be much smaller than for taps near the middle. This demonstrates that for many cases of interest, a tapped delay line model can be developed using the assumption that the fading at each tap is uncorrelated with the fading at any other tap.

If a more exact representation is desired, or the assumption of independent fading at each tap is not valid, then the correlations $\rho_{mn}(\delta)$ must be computed by equation (2-7) above. Then, if the assumption that $\rho(\tau, \delta)$ is separable is made (i.e., $\rho(\tau, \delta) = r(\tau) \cdot q(\delta)$), a straightforward method of creating tap gain functions with the proper correlations from a set of independent Gaussian random processes can be found. The assumption that $\rho(\tau, \delta)$ is separable can be shown to be equivalent to the assumption that the velocity distribution of the scatterers is identical for each group of scatterers contributing a given delay τ .² It also appears that the present state of theoretical and experimental knowledge concerning existing scatter-multipath channels is not sufficiently well developed to provide a more detailed indication of the precise form of the channel correlation function.

With the assumption that $\rho(\tau, \delta)$ is separable, we find

$$\rho(\tau, \delta) = r(\tau) \cdot q(\delta) \quad (2-8)$$

and

$$\rho(\tau, 0) = r(\tau) \cdot q(0)$$

Without loss of generality, we can let $q(0) = 1$. Therefore, $r(\tau)$ must be positive for all τ . Now from equation (2-7) we find that the correlations for the tap multipliers are of the form

$$\begin{aligned} \rho_{mn}(\delta) &= \int_{-\infty}^{\infty} \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - u)}{\pi^2 (m\Delta - u)(n\Delta - u)} \rho(u, \delta) du \\ &= q(\delta) \int_{-\infty}^{\infty} \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - u)}{\pi^2 (m\Delta - u)(n\Delta - u)} r(u) du \end{aligned}$$

$$= q(\delta) r_{mn} \quad (2-9)$$

where

$$r_{mn} = \int_{-\infty}^{\infty} \frac{\sin \pi B(m\Delta - u) \sin \pi B(n\Delta - u)}{\pi^2 (m\Delta - u)(n\Delta - u)} r(u) du \quad (2-10)$$

This means that all the tap multiplier auto- and cross-correlation functions have the same functional dependence on δ . Therefore, the tap multiplier functions have identical forms for their auto- and cross-power spectral density functions. Thus, the set of tap multiplier functions can be generated from a set of $2[BT_m] + 1$ independent Gaussian random processes whose spectral densities are all identical to those of the $b_m(t)$'s by forming suitable linear combinations of these processes. The appropriate linear combination can be determined in the following manner. Let the independent processes to be combined be $a_i(t)$, $i = -[BT_m], \dots, -1, 0, 1, \dots, [BT_m]$. Since the $a_i(t)$ are independently generated, we will have

$$a_m(t) a_n^*(t+\delta) = \delta_{mn} q(\delta) \quad (2-11)$$

$$\delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

Here we have assumed that $a_i^2(t) = 1$ for convenience. As stated above, the $b_m(t)$'s are obtained from a linear combination of the $a_i(t)$'s. Thus we have

$$b_m(t) = \sum_{k=-[BT_m]}^{[BT_m]} c_{mk} a_k(t) \quad (2-12)$$

where the c_{mk} 's will be complex constants. The correlation of the $b_m(t)$'s can now be written as

$$\rho_{mn}(\delta) = b_m(t) b_n^*(t+\delta) = \sum_k \sum_j c_{mk} c_{nj}^* a_k(t) a_j^*(t+\delta)$$

$$= \sum_k \sum_j c_{nk} c_{nj}^* \delta_{k,j} q(\delta)$$

$$\rho_{mn}(\delta) = q(\delta) \left[\sum_k c_{mk} c_{nk}^* \right] \quad (2-13)$$

Now from equation (2-9) we know that

$$\rho_{mn}(\delta) = r_{mn} q(\delta)$$

Thus

$$r_{mn} = \sum_k c_{mk} c_{nk}^* \quad \text{for all } m, n \quad (2-14)$$

Now let us define the following matrices*:

$$\begin{aligned} R &= [r_{mn}] \\ C &= [c_{mn}] \\ C^+ &= (C^T)^* \quad (\text{conjugate transpose}) \end{aligned} \quad (2-15)$$

Then equation (2-14) becomes

$$R = CC^+ \quad (2-16)$$

Now from (2-10) it can be seen that since $r(\tau)$ is real, the r_{mn} are also.

Furthermore, it can be seen that $r_{mn} = r_{nm}$. Hence, R is a symmetric matrix and is therefore Hermitian. Therefore, (2-13) will always have a solution, but the solution C is not unique. Of the infinity of solutions C , we can identify one which is a real matrix (implying the c_{ij} will be only gain factors and will not require phase shifts). Since R is real, there exists a matrix O which diagonalizes R . That is

$$O^+ R O = \Lambda \quad (2-17)$$

*Note that these definitions do not conform to the common definitions of matrices in that the subscripts m and n range over negative as well as positive values (e.g. the upper left-hand corner element of R will have the subscripts $-[BT_m]$, $-[BT_m]$). Obviously, this fact does not effect any of the matrix operations involved.

Where Λ is a diagonal matrix whose elements λ_i are the eigenvalues of R .

O can be shown to be the matrix whose columns are the eigenvectors of R . Now, since R is a covariance matrix, it is positive semidefinite, and therefore all its eigenvalues are positive and real. Thus, one solution for C is given by

$$C = O \Lambda^{\frac{1}{2}} O^+ \quad (2-18)$$

where $\Lambda^{\frac{1}{2}}$ is the diagonal matrix whose entries are $\sqrt{\lambda_i}$. Thus, the weighting coefficients required to produce the tap multiplier functions from a set of independent Gaussian random processes can be found either analytically or by numerical methods.

III. Statistical Properties of Tapped Delay Line Systems

In the previous section we have discussed the statistical properties required of a tapped delay line system which is to model a given time-variant channel. It is now of interest to determine how closely such a tapped delay line system can approximate the given time-variant system. As a basis for a study of this question, we will now investigate several statistical properties of a general tapped delay line system. This analysis will be used in Section V to provide an indication of how effective the tapped delay line method is for simulation of a certain idealized tropospheric scatter communication channel.

It is of interest to compute the time-frequency correlation function for the tapped delay line system. This correlation function is defined as

$$\phi_{\beta}(\tau, \tau, t, \delta) = E[\beta(\tau, t) \beta^*(\tau + \tau, t + \delta)] \quad (3-1)$$

Now for the tapped delay line, $\beta(\tau, t)$ is given by

$$\beta(\tau, t) = \sum_{n=-M}^M b_n(t) \delta(\tau - n\Delta) \quad (3-2)$$

where we have let $[BT_m] = M$

Thus

$$\begin{aligned} \phi_{\beta}(\tau, \tau, t, \delta) &= E[\beta(\tau, t) \beta^*(\tau + \tau, t + \delta)] \\ &= E\left[\sum_{n=-M}^M b_n(t) \delta(\tau - n\Delta) \sum_{k=-M}^M b_k^*(t + \delta) \delta(\tau + \tau - k\Delta) \right] \\ &= \sum_{n=-M}^M \sum_{k=-M}^M E[b_n(t) b_k^*(t + \delta)] \delta(\tau - n\Delta) \delta(\tau + \tau - k\Delta) \end{aligned} \quad (3-3)$$

Now $E[b_n(t) b_k^*(t + \delta)]$ was shown in Section II to be equal to

$$E[b_n(t) b_k^*(t + \delta)] = q(\delta) r_{nk}$$

Thus

$$\Phi_\beta(\tau, \epsilon, t, \delta) = q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \delta(\tau - n\Delta) \delta(\tau + \epsilon - k\Delta) \quad (3-4)$$

Note that the product $\delta(\tau - n\Delta) \delta(\tau + \epsilon - k\Delta)$ is a product of delta functions which is zero everywhere on the (τ, ϵ) plane except at $(\tau, \epsilon) = (n\Delta, k\Delta - \tau)$ or $(\tau, \epsilon) = (n\Delta, (k-n)\Delta)$. Thus it is equivalent to the product $\delta(\tau - n\Delta) \delta(\epsilon - (k-n)\Delta)$.

Thus

$$\Phi_\beta(\tau, \epsilon, t, \delta) = q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \delta(\tau - n\Delta) \delta(\epsilon - (k-n)\Delta)$$

We note that Φ_β is independent of t , so we can write

$$\Phi_\beta(\tau, \epsilon, \delta) = q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \delta(\tau - n\Delta) \delta(\epsilon - (k-n)\Delta) \quad (3-5)$$

The frequency correlation function for the tapped delay line system can now be found. Recall from Eq (1-8) of Section I that the time variant transfer function associated with a system is given by

$$H(f, t) = \int_{-\infty}^{\infty} \beta(\tau, t) e^{-j2\pi f\tau} d\tau \quad (3-6)$$

Now the frequency correlation function associated with the tapped delay line is defined as

$$\Phi_H(f, \Omega, t, \delta) = E[H(f, t) H^*(f + \Omega, t + \delta)] \quad (3-7)$$

Substituting (3-6) into (3-7), we have (compare with Eq. (1-12))

$$\begin{aligned}
\phi_H(f, \Omega, t, \delta) &= E \left[\int_{-\infty}^{\infty} \beta(\tau, t) e^{-j2\pi f \tau_1} d\tau_1 \int_{-\infty}^{\infty} \beta^*(\tau_2, t+\delta) e^{j2\pi(f+\Omega)\tau_2} d\tau_2 \right] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\beta(\tau, t) \beta^*(\tau_2, t+\delta)] e^{-j2\pi f \tau_1} e^{j2\pi(f+\Omega)\tau_2} d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{\beta}(\tau_1, \tau_2 - \tau_1, t, \delta) e^{-j2\pi f \tau_1} e^{j2\pi(f+\Omega)\tau_2} d\tau_1 d\tau_2
\end{aligned} \tag{3-8}$$

Thus ϕ_H and ϕ_{β} are a two-dimensional Fourier transform pair.

Substituting (3-4) into (3-8), we find

$$\begin{aligned}
\phi_H(f, \Omega, t, \delta) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \delta(\tau_1 - n\Delta) \delta(\tau_2 - k\Delta) \\
&\quad e^{-j2\pi f \tau_1} e^{j2\pi(f+\Omega)\tau_2} d\tau_1 d\tau_2 \\
&= q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \int_{-\infty}^{\infty} \delta(\tau_1 - n\Delta) e^{-j2\pi f \tau_1} d\tau_1 \\
&\quad \int_{-\infty}^{\infty} \delta(\tau_2 - k\Delta) e^{j2\pi(f+\Omega)\tau_2} d\tau_2 \\
&= q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} e^{-j2\pi n\Delta f} e^{j2\pi k\Delta(f+\Omega)} \\
\phi_H(f, \Omega, t, \delta) &= q(\delta) \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} e^{-j2\pi\Delta(n-k)f} e^{j2\pi\Delta k\Omega}
\end{aligned} \tag{3-9}$$

Note that ϕ_H can be considered a two-dimensional Fourier series in f and Ω .

In connection with the measurement of the frequency correlation functions it is of interest to compute another statistical quantity. To understand the need for this quantity, let us consider the experimental method which would be used to determine the frequency correlation function for a channel. The method is shown in Figure 2.

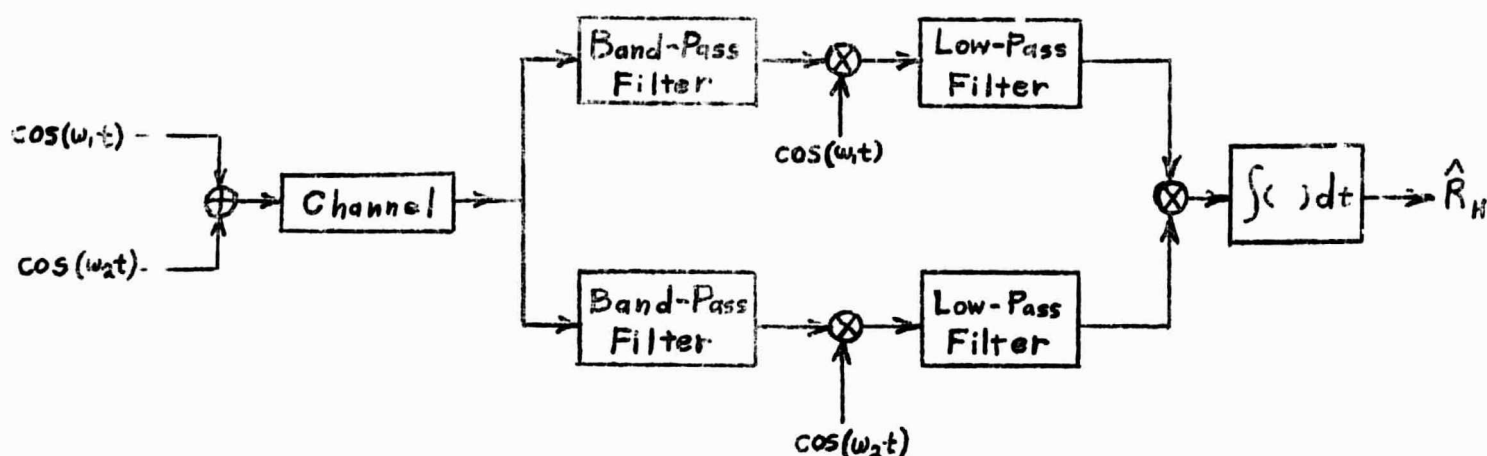


Figure 2

In this experimental system, two sinusoidal signals are transmitted through the channel and the responses to these excitations are separated at the channel output. The amplitudes of the two channel responses are then cross-correlated to produce an estimate of the channel's frequency correlation function for frequency separation $\omega_2 - \omega_1$. To perform this experiment with a low-pass equivalent channel model, the excitation functions must be $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$. The response to such an excitation will be (see Eq. (1-5)).

$$\begin{aligned}
 y_1(t) &= \int_{-\infty}^{\infty} \beta(\tau, t) e^{j\omega_1(t-\tau)} d\tau \\
 &= e^{j\omega_1 t} \int_{-\infty}^{\infty} \beta(\tau, t) e^{-j\omega_1 \tau} d\tau \\
 &= A_1(t) e^{j\omega_1 t}
 \end{aligned} \tag{3-10}$$

where

$$A_1(t) = \int_{-\infty}^{\infty} \beta(\tau, t) e^{-j\omega_1 \tau} d\tau = R(\omega_1/2\pi, t) \tag{3-11}$$

is the amplitude of the channel's response to the excitation $e^{j\omega_1 t}$. The above experiment to determine the frequency correlation function of the

channel will yield the quantity $E[\text{Re}\{A_1(t)\} \text{Re}\{A_2(t)\}]$. For the tapped delay line system, $E[\text{Re}\{A_1(t)\} \text{Re}\{A_2(t)\}]$ can be computed as follows. We note that

$$\begin{aligned} \text{Re}\{A_1(t)\} &= \text{Re}\left\{ \int_{-\infty}^{\infty} \beta(\tau, t) e^{-j\omega_1 \tau} d\tau \right\} \\ &= \text{Re}\left\{ \int_{-\infty}^{\infty} [\hat{\beta}(\tau, t) + j \tilde{\beta}(\tau, t)] [\cos \omega_1 \tau - j \sin \omega_1 \tau] d\tau \right\} \end{aligned}$$

where $\hat{\beta}(\tau, t) = \text{Re}\{\beta(\tau, t)\}$ and $\tilde{\beta}(\tau, t) = \text{Im}\{\beta(\tau, t)\}$

$$\begin{aligned} \text{Re}\{A_1(t)\} &= \text{Re}\left\{ \int_{-\infty}^{\infty} [\hat{\beta}(\tau, t) \cos \omega_1 \tau + \tilde{\beta}(\tau, t) \sin \omega_1 \tau + j \tilde{\beta}(\tau, t) \cos \omega_1 \tau \right. \\ &\quad \left. - j \hat{\beta}(\tau, t) \sin \omega_1 \tau] d\tau \right\} \\ &= \int_{-\infty}^{\infty} [\hat{\beta}(\tau, t) \cos \omega_1 \tau + \tilde{\beta}(\tau, t) \sin \omega_1 \tau] d\tau \end{aligned} \quad (3-12)$$

We note that for the tapped delay line system

$$\beta(\tau, t) = \sum_{n=-M}^M b_n(t) \delta(\tau - n\Delta)$$

Thus

$$\hat{\beta}(\tau, t) = \sum_{n=-M}^M \hat{b}_n(t) \delta(\tau - n\Delta) \quad (3-13a)$$

$$\tilde{\beta}(\tau, t) = \sum_{n=-M}^M \tilde{b}_n(t) \delta(\tau - n\Delta) \quad (3-13b)$$

where $\hat{b}_n(t) = \text{Re}\{b_n(t)\}$ and $\tilde{b}_n(t) = \text{Im}\{b_n(t)\}$

We now have

$$\text{Re}\{A_1(t)\} = \int_{-\infty}^{\infty} \left[\sum_{n=-M}^M \hat{b}_n(t) \delta(\tau - n\Delta) \cos \omega_1 \tau + \sum_{n=-M}^M \tilde{b}_n(t) \delta(\tau - n\Delta) \sin \omega_1 \tau \right] d\tau$$

$$= \sum_{n=-M}^M \left[\hat{b}_n(t) \cos(\omega_1 n \Delta) + \tilde{b}_n(t) \sin(\omega_1 n \Delta) \right] \quad (3-14)$$

Thus

$$\begin{aligned} E \left[\text{Re} \{ A_1(t) \} \text{Re} \{ A_2(t) \} \right] &= E \left[\left(\sum_{n=-M}^M \left[\hat{b}_n(t) \cos(\omega_1 n \Delta) + \tilde{b}_n(t) \sin(\omega_1 n \Delta) \right] \right) \right. \\ &\quad \left. \left(\sum_{k=-M}^M \left[\hat{b}_k(t) \cos(\omega_2 k \Delta) + \tilde{b}_k(t) \sin(\omega_2 k \Delta) \right] \right) \right] \end{aligned}$$

Now the tapped delay line system is constructed such that

$$E \left[\hat{b}_n(t) \hat{b}_k(t) \right] = E \left[\tilde{b}_n(t) \tilde{b}_k(t) \right] = \frac{r_{nk}}{2} \quad \text{all } n, k$$

$$E \left[\hat{b}_n(t) \tilde{b}_k(t) \right] = E \left[\tilde{b}_n(t) \hat{b}_k(t) \right] = 0 \quad \text{all } n, k$$

Thus

$$\begin{aligned} E \left[\text{Re} \{ A_1(t) \} \text{Re} \{ A_2(t) \} \right] &= \sum_{n=-M}^M \sum_{k=-M}^M \left(E \left[\hat{b}_n(t) \hat{b}_k(t) \right] \cos(\omega_1 n \Delta) \right. \\ &\quad \left. \cos(\omega_2 k \Delta) + E \left[\tilde{b}_n(t) \tilde{b}_k(t) \right] \sin(\omega_1 n \Delta) \sin(\omega_2 k \Delta) + 0 + 0 \right) \\ &= \sum_{n=-M}^M \sum_{k=-M}^M \frac{r_{nk}}{4} \left[\cos((\omega_1 n + \omega_2 k) \Delta) + \cos((\omega_1 n - \omega_2 k) \Delta) \right] \\ &\quad + \frac{r_{nk}}{4} \left[\cos((\omega_1 n - \omega_2 k) \Delta) - \cos((\omega_1 n + \omega_2 k) \Delta) \right] \\ E \left[\text{Re} \{ A_1(t) \} \text{Re} \{ A_2(t) \} \right] &= 1/2 \sum_{n=-M}^M \sum_{k=-M}^M r_{nk} \cos[(\omega_1 n - \omega_2 k) \Delta] \quad (3-15) \end{aligned}$$

In Section V, Eqs. (3-9) and (3-15) are evaluated for a specific time-variant channel, and Eq. (3-15) is compared with experimental results from a tapped delay line model.

IV. Distortion Parameters in Time-Variant Channels

The work of Sunde³ has shown the importance of the instantaneous values of the linear delay distortion and linear amplitude distortion in time-variant channels. It appears that these types of distortion may be the most important effects which such channels impose on communication signals. Linear delay distortion and linear amplitude distortion can be defined in terms of $H(f,t)$, the time-variant transfer function of a channel. $H(f,t)$ can be written in polar form as follows

$$\begin{aligned}
 H(f,t) &= \hat{H}(f,t) + j\tilde{H}(f,t) = R(f,t)e^{j\psi(f,t)} \\
 \text{where } \hat{H}(f,t) &= \text{Re} \{H(f,t)\} \\
 \tilde{H}(f,t) &= \text{Im} \{H(f,t)\} \\
 R(f,t) &= [\hat{H}^2(f,t) + \tilde{H}^2(f,t)]^{\frac{1}{2}} \\
 \psi(f,t) &= \tan^{-1} [\tilde{H}(f,t)/\hat{H}(f,t)]
 \end{aligned} \tag{4-1}$$

The linear amplitude distortion, α , associated with $H(f,t)$ at frequency f_0 is defined to be the first f -derivative of $R(f,t)$ evaluated at $f=f_0$, i.e.

$$\alpha = \left. \frac{\partial}{\partial f} R(f,t) \right|_{f=f_0} \tag{4-2}$$

The linear delay distortion, d , associated with $H(f,t)$ is defined as the negative of the first f -derivative of the group or envelope delay function associated with $\psi(f,t)$ at f_0 , i.e.

$$d = -\frac{\partial}{\partial f} \psi(f,t) \tag{4-3}$$

The results of Rice⁴ and extensions of them³ provide first-order statistics of the linear delay and amplitude distortion parameters if (as has been assumed in this work) the channel transfer function is assumed to be a complex gaussian process. We assume that the correlation function of the time-variant transfer function is given by

$$R_H(\Omega, \delta) = \overline{H(f, t) H^*(f + \Omega, t + \delta)} = w(\Omega) q(\delta)$$

and also

$$R_B(\tau, \mu, \delta) = \delta(\mu) r(\tau) q(\delta)$$

where $r(\tau)$ is given by (see Eq. (1-15))

$$r(\tau) = \int_{-\infty}^{\infty} w(\Omega) e^{-j2\pi\Omega\tau} d\Omega$$

It is also assumed that $r(\tau)$ is an even function of τ .

It is known that under the above conditions σ , the linear amplitude distortion of the channel, is a normal random variable with density function

$$p(\sigma) = \frac{1}{\sqrt{2\pi}\sigma_\sigma} e^{-\frac{1}{2}(\sigma/\sigma_\sigma)^2} \quad (4-4)$$

$$\text{where } \sigma_\sigma = \left[\frac{1}{2} (b_2 - b_1^2/b_0) \right]^{\frac{1}{2}} \quad (4-4a)$$

and where

$$b_n = \int_{-\infty}^{\infty} r(\tau) \tau^n d\tau \quad n = 0, 1, 2, \dots \quad (4-4b)$$

Furthermore, the distribution function associated with the linear delay distortion, d , can be expressed as follows

$$\begin{aligned} \Pr \left[|d| \geq k b_2/b_0 \right] &= 1 - \frac{2k}{\pi} \int_0^{\infty} \frac{dx}{[k^2 + g(x)] g(x)} \\ &\quad - \frac{2}{\pi} \int_0^{\infty} \frac{\tan^{-1}(k/\sqrt{g(x)})}{(1+x^2)^{3/2}} dx \end{aligned} \quad (4-5)$$

$$\text{where } g(x) = (b_0 b_4/b_2^2 - 1 + 4x^2)(1+x^2)$$

We will now derive the equations necessary to compute the linear amplitude and delay distortions associated with a tapped delay line system. We first consider linear amplitude distortion. From Eqs. (4-1) and (4-2) we have

$$\begin{aligned}
 \sigma &= \frac{\partial}{\partial f} R(f, t) = \frac{\partial}{\partial f} \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right]^{\frac{1}{2}} \\
 &= \frac{1}{2} \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right]^{-\frac{1}{2}} \left[2\hat{H}(f, t) \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \right. \\
 &\quad \left. + 2\tilde{H}(f, t) \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \right] \\
 &= \frac{\hat{H}(f, t) \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) + \tilde{H}(f, t) \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right)}{\left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right]^{1/2}} \quad (4-6)
 \end{aligned}$$

Similarly, using Eqs. (4-1) and (4-3) we have for the linear delay distortion

$$\begin{aligned}
 d &= \frac{\partial^2}{\partial f^2} \psi(f, t) = \frac{\partial^2}{\partial f^2} \tan^{-1} [\tilde{H}(f, t) / \hat{H}(f, t)] \\
 &= \frac{\partial}{\partial f} \frac{1}{\left[1 + \tilde{H}^2(f, t) / \hat{H}^2(f, t) \right]} \cdot \frac{\left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \hat{H}(f, t) - \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \tilde{H}(f, t)}{\hat{H}^2(f, t)} \\
 &= \frac{\partial}{\partial f} \frac{\left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \hat{H}(f, t) - \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \tilde{H}(f, t)}{\hat{H}^2(f, t) + \tilde{H}^2(f, t)} \\
 &= \left[\left\{ \left(\frac{\partial^2}{\partial f^2} \tilde{H}(f, t) \right) \hat{H}(f, t) + \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) - \left(\frac{\partial^2}{\partial f^2} \hat{H}(f, t) \right) \tilde{H}(f, t) \right. \right. \\
 &\quad \left. \left. - \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \right\} \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right] - \left\{ 2\hat{H}(f, t) \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \right. \right. \\
 &\quad \left. \left. + 2\tilde{H}(f, t) \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \right\} \left\{ \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \hat{H}(f, t) - \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \tilde{H}(f, t) \right\} \right] \\
 &\quad \div \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right]^2
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 d &= \left[\left\{ \left(\frac{\partial^2}{\partial f^2} \tilde{H}(f, t) \right) \hat{H}(f, t) - \left(\frac{\partial^2}{\partial f^2} \hat{H}(f, t) \right) \tilde{H}(f, t) \right\} \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right] \right. \\
 &\quad \left. - \left\{ 2\hat{H}(f, t) \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) + 2\tilde{H}(f, t) \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \right\} \right. \\
 &\quad \left. \cdot \left\{ \left(\frac{\partial}{\partial f} \tilde{H}(f, t) \right) \hat{H}(f, t) - \left(\frac{\partial}{\partial f} \hat{H}(f, t) \right) \tilde{H}(f, t) \right\} \right] \div \left[\hat{H}^2(f, t) + \tilde{H}^2(f, t) \right]^2 \quad (4-7a)
 \end{aligned}$$

$$\begin{aligned}
a = & \left[\left\{ \left(\frac{\partial^2}{\partial f^2} \tilde{H}(f,t) \right) \hat{H}(f,t) - \left(\frac{\partial^2}{\partial f^2} \hat{H}(f,t) \right) \tilde{H}(f,t) \right\} \left[\tilde{H}^2(f,t) + \hat{H}^2(f,t) \right] \right. \\
& - 2 \left\{ \left(\frac{\partial}{\partial f} \hat{H}(f,t) \right) \left(\frac{\partial}{\partial f} \tilde{H}(f,t) \right) \left[\hat{H}^2(f,t) - \tilde{H}^2(f,t) \right] \right. \\
& \left. \left. + \hat{H}(f,t) \tilde{H}(f,t) \left[\left(\frac{\partial}{\partial f} \tilde{H}(f,t) \right)^2 - \left(\frac{\partial}{\partial f} \hat{H}(f,t) \right)^2 \right] \right\} \right] \div \left[\tilde{H}^2(f,t) + \hat{H}^2(f,t) \right]^2
\end{aligned}
\tag{4-7b}$$

It is now necessary to determine the quantities $\hat{H}(f,t)$, $\tilde{H}(f,t)$, $\frac{\partial}{\partial f} \hat{H}(f,t)$, etc. in terms of the values of the tap gain functions $b_n(t)$.

Recall that for the tapped delay line

$$\beta(\tau, t) = \sum_{n=-M}^M b_n(t) \delta(\tau - n\Delta)$$

and that $H(f,t)$ is given by

$$H(f,t) = \int_{-\infty}^{\infty} \beta(\tau, t) e^{-j2\pi f\tau} d\tau$$

Thus, for the tapped line

$$\begin{aligned}
H(f,t) &= \int_{-\infty}^{\infty} \sum_{n=-M}^M b_n(t) \delta(\tau - n\Delta) e^{-j2\pi f\tau} d\tau \\
&= \sum_{n=-M}^M b_n(t) \int_{-\infty}^{\infty} \delta(\tau - n\Delta) e^{-j2\pi f\tau} d\tau \\
H(f,t) &= \sum_{n=-M}^M b_n(t) e^{-j2\pi f n\Delta}
\end{aligned}
\tag{4-8}$$

Now recall that we have written $b_n(t)$ as

$$b_n(t) = \hat{b}_n(t) + j\tilde{b}_n(t)$$

where

$$\hat{b}_n(t) = \text{Re} \{ b_n(t) \} \quad \text{and} \quad \tilde{b}_n(t) = \text{Im} \{ b_n(t) \}$$

so we can rewrite (4-8) as

$$\begin{aligned} H(f, t) &= \sum_{n=-M}^M \left[\hat{b}_n(t) + j\tilde{b}_n(t) \right] \left[\cos(2\pi f n \Delta) - j \sin(2\pi f n \Delta) \right] \\ &= \sum_{n=-M}^M \left[\hat{b}_n(t) \cos(2\pi f n \Delta) + \tilde{b}_n(t) \sin(2\pi f n \Delta) \right] \\ &\quad + j \left[\tilde{b}_n(t) \cos(2\pi f n \Delta) - \hat{b}_n(t) \sin(2\pi f n \Delta) \right] \end{aligned}$$

Thus

$$\hat{H}(f, t) = \sum_{n=-M}^M \left[\hat{b}_n(t) \cos(2\pi f n \Delta) + \tilde{b}_n(t) \sin(2\pi f n \Delta) \right] \quad (4-9a)$$

and

$$\tilde{H}(f, t) = \sum_{n=-M}^M \left[\tilde{b}_n(t) \cos(2\pi f n \Delta) - \hat{b}_n(t) \sin(2\pi f n \Delta) \right] \quad (4-9b)$$

From these we obtain

$$\frac{\partial}{\partial f} \hat{H}(f, t) = 2\pi \Delta \sum_{n=-M}^M n \left[-\hat{b}_n(t) \sin(2\pi f n \Delta) + \tilde{b}_n(t) \cos(2\pi f n \Delta) \right] \quad (4-10a)$$

$$\frac{\partial}{\partial f} \tilde{H}(f, t) = -2\pi \Delta \sum_{n=-M}^M n \left[\tilde{b}_n(t) \sin(2\pi f n \Delta) + \hat{b}_n(t) \cos(2\pi f n \Delta) \right] \quad (4-10b)$$

$$\frac{\partial^2}{\partial f^2} \hat{H}(f, t) = - (2\pi\Delta)^2 \sum_{n=-M}^M n^2 \left[\hat{b}_n(t) \cos(2\pi f n \Delta) + \tilde{b}_n(t) \sin(2\pi f n \Delta) \right]$$

(4-11a)

$$\frac{\partial^2}{\partial f^2} \tilde{H}(f, t) = - (2\pi\Delta)^2 \sum_{n=-M}^M n^2 \left[\tilde{b}_n(t) \cos(2\pi f n \Delta) - \hat{b}_n(t) \sin(2\pi f n \Delta) \right]$$

(4-11b)

Substitution of Eqs. (4-9) through (4-11) into Eqs. (4-6) and (4-7) yields the values of linear amplitude and delay distortion at any frequency f in terms of the values of the tap delay line gain functions at time t . Considerable simplification occurs if these distortion factors are evaluated for $f = 0$. In this case we find

$$H(f, t) = \sum_{n=-M}^M b_n(t)$$

$$\hat{H}(f, t) = \sum_{n=-M}^M \hat{b}_n(t) \quad (4-12a)$$

$$\tilde{H}(f, t) = \sum_{n=-M}^M \tilde{b}_n(t) \quad (4-12b)$$

$$\frac{\partial}{\partial f} \hat{H}(f, t) = 2\pi\Delta \sum_{n=-M}^M n \tilde{b}_n(t) \quad (4-13a)$$

$$\frac{\partial}{\partial f} \tilde{H}(f, t) = - 2\pi\Delta \sum_{n=-M}^M n \hat{b}_n(t) \quad (4-13b)$$

$$\frac{\partial^2}{\partial f^2} \hat{H}(f,t) = - (2\pi\Delta)^2 \sum_{n=-M}^M n^2 \hat{b}_n(t) \quad (4-14a)$$

$$\frac{\partial^2}{\partial f^2} \tilde{H}(f,t) = - (2\pi\Delta)^2 \sum_{n=-M}^M n^2 \tilde{b}_n(t) \quad (4-14b)$$

In Section V, the results of an experiment are presented where Eqs. (4-12) through (4-14) were used with Eq. (4-7) to compute the linear delay distortion associated with a randomly time-variant channel simulated with a digital computer. The distribution function for this quantity is then compared with a plot of the theoretical distribution function for σ as given in equation (4-5).

V. Application to Simulation of a Tropospheric Scatter Communication Channel

In this section, a theoretical model for a tropospheric scatter communication channel is used together with the theory presented in Section II to construct an experimental troposcatter channel simulation for the digital computer. Some experimental results obtained from this simulator are also presented.

Sunde³ has derived an idealized model for a tropospheric scatter channel for which he has deduced that the response to a transmitted sinusoid is a Gaussian random process. For this model, the correlation function $R_H(\Omega, 0)$ can be shown to be

$$R_H(\Omega, 0) = 2R_0 T_{\max} \left(\frac{\sin 2\pi \Omega T_{\max}}{2\pi \Omega T_{\max}} \right) \quad (5-1)$$

Thus, we find from equations (1-15) and (2-8)

$$\begin{aligned} \rho(\tau, 0) &= r(\tau) = \int_{-\infty}^{\infty} R_H(\Omega, 0) e^{-j2\pi \Omega \tau} d\Omega \\ &= \int_{-\infty}^{\infty} 2R_0 T_{\max} \frac{\sin 2\pi \Omega T_{\max}}{2\pi \Omega T_{\max}} e^{-j2\pi \Omega \tau} d\Omega \\ r(\tau) &= \begin{cases} R_0, & -T_{\max} < \tau < T_{\max} \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \quad (5-2)$$

Experimental evidence has also been found³ indicating that

$$q(\delta) = e^{-\sigma^2 \delta^2 / 2} \quad (5-3)$$

Thus for this channel, we have

$$\rho(\tau, \delta) = \begin{cases} R_0 e^{-\sigma^2 \delta^2 / 2}, & -T_{\max} < \tau < T_{\max} \\ 0, & \text{elsewhere} \end{cases} \quad (5-4)$$

and

$$R_B(\tau, \mu, \delta) = \begin{cases} R_0 \delta(\mu) e^{-\sigma^2 \delta^2 / 2}, & -T_{\max} < \tau < T_{\max} \\ 0, & \text{elsewhere} \end{cases} \quad (5-5)$$

Therefore, in the tapped delay line model for this channel, we find

$$\begin{aligned}
 r_{mn} &= \int_{-\infty}^{\infty} \frac{\sin \pi B(\frac{m}{B} - u) \sin \pi B(\frac{n}{B} - u)}{\pi^2 (\frac{m}{B} - u) (\frac{n}{B} - u)} r(u) du \\
 &= R_0 \int_{-T_{\max}}^{T_{\max}} \frac{\sin \pi B(\frac{m}{B} - u) \sin \pi B(\frac{n}{B} - u)}{\pi^2 (\frac{m}{B} - u) (\frac{n}{B} - u)} du \\
 &= BR_0 \int_{-BT_{\max}}^{BT_{\max}} \frac{\sin \pi(m - x) \sin \pi(n - x)}{(m - x) (n - x)} dx \quad (5-6)
 \end{aligned}$$

Note that for given m and n , r_{mn} is a function only of the "time-bandwidth" product $BT_{\max}$ except for the unimportant multiplicative constant.

In this section, results are reported concerning tapped delay line simulation of a tropospheric scatter channel possessing the correlation function of Eq. (5-5). From equations (5-5) and (5-6) it can be seen that four parameters are required to specify the tapped delay line model. These are B , σ , $T_{\max}$ and R_0 .

In the system to be described, B was chosen to be .50 Hertz. Thus, the spacing between taps on the delay line was 2 sec. σ was chosen so that the time correlation function width (distance between e^{-1} points) would be at least 10 time samples (20 seconds). Therefore, σ was chosen to be .25 sec.⁻¹. R_0 was chosen to be 2 in order to make the constant multiplier in (5-6) equal to unity. The value of $T_{\max}$ was chosen in order to insure that the absolute value of the linear delay distortion parameter associated with the channel would exceed $6/\pi$ sec²/rad. 40% of the time. The relationship between $T_{\max}$ and the probability

distribution function of the linear delay distortion parameter, d , is found in equation (4-5). However, equation (4-5) is given in terms of the moments b_0 and b_2 which must be related to $T_{\max}$.

Figure 7 of Reference 3 contains a graph of equation (4-5) which indicates that $|d|$ exceeds the quantity kb_2/b_0 40% of the time if k is approximately equal to 2. Thus, to have $|d|$ exceed $6/\pi \text{ sec}^2/\text{rad}$. 40% of the time, we must have

$$\frac{6}{\pi} = 2 \frac{b_2}{b_0} \quad (5-7)$$

Now b_2 and b_0 are defined in equation (4-4b) while $r(\tau)$ is given by equation (5-2). Combining these equations we find that

$$b_0 = 2T_{\max}$$

$$b_2 = \frac{2T_{\max}^3}{3}$$

thus,

$$\frac{b_2}{b_0} = \frac{T_{\max}^2}{3}$$

herefore, from (5-7) we must have

$$\frac{2T_{\max}^2}{3} = \frac{6}{\pi}$$

$$T_{\max}^2 = \frac{9}{\pi} = 2.865$$

$$T_{\max} = 1.692 \text{ sec.}$$

The first step in specifying the form of the tapped delay line is the computing of the tap gain function correlations, the r_{mn} 's of eq. (5-6). For this system Eq. (5-6) was evaluated using numerical integration techniques on a digital computer. Because of the discrete

nature of digital computations, the desired value of $T_{\max}$ was rounded off to 1.70 sec. Thus, the parameter $BT_{\max}$ was 0.85. Equation (5-6) was evaluated for values of m and n varying from -8 to +8 (i.e., the matrix R is a 17 x 17 matrix). Appendix A gives the results of this computation.

The next step required for setting up the tapped delay line system was to determine the weighting coefficients, the c_{ij} 's, by solution of Equation (2-16) for the matrix C . As discussed in Section II, this requires the determination of Λ , a diagonal matrix of eigenvalues of R , and O , the matrix of eigenvectors of R . For the system to be discussed, it was felt that a delay line of 11 taps would be sufficient since the entries in R become quite small at a distance of five elements from the center of the matrix. Thus, the O and Λ matrices were computed for the center 11 x 11 section of the original 17 x 17 R matrix. Appendix B lists the eigenvalues, the O matrix, and the resulting C matrix corresponding to the center 11 x 11 section of the R matrix of Appendix A.

In order to complete the design of the tapped delay line simulation system, a linear filter must be found which will impart the proper autocorrelation function to each of the independent random processes to be passed through the weighting coefficients. This filter is needed since the digital computer produces uncorrelated random numbers for samples of random processes. These uncorrelated samples must be passed through a linear filtering operation to produce sequences with the proper sample-to-sample correlation. In the present case, we desire

to have the following autocorrelation function

$$q(\delta) = e^{-\sigma^2 \delta^2 / 2}$$

or equivalently, we seek the following spectral density

$$S(\omega) = \frac{\sqrt{2\pi}}{\sigma} e^{-\omega^2 / 2 \sigma^2}$$

Now it is known that a linear filter which will produce a random process with spectral density $S(\omega)$ at its output when white noise is applied to its input possesses a transfer function $H(\omega)$ which satisfies

$$|H(\omega)|^2 = S(\omega)$$

Thus we can use a linear filter having transfer function

$$H(\omega) = \frac{\sqrt[4]{2\pi}}{\sqrt{\sigma}} e^{-\omega^2 / 4 \sigma^2}$$

Taking the inverse Fourier transform of $H(\omega)$, we find that the required impulse response is

$$h(t) = \frac{\sqrt{\sigma}}{\sqrt[4]{\pi/2}} e^{-\sigma^2 t^2}$$

In the example under discussion, σ has been chosen to be $.25 \text{ sec}^{-1}$, so $h(t)$ becomes

$$h(t) = \frac{\sqrt{.25}}{\sqrt[4]{\pi/2}} e^{-.0625t^2} = \frac{.5}{1.1892} e^{-.0625t^2} = .42e^{-.0625t^2}$$

With the C matrix and $h(t)$ specified, it is possible to generate the 11 required tap gain functions from 11 independent white gaussian random processes. Figure 3 is a block diagram which indicates the method for doing this. The block diagram of Figure 3 was used as the

basis of a digital computer program to generate samples of the tap gain functions for a delay-line simulation of the tropospheric scatter channel.

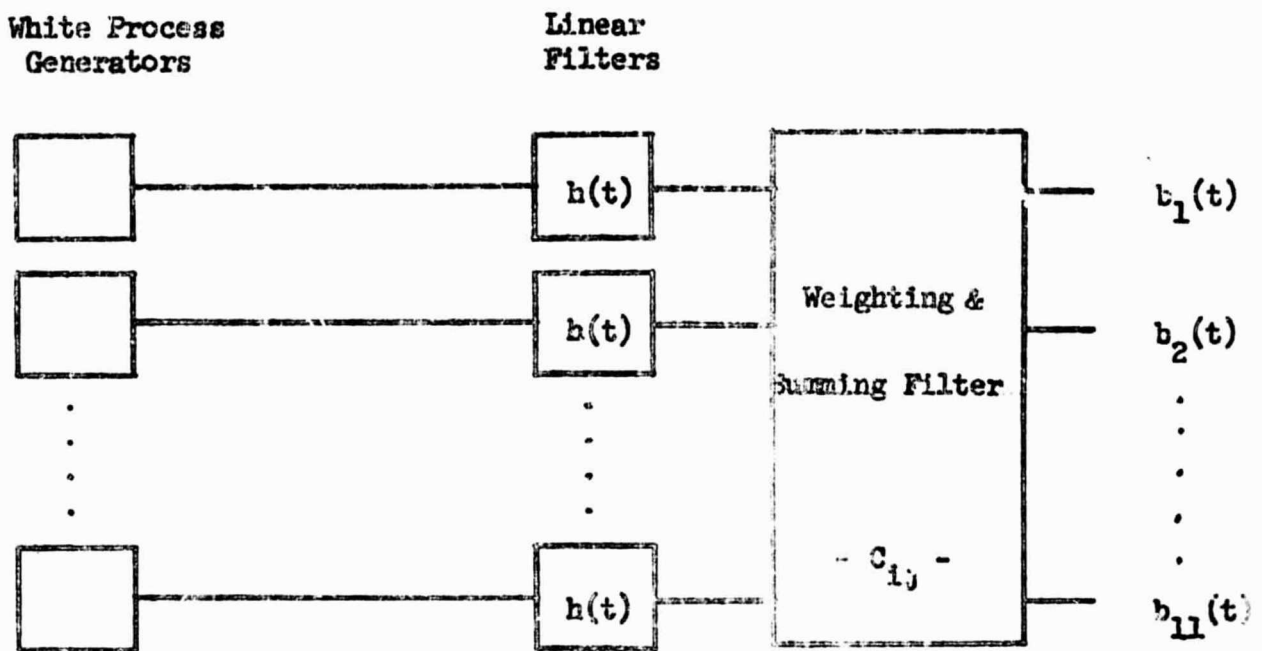


Figure 3

It should be noted that the R matrix computed from Equation (5-6) allows the expression for $\Phi_H(f, \Omega, 0)$ and $E[\text{Re}\{A_1(t)\}\text{Re}\{A_2(t)\}]$ given in equations (3-9) and (3-15) to be evaluated. This evaluation has been equations (3-9) and (3-15) to be evaluated. This has been done for both equations with the help of a digital computer. The values for the 11 x 11 center section of the R matrix given in Appendix A were substituted into equations (3-9) and (3-15) for these evaluations. Plots of the results are shown in Figures 4 through 6.

In Figure 4, the frequency correlation function for the mathematical channel model is plotted together with $\phi_H(f, \Omega, 0)$ (Equation (3-9)) for $f = 0, 0.05$, and 0.10 Hertz. These curves show the effect of approximating the physical bandlimited channel with a delay line containing only a finite number of taps. The mathematical model was specified to have a frequency correlation function $R_H(f, \Omega, 0)$ which is independent of the frequency of evaluation, f , and dependent only on the frequency separation Ω . For the tapped delay line, however, we find that for $f = 0.05$ and 0.10 Hertz, the frequency correlation function shifts and loses the $\sin x/x$ shape. Thus, for center frequencies not close to zero, the 11-tap delay line does not exhibit a frequency correlation function which is close to that of the mathematical model. Figures 5 and 6 show the desired theoretical frequency correlation function ($\sin x/x$ form) together with the theoretical prediction of a physical measurement of this function, $E[R_e\{A_1(t)\} R_e\{A_2(t)\}]$, obtained from equation (3-15). In Figure 5 the center frequency, say ω_1 , is zero, while in Figure 6, $\omega_1 = 0.10$ Hertz. Both figures are plotted as a function of $\Omega = (\omega_1 - \omega_2)/2\pi$, the frequency separation.

In Figure 5, the theoretical plot of $E[R_e\{A_1(t)\} R_e\{A_2(t)\}]$ is identical to the plot of $\phi_H(\Omega)$ for $f_0 = 0$ in Figure 4. Thus we see that for $\omega_1 = 0$, the measured correlation function is a good approximation to the desired $\sin x/x$ curve. In Figure 6, the theoretical plot has several interesting aspects. First, for values of Ω , greater than $+0.12$ Hertz, the theoretical curve is markedly different from the desired correlation function. However, for negative values of Ω , the theoretical curve is seen to be a good approximation to the desired function. The reason for this behavior is that a measurement taken for $\omega_1 = .1$ and Ω near to $.15$ Hertz implies that one of the sinusoids being passed through the channel has a frequency near $.25$ Hertz, which is the

upper limit for frequency components to be passed through the system. Hence, as could be expected, the system does a poor job of processing signals whose frequencies are near to the upper limit for the system. It is also interesting to note that the theoretical function's peak occurs at $D = 0$ for $\rho_p = .10$, and that the shape of the theoretical curve remains much closer to that of a $\sin x/x$ curve than does the curve of Figure 4 for $\rho_p = .10$.

Figures 5 and 6 also contain experimental results obtained from the tapped delay line simulator implemented on the digital computer. These results were obtained with the use of 3000 time samples of the channel output at various frequencies. In Figure 5, it can be seen that the shape of the experimental curve compares well with that of the desired curve, but the experimental values are about 1/3 lower than the theoretical ones. It appears that this discrepancy is due to measurement inaccuracy caused by relatively small number of time samples used in the experiment. In Figure 6, it can be seen that the experimental values obtained agree quite well with the theoretically predicted values and that the desired $\sin x/x$ form is achieved.

Figure 7 shows the results of an experiment to determine the distribution function for the linear delay distribution parameter associated with the tapped delay line simulator. The theoretical plot is a plot of equation (4.5), which gives the theoretical distribution function of the linear delay distribution parameter for the mathematical model being used. The experiment consisted of using equations 4-7a, 4-12, 4-13 and 4-14 to compute the value of a from the two gain function values for each instance of time. The values of a

obtained were used to form a histogram which was then converted into an exponential distribution function. Figure 7 indicates some of the points from this experimental distribution function. It can be seen that the experimental and theoretical distribution functions compare quite well.

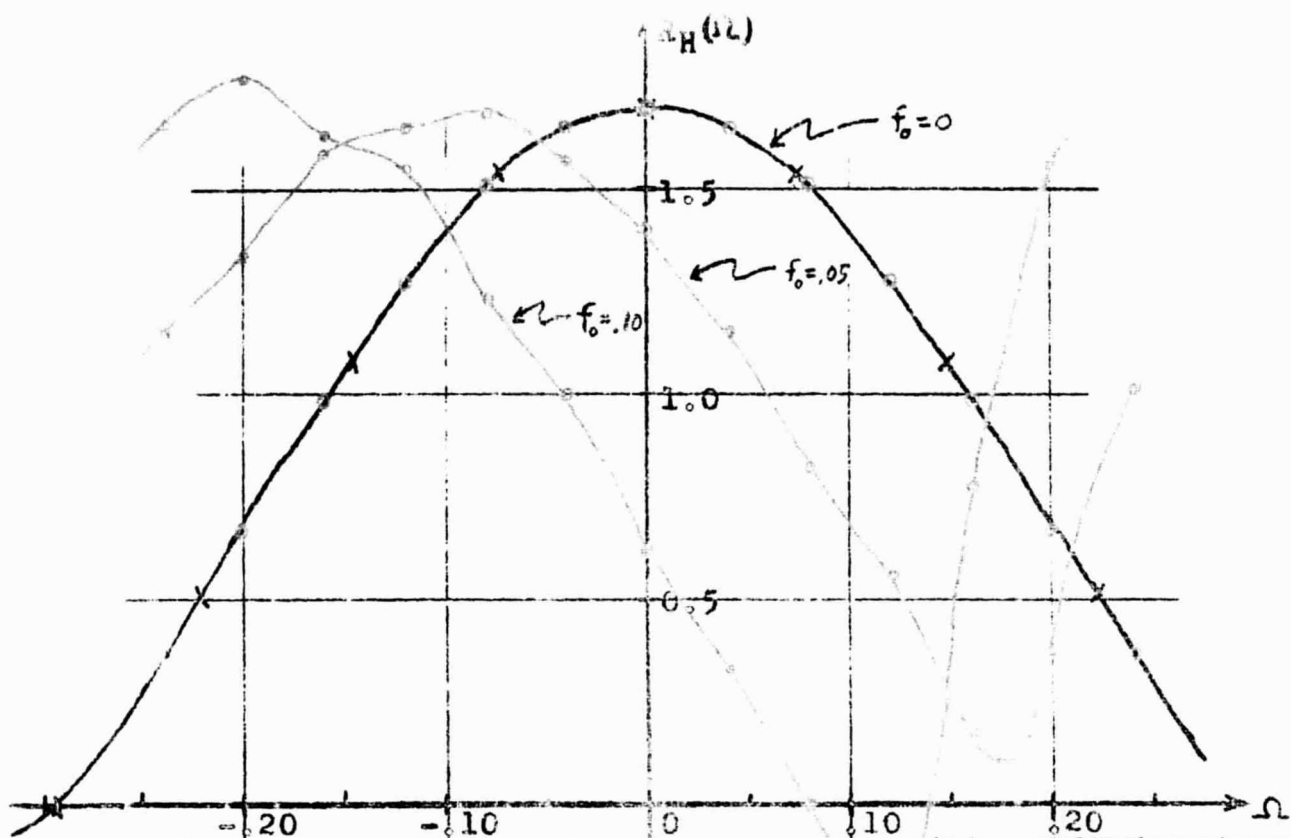


Figure 4 -- Frequency correlation function $R_H(\Omega) \equiv \overline{H(f_0, t)H^*(f_0 + \Omega, t)}$
 $\times$ -- desired $\sin(x)/x$ function, $\circ$ -- tapped delay line frequency correlation function for center frequencies of 0, 0.05 and 0.10 Hertz 9Eq. 3-9).

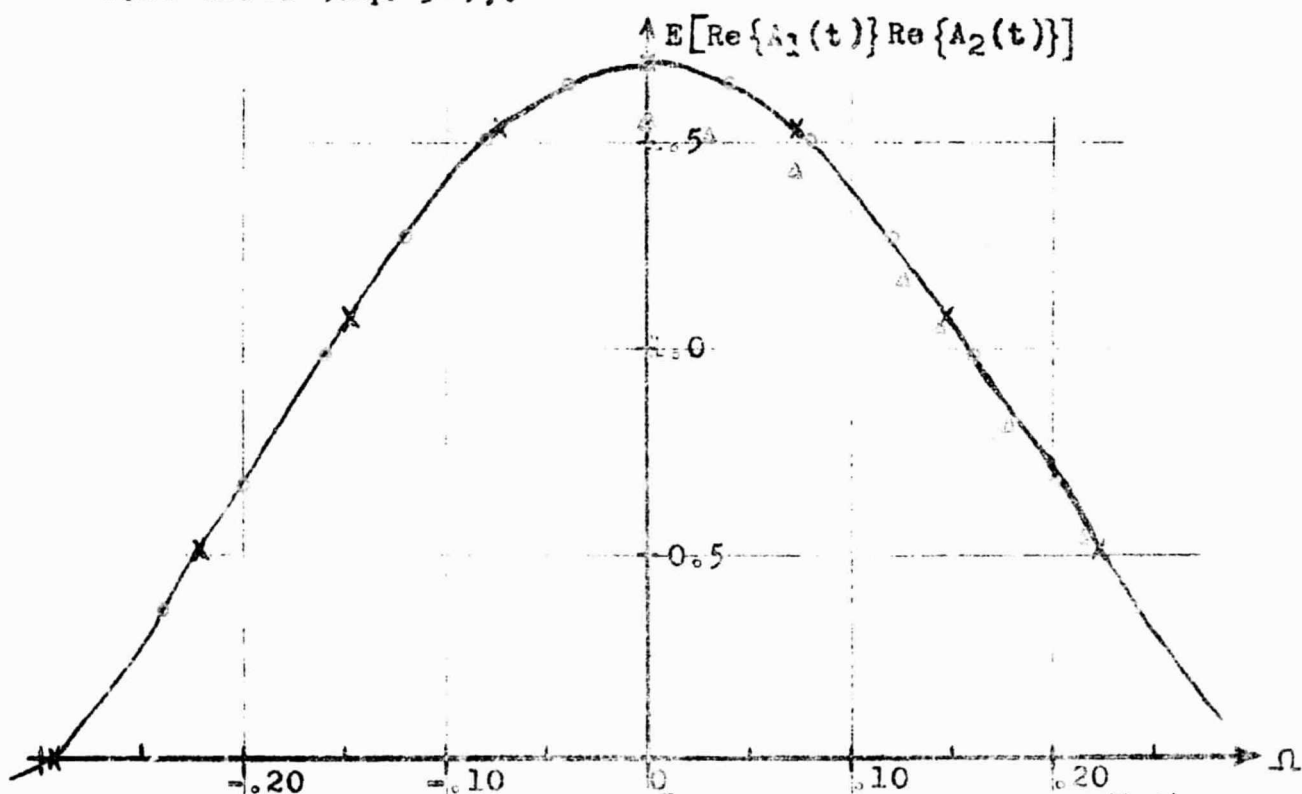


Figure 5 - $E[\text{Re}\{A_1(t)\} \text{Re}\{A_2(t)\}]$ as a function of $\Omega = \frac{\omega_1 - \omega_2}{2\pi}$ for $\omega_1 = 0$
 $\times$ -- desired $\sin(x)/x$ function, $\circ$ -- theoretical function for tapped delay line (Eq. 3-15), Δ -- experimental results from tapped delay line simulator.

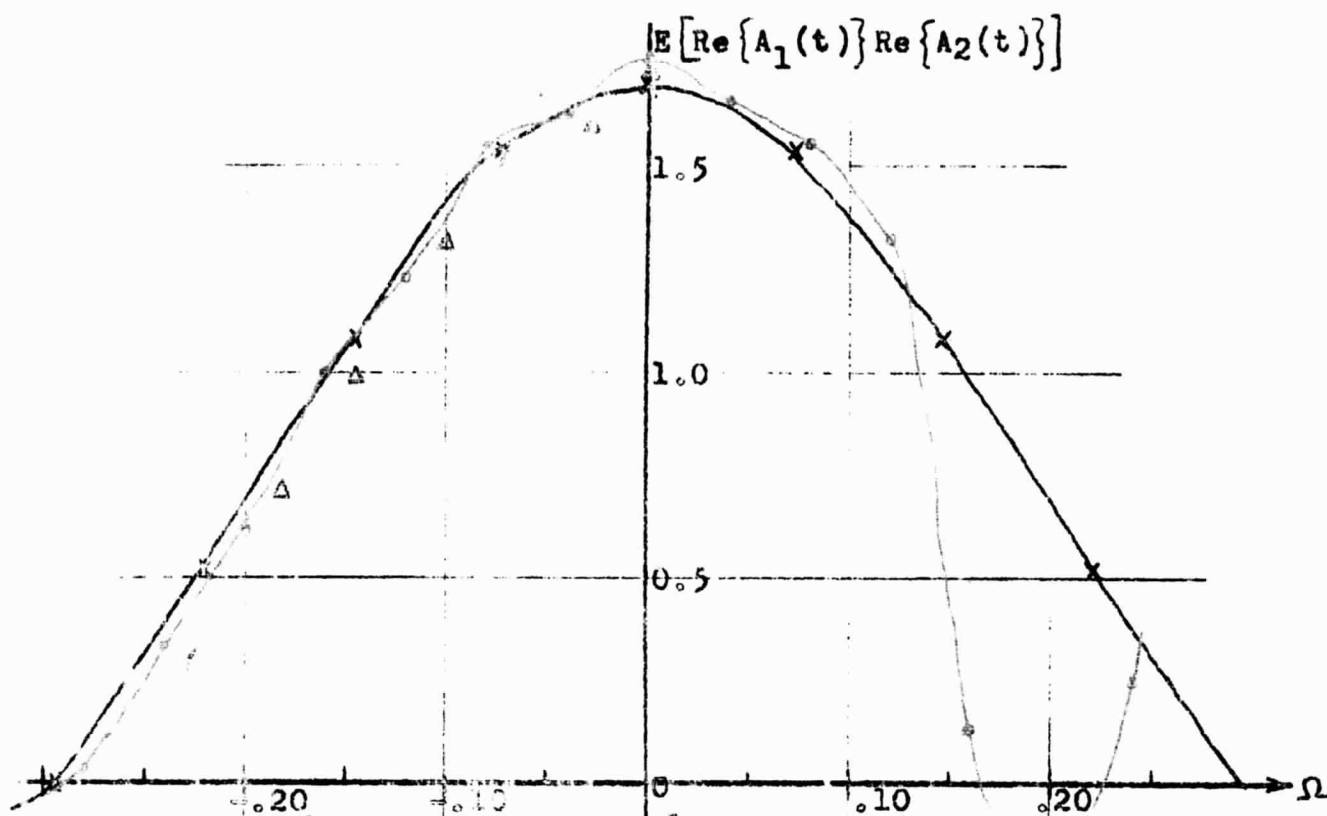


Figure 6 - $E[\text{Re}\{A_1(t)\} \text{Re}\{A_2(t)\}]$ as a function of $\Omega = \frac{\omega_2 - \omega_1}{2\pi}$ for $\omega_1 = 10$ Hertz; x - desired $\sin(x)/x$ function, o - theoretical function for tapped delay line (Eq. 3-15), Δ - experimental results from tapped delay line simulator.

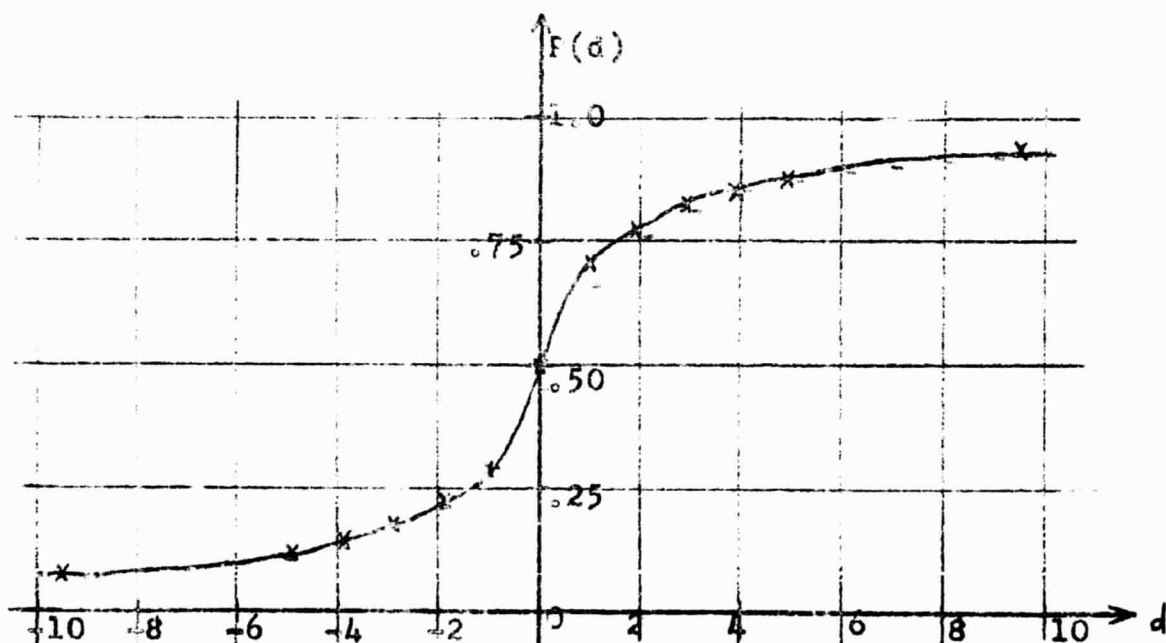


Figure 7 - Distribution function for linear delay distortion parameter of tropospheric scatter channel; x - theoretical distribution function, - - experimental distribution function.

VI. Conclusions

It is concluded that the methods presented in this report can be used for designing acceptable tapped delay line models of randomly time-variant channels. Further it has been shown that digital computer simulation of the tapped delay line model is practical. In this report an example is presented for which several experimentally measured quantities associated with the system gave good agreement with those of the desired mathematical model. It is also seen that the tapped delay line simulator can give poor results if signals with frequencies very close to the theoretical limits of the simulator are used. It appears that a digital time-variant channel simulator of the type described in this memorandum can be useful for testing of modulation and demodulation schemes for use with time-variant channels.

Appendix A

The following are the entries for the R matrix computed from equation (5-6) for $BT_{\max} = .85$ and $BR_0 = 1$.

	<u>Column Number</u>				
	-8	-7	-6	-5	-4
-8	0.001569	-0.001797	0.002103	-0.002535	0.003194
-7	-0.001797	0.002058	-0.002408	0.002903	-0.003659
-6	0.002103	-0.002408	0.002818	-0.003398	0.004285
-5	-0.002535	0.002903	-0.003398	0.004099	-0.005171
-4	0.003194	-0.003659	0.004285	-0.005171	0.006527
-3	-0.004330	0.004963	-0.005815	0.007023	-0.008875
-2	0.006811	-0.007813	0.009165	-0.011087	0.014045
-1	-0.019131	0.022020	-0.025942	0.031578	-0.040380
0	-0.001556	0.002035	-0.002776	0.004011	-0.006306
1	0.017646	-0.020076	0.023288	-0.027736	0.034318
2	-0.006576	0.007507	-0.008746	0.010481	-0.013089
3	0.004232	-0.004835	0.005639	-0.006769	0.008475
4	-0.003139	0.003588	-0.004188	0.005031	-0.006306
5	0.002500	-0.002858	0.003337	-0.004011	0.005031
6	-0.002079	0.002377	-0.002776	0.003337	-0.004188
7	0.001780	-0.002035	0.002377	-0.002858	0.003588
8	-0.001556	0.001780	-0.002079	0.002500	-0.003139

Row Number

Column Number

	-3	-2	-1	0	1
-8	-0.004330	0.006811	-0.019131	-0.001556	0.017646
-7	0.004963	-0.007813	0.022020	0.002035	-0.020076
-6	-0.005815	0.009165	-0.025942	-0.002776	0.023288
-5	0.007023	-0.011087	0.031578	0.004011	-0.027736
-4	-0.008875	0.014045	-0.040380	-0.006306	0.034318
-3	0.012092	-0.019214	0.056133	0.011366	-0.045116
-2	-0.019214	0.030793	-0.093051	-0.026657	0.066559
-1	0.056133	-0.093051	0.328303	0.146364	-0.146364
0	0.011366	-0.026657	0.146364	0.900085	0.146364
1	-0.045116	0.066559	-0.146346	0.146364	0.328303
2	0.017482	-0.026657	0.066559	-0.026657	-0.093051
3	-0.011366	0.017482	-0.045116	0.011366	0.056133
4	0.008475	-0.013089	0.034318	-0.006306	-0.040380
5	-0.006769	0.010481	-0.027736	0.004011	0.031578
6	0.005639	-0.008746	0.023288	-0.002776	-0.025942
7	-0.004835	0.007507	-0.020076	0.002035	0.022020
8	0.004232	-0.006576	0.017646	-0.001556	-0.019131

Row Number

Column Number

2

3

4

5

6

Row Number

-8	-0.006576	0.004232	-0.003139	0.002500	-0.002079
-7	0.007507	-0.004834	0.003588	-0.002858	0.002377
-6	-0.008746	0.005639	-0.004188	0.003337	-0.002776
-5	0.010481	-0.006769	0.005031	-0.004011	0.003337
-4	-0.013089	0.008475	-0.006306	0.005031	-0.004188
-3	0.017482	-0.011366	0.008475	-0.006769	0.005639
-2	-0.026657	0.017482	-0.013089	0.010481	-0.008746
-1	0.066559	-0.045116	0.034328	-0.027736	0.023288
0	-0.026657	0.011366	-0.006306	0.004011	-0.002776
1	-0.093051	0.056133	-0.040330	0.031578	-0.025942
2	0.030793	-0.019214	0.014045	-0.011087	0.009165
3	-0.019214	0.012092	-0.008875	0.007023	-0.005815
4	0.014045	-0.008875	0.006527	-0.005171	0.004285
5	-0.011087	0.007023	-0.005171	0.004099	-0.003398
6	0.009165	-0.005815	0.004285	-0.003398	0.002818
7	-0.007813	0.004963	-0.003659	0.002903	-0.002408
8	0.006811	-0.004330	0.003194	-0.002535	0.002103

Column Number

7

8

Row Number

-8	0.001780	-0.001556
-7	-0.002035	0.001780
-6	0.002377	-0.002079
-5	-0.002858	0.002500
-4	0.003588	-0.003139
-3	-0.004834	0.004232
-2	0.007507	-0.006576
-1	-0.020076	0.017646
0	0.002035	-0.001556
1	0.022020	-0.019131
2	-0.007813	0.006811
3	0.004963	-0.004330
4	-0.003659	0.003194
5	0.002903	-0.002535
6	-0.002408	0.002103
7	0.002058	-0.001797
8	-0.001797	0.001569

Appendix B

In this appendix, the eigenvalues and the matrix of eigenvectors are given for the center 11 x 11 section of the R matrix given in Appendix A. The C matrix which satisfies equation (2-16) is also given. The relationship between the eigenvalues, matrix of eigenvectors and C is given by equation (2-18).

The eigenvalues of R are:

.957946
 .570340
 .129005
 .617526 x 10⁻²
 .245153 x 10⁻³
 .238504 x 10⁻⁵
 .434760 x 10⁻⁷
 .122817 x 10⁻⁷
 .120857 x 10⁻⁸
 -.992990 x 10⁻⁸
 -.101912 x 10⁻⁷

Since the R matrix is a covariance matrix, it is positive definite. Therefore it should not possess negative eigenvalues. It appears that the two negative eigenvalues shown above are due to round-off errors in the digital computation.

The matrix of eigenvectors of R, the C matrix, is given on pages B-2 and B-3. The C matrix corresponding to the center 11 x 11 section of the R matrix given in Appendix A is shown on pages B-4 and B-5.

Column NumberRow Number

	-5	-4	-3	-2	-1	0
-5	-.004798	-.081679	-.012412	-.201117	.101332	-.315848
-4	.007559	.102611	.019633	.249266	-.157992	.328488
-3	-.013642	-.139285	-.035865	-.325824	.279401	-.213429
-2	.032117	.219257	.087598	.454795	-.613136	-.482072
-1	-.183278	-.644388	-.675052	.290562	-.103386	-.018991
0	-.964481	-.000002	.263837	.000000	-.012870	.000000
1	-.183277	.644349	-.675053	-.290564	-.103386	.018996
2	.032118	-.219490	.087598	-.454805	-.613131	.482366
3	-.013641	.139391	-.035865	.325809	.279403	.243873
4	.007558	-.102872	-.019633	-.249268	-.157996	-.329051
5	-.004804	.081696	-.012412	.201108	.101335	.315496

	1	2	3	4	5
-5	-.329327	-.580234	-.254272	.147341	-.582072
-4	-.259644	-.351754	.491144	.592151	-.296764
-3	-.462944	.171039	.128096	.431362	.237570
-2	-.321418	.056954	.025565	.058140	.035875
1	-.004775	.000260	-.000342	.000032	.000828
0	.000184	.000000	.000018	.000011	-.000028
1	-.004772	-.000253	-.000346	-.000675	.000705
2	-.330489	-.051188	.026298	-.115577	.056363
3	.459531	.197257	.434989	-.540041	.072977
4	.256065	.331558	.515911	-.252345	-.437233
5	-.337435	.595411	-.236471	.304576	-.556446

	-5	-4	-3	-2	-1	0
-5	.003703	-.010769	.014536	.021777	.033872	.053333
-4	-.010769	.013673	-.018372	.028011	-.050412	-.009244
-3	.014536	.018372	.005029	-.038726	.071042	.009423
-2	-.021777	.028011	-.038726	.062595	-.122309	-.021099
-1	.033872	-.050412	.071042	-.122309	.521612	.050364
0	.053333	.009244	.009423	-.009093	.109064	.935195
1	-.051157	.038467	-.049656	.070286	-.123488	.109060
2	.018999	-.023357	.030239	-.043284	.070109	-.021894
3	-.013171	.016241	-.021204	.030333	-.049711	.009421
4	.010050	-.012449	.016250	-.023346	.028520	-.005242
5	-.008109	.010059	-.013154	.018990	-.031467	.003338

Box-Mueller

Column Heights

	1	2	3	4	5
5	-.051457	.018999	-.013171	.010050	-.008109
4	.038987	-.023357	.026241	-.012443	.010057
3	-.049646	.050199	-.021204	.016949	-.013154
2	.070288	-.043264	.030333	-.023246	.018990
1	-.123488	.070408	-.049711	.038520	-.031467
0	.109060	-.021874	.009421	-.005242	.003338
4	.516904	-.122417	.071089	-.050238	.038879
3	-.122417	.042675	-.036708	.028069	-.021997
2	.071089	-.053768	.025058	-.018394	.011521
1	-.053768	.014068	-.005034	.001994	-.001361
0	.014068	-.021997	.014531	-.010802	.008715

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