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Gravitational Equilibria of a Rigid
Body Containing Symmetric Rotors¹

by

Robert E. Roberson² and William W. Hooker³

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INTRODUCTION

The existence and stability of equilibrium orientations of an aspherical body in a nonuniform gravitational field is a classical problem.⁴ Various generalizations of it also have direct applications to modern astronautical problems. This paper considers one of these: the problem of a single rigid body containing a symmetric, constant speed, fixed axis rotor -- i.e. a gyrostat. We consider the existence and determination of gravitational equilibria for three basic cases for gyrostats on circular orbits in an inverse square field. Space does not permit discussion of stability problems or of computational algorithms for the numerical solution of the most general case. Both of these questions will be elaborated elsewhere.

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² Professor of Engineering, University of California, Los Angeles

³ Mathematical Specialist, Philco Western Development Laboratories, Palo Alto

⁴ Some of the pertinent literature is cited in [1]

DYNAMICAL BASIS and EQUILIBRIUM CONDITIONS

Let⁵ $I = [I_{\alpha\beta}]$ be the inertia matrix of the whole body, including rotors, $\omega = [\omega_\alpha]$ be the 3x1 matrix of body components of inertial angular velocity, $\tilde{\omega} = [\epsilon_{\alpha\lambda\beta} \omega_\lambda]$ ($\epsilon_{\alpha\beta\gamma}$ the "Epsilon symbols" of tensor analysis), $h = [h_\alpha]$ be the 3x1 matrix of body components of internal angular momentum from the rotors, ω_0 be the (circular) orbital angular velocity of the gyrostat.

The orthonormal body frame is $\{\underline{X}_\alpha\}$, and $\{\underline{\xi}_\alpha\}$ is a second orthonormal frame such that $\underline{\xi}_3$ is the geocentric vertical, $\underline{\xi}_2$ is normal to the orbit plane, $\underline{\xi}_1$ is "forward". These are related by the direction cosine matrix $\Theta = [\Theta_{\alpha\beta}]$ as

$$\underline{X}_\alpha = \Theta_{\alpha\beta} \underline{\xi}_\beta \quad (1)$$

Introduce 3x1 matrices $\xi_\lambda = [\Theta_{\alpha\lambda}]$, matrices of components of $\underline{\xi}_\lambda$ resolved in body axes. Define $\tilde{\xi}_\lambda = [\epsilon_{\alpha\gamma\beta} \Theta_{\gamma\lambda}]$.

As a starting point we use the Euler rotational equations in matrix form (see methodology of [2] or [3]):

$$I\dot{\omega} + \tilde{\omega} (I\omega + h) = 3\omega_0^2 \tilde{\xi}_3 I \xi_3 \quad (2)$$

With the rationale of [1], we argue that a necessary and sufficient condition for equilibrium is that $\omega = \omega_0 \xi_2$ (constant), i.e. that ξ_2, ξ_3 satisfy

$$\tilde{\xi}_2 (I \xi_2 + h/\omega_0) = 3 \tilde{\xi}_3 I \xi_3 \quad (3)$$

⁵Greek subscripts have range 1,2,3 and the summation convention is used.

As in [1], note that the ξ_λ form a basis for our (physical) vector space, so every 3×1 matrix representing a vector in it is a linear combination of the ξ_λ .

Put

$$I \xi_1 = \lambda \xi_1 + a_2 \xi_2 + a_3 \xi_3 \quad (4a)$$

$$I \xi_2 = b_1 \xi_1 + \mu \xi_2 + b_3 \xi_3 \quad (4b)$$

$$I \xi_3 = c_1 \xi_1 + c_2 \xi_2 + \nu \xi_3 \quad (4c)$$

$$h/\omega_0 = J_\alpha \xi_\alpha \quad (4d)$$

Note that h usually is the known quantity in Eq. 4d, the J_α being unknown except as determined along with ξ_α . It is helpful to approach the problem by thinking of them as known, however. Equations 4 in 3 imply $c_1=0$, $b_1=-J_1$, $b_3=-J_3-3c_2$. Next, $\xi_\alpha^T I \xi_\beta = \xi_\beta^T I \xi_\alpha$ imply $a_2=b_1$, $a_3=c_1$, $b_3=c_2$. This leaves

$$(I - \lambda E) \xi_1 + J_1 \xi_2 = 0 \quad (5a)$$

$$J_1 \xi_1 + (I - \mu E) \xi_2 + \frac{1}{4} J_3 \xi_3 = 0 \quad (5b)$$

$$\frac{1}{4} J_3 \xi_2 + (I - \nu E) \xi_3 = 0 \quad (5c)$$

, where E is the 3×3 unit matrix, as the equilibrium conditions. Note that J_2 does not appear, as might be anticipated on physical grounds.

Recognize that $\xi_\alpha^T \xi_\alpha = 1$ (α not summed) and $\xi_\alpha^T \xi_\beta = 0$ ($\alpha \neq \beta$) because of the orthonormality of $\{\xi_\lambda\}$. Then the following theorems are obvious, in view of the eigenvector character of a principal axis as regards the inertia matrix:

- TH. 1 The condition that $\underline{\xi}_1$ coincide with a principal axis of inertia implies and is implied by $J_1 = 0$.
- TH. 2 The condition that $\underline{\xi}_3$ coincide with a principal axis of inertia implies and is implied by $J_3 = 0$.
- TH. 3 The condition that $\underline{\xi}_2$ be a principal axis implies and is implied by $J_1 = J_3 = 0$.

BASIC CASES

The case $J_1 = J_2 = J_3 = 0$ is the classical libration problem, which we do not consider further. The case $J_1 = J_3 = 0, J_2 \neq 0$ is well defined by TH. 3 and Eqs.5: all principal axes coincide with $\underline{\xi}_\lambda$ as in the classical problem, the principal axis carrying the internal angular momentum necessarily coincident with the normal to the orbit plane. It can be shown by listing exclusive and exhaustive physical situations that the only other cases that need be considered are $J_1 = 0, J_3 \neq 0$; $J_1 \neq 0, J_3 = 0$; $J_1 J_3 \neq 0$. Even though the J_α are not "known" parameters, no other physical equilibrium possibilities exist.

THE CASE $J_3 = 0, J_1 \neq 0$

The case with $J_1 = 0, J_3 \neq 0$ can be argued by analogy. By TH.2, $\underline{\xi}_3$ is aligned with a principal axis along which $\underline{h}$ has no component. For definiteness, call it axis $\underline{X}_3$. In this case

$\nu = I_3$ in Eq.5c, and Eqs.52,b take the scalar form

$$(I_\alpha - \lambda) \xi_{1\alpha} + J_1 \xi_{2\alpha} = 0 \quad (6a)$$

$$J_1 \xi_{1\alpha} + (I_\alpha - \mu) \xi_{2\alpha} = 0 \quad (6b)$$

(not summed on α). Eliminating $\xi_{2\alpha}$ necessary and sufficient equilibrium conditions are

$$[(I_\alpha - \lambda)(I_\alpha - \mu) - J_1^2] \xi_{1\alpha} = 0 \quad (7)$$

(not summed). Now $\xi_{13} = 0$ because $\xi_{33} = 1$, and neither ξ_{11} nor ξ_{12} are zero, for otherwise $\underline{e}_1$ is a principal axis contrary to TH.2. Thus for $\alpha = 1, 2$ the bracketed quantity in Eq.7 must vanish.

Suppose $D = I_1 - I_2 \neq 0$. Then

$$\lambda = \frac{1}{2} (I_1 + I_2 \pm \sqrt{D^2 - 4J_1^2}) \quad (8a)$$

$$\mu = \frac{1}{2} (I_1 + I_2 \mp \sqrt{D^2 - 4J_1^2}) \quad (8b)$$

Evidently⁶:

TH. 4 If $|I_1 - I_2| < 2|J_1|$, no real equilibrium exists for which $\underline{h}$ is in the plane of the local horizontal (i.e. the $\xi_1 \xi_2$ plane). Here I_1, I_2 are principal moments in the principal plane containing $\underline{h}$.

⁶

This has an immediate analogue for the case where $J_1 = 0, J_3 \neq 0$, again calling $\underline{X}_3$ the axis with no $\underline{h}$ -component:

TH.5 If $|I_1 - I_2| < 2|J_3/4|$, no real equilibrium exists for which $\underline{h}$ is in the plane normal to the orbit (i.e. the $\xi_2 \xi_3$ plane).

Refer to Fig. 1 for the definition of angles θ, φ and put $H = |\underline{h}|/\omega_0 = \sqrt{J_\alpha J_\alpha'}$. The angle φ and magnitude H are considered known, and J_1, θ are to be determined from Eqs. 4d, 6a (with $\alpha=1$, say): i.e. from $J_1 = H \cos(\theta + \varphi)$ and

$$\frac{1}{2} [D \mp \sqrt{D^2 - 4J_1^2}] \cos \theta + J_1 \sin \theta = 0. \quad \text{Define } \psi = \theta - \pi/2$$

and

$$\kappa = -D/2H = (I_2 - I_1)/2H \quad (9)$$

By labeling axes so $I_2 \geq I_1$ we also can assure $\kappa \geq 0$. Then it is straightforward to show that ψ (hence θ) is to be determined from

$$\sin(\psi + \varphi) = \kappa \sin 2\psi \quad (10)$$

This establishes the qualitative relationship between φ and ψ shown in Fig. 2. If $0 \leq \kappa \leq \frac{1}{2}$ ($0 \leq I_2 - I_1 \leq H$) there is exactly one equilibrium orientation for each φ -value. For $1/2 < \kappa \leq 1$ there are two equilibria for each φ in small ranges near $\varphi=0$ and $\varphi = -\pi$ (dark line segments in Fig. 2.) An illustrative pair of such equilibria is shown in Fig. 3. The case $D = 0$ is approached uniformly as κ approaches zero, and one finds that the corresponding physical equilibrium is the one found previously with $J_1 = J_3 = 0$.

THE GENERAL CASE

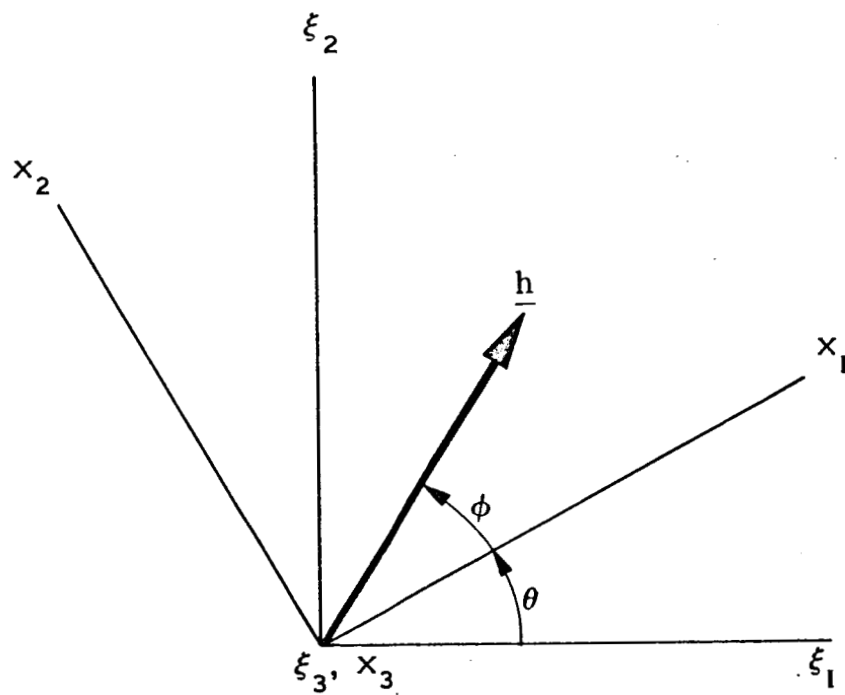
Assume $J_1 \neq 0, J_3 \neq 0$. The analogs of Eqs. 6 are

$$(I_\alpha - \lambda) \xi_{1\alpha} + J_1 \xi_{2\alpha} = 0 \quad (11a)$$

$$J_1 \xi_{1\alpha} + (I_\alpha - \mu) \xi_{2\alpha} + \frac{1}{4} J_3 \xi_{3\alpha} = 0 \quad (11b)$$

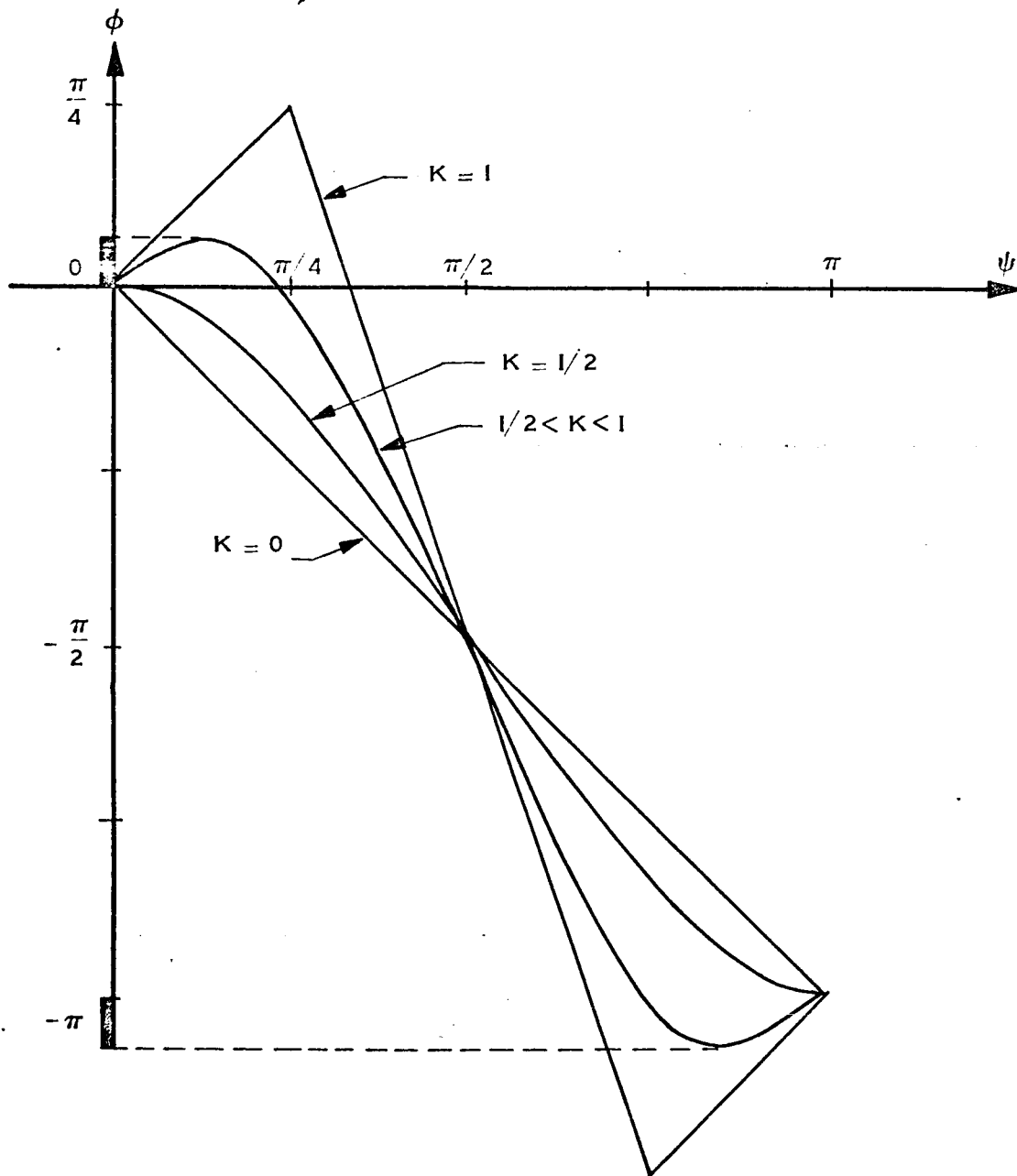
$$\frac{1}{4} J_3 \xi_{2\alpha} + (I_\alpha - \gamma) \xi_{3\alpha} = 0 \quad (11c)$$

(α not summed). Necessary and sufficient conditions for nontrivial



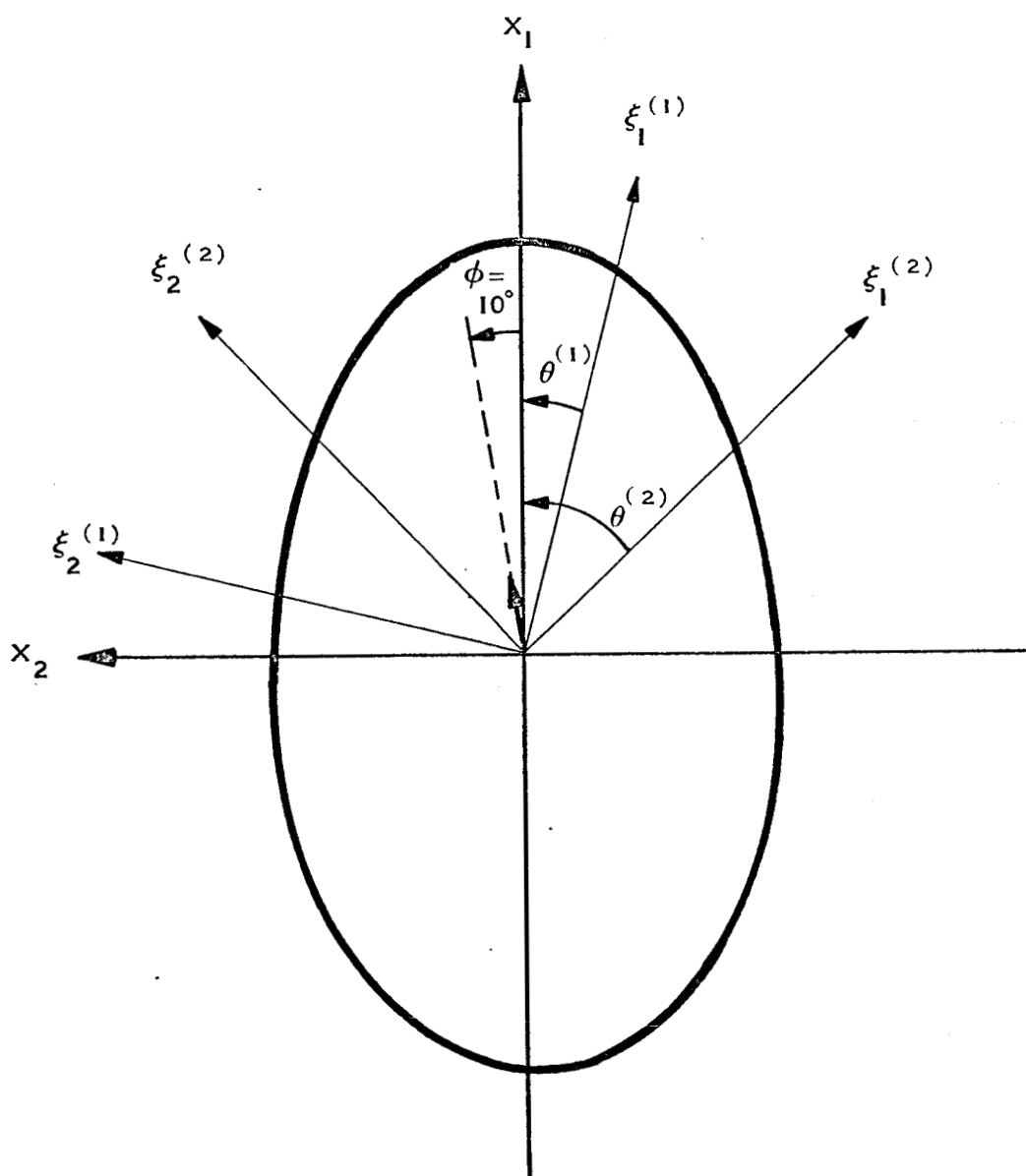
ROTOR ANGULAR MOMENTUM IN THE PLANE OF THE
LOCAL HORIZONTAL

FIGURE 1



SOLUTION POSSIBILITIES FOR EQUATION 13

FIGURE 2



ILLUSTRATIVE EQUILIBRIA FOR $\kappa = 0.8$, $B_2/B_1 = 25/16$, $\phi = 10^\circ$

FIGURE 3

solutions are the vanishing of the three determinants of coefficients:

$$\begin{vmatrix} I_\alpha - \lambda & J_1 & 0 \\ J_1 & I_\alpha - \mu & \frac{1}{4} J_3 \\ 0 & \frac{1}{4} J_3 & I_\alpha - \nu \end{vmatrix} = 0 \quad (12)$$

It is convenient to suppose that body axes are labeled such that $I_1 \geq I_2 \geq I_3$.

Define parameters

$$\begin{aligned} \kappa_1 &= (I_1 - I_3)/2H & j_1 &= J_1/H \\ \kappa_2 &= (I_2 - I_3)/2H & j_3 &= J_3/4H \end{aligned} \quad (13)$$

and for brevity use $x = (\lambda - I_3)/H$, $y = (\mu - I_3)/H$, $z = (\nu - I_3)/H$. With some manipulation it can be shown that Eqs. 12 for $\alpha = 1, 2, 3$ can be written as

$$j_1^2 = x(2\kappa_1 - x)(2\kappa_2 - x)/(z - x) \quad (14a)$$

$$j_3^2 = z(2\kappa_1 - z)(2\kappa_2 - z)/(x - z) \quad (14b)$$

$$y = (j_1^2/x) + (j_3^2/z) \quad (14c)$$

If the j_1, j_3 were given, these would determine the eigenvalue-like quantities x, y, z , whence ξ_λ could be determined from Eqs. 11. In fact the j_1, j_3 are not known, so a certain set of equations must be solved simultaneously. Specifically, Eqs. 11a and 11c imply

$$\xi_{11} = j_1 \xi_{21}/(x - 2\kappa_1) ; \xi_{12} = j_1 \xi_{22}/(x - 2\kappa_2) ; \xi_{13} = j_1 \xi_{23}/x \quad (15a)$$

$$\xi_{31} = j_3 \xi_{21}/(z - 2\kappa_1) ; \xi_{32} = j_3 \xi_{22}/(z - 2\kappa_2) ; \xi_{33} = j_3 \xi_{23}/z \quad (15b)$$

Using Eq.15b, for example, together with the conditions $\xi_{3\alpha}\xi_{3\alpha} = 1$, $\xi_{3\alpha}\xi_{2\alpha} = 0$, $\xi_{2\alpha}\xi_{2\alpha} = 1$, one gets three algebraic equations linear in the $(\xi_{2\alpha})^2$ whose solutions are

$$(\xi_{21})^2 = (2\kappa_1 - x)(2\kappa_1 - z) / 4\kappa_1(\kappa_1 - \kappa_2) \quad (16a)$$

$$(\xi_{22})^2 = (2\kappa_2 - x)(2\kappa_2 - z) / 4\kappa_2(\kappa_1 - \kappa_2) \quad (16b)$$

$$(\xi_{23})^2 = xz / 4\kappa_1\kappa_2 \quad (16c)$$

Finally, if $\eta = \hbar / H\omega_0$ is a known matrix of direction cosines of $\underline{h}$, Eq. 4d gives

$$\eta_\alpha = j_1 \xi_{1\alpha} + j_2 \xi_{2\alpha} + j_3 \xi_{3\alpha} \quad (17)$$

The fourteen equations Eqs. 14a, b, 15a,b, 16a,b,c, and 17 relate the fourteen unknowns $x, z, j_\alpha, \xi_{\alpha\beta}$. They can, in principal, be solved numerically for the equilibrium conditions given only the three components η_α and the parameters κ_1, κ_2 . A discussion of solution algorithms is impossible within the limitations on this paper and will be given elsewhere. However, it is possible to examine some of the qualitative solution possibilities here.

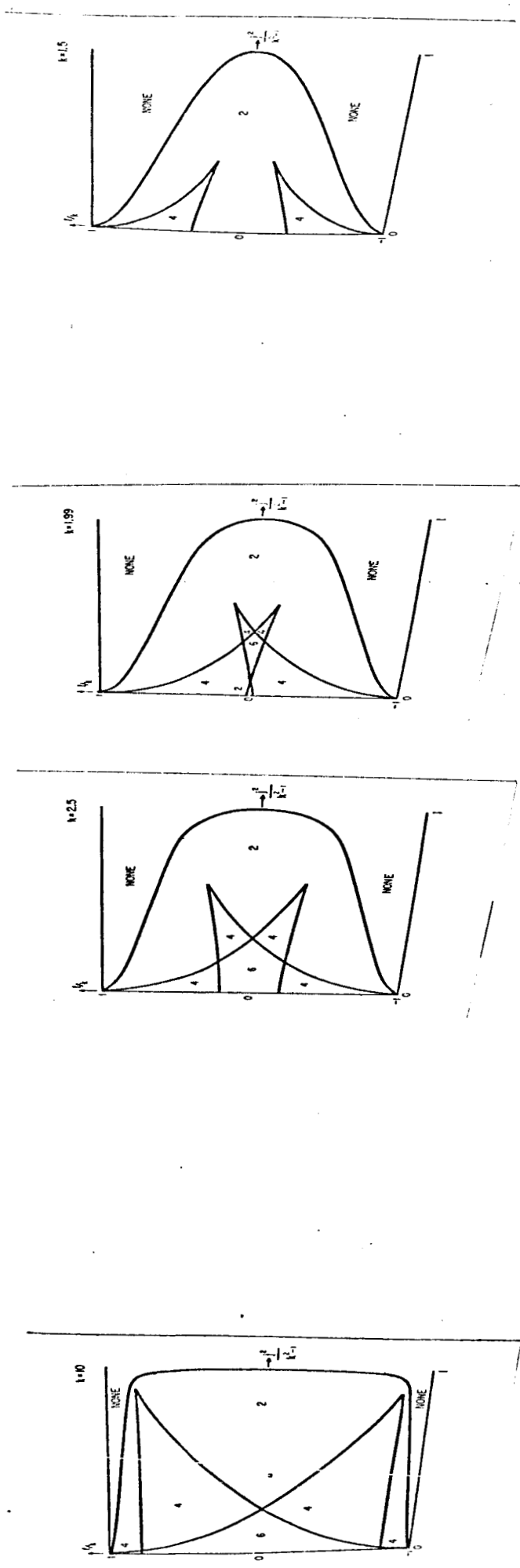
First, note in Eqs.14 that the need for j_1^2 and j_3^2 to be positive means that x and z -values must lie only in the regions $(2\kappa_2 < x < 2\kappa_1, 2\kappa_2 < z < 0)$ and $(0 < x < 2\kappa_2, 2\kappa_2 < z < 2\kappa_1)$.

Next, observe that Eqs. 14a, b can be written as a sixth order algebraic equation $P(z; \kappa_1, \kappa_2, j_1, j_3) = 0$ by eliminating x . If this equation is solved simultaneously with $\partial P / \partial z = 0$ one finds z -values which separate regions in the parameter space into regions within which different numbers of real solutions exist. (The fact that j is absent from $\partial P / \partial z$ enables some of the results to be gotten analytically.) Some additional parameter groupings are defined, in terms of which convenient symmetry properties exist: i.e. $k = \kappa_1 / |j_1|$, $l = (2\kappa_2 - \kappa_1) / |j_1|$, $j^2 = (j_1 / j_3)^2$.

Leaving aside the details of this investigation, which utilizes digital computer solutions, the results shown in Fig. 4 are obtained. Parameter regions are seen to exist within which exist 0, 2, 4 or 6 real pairs of eigenvalue-like quantities x, z . Further study of these equilibria and their stability is planned.

CONCLUDING REMARKS

Two classes of problems fall within the scope of this work. In "design problems" a desired orientation (i.e. the scalars $\xi_{\alpha\beta} \equiv \Theta_{\beta\alpha}$) and matrix I are specified. Then from Eqs. 52, c follow $\lambda = \xi_1^T I \xi_1$, $\nu = \xi_3^T I \xi_3$ whence J_1, J_3 are determined from the same equations. Thereafter, Eq. 4d determines h except as regards component J_2 which is entirely arbitrary. In "analysis problems" one is given κ_1, κ_2 and h , and must determine the equilibria. This problem now has been solved completely for the case where h is parallel or perpendicular to a principal axis. In other cases, a set of simultaneous equations has been given from which equilibrium orientations can be determined.



SEPARATRICES AND NUMBER OF REAL ROOTS IN EACH REGION OF THE PARAMETER SPACE $(1/k, j^2/(k^2-1))$, SHOWN BY THE CROSS SECTIONS AT $k=10, 2.5, 1.99$, and 1.5 .

FIGURE 4

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