WYLE LABORATORIES - RESEARCH STAFF

TECHNICAL MEMORANDUM 65-19

PROGRESS REPORT ON

JOINT ACCEPTANCE FOR CYLINDERS IN REVERBERANT ACOUSTIC FIELD

Submitted on

Brown Engineering Company Work Order No. 933-50-19-9150

Technical Directive W-12

NASA Contract No. NAS8-20073-1

Prepared by A. Wenzel

A. Wenzel

Approved by

K. McK. Eldred

Director or Research

Prepared by

L.C. Sutherland

September, 1965

JOINT ACCEPTANCE FOR CYLINDERS IN A REVERBERANT FIELD

1.0 INTRODUCTION

The problem of finding joint acceptance for any body undergoing random vibration can be divided into two parts. The first part involves finding the normal modes of vibration of the body and writing the equation for the joint acceptance squared as a function of the normal modes and the cross spectral density of the(random)forcing function. The second part involves finding or approximating the cross spectral density of the forcing function. The problem of the vibration of a thin cylinder or shell in an acoustic field is complicated by the fact that a scattering problem must be solved in order to obtain the cross spectral density of the forcing function. In such problems the second part usually presents much more difficulty than the first.

In the discussion which follows we shall consider each part of the problem separately. A brief derivation of the equation for the joint acceptance squared is given for the case of a general body undergoing random vibration. The discussion of the scattering problem will be limited to the case of a plane acoustic wave impinging on an infinite rigid cylinder.

1.1 Derivation of Equation for Joint Acceptance

The equation of motion of a body undergoing forced vibration can be written in the form

$$\frac{1}{c^2} U_{tt} + dU_t + LU = -f \quad , \qquad (1)$$

where $U(x,t)$ is the displacement of the body, c is the characteristic wave speed of the body, d is the damping coefficient, f is the external force, and L is a symmetric, positive definite linear differential operator involving only space derivatives. It is understood that the variable x is actually a vector $x = (x_1, --, x_n)$, where $n = 1, 2,$ or 3 depending on the dimension of the problem. We assume that the body occupies a domain $\mathcal{D}$ in x space, so that equation (1) holds for x in $\mathcal{D}$, and U must satisfy the necessary boundary conditions on the boundary of $\mathcal{D}$.

If $f(x)$ is of the form

$$f(x,t) = g(x) \; e^{i\omega t}$$

we assume a solution of the form

$$U(x,t) = V(x) \; e^{i\omega t} \quad .$$

Substituting into (1) we obtain

$$(-\omega^2/c^2 + i\omega d)\ V + LV = -\bar{g}\ , \text{ or}$$

$$(\omega^2/c^2 - i\omega d)\ V - LV = g\ .$$

Letting $\lambda = \omega^2/c^2 - i\omega d$ we have

(2) $$(\lambda - L)\ V = g$$

as the equation to be solved, subjected to the same boundary conditions as before.

The solutions λ_n and $\phi_n(x)$, $n = 1, 2, --$, of (2), obtained by setting $g = 0$, are the eigenvalues and orthogonal eigenvectors, respectively, of the system, where the ϕ_n's are normalized by setting

$$(\phi_m, \phi_n) = \int_{\mathcal{D}} \overline{\phi_m(x)}\ \phi_n(x)\, dx = \delta_{mn}\ .$$

Here δ_{mn} is the Kronecker delta, while the bar denotes complex conjugate.

We are interested in obtaining a solution of (1) when $f(x,t) = \delta(x - \xi)\ \delta(t)$, where δ is the Dirac delta function, and ξ is in $\mathcal{D}$. Substituting this into (1) and taking the Fourier transform of both sides we obtain

$$(-\omega^2/c^2 + i\omega d)\ \hat{G}(x, \xi, \omega) + L\ \hat{G}(x, \xi, \omega) = -\frac{1}{\sqrt{2\pi}}\ \delta(x - \xi)$$

i.e.,

(3) $$(\lambda - L)\ \hat{G} = \frac{1}{\sqrt{2\pi}}\ \delta(x - \xi)\ ,$$

Where $\hat{G}(x, \xi\ \omega)$ is the transform of the desired solution $G(x,t)$, and ω is the transform variable. Assuming a solution of (3) of the form

$$\hat{G} = \sum_{n=1}^{\infty} c_n \phi_n \tag{4}$$

where the c_n's are constants to be determined, we obtain, after substituting into (3),

$$(\lambda - L)\,\hat{G} = \sum_{n=1}^{\infty} c_n (\lambda - L)\,\phi_n = \frac{1}{\sqrt{2\pi}}\,\delta(x - \xi) \;.$$

Now

$$\sum_{n=1}^{\infty} c_n (\lambda - L)\,\phi_n = \sum_{n=1}^{\infty} c_n (\lambda - \lambda_n + \lambda_n - L)\,\phi_n =$$

$$\sum_{n=1}^{\infty} c_n (\lambda - \lambda_n)\,\phi_n \quad ,$$

since $(\lambda_n - L)\,\phi_n = 0$. Therefore we have that

$$\sum_{n=1}^{\infty} c_n (\lambda - \lambda_n)\,\phi_n = \frac{1}{\sqrt{2\pi}}\,\delta(x - \xi) \;.$$

Multiplying both sides of the expression by $\phi_m(x)$ and integrating over $\mathcal{D}$ we obtain

$$c_m (\lambda - \lambda_m) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{D}} \overline{\phi_m(x)}\,\delta(x - \xi)\,dx = \frac{\overline{\phi_m(\xi)}}{\sqrt{2\pi}} \quad ,$$

so that

$$c_m = \frac{1}{\sqrt{2\pi}} \; \frac{\overline{\phi_m(\xi)}}{\lambda - \lambda_m} \quad , \quad m = 1, 2, \ldots \tag{5}$$

We now assume that $f(x,t)$ is a random forcing function, and proceed to obtain an expression for the cross spectral density $S_u(x,y,\omega)$ of the response as a function of the cross spectral density $S_f(x,y,\omega)$ of the forcing function. The formula for $S_u(x,y,\omega)$ in terms of $S_f(x,y,\omega)$ and $G(x,\xi,\omega)$ is (see Reference 1), where x and y are any two points in $\mathscr{D}$

(6)
$$S_u(x,y,\omega) = 2\pi \int_{\mathscr{D}}\int_{\mathscr{D}} \overline{\hat{G}(x,x_1,\omega)}\ \hat{G}(y,x_2,\omega)\ S_f(x_1,x_2,\omega)\, dx_1\, dx_2 \quad .$$

From (4) and (5) we have that

$$\hat{G}(x,\xi,\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\overline{\phi_n(\xi)}\ \phi_n(x)}{\lambda - \lambda_n}$$

so that

$$\overline{\hat{G}(x,x_1,\omega)}\ \hat{G}(y,x_2,\omega) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\phi_m(x_1)\ \overline{\phi_m(x)}\ \overline{\phi_n(x_2)}\ \phi_n(y)}{\overline{(\lambda-\lambda_m)}\ (\lambda-\lambda_n)}$$

Substituting this into (6) we obtain

(7)
$$S_u(x,y,\omega) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\int_{\mathscr{D}}\int_{\mathscr{D}} S_f(x_1,x_2,\omega)\ \phi_m(x_1)\ \phi_n(x_2)\ dx_1 dx_2 \right] \frac{\overline{\phi_m(x)}\ \phi_n(y)}{\overline{(\lambda-\lambda_m)}(\lambda-\lambda_n)}$$

The quantity $\int_{\mathscr{D}}\int_{\mathscr{D}} S_f(x_1, x_2,\omega)\ \phi_m(x_1)\ \overline{\phi_n(x_2)}\ dx_1\ dx_2 \Big/ \left|S_f(x_0,x_0,\omega)\right| \left(\int_{\mathscr{D}} dx\right)^2$,

where x_0 is an arbitrary reference point, is known as the joint acceptance squared $j_{mn}^2(\omega)$ of the body for this particular type of loading.

Since the mode shapes and frequencies of a vibrating cylinder are well known (see Reference 2), the only remaining problem is to compute or estimate the cross spectral density $S_f(x,y,\omega)$.

1.2 Calculation of Cross Spectral Density

In order to calculate the cross spectral density of the forcing function, it is necessary to solve the problem of scattering by a cylinder of a plane acoustic wave whose incident ray is not necessarily perpendicular to the axis of the cylinder. The simplest case occurs when the cylinder is infinitely long, since then there are no boundary conditions at the ends of the cylinder. This problem has been solved for the case when the incident ray is perpendicular to the axis of the cylinder by the method of separation of variables (see Reference 3). During the past month this method was extended to the case of the oblique incident wave as follows: In cylindrical coordinates the equation to be solved for the scattered wave is

$$(\nabla^2 + k^2)\ \phi_s = 0 \quad , \text{ i.e.,}$$

$$(8) \qquad \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\phi_s = 0 \quad .$$

The equation for the incident wave takes the form

$$\phi_i = Ae^{ik(x\cos\beta + z\sin\beta)} = Ae^{ik(r\cos\theta\cos\beta + z\sin\beta)} \quad ,$$

where β is the angle between the normal to the incident wave front and the axis of the cylinder. Taking the velocity potential ϕ of the total wave to be $\phi_i + \phi_s$, the boundary condition for ϕ is $\frac{\partial\phi}{\partial n} = 0$ on the surface of the cylinder. This implies that for $r = a$,

$$\frac{\partial\phi_s}{\partial r} = -\frac{\partial\phi_i}{\partial r} \quad , \text{ i.e.,}$$

$$(9) \qquad \frac{\partial\phi_s}{\partial r} = -ik\cos\theta\cos\beta\, Ae^{ik(a\cos\theta\cos\beta + z\sin\beta)}$$

when $r = a$. If we assume a solution of (8) of the form $\phi_s = G(r,\theta)\,F(z)$, then Equation (8) takes the form

$$\frac{F}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{F}{r^2}\frac{\partial^2 G}{\partial\theta^2}+G\frac{d^2F}{dz^2}+k^2FG=0\quad,$$

i.e.,

$$\frac{1}{G}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{G}\cdot\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2}+\frac{1}{F}\frac{d^2F}{dz^2}+k^2=0\quad,$$

or

$$\frac{1}{G}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{G}\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2}+k^2=-\frac{1}{F}\frac{d^2F}{dz^2}=\text{constant}=$$

k_3^2 say, so that

(10) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2}+(k^2-k_3^2)\;G=0\quad,$$

(11) $$\frac{d^2F}{dz^2}+k_3^2\;F=0\quad.$$

A solution of (11) is $F(z)=e^{ik_3z}$, and from the boundary condition (9) we see that we must take $k_3=k\sin\beta$. Then letting $K^2=k^2-k_3^2$ we have

$$K^2=k^2-k^2\sin^2\beta=k^2(1-\sin^2\beta)=k^2\cos^2\beta\quad,$$

so that equation (10) becomes

(12) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G}{\partial\theta^2}+K^2G=0\quad.$$

The boundary condition (9) takes the form

$$\frac{\partial G}{\partial r}(a,\theta)\,F(z) = -K \cos\theta \, Ae^{iKa\cos\theta} \, e^{ik_3 z} \quad , \text{ or,}$$

since $F(z) = e^{ik_3 z}$,

$$\frac{\partial G}{\partial r}(a,\theta) = -iK \cos\theta \, Ae^{iKa\cos\theta} \quad . \tag{13}$$

The differential equation (12) and the boundary condition (13) are seen to be identical (if we replace k by K) with those of the plane wave at normal incidence, the solution of which is known. Therefore, by taking this solution and multiplying by $F(z) = e^{ik(\sin\beta)z}$ we obtain the solution for the oblique incident wave.

REFERENCES

1. Robson, J.D., "Random Vibration", Elsevier Publishing Co., New York, N.Y., 1963.

2. Bleich, H.H., and Baron, M.L., "Free and Forced Vibrations of an Infinitely Long Cylindrical Shell in an Infinite Acoustic Medium," Journal of Applied Mechanics, Vol. 21, p. 167-177, 1954.

3. Morse, P.M., "Vibration and Sound", McGraw-Hill Book Co., New York, 1948.

JOINT ACCEPTANCE FOR A SIMPLY SUPPORTED BEAM IN A DIFFUSE SOUND FIELD

The final solution for the joint acceptance of a cylindrical shell in a diffuse sound field will be developed along the lines outlined in the first section.

The joint acceptance for this case will differ from known solutions for a flat plate in a progressive wave sound field for two obvious reasons. The structure is cylindrical instead of flat and located in a diffuse field instead of a progressive wave field. To indicate the significance of the latter aspect, the joint acceptances for a simple supported beam in a diffuse and progressive wave field are compared in the following.

The general expression for the joint acceptance of any structure may be given in the form

$$j^2 = \frac{1}{A^2} \oint \oint R(x,x') \; \phi(x) \, \phi(x') \, dx, \, dx' \qquad (1)$$

where x and x' represent two elemental areas on the surface with modal deflections $\phi(x)$ and $\phi(x')$ respectively. The quantity $R(x,x')$ is the normalized space correlation coefficient.

For a simply supported beam of length w

$$\phi(x) = \sin n\pi x/w \qquad (2)$$

For a homogeneous plane progressive wave sound field, the space correlation coefficient is

$$R(x,x') = \cos K(x - x') \qquad (3)$$

where $K = 2\pi/\lambda$ and λ is the trace wavelength of the acoustic field along the length of the beam.

For a diffuse sound field, the space correlation coefficient is

$$R(x,x') = \frac{\sin K(x - x')}{K(x - x')} \qquad (4)$$

For the plane progressive wave, Equations 1, 2, and 3 give the closed solution

$$j_n^2 = \frac{2}{(\pi n)^2} \left[\frac{1}{1 - (Kw/\pi n)^2} \right]^2 \left[1 - \cos(\pi n) \cos Kw \right] \qquad (5)$$

For a diffuse field, a closed solution has not been found and the result must be left as the integral

$$j_n^2 = \frac{1}{w^2} \int_0^w \int_0^w \frac{\sin K(x - x')}{K(x - x')} \sin \frac{n\pi x}{w} \sin \frac{n\pi x'}{w} \, dx \, dx' \qquad (6)$$

This integral has been programmed for a numerical integration.

Figure 1 shows a comparison of the results computed for the first three modes from Equations 5 and 6. for a particular beam where $w = 25''$.

The results show that the diffuse field serves to smooth out the sharp nulls in joint acceptance present for a progressive field. In addition, the joint acceptance for a diffuse field is much higher, at high frequencies and tends to become independent of mode number.

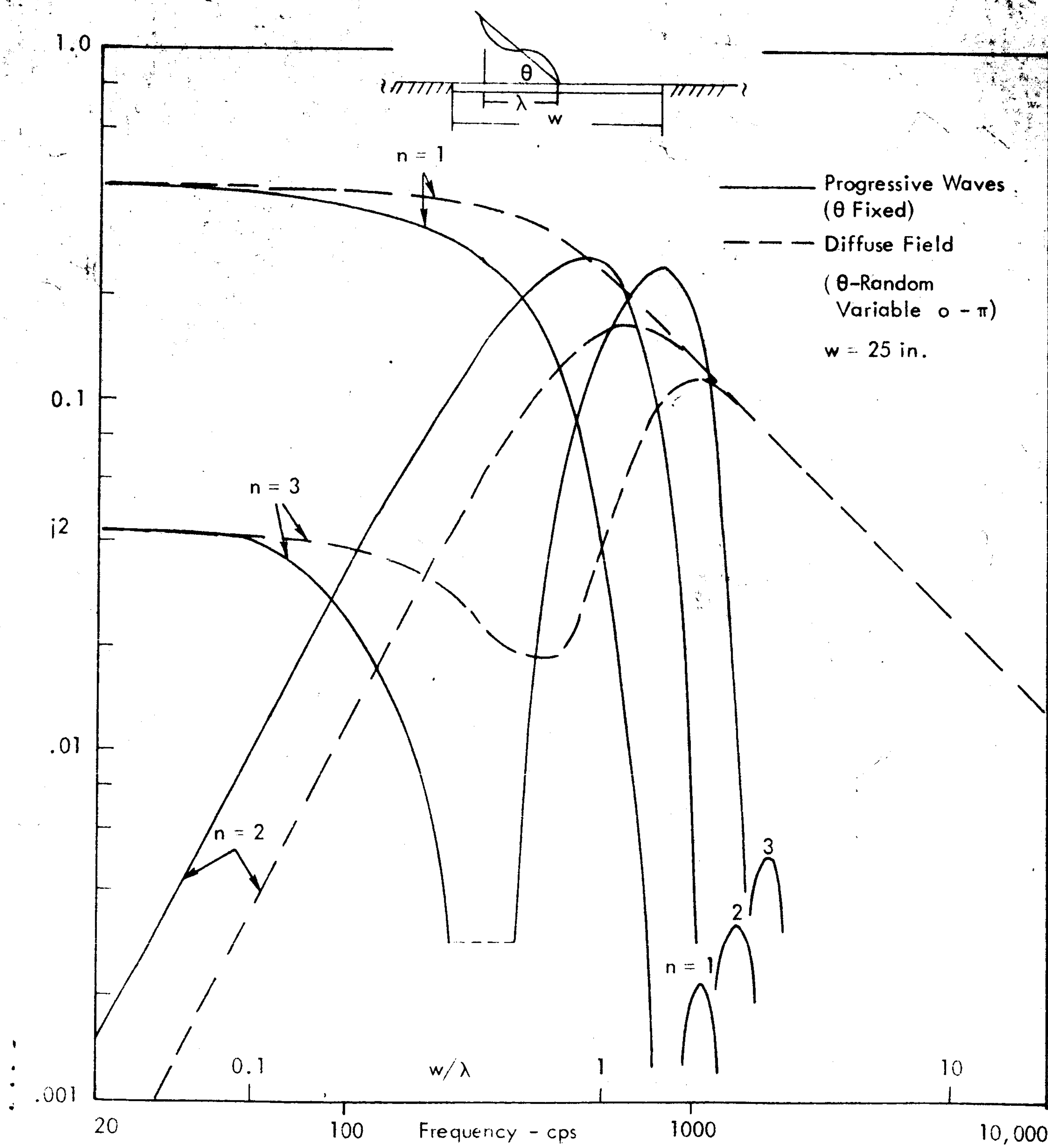

Figure 1. Comparison of Joint Acceptance for First Three Modes of a Simply Supported Beam in a Plane Progressive Sound Field and a Diffuse Sound Field