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Non-linear waves in Plasmas and disk-like Galaxies

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# Non-linear waves in Plasmas and disk-like Galaxies

The object of this paper is to draw attention to the fact that the analogy between cooperative waves in plasmas and in systems involving gravitational interaction extends beyond the commonly explored [(1)(2)] domain of linear approximations. We shall show on a simple model[(3)] for a disk-like galaxy, the counterpart to a "nonlinear cold plasma". A characteristic feature of the associated waves is (see fig. 1 ) that even for infinitesimally small amplitudes one may have arbitrarily large gradients and hence very strong bunching up of particles. For gravitational attraction, this may lead to local instabilities growing at much faster rate than predicted by the linear analysis. It also turns out that these nonlinear modes in general display azimuthal asymmetries of the spiral type.[(4)] It is thus expected that their introduction will have bearing on current endeavors on the exploration of stability[(5)-(12)] and of the nature of spiral arms [(13)-(18)] of galactic models. Another application of the method would be to radial and nonradial, non-linear oscillations of variable stars.[(19)-(24)] However, a complete solution of the nonlinear problem is of course formidable even in the simplest examples, and at one point or another one has to resort to some approximation procedures. We here propose therefore to introduce the method only as far as it can be carried exactly and merely to indicate the possible approximation procedures which may then be taken in order to continue the solution. A more systematic attack on the latter is hoped to be dealt with in another paper, in conjunction with C. Cuvaj.

In Part I we shall re-formulate the problem of a nonlinear cold plasma in a way which is somewhat different from previous formulations.*
(see next page for footnote)

Most of the results here obtained have been established, in portions, in a number of publications ranging over the past 8 or 9 years.[26]-[28] However, our main result, equations (11), (13), (19), which is particularly convenient for fast iteration, seems to have remained unnoticed. Also, the exact solution (23) for the three dimensional case is, as far as we could ascertain, new. The main point in the present representation is that it works from the start - and obtains the exact solutions - in the Eulerian representation, whereas other treatments use the Lagrangian representation in varying degrees.

The convenient form (11) in which the Eulerian solutions are obtained should save much work which has often been left to electronic computers.[29]

* (footnote for page #1) The results of Part I were obtained in 1957/58, while the author was a graduate student at Bristol University, England, but remained unpublished.[25] In connection with the work of Part I, the author wishes to thank Prof. D. Bohm for criticism and discussions, Prof. M.H.L. Pryce for the hospitality at the H.H. Wills laboratories and the N.S.D.I.R. of the British Government for a grant at that period.

# Part I : Non linear waves in a Plasma

## I.1. The equations of a cold Plasma

We consider the model of a negatively charged, continuous fluid, representing approximately a nondegenerate gas of electrons of mass m and charge $e = -|e|$. This fluid is immersed in a background of a uniform, stationary positive charge distribution of charge density $|n_o|$ which is equal in magnitude to the average charge density $n_o$ of the electron fluid. We assume that the fluid is cold, i.e., at each point in space, $\vec{x}$, only one value of velocity exists, $\vec{v}(\vec{x},t)$ (as far as the hydrodynamical description is single-valued). Let D/Dt be the convective derivative for the fluid, $\vec{E}(\vec{x},t)$ the electric field produced at $\vec{x},\vec{t}$ by all charges present, $\rho(x,t) + n_o$ the density of the electron fluid. If we assume the velocities to be slow enough for magnetic effects to be negligible, the basis equations of the model are:

$$m \frac{D\underset{\sim}{v}}{Dt} = e\, \underset{\sim}{E}, \tag{1a}$$

$$\text{div}\, \underset{\sim}{E} = 4\pi\rho \quad (\rho > 0 \text{ means deficiency of electrons}), \tag{1b}$$

$$\frac{d\underset{\sim}{E}}{dt} + 4\pi(\rho + n_o)\, \underset{\sim}{v} = 0. \tag{1c}$$

(1c) is the third Maxwell equation in the absence of magnetic field; its div gives the conservation equation for $\rho + n_o$. Taking D/Dt of (1a) one obtains

$$\frac{\partial \underset{\sim}{E}}{\partial t} + (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{E} = \frac{m}{e} \frac{D^2 v}{Dt2},$$

and by eliminating $\rho$ from (1b) and (1c) one obtains

$$\frac{\partial \underset{\sim}{E}}{\partial t} + \underset{\sim}{v}\, \text{div}\, \underset{\sim}{E} = 4\pi\, n_o\, \underset{\sim}{v}.$$

From these two equations one obtains, by subtraction

$$\frac{D^2\underset{\sim}{v}}{Dt^2} + \omega_\rho^2 \underset{\sim}{v} = (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \frac{D\underset{\sim}{v}}{Dt} - \underset{\sim}{v} \operatorname{div} \frac{D\underset{\sim}{v}}{Dt} \; ; \tag{2}$$

where $$\omega_\rho^2 = 4 \pi n_o e/m \tag{3}$$

This is an exact non-linear equation for $\underset{\sim}{v}(\underset{\sim}{x},t)$ alone. An equation for $\rho$, which, however, involves $\underset{\sim}{v}$ too, can also be obtained: the conservation equation $D\rho/Dt + (n_o + \rho) \operatorname{div} \underset{\sim}{v} = 0$ can be written in the form

$$\frac{D}{Dt} \ln (\rho + n_o) = \operatorname{div} \underset{\sim}{v}.$$

If the D/Dt of this equation is taken, and

$$\frac{D}{Dt} \operatorname{div} \underset{\sim}{v} = \operatorname{div} \frac{D\underset{\sim}{v}}{Dt} - \operatorname{Tr}(S^2) \tag{4}$$

is used, where Tr is the trace and S is the "dissipation matrix"

$$S = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_2}{\partial x} & \frac{\partial v_3}{\partial x} \\ \frac{\partial v_1}{\partial y} & \frac{\partial v_2}{\partial y} & \frac{\partial v_3}{\partial y} \\ \frac{\partial v_1}{\partial z} & \frac{\partial v_2}{\partial z} & \frac{\partial v_3}{\partial z} \end{bmatrix} ,$$

one obtains

$$\frac{D^2}{Dt^2} \ln (\rho + n_o) + \frac{\omega_\rho^2}{n_o} \rho = \operatorname{Tr}(S^2) \tag{5}$$

<u>I.2. The exact solution for the one-dimensional case</u>.

In the one-dimensional case, (4) and (5) reduce to

$$\frac{D^2\underset{\sim}{v}}{Dt^2} + \omega_\rho^2 \underset{\sim}{v} = 0, \tag{6}$$

$$\frac{D^2 \ln(\rho + n_o)}{Dt^2} - \frac{D \ln(\rho + n_o)}{Dt} + \frac{\omega_\rho^2}{n_o} \rho = 0, \tag{7}$$

because in one dimension $\underset{\sim}{v}\cdot\underset{\sim}{\nabla} = \underset{\sim}{v}$ div and

$Tr(S^2) = (\partial v/\partial x)^2 = [(1/\rho+n_o)D\rho/Dt]^2 = [D \ln(\rho+n_o)/Dt]^2$

by virtue of the conservation equation for $\rho$. (7) can be further simplified:

$$\frac{D^2\rho}{Dt} + \frac{\omega_\rho^2}{n_o}\rho(\rho+n_o) = 0. \tag{7a}$$

(6) seems to have been obtained first by Polovin et al[30] (from the one-dimensional equations of motion) who, however, did not attempt an exact solution. Sturrock[31] obtained the equivalent of (3) and (5) by working from the start in the fourier representation, and he solved them by an approximation procedure in that representation up to the second approximation.

We shall first look for complex solutions of (6). Led by the analogy that $Q_k \sim \sum_i \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}^i)/(1 - \frac{\underset{\sim}{k}\cdot\underset{\sim}{v}^i}{\omega})$ solves the case of linear (but non-continuous) plasma,( ) one may consider the denominator of $Q_k$ as the first order expansion of $\exp(-\underset{\sim}{k}\cdot\underset{\sim}{v}^i/\omega_\rho)$, and one is thus led to try solving (6) by

$$v(x_1 t) = \alpha e^{i(kx - \omega_\rho t)}\, e^{kv/\omega_\rho}, \tag{8}$$

where $\alpha$ is a constant. Indeed, since

$(\partial/\partial t + v\partial/\partial x) \exp[i(kx - \omega_\rho t)] = -i(\omega_\rho - kv) \exp[i(kx - \omega_\rho t)]$,

the D/Dt of (8) gives

$$\frac{Dv}{Dt} = -i(\omega_\rho - kv)\, v + \frac{k}{\omega_\rho}\frac{Dv}{Dt} v, \qquad \text{i.e.,}$$

$$\frac{Dv}{Dt}(\omega_\rho - kv) = -i\,\omega_\rho(\omega_\rho - kv)\, v,$$

so that, as long as $\omega_\rho \neq kv$,

$$\frac{Dv}{Dt} + i\,\omega_\rho\, v = 0. \tag{9}$$

But since

$$\frac{D^2 v}{Dt^2} + \omega_\rho^2\, v = \left(\frac{D}{Dt} + i\,\omega_\rho\right)\left(\frac{D}{Dt} - i\,\omega_\rho\right) v,$$

(8) solves also the equation (6). (8) defines v only implicitly, since v appears also on the right-hand side. For small $kv/\omega_\rho$ we have $v \approx \alpha \exp\,(i(kx-\omega_\rho t))$, i.e., we have the linear solution. For larger $kv/\omega_\rho$, the factor $\exp\,(kv/\omega_\rho)$ modulates the amplitude in a way which depends on itself, i.e., in a typical non-linear way. (8) can then be solved by an iteration procedure, starting with the linear solution. However, (8) can easily be thrown into the form $z \exp z = a$, where $z = (-kv/\omega_\rho)$ and $a = (-k\alpha/\omega_\rho) \exp\,[i(kx-\omega_\rho t)]$. This equation has been investigated for over a hundred years by various mathematicians (see Wright)[32] and their results can be used to obtain further physical properties.*

As can be seen by straightforward differentiation, the solution (8) can be generalized: $\alpha_i$ and $k_i$ being arbitrary, also

$$v(x_1 t) = \sum_i \alpha_i e^{i(k_i x - \omega_\rho t)}\, e^{(k_i v/\omega_\rho)} \tag{10}$$

is a solution of (6).

So far, we have disregarded the fact that (8) is a complex solution, whereas v is to be real. Hence, we shall have to modify

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* I am indebted to A.A. Cottey for drawing my attention to the fact that the solution (8) can also be obtained in the following way:
Assuming a plane-wave solution $v = v(\xi)$ where $\xi = kx - \omega_\rho t$ we have that

$$(kv - \omega_\rho)\frac{dv}{d\xi} + i\,\omega_\rho v = 0$$

and this integrates to give $kv - \omega_\rho \ln v + i\,\omega_\rho \xi = \text{const}$, from which (8) follows.

our method of solution somewhat. By taking the D/Dt of (6) and using $E = (m/e)\, Dv/Dt$ we notice that $E$ too fulfills $D^2E/Dt^2 + \omega_\rho^2 E = 0$. To obtain a real solution for both, we write

$$v = \alpha \cos\left(kx - \omega_\rho t + \frac{kE}{\omega_\rho^2} + \theta\right), \tag{11a}$$

$$E = \omega_\rho\, \alpha \sin\left(kx - \omega_\rho t + \frac{kE}{\omega_\rho^2} + \theta\right), \tag{11b}$$

where $\theta$ is a constant phase factor.

Indeed (denoting Df/Dt from now on by $\dot{f}$),

$$\dot{v} = (\omega_\rho - kv)\frac{E}{\omega_\rho} - \frac{k\dot{E}E}{\omega_\rho^3},$$

$$\frac{\dot{E}}{\omega_\rho} = -(\omega_\rho - kv)\, v + \frac{k\dot{E}v}{\omega_\rho^2},$$

i.e., $\dot{E}(\omega_\rho - kv) = -\omega_\rho - kv)\, v$,

so that as long as $\omega_\rho \neq kv$

$$\begin{matrix} \dot{E} = -\omega_\rho^2 v & & \ddot{E} + \omega_\rho^2 E = 0 \\ & \text{giving} & \\ \dot{v} = E & & \ddot{v} + \omega_\rho^2 v = 0 \end{matrix} \quad . \tag{12}$$

Hence (11) solves our basic set of equations (1). Again it can be shown that also

$$v = \sum_i \alpha_i \cos\left(k_i x - \omega_\rho t + \frac{k_i E}{\omega_\rho^2} + \theta_i\right), \tag{13a}$$

$$E = \omega_\rho \sum_i \alpha_i \sin\left(k_i x - \omega_\rho t + \frac{k_i E}{\omega_\rho^2} + \theta_i\right), \tag{13b}$$

solves (1).

(11) can be condensed into the complex form (where v and E are

now real)

$$v + \frac{iE}{\omega_\rho} = \alpha e^{\,i(kx - \omega_\rho t)}\, e^{\frac{i\,k\,E}{\omega_\rho^2}} \qquad (14)$$

If we write $z = x + iv/\omega_\rho$, $w = \dot{z} = v + iE/\omega_\rho$,

(14) can be written

$$w = \alpha e^{i(kz - \omega_\rho t)}\, e^{(kw/\omega_\rho)}. \qquad (15)$$

The implicit solution (11b) for E is independent of v; denoting the phase $kx = \omega_\rho t$ by $\xi$, we have $E = \omega_\rho \alpha \sin(\xi + kE/\omega_\rho^2)$.

The graphic representation of $E(\xi)/\alpha$ is a sine-curve which is tilted to the right, as shown in fig. 1:

If $\alpha k/\omega_\rho = 1$ the derivative $\partial E/\partial \xi$ becomes infinite at the points $\xi = 0$ and if $\alpha k/\omega_\rho > 1$ the graph becomes at first three-valued in a certain range, and if $\alpha k/\omega_\rho$ is further increased, it becomes 5,7,9...-valued. This indicates that shock conditions are obtained when $\alpha_k/\omega_\rho = 1$, i.e., continuity is destroyed.

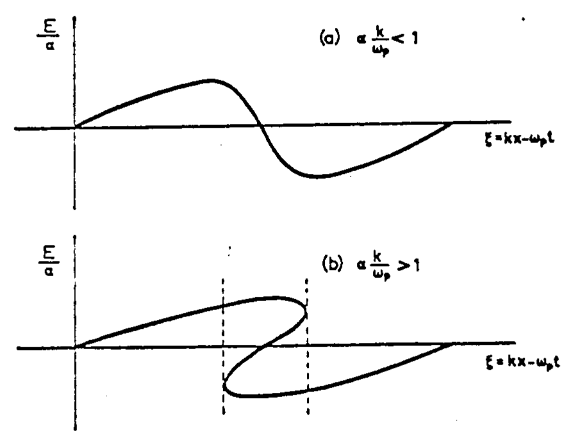


FIG. 1. Graphical representation of $E = \alpha \sin(kx - \omega_p t + \frac{kE}{\omega_p^2})$

Fig. 1. Graphical representation of

$$E = \alpha \sin\left(kx - \omega_\rho t + \frac{kE}{\omega_\rho^2}\right)$$

By taking $k_2 = -k$ and $\alpha_1 = \alpha_2$ in (13), one verifies that also

$$v = \alpha \cos \omega_\rho t \cos k\, (x + \frac{E}{\omega_\rho^2}),$$

$$E = \omega_\rho\, \alpha \sin \omega_\rho t \cos k\, (x + \frac{E}{\omega_\rho^2}),$$

are solutions, and by "superposing" these solutions for different k's in the above sense, one obtains the general one-dimensional solution of (1) in the form

$$E = \omega_\rho f(x + \frac{E}{\omega_\rho^2}) \sin\, (\omega_\rho t + \theta),$$

$$v = f(x + \frac{E}{\omega_\rho^2}) \cos\, (\omega_\rho t + \theta), \qquad (16)$$

where f is an arbitrary function. This can also be verified by straightforward differentiation. We shall now proceed to interpret these solutions physically.

Integrating $\dot{v} + \omega_\rho^2\, v = 0$, we have

$$\frac{D^2 x}{Dt^2} + \omega_\rho^2\, (x - x_o) = 0, \qquad (17)$$

where $\frac{Dx_o}{Dt} = 0$.

Here $x = x(t,x_o)$ is the position at time t of a fluid-element, whose equilibrium position is $x_o$. The various elements execute harmonic oscillations, each with another center and another maximum amplitude. These centers and amplitudes are arbitrary (except for certain continuity conditions), and therefore the elements may crowd up, at various phases of their motion, in a way which gives the space functions $\rho(x,t)$ and $v(x,t)$ their non-linear character.

(17) also explains why (16) is a solution: since $m\ddot{x} = eE$ and $E = -\,\omega_\rho^2\, (x-x_o)$, the argument $x + E/\omega_\rho^2)$ fulfills $(D/Dt)(x+(E/\omega_\rho^2))=0$,

and therefore it plays the role of a constant when (16) is differentiated by D/Dt. From the point of view of the theory of characteristics of partial differential equations, D/Dt is a differentiation in a characteristic direction in the space x,t,E,v[33] and the solution (16) is essentially written in terms of the parameters $\alpha = x + (E/\omega_\rho^2)$, $\beta = t$ along the characteristic lines. The method of characteristics has been used in the present case by Dolph.[28]

Since $x + (E/\omega_\rho^2) = x_o$ with $Dx_o/Dt = 0$, (13b) can also be written in the form

$$x(t,x_o) = x_o + A(x_o) \sin \omega_\rho t + B(x_o) \cos \omega_\rho t. \tag{18}$$

This is the solution in the Lagrangian representation; it first appeared in literature in a papér by Dawson[26] who obtained it from physical considerations: if a fluid particle is displaced by $x - x_o$ to the right, say, and if overtaking of one particle by another is excluded, it will feel a restoring force $(- \omega_\rho{}^2)(x-x_o)$ coming from the left (since the amount of negative charge on the left has not changed, but the amount of background charge has increased by $-n_o(x - x_o)$ ).

Before we proceed to the three-dimensional case, we should like to mention some more properties of the one-dimensional case.

1) Equation (7a) for $\rho$ can be solved straightforwardly in the $x_o$ frame (i.e., in the Lagrange representation of the fluid): multiplying (7a) by $\dot{\rho}$, we find the first integral

$$\dot{\rho}^2 + \omega_\rho^2 \rho^2 + \frac{2}{3} \frac{\omega_\rho^2}{n_o} \rho^3 = \text{constant},$$

which shows the characteristic behavior for non-linear oscillations: if the $\rho^3$ term were absent, the corresponding curves in $\rho$, $\dot{\rho}$ -space would be ellipses; the $\rho^3$ -term distorts these ellipses towards negative values of $\rho$ so that the maximum excess of electrons is greater than the maximum electron deficiency.

2) Multiplying $\dot{E} + \omega_\rho^2 v = 0$ by E and integrating, we obtain $E^2 + \omega_\rho^2 v^2 = h^2$, where $Dh^2/Dt = 0$. This expresses the energy conservation per particle. Conversely, we could have obtained our solution (11) from this conservation equation, since $E^2/h^2 + \omega_\rho^2 v^2/h^2 = 1$ admits $E^2 = h^2 \sin^2 \omega_\rho t$, $\omega_\rho^2 v^2 = h^2 \cos^2 \omega_\rho t$.

3) The question naturally arises, whether our solution can be generalized to include also thermal motion. As a first step in that direction one might think of the beams-method of Bohm and Gross.[34] One is thus led to consider the case of two charged fluids, $\rho_1, v_1, E_1$ and $\rho_2, v_2, E_2$ superposed on the same background $n_o$. Both produce the field $E = E_1 + E_2$ through

$$\mathrm{div}\,(E_1 + E_2) = 4\pi\,(\rho_1 + \rho_2 - n_o)$$

($\rho_i$ is now the total charge density of the i'th fluid, and not its excess over $n_o$); they fulfill separate conservation equations

$$\frac{\partial \rho_1}{\partial t} + \mathrm{div}\,\rho_1 v_1 = 0\ ;\ \frac{\partial \rho_2}{\partial t} + \mathrm{div}\,\rho_2 v_2 = 0,$$

and they are driven by the total field

$$\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} = E_1 + E_2\ ;\ \frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x} = E_1 + E_2.$$

However, no simple solution to these equations, corresponding to the one-fluid case, was found. Also the method of characteristics seems to give no straightforward solution. The physical reason for the complications can be explained by the fact that the corresponding $x^i(t)$

equations are $\ddot{x}_1 = -\omega_\rho^2(x_1 - f_1) - \omega_\rho^2(x_1 - f_2)$ where $(\partial f_1/\partial t)+v_1(\partial f_1/\partial x)= 0$ and $(\partial f_2/\partial t) + v_2(\partial f_2/\partial x) = 0$. Thus, the particle $x_1$ of the first fluid is subject not only to a restoring force towards a constant center $f_1$, but also to a restoring force towards a varying center $f_2$. The varying values of $f_2$ are the centers of the particles of the second fluid which the first particle passes on its way. The motions of the particles are therefore coupled in a complicated way and no simple solution should be expected.

## I.3. Some exact solutions of the three-dimensional case

Our results for the three-dimensional case, i.e., for the solution of (3), can be stated very shortly: All the solutions (11), (15), (16), (17), (18), hold also for the three-dimensional case; the more general solutions (10) and (13), however, do not hold any longer, except for some extreme cases of parallelism or perpendicularity of the corresponding modes.

We shall prove this for the solution

$$v + \frac{iE}{\omega_\rho} = \alpha\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega_\rho t)}\, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{E})/\omega_\rho^2}. \tag{19}$$

We have

$$\dot{\underset{\sim}{v}} + \frac{i\dot{\underset{\sim}{E}}}{\omega_\rho} = -i(\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{v} + \frac{i\underset{\sim}{E}}{\omega_\rho}) + \frac{i\underset{\sim}{k}\cdot\dot{\underset{\sim}{E}}}{\omega_\rho^2}(\underset{\sim}{v} + \frac{i\underset{\sim}{E}}{\omega_\rho}).$$

The real part gives

$$\dot{\underset{\sim}{v}} = (\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v})\frac{\underset{\sim}{E}}{\omega_\rho} - (\frac{\underset{\sim}{k}\cdot\underset{\sim}{E}}{\omega_\rho^3})\underset{\sim}{E},$$

and the imaginary part gives

$$\dot{\underset{\sim}{E}} = -(\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v})\,\omega_\rho\underset{\sim}{v} + (\frac{\underset{\sim}{k}\cdot\underset{\sim}{E}}{\omega_\rho})\underset{\sim}{v}.$$

This means that $\dot{\underset{\sim}{E}}$ is parallel to $\underset{\sim}{v}$, and hence $(\underset{\sim}{k}\cdot\underset{\sim}{E})\,\underset{\sim}{v} = (\underset{\sim}{k}\cdot\underset{\sim}{v})\,\dot{\underset{\sim}{E}}$

and

$$\dot{\underset{\sim}{E}} = -\,(\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v})\,\omega_\rho\,\underset{\sim}{v} + \left(\frac{\underset{\sim}{k}\cdot\underset{\sim}{v}}{\omega_\rho}\right)\dot{\underset{\sim}{E}},\ \text{i.e.,}$$

$$\dot{\underset{\sim}{E}}\,(\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v}) = -\,(\omega_\rho - \underset{\sim}{k}\cdot\underset{\sim}{v})\,\omega_\rho^2\,\underset{\sim}{v},\ \text{i.e.,}$$

$$\dot{\underset{\sim}{E}} = -\,\omega_\rho^2\,\underset{\sim}{v}.$$

Inserted in the $\dot{\underset{\sim}{v}}$ -equation, this gives $\dot{v} = E$. But $(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})(D\underset{\sim}{v}/Dt) - \underset{\sim}{v}\,\mathrm{div}(D\underset{\sim}{v}/Dt)$ can be seen to vanish for (19) since, as a simple calculation shows, both terms give $\omega_\rho(\underset{\sim}{k}\cdot\underset{\sim}{v}/(1 - \underset{\sim}{k}\cdot\underset{\sim}{v}/\omega_\rho))\underset{\sim}{v}$. The system (1) reduces therefore to $\dot{\underset{\sim}{E}} = -\,\omega_\rho^2\,\underset{\sim}{v}$, $\dot{\underset{\sim}{v}} = \underset{\sim}{E}$ q.e.d. This result should have been expected: a mode which is of the plane-wave form will remain so, by the conservation of the circulation. On the other hand, plane-wave modes of different directions will interact and will dissipate their energy by exciting all the other plane-wave modes. This fact, which is indicated by the dissipation* -term $\mathrm{Tr}(s^2)$ in (5), was proved by Sturrock.[31] Therefore the superposition property (10) will no longer hold. Indeed, $\underset{\sim}{v} = \sum_i \underset{\sim}{\alpha}_i \cos\xi_i$ and $\underset{\sim}{E} = \omega_\rho \sum_i \alpha_i \sin\xi_i$: will still be a solution of $\dot{\underset{\sim}{E}} = -\omega_\rho^2\,\underset{\sim}{v}$ and $\dot{\underset{\sim}{v}} = \underset{\sim}{E}$ but now $(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\,D\underset{\sim}{v}/Dt - \underset{\sim}{v}\,\mathrm{div}\,(D\underset{\sim}{v}/Dt) \neq 0$.

It is quite difficult to obtain exact solutions in three-dimensions for which the right-hand side of (3) does not vanish. Various considera-tion, which will not be entered here, led us to expect that the "Ansatz"

$$\begin{aligned}\underset{\sim}{v} &= X\,(\underset{\sim}{a}_1 \cos\phi - \underset{\sim}{\alpha}_2 \sin\phi),\\ \underset{\sim}{E} &= \dot{\underset{\sim}{v}} = \dot{X}\,(\underset{\sim}{\alpha}_1 \cos\phi - \underset{\sim}{\alpha}_2 \sin\phi) - X\,\dot{\phi}\,(\underset{\sim}{\alpha}_1 \sin\phi + \underset{\sim}{\alpha}_2 \cos\phi),\end{aligned} \tag{20}$$

would lead to a solution, where $X(\underset{\sim}{x}_1 t)$ and $\phi(\underset{\sim}{x}_1 t)$ are scalar functions to be determined by the equations (2). This can be demonstrated in

* $\mathrm{Tr}(s^2)$ appears also in the entropy-relation for viscous fluids, cf., c.g., Lamb, Hydrodynamics.

some simplified cases. For instance, if the assumptions

$$\frac{DX}{Dt} = 0, \tag{21}$$

$$\frac{D_\epsilon X}{Dt} \equiv \frac{\partial X}{\partial t} + (\underset{\sim}{\alpha}_1 \sin \phi + \underset{\sim}{\alpha}_2 \cos \phi) \cdot \underset{\sim}{\nabla} X = 0,$$

are made, one obtains for X and $\phi$ the following equations:

$$\begin{aligned} &\frac{\partial}{\partial t} (X\dot{\phi}) = 0, \\ &\omega_\rho^2 - \dot{\phi}^2 - X \frac{D_\epsilon \dot{\phi}}{Dt} = X (\underset{\sim}{\alpha}_1 \sin \phi + \underset{\sim}{\alpha}_2 \cos \phi) \cdot \underset{\sim}{\nabla}(\dot{\phi}), \end{aligned} \tag{22}$$

where $\underset{\sim}{\epsilon} = \underset{\sim}{\alpha}_1 \sin \phi + \underset{\sim}{\alpha}_2 \cos \phi$ and $D_\epsilon/Dt = \partial/\partial t + (\underset{\sim}{\epsilon} \cdot \underset{\sim}{\nabla})$.

The second of these equations is separable, $D_\epsilon \dot{\phi} / (\omega_\rho^2 - \dot{\phi}^2) = (1/X)\, Dt$ and gives $\dot{\phi} = \mathrm{tgh}\ (\omega_\rho / X)(t - t_o)$, where $D_\epsilon t_o/Dt = 0$.

If it is assumed that $t_o$ is a pure constant, one obtains

$$\phi = X\, \ln \cos h \frac{\omega_\rho}{X} (t - t_o) + \phi_o, \text{ with } \dot{\phi}_o = 0. \tag{23}$$

This provides a special exact solution of (3), for which the right-hand side of (3) is not zero. It should be noted that for $\underset{\sim}{x} \to \infty$, $\phi$ tends to $(1/2)\, \omega_\rho t)$. In attempting to find more general solutions, the following lemma (which can easily be proved), may serve as a time-saver:

If $\underset{\sim}{E}(h)$ and $\underset{\sim}{v}(h)$ are a solution of the set (2) where $h = h(\underset{\sim}{x}, t, \underset{\sim}{E})$ is any function, then the right-hand side of (3) vanishes for this solution, i.e., (2) reduces to

$$\frac{e}{m} \dot{\underset{\sim}{E}} + \omega_\rho^2 \underset{\sim}{v} = 0, \; \dot{\underset{\sim}{v}} = \frac{e}{m} \underset{\sim}{E}.$$

## Part II. Nonlinear density waves in a disk-model for a galaxy

### II.1. Discussion of the model and a general physical picture of the modes.

The model of an infinitesimally thin disk of matter, whose parts are subject only to mutual gravitational attraction and to centrifugal forces, has frequently been used to represent the star-distribution in SO-galaxies and for the major portions of all ordinary spiral galaxies. Especially during the last 2-3 years, the question of the gravitational instabilities [5]-[12] of such disks and the possibility of spiral arms[13]-[18] has received much attention, whereas previously efforts were directed mainly at finding consistent distributions of mass, potential and angular velocity over such disks.[35]-[41]

Such models seem to be quite realistic because of the high (~ 10:1) degree of flattening observed in such galaxies; because it now appears that interstellar matter makes only a few percent of their total mass; because stellar encounters are too rare to make consideration of collision necessary, except at the center, and because magnetic forces seem to be small. The only other factor which need be considered is the random motion of the stars; this is a stabilizing factor which introduces a lower limit to the characteristic lengths of instabilities, but may otherwise be left out.[5]

Mestel[42] divides disk-like galaxies roughly into two classes: those like M 31 or our own whose rotational (linear) velocity, V(r), is roughly uniform over a wide range of distances from the center*, and

* (quoted from (42) ): "According to Kwee et al., a value of 200 km/sec is correct [in our galaxy] to within 12 or 13 percent over the range of 1.9-8.2 kpc.; between 2.93 and 8.2 kpc, a value of 210 km/sec is correct to about 7 percent."

those whose angular velocity, $\Omega(r)$, is roughly uniform (solid-body rotation). In a careful mathematical analysis, Hunter[43] finds that these two cases are only the two extreme ends (with $n = \infty$ for the $V$ = const-case and $n = 1$ for $\Omega$-const) of a whole chain, $V_n$, $\Omega_n$ of models, in which the density functions $\sigma_n(r)$ are proportional to the m-n-th associated Legendre polynomials $P_n^m(\xi)$ of $\xi = \sqrt{1 - \frac{r^2}{R_o^2}}$ (the variation with the azimuthal angle $\theta$ being given by a factor $e^{im\theta}$).

We shall here be concerned with the case $n = 1$, i.e., (taking the disk to be in the x,y-plane with polar coordinates $r,\theta$)

$$\mathbf{v}(r) = \left(\frac{\pi GM}{2R_o^3}\right)^{1/2} r \tag{1}$$

$$\Omega(r) = \left(\frac{\pi GM}{2R_o^3}\right)^{1/2} = \text{const} \tag{2}$$

where $G$ is the gravitational constant, $M$ the total Mass of the disk and $R_o$ its radius. $\Psi$, the gravitational potential in the plane of the disk, can be found by equating the gravitational force per unit mass in the plane,

$$\vec{g} = \frac{\partial \Psi}{\partial r}\,\hat{i}_r \tag{3}$$

to minus the centrifugal force per unit mass, $\Omega^2 r$:

$$\Omega^2 r = -\frac{\partial \Psi}{\partial r} \tag{4}$$

giving

$$\Psi = \frac{\pi MG}{4R_o^3}\,(2\,R_o^2 - r^2) \qquad (r < R_o) \tag{5}$$

$$g = -\frac{\pi MG}{2R_o^3}\,r \qquad (r < R_o) \tag{6}$$

where we used Hunters choice of integration constant. The density $\sigma(r)$

can not be found from this by taking the planar $\vec{\nabla}\cdot\vec{G}$, since the mass element in question sends force-lines also in the z-direction (perpendicular to the disk). The density distribution for this model seems to have been found first by Wyse and Mayall[3] as a limit of a homogeneous oblate spheroid:

$$\sigma = \frac{2M}{R_o^2}\left(1 - \frac{r^2}{R_o^2}\right)^{1/2} \tag{7}$$

A galaxy model of this kind may therefore be considered as the final stage of collapse of a "primeval" uniform spherical mass distribution which rotated about its z-axis and therefore offered no counterforce to the flattening in that direction (as in the disk case , we are justified in neglecting pressure forces). Case $n = \infty$ would have resulted only if an additional collapse, in the plane, of mass towards the center of the disk, had taken place.

We take the model given by equations (1) to (7), in which gravitational pull and centrifugal force balance each other, as our unperturbed disk ("equilibrium disk"). This would correspond, in the electron plasma case of Part I, to a homogeneous electron-fluid distribution, in which the forces from the negative and positive charges balance each other at each point. We saw in the plasma case that a deviation from homogeneity tends to correct itself, producing a wave-like motion, the non-linear character of whose basic modes is depicted by Fig. 1 of Part I. We also saw there that one may understand this behavior in terms of the individual motions of the "particles" (fluid elements), i.e., in the Lagrangian picture: A particle displaced from its equilibrium position will feel an excess of positive charge over the negative charge, pulling it back. Similarly in our case, a particle perturbed to a larger r will feel an excess of gravitational pull towards the center over the centrifugal force (the latter decreases* whereas the former increased because of the greater mass now enclosed by the new circle r). The particle will be pulled back, acquiring momentum on the way, whereby it overshoots its mark and the process reverses. Thus, oscillations are set up, with a typical frequency which corresponds to the plasma frequency of Part I. However, an oscillation in the r direction must be accompanied by an oscillation, which is $90^{\circ}$ out of phase, in the $\theta$ direction: As the particle swings out to larger r, its $\theta$ motion has to slow down, in order to keep the angular momentum of the particle constant, and, conversely, it must accerate while the particle swings inward. Thus, instead of being an oscillation, the perturbed motion of the particle is in reality a circular motion, about the tip of its equilibrium position-vector, which latter rotates with the disk.

As in the plasma case, we may, for different particles, choose arbitrarily related amplitudes and initial phases for this circular perturbation-motion. By choosing certain simple periodic relationships between the amplitudes and phases of different particles, one obtains the basic "modes" of the density waves produced thereby. Since it will, in general, happen that the particles will bunch up, in a nonlinear way, at various regular intervals of space and time, one will find that the modes so defined have the typical non-linear character of Fig. 1, Part I, even before we take into account the gravitational effect of these density waves in further perturbing the motions of the particles.

This type of nonlinearity, which is obtained on the assumption of an unperturbed Force law acting on particles (which is a good

* Since we are discussing radial oscillations, we assume that the particle is perturbed in the radial direction only, i.e., the angular momentum $\sim \Omega\ r^2$ remains unperturbed; hence the centrifugal force, $\sim \Omega^2 r$, changes like $1/r^3$.

assumption as long as the perturbation is small) will be called "kinematic nonlinearity". The nonlinearity resulting from the effect of the perturbed density on the motions of the particles which produce this density perturbation, will be called "dynamic nonlinearity". In retrospect, all nonlinearities found on the one-dimensional cold plasma case were strictly of the kinematical type; that they could not be anything else follows immediately from the fact that Coulombs law in one dimension gives a force that is independent of distance; hence bunching up of particles will have no further effect on other particles.

Our task in the present model is therefore considerably more difficult, as we have to take into consideration also the dynamical nonlinearities. However, the discussion of the kinematical nonlinearities is of considerable value in itself because it clarifies the framework upon which a more complete nonlinear theory has to be built, by providing the form of the basic modes relevant to such a theory, and by providing a much more suitable starting point to an approximation hierarchy (such as a perturbation series). (In this sense this is analogous to "renormalized" particles providing a much better starting point than "bare" particles in quantum field theory.) The main object of the present paper is to set up such a kinematical framework; it will be left to later work to carry out some of the dynamical part of the program, although indications will be given in Section 4 of this paper as to possible ways of attack on the dynamical problem.

The way in which this object will be carried out will, in a sense, be reverse to the logical development in Part I. We shall start, in Section II.2, with the discussion of the individual motions, i.e., with the Lagrangian picture. In Section II.3 we shall then eliminate the

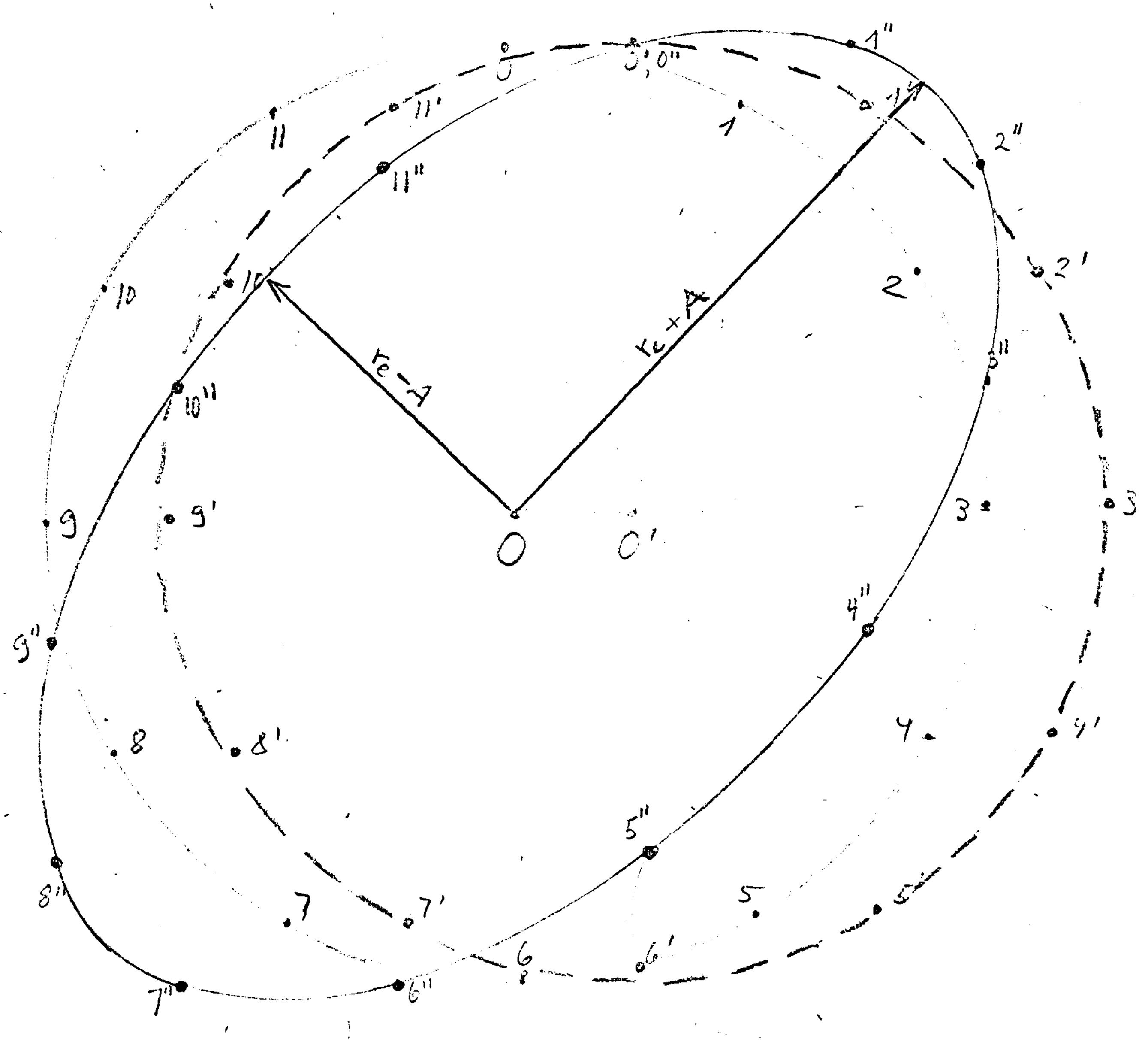


FIG. 2

The motion of a perturbed point in a uniformly rotating flat disk of density $\sigma = \text{const}\,\sqrt{1 - \frac{r^2}{k_o^2}}$. 1,2,3,... are consecutive positions of the unperturbed point;

1",2",3",... are the corresponding positions of the perturbed point. The broken circle would be described if the oscillatory perturbation were of frequency $\Omega$ instead of $2\,\Omega$.

Lagrangian variables and obtain the nonlinear waves for the field G, density $\rho$ and velocity $\vec{v}$, i.e., this will be in the Eulerian picture. Finally, in II.4, we shall indicate the treatment of dynamical nonlinearities.

## II.2. The perturbed individual-motions of particles in the disk

### (a) The equations of motion

In this section we shall develop the details, in the Lagrangian representation, of the general physical picture of nonlinear modes introduced in Section 1. Using the notation introduced in connection with equations (1)-(7), the motions of particles in the disk will be governed by the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\alpha}{2} r^2 \tag{8}$$

where

$$\alpha = \frac{\pi MG}{2R_o^3} \tag{9}$$

and $p_r$, $p_\theta$ are the radial and azimuthal canonical momentae. The equations of motion,

$$\dot{r} = \frac{p_r}{m}, \quad \dot{p}r = \frac{p_\theta}{mr^3} - \alpha r$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{p}_\theta = 0$$

give

$$m\ddot{r} = -\alpha r + \frac{p_\theta^2}{mr^3} \tag{10}$$

$$p_\theta = \text{const} \tag{11}$$

$$-m\ddot{\theta} = \frac{2p_\theta}{r^3}\dot{r} \tag{12}$$

$p_\theta^2/mr^3$ is the centrifugal force; $-2p_\theta \dot{r}/r3$ is the gyroscopic force which produces an oscillation in the $\theta$-direction if there is an oscillation in the r direction. To treat oscillations in the r direction, we define an equilibrium radius, $r_e$, by

$$m\ddot{r}_e = 0 = -\alpha r_e + \frac{p_\theta^2}{mr_e^3}\ ,\ \text{i.e.,}\ r_e = \left(\frac{p_\theta}{m\alpha}\right)^{1/4} \tag{13}$$

(b) Small radial oscillations

We may define

$$\delta r = r - r_e \tag{14}$$

and in this section we shall assume this to be small enough for $\delta r^2$ to be negligible in the force equations. (We shall later see that we might as well solve equations (10)-(13) exactly, but the linear treatment is of interest for several reasons.) Expanding (10) about $r_e$, we obtain

$$m\delta\ddot{r} = -\left(\alpha + \frac{3p_\theta^2}{mr_e^4}\right)\delta r = -\frac{4p_\theta^2}{mr_e^4}\ \delta r \tag{15}$$

i.e., $\delta r$ oscillates harmonically,

$$\delta r = A \cos\ (\omega t + \beta) \tag{16}$$

with frequency

$$\omega = \frac{2p_\theta}{mr_e^2} = 2\sqrt{\frac{\alpha}{m}} = 2\,\Omega \tag{17}$$

i.e., this is twice the frequency $\Omega$ at which the disk rotates. Expanding $\theta = p_\theta/mr^2$ to first order in $\delta r$, we obtain

$$\dot{\theta} = \frac{p_\theta}{mr_e^2}\left(1 - 2\,\frac{\delta r}{r_e}\right) \tag{18}$$

so that, by inserting (16) and integrating,

$$\theta = \frac{p_\theta}{mr_e^2}\,t - \frac{A}{r_e}\,\sin\ (\omega t + \beta) \tag{19}$$

The first part of (19) describes the uniform rotation of the disk, with angular frequency $\Omega$. The second part is an oscillation in the azimuthal direction which is $90^\circ$ in advance of the oscillation $\delta r$. The maximum amplitude of $r_e\delta\theta$ is the same as for $\delta r$. Hence, in the frame moving anti-clockwise with the uniform rotation $\theta_e = \frac{p_\theta}{mr_e^2} t$, the two components of the oscillation make a clockwise rotation, of radius A and of twice the frequency as that of the disk. As seen in the laboratory-frame, the point describes the elongated ellipse in Fig. 2. Its distance from the center is given by

$$r = (r_e^2 + A^2 - 2r_e A \sin \omega t)^{1/2}$$

Introducing the position-vector $\vec{x}_e$ of the unperturbed particle,

$$\vec{x}_e = r_e \, \hat{i}_{ro} \tag{20}$$

where $\hat{i}_{ro}$, $\hat{i}_{\theta o}$ are the radial and azimuthal unit vectors rotating with the disk, i.e.,

$$\frac{d\hat{i}_{ro}}{dt} = \frac{\omega}{2} \hat{i}_{\theta o} , \quad - \frac{d\hat{i}_{\theta o}}{dt} = \frac{\omega}{2} \hat{i}_{ro} \tag{21}$$

one finds for $\vec{x}_e$ and for the perturbed motion $\vec{x} = \vec{x}_e + \delta\vec{x}$,

$$m \frac{d^2\vec{x}e}{dt^2} = - \alpha r_e i_{ro} = - \alpha\vec{x}_e$$

$$m \frac{d^2\vec{x}}{dt^2} = - \alpha \, \vec{x}$$

and hence

$$\frac{d^2\delta\vec{x}}{dt^2} = - (\frac{\omega}{2})^2 \, \delta\vec{x} \tag{22}$$

which is in agreement with $\delta\ddot{r} = - \omega^2 \delta r$ because of (21). The solution we found for small perturbations can be written as

$$\vec{x} = \vec{x}_e + A \, (\cos \, (\omega t + \beta) \, \hat{i}_{ro} - \sin \, (\omega t + \beta) \, \hat{i}_{\theta o})\ldots \tag{23}$$

By choosing a suitable distribution of $\vec{x}_e$, A, $\beta$-values over the various particles, the Lagrangian variables may be eliminated between (22) and (23) and one obtains nonlinear waves in the Eulerian representation. This will be done in Section 3. However, we have to convince ourselves that the terms neglected in the present subsection (i.e., $(\delta r)^2$ etc.) are not of the same order as the nonlinear part of these waves. We shall therefore proceed to solve the system (10)-(12) exactly, and we shall see that the nonlinear features of the Eulerian representation have very little to do with the nonlinear terms of the Lagrangian representation. Thus, for most purposes the above solution will be sufficient.

(c) Large radial oscillations

The first integral of 10 is

$$\dot{r}^2 + \frac{\alpha}{m} r^2 + \frac{p_\theta^2}{m^2 r^2} = \frac{2E}{m} = \epsilon \tag{24}$$

this leads to

$$t - t_o = \int (\epsilon - \frac{\alpha}{m} r^2 - p_\theta^2 / m^2 r^2)^{-1/2} \, dr \tag{25}$$

By the transformation $r^2 = u + \epsilon m/2\alpha$, the integration gives

$$r^2 = \frac{m}{2\alpha} [\epsilon + (\epsilon^2 - 4\alpha \, p_\theta^2 / m^3)^{1/2} \sin \omega(t - to)]$$

Denoting the unperturbed value of $\epsilon$ and $E$ by $\epsilon_e$ and $E_e$ respectively, and introducing f:

$$\epsilon_e \equiv \frac{\omega^2}{2} \, r_e^2, \quad E_e = \frac{m \, \omega^2}{4} r_e^2$$
$$f = \frac{\epsilon}{\epsilon_e} = \frac{E}{E_e} \tag{26}$$

$$r^2 = r_e^2 \, [f + (f^2 - 1)^{1/2} \sin (\omega t + \phi_o)] \tag{27}$$

$$\sin \phi_o = \frac{r_o^2 - f}{(f^2 - 1)^{1/2}} \tag{27'}$$

$\dot{\theta} = p_\theta / mr^2$ now integrates to

$$\theta = \frac{1}{2} \int \frac{d(\omega t + \phi_o)}{f + \sqrt{f^2 - 1} \, \sin (\omega t + \phi_\theta} + \theta'$$

where $\theta'$ is a constant, and this gives

$$\text{tg}\,(\theta - \theta') = f\,\text{tg}\,(\tfrac{\omega}{2} + \tfrac{1}{2}\,\phi_o) + \sqrt{f^2 - 1} \tag{28}$$

and by $\theta(o) = \theta_o$,

$$\text{tg}\,(\theta_o - \theta') = f\,\text{tg}\,\frac{\phi_o}{2} + \sqrt{f^2 - 1} \tag{28'}$$

(28) can also be written

$$\text{tg}\,(\theta - \theta') = \sqrt{2f^2-1}\;\frac{\sin\,(\frac{\omega t}{2} + \frac{\phi_o}{2} + \Psi)}{\cos\,(\frac{\omega t}{2} + \frac{\phi_o}{2})} \tag{29}$$

where $\cos\Psi = f/\sqrt{2f^2-1}$.

In vector form, the equations of motion of a particle are

$$\frac{D^2\vec{x}}{Dt^2} = -\frac{\alpha}{m}\,\vec{x} = -\frac{\omega^2}{4}\,\vec{x} \tag{30}$$

In the following, we shall have to distinguish between the starionary (-"laboratory"-) frame and the frame which is fixed in the unperturbed disk, i.e., which rotates with uniform angular velocity $\Omega = \omega/2$. Vectors which are constant in the stationary frame will be denoted by $\vec{A},\vec{B},\vec{C},..,\vec{X}_o,\vec{V}_o,...$ and the convective derivative in this frame will be denoted by D/Dt. Vectors which are constant in the rotating frame will be denoted by $\vec{a},\vec{b},\vec{c},...,\vec{x}_o,\vec{v}_o,...$ and the convective derivative in that frame by d/dt. The position vector of a particle will in general be denoted by $\vec{x}$ in both frames, unless we have reason to emphasize that we wish to consider the components of $\vec{x}$ in the stationary frame in which case we shall use X. Similarly, any vector $\vec{f}$ which is variable in both frames will in general have only one notation. The Cartesian axes in the stationary frame are X,Y in the rotating frame x,y; thus

$$\vec{x} = x\,\hat{i}_x + y\,i_y = X\,\hat{i}_X + Y\,\hat{i}_Y = \vec{X}$$

The solution of (30) in the stationary frame is of course the ellipse

$$\vec{x} = A_1 \cos \left(\frac{\omega}{2} t + \alpha_1\right) \hat{i}_X + A_2 \cos \left(\frac{\omega}{2} t + \alpha_2\right) i_Y \tag{31}$$

and by putting in the initial values $\vec{X}_o, \vec{Y}_o$ one obtains

$$\begin{aligned} A_1 &= (X_o^2 + 4\, V_{oX}/\omega^2)^{1/2}, \quad \mathrm{tg}\, \alpha_1 = -\frac{2}{\omega} \frac{V_{oX}}{X_o} \\ A_2 &= (Y_o^2 + 4\, V_{oY}/\omega^2)^{1/2}, \quad \mathrm{tg}\, \alpha_2 = -\frac{2}{\omega} \frac{V_{oY}}{Y_o} \end{aligned} \tag{32}$$

However, our purpose is to go over to the Eulerian representation and for that the rotating frame is a more suitable starting point. If $\vec{\Omega}$ is of magnitude $\Omega$ and in the direction of the angular momentum of the disk (+z direction, since we assumed the disk to rotate counter-clockwise), we have, for any $\vec{a}$,

$$\frac{D\vec{a}}{Dt} = \vec{\Omega} \times \vec{a}, \quad \frac{d\vec{a}}{dt} = 0$$

(here and in what follows, x denotes the vector product)

and for any variable vector $\vec{f}$

$$\frac{D\vec{f}}{Dt} = \frac{d\vec{f}}{dt} + \vec{\Omega} \times \vec{f}, \tag{33}$$

in particular,

$$\frac{D^2\vec{x}}{Dt^2} = \frac{d^2\vec{x}}{dt^2} - \Omega^2\vec{x} + 2\,\vec{\Omega} \times \frac{d\vec{x}}{dt} \tag{34}$$

Using (34), we obtain from (30) the equation of motion in the rotating frame

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\omega} \times \frac{d\vec{x}}{dt}$$

where $\vec{\omega} = 2\,\vec{\Omega}$, and from this

$$\frac{d\vec{x}}{dt} = -\vec{\omega} \times \vec{x} + \vec{c}$$

where, according to our notation, $\vec{c}$ is fixed in the rotating frame. Taking the last equation at $t = 0$, and using (32) to express $\left(\frac{dx}{dt}\right)_{t=0}$, we find

$$\vec{c} = \vec{v}_o + \frac{\omega}{2} \times \vec{x}_o$$

where $\vec{x}_o$, $\vec{v}_o$ are vectors fixed in the rotating frame which coincide at t = 0 with the initial conditions $\vec{X}_o$, $\vec{V}_o$ of the particle. $\vec{c}$ can further be re-written

$$\vec{c} = \vec{\omega} \times \left(\frac{\vec{x}_o}{2} + \frac{\vec{v}_o \times \vec{\omega}}{\omega^2}\right)$$

so that if we define

$$a = \frac{\vec{x}_o}{2} + \frac{\vec{v}_o \times \vec{\omega}}{\omega^2} \tag{35}$$

We obtain for the equation of motion in the rotating frame

$$\frac{d\vec{x}}{dt} = - \vec{\omega} \times (\vec{x} - \vec{a}), \quad \frac{d\vec{a}}{dt} = 0, \text{ or}$$

$$\frac{d(\vec{x}-\vec{a})}{dt} = - \vec{\omega} \times (\vec{x} - \vec{a}), \tag{36}$$

$$\frac{d^2\vec{x}}{dt^2} = - \omega^2 (\vec{x} - \vec{a})$$

$\vec{a}$ replaces the vector $x_e = r_e \hat{i}_r$ of subsection (b); it is, in fact, different from $r_e \hat{i}_r$: denoting by $\hat{i}_{ro}$, $\hat{i}_{\theta o}$ the polar unit vectors associated with $\vec{x}_o$, and using $v_{o\theta} = p_\theta / mr_o = \omega\, r_e^2 / r_o$,

$$\frac{\vec{v}_o \times \vec{\omega}}{\omega^2} = \frac{1}{\omega^2} (v_{o\theta} \hat{i}_{\theta o} + v_{or} \hat{i}_{ro}) \times \vec{\omega} = \frac{r_e^2}{2r_o} \hat{i}_{ro} - \frac{v_{or}}{\omega} \hat{i}_{\theta o}$$

$$= \frac{r_e^2}{2r_o^2} \vec{x}_o - \frac{v_{or}}{\omega} \hat{i}_{\theta o}$$

where $r_o = |\vec{x}_o|$; thus

$$\vec{a} = \left(\frac{1}{2} + \frac{r_e^2}{2r_o^2}\right) \vec{x}_o - \frac{v_{or}}{\omega} \hat{i}_{\theta o} \tag{37}$$

The general solution of (36) can be written in the form

$\vec{x} - \vec{a} = b\, \vec{\nu}$, where b is a const. and

$$\hat{\nu} = \cos(-\omega t + \beta)\, \hat{i}_x + \sin(-\omega t + \beta)\, \hat{i}_y \tag{38}$$

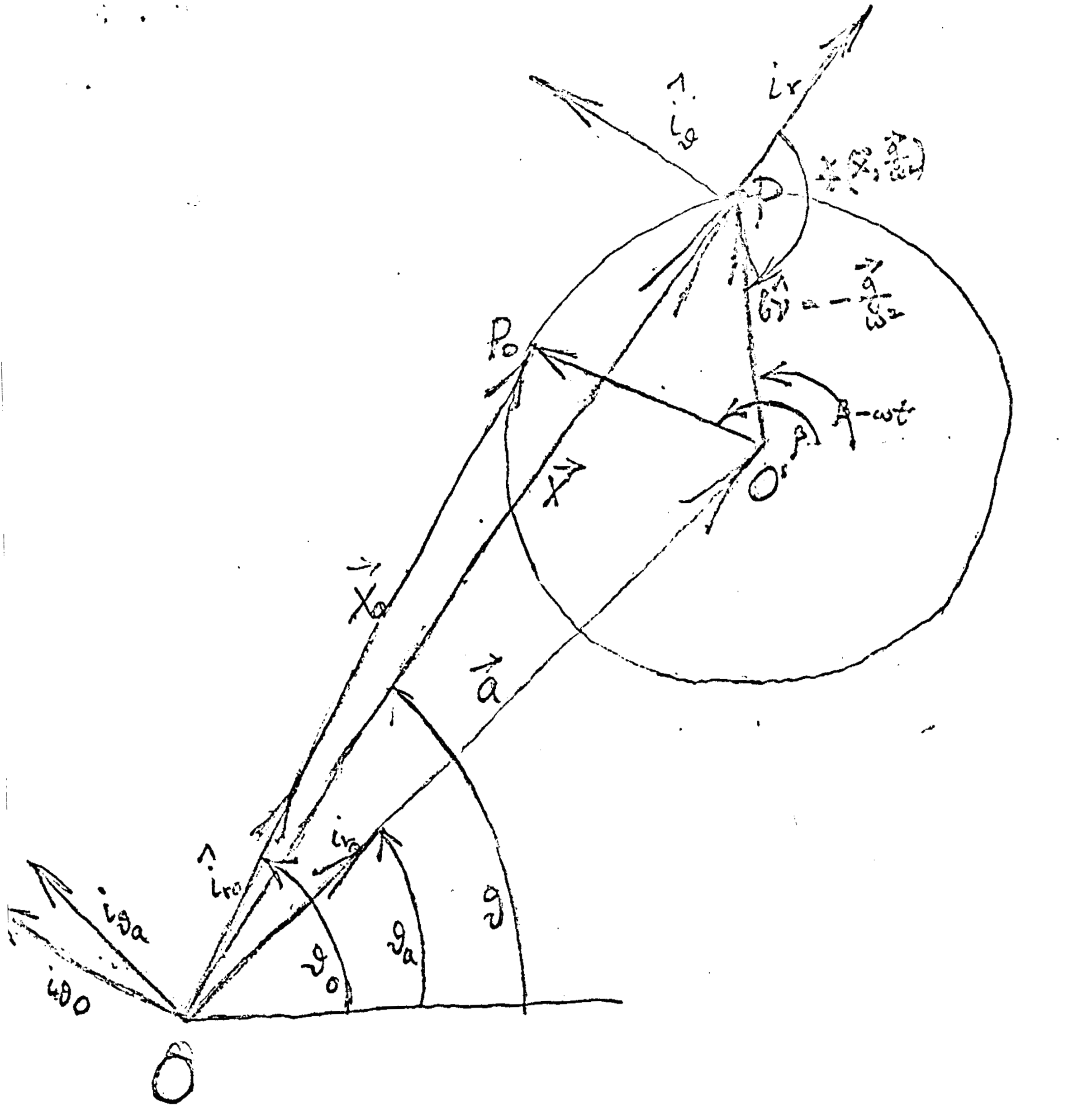


FIG. 3

O is the center of the (counter-clockwise) rotating disk; $\vec{a}, \vec{x}_o$ and the x axis are fixed in the disk, $P_o$ is the initial position of the particle, which rotates clockwise around the perburbation-center O', with angular frequency $\omega = 2\,\Omega$ ($\Omega$ = rotation frequency of disk).

i.e., $\hat{\nu}$ is a unit vector which rotates counter-clockwise, with angular frequency $\omega$, as seen from a Cartesian system $\hat{i}_x$, $\hat{i}_y$ which is fixed in the rotating disk; in terms of the stationary frame we would have

$$\nu = \cos\left(-\frac{\omega}{2}t + \beta\right)\hat{i}_X + \sin\left(-\frac{\omega}{2}t + \beta\right)\hat{i}_Y$$

We thus obtain the same type of motion as in subsection (b) and Fig. 2 applies here as well. The only difference is that now $\vec{x}$ does not rotate about the tip of $\vec{x}_e = r_c\,\hat{i}_r$ but about that of $\vec{a}$, (37).*
The relations between the various unit-vectors and angles are illustrated in Fig. 3.

## II.3. The perturbed motions in the Eulerian representation; kinematic nonlinearities.

Equations (38) (or (23) ) represented the motion of a particle which was by some cause perturbed from its equilibrium position (which latter is fixed in the rotating disk). The motion contains the four constants $\vec{a}$, b and $\beta$ which, so far, were arbitrary (corresponding to an arbitrary choice of $\vec{x}_o, \vec{v}_o$ ). In reality, we will be interested in perturbations produced on the particles by, e.g., an initial deviation from homogeneity of the density. The values of the constants $\vec{a}$,b,$\beta$ of the various particles will then be correlated by the effect upon them of this deviation from homogeneity and by the fact that this deviation will have some smooth functional form vs. position and time.

In "linear" treatments it has been customary to concentrate in particular on, e.g., sinusoidal deviations from homogeneity; the response of the particles to this perturbation is, in itself, of relatively quite complex nature and would produce a further change in

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* We do not denote $b\hat{\nu}$ by $\vec{b}$ because it is not constant in the rotating frame.

the functional form of the density which is quite different from the original perturbation. However, one usually analyzes this response again in terms of sinusoidal functions. Although, through Fourier Composition, one in principle has the tool for controlling all changes, in practice one looses this control again by the neglect of higher order terms. Thus, a solution of the form of Fig. 1, which, as we shall see, is quite simple and natural from the point of view of our analysis, would be quite inaccessible from a linear treatment.

Thus, we shall look into the system itself for the definition of "natural" functional forms, in terms of which the analysis will be carried out, rather than imposing such forms arbitrarily from the outside. The first step in this direction is by formulating simple modes which describe the waves of force, velocity anddensity, in an approximation which is of zeroth order in the dynamical effects of these waves on the motions of the particles.

To define such modes, we assume simple relationships between the constants $\vec{a}$,b,$\beta$ of different particles.

Since $\vec{a}$ is the position vector of the point of equilibrium about which the perturbed motion of the particle rotates, and since this point is simply related to the position of the particle before the perturbation was "switched on", we shall assume that we have to deal only with cases in which the tips of the various $\vec{a}$ vectors are distributed as the unperturbed density distribution, $\sigma(r)$ of eq. (7), of the disk; further, that to each particle there is one $\vec{a}$ and vice versa, so that $\vec{a}$ may be taken as a single valued Lagrangian variable for the particles.

Second, we define fundamental modes by choosing relationships b($\vec{a}$), e.g.,

$$\begin{aligned} b(\vec{a}) &= \alpha_k \cos\,(\pm\,|\vec{k} \times \vec{a}|) \\ \text{where} \quad \vec{k} &= k\,\hat{i}_z \end{aligned} \tag{39}$$

Thus, different modes will have different "wave numbers" k and "amplitudes" $\alpha_k$. The utility of such a choice will depend on whether we shall be able to fulfill a suitable superposition process with such "modes", so that a general profile can be resolved this way. We shall see that this is indeed the case.

Taking (39) as one possible relationship (other useful ones can easily be thought of) and assuming first for simplicity that $\beta$ of eqtn. (38) in zero for all particles, we may now use the equations of motion (36) in order to eliminate $\vec{a}$ from equation (38). This will give us the Eulerian description of the k-th mode. Denoting by $\vec{g}$ the force per unit mass ($\vec{g}$ here takes a similar role to that of $\vec{E}$ in Part I):

$$\vec{g}\,(\vec{x},t) \equiv \frac{d^2\vec{x}}{dt^2} \tag{40}$$

we obtain from (36)

$$\vec{a} = \vec{x} + \frac{\vec{g}}{\omega^2} \tag{41}$$

and hence from (38) and (39)

$$\begin{aligned} \vec{g}(\vec{x},t) &= -\,\omega^2\, b(\vec{a})\,\hat{\nu} \\ &= -\,\omega^2\,\alpha_k \cos\,(\pm|\vec{k} \times \vec{x} + \frac{\vec{k} \times \vec{g}}{\omega^2}|)\,\hat{\nu} \end{aligned} \tag{42}$$

where now*

$$\hat{\nu} = \cos\,(-\,\omega t)\,\hat{i}_x + \sin\,(-\,\omega t)\,\hat{i}_y \tag{43}$$

(42) implicitly defines the field strength $\vec{g}$ as a function of $\vec{x}$ and t, in much the same way as (IIb) did so for $\vec{E}$. To see the similarity with Fig. I, we write

*More generally, we may choose some functional form $\beta(\vec{a})$.

$$|\vec{k} \times \vec{x} + \frac{\vec{k} \times \vec{g}}{\omega^2}|^2 = k^2 \; |x + \frac{g}{\omega^2}|^2 = k^2(r^2 + \frac{g^2}{\omega^4} + \frac{2rg}{\omega^2} \cos\;(wt + \theta)) \quad (44)$$

where we have used (see Fig. 3)

$$\cos \sphericalangle \; (x, \frac{g}{\omega^2}) = \cos \sphericalangle \; OPO'; \sphericalangle \; OPO' = \beta - \omega t - \theta = - \omega t - \theta$$

Thus

$$g = \omega^2 \alpha_k \cos\; (\pm \; k \sqrt{r^2 + \frac{g2}{\omega^4} + \frac{2rg}{\omega^2} \cos\;(\omega t + \theta)}\;) \qquad (45)$$

to see the form of g, denote $G = kg/\omega^2$, $R = kr$, and $\phi = \omega t + \theta$, and choose the + sign:

$$G = \alpha \cos \sqrt{R^2 + G^2 + 2\;RG \cos\phi}$$

One easily finds the following properties of G:

(a) For $R = 0$, $G = \alpha \cos G$, which gives one solution as long as $|\alpha| \ll \pi$ but may be 2- or 3-valued when $|\alpha|$ approaches $\pi$.

(b) $G = 0$ implies $R = (2n + 1)\frac{\pi}{2}$ for some integer n; however, R being of this form does not necessarily imply $G = 0$, i.e., the curve $G(R)$ (for given $\phi$) need not pass through all points $R = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2} \ldots$ on the abcissa.

(c) $\frac{dG}{dR} = 0$ implies either $\sqrt{R^2 + G^2 + 2\;RG \cos\phi} = 2n\,\pi$, in which case $G = \pm\,\alpha$ and one has $\sqrt{R^2 + \alpha^2 \pm 2\alpha R \cos\phi} = 2n\,\pi$, or it implies $R + G \cos\phi = 0$.

Taking $\cos\phi = 0$, one obtains

$$G = \alpha \cos \sqrt{R^2 + G^2}$$

which is given in Fig. 4 for the case of sufficiently small $\alpha$ (so that multivaluedness near $R = 0$ is excluded). We see that for low n values, G resembles Fig. I, but as n increases the distortion becomes less and less and the curve resembles an ordinary cos R more and more.

These results can be generalized to the case $\cos\phi \neq 0$, and they will be of the same nature.

Again, the solution (42) can be generalized to

$$\vec{g}(\vec{x},t) = -\omega^2 \sum_k \alpha_k \cos\left(\pm\left|\vec{k} \times \vec{x} + \frac{\vec{k} \times \vec{g}}{\omega^2}\right|\right) \hat{v} \tag{46}$$

Since this is just a way of stating that the function $b(\vec{a})$ may be chosen arbitrarily, the most general solution can therefore be written

$$g(x,t) = -\omega^2 f\left(\vec{x} + \frac{\vec{g}}{\omega^2}\right) \hat{v} \tag{47}$$

where $f(\vec{\xi})$ is an arbitrary scalar function of $\vec{\xi}$. In treating dynamic nonlinearities within a certain approximation (to be defined), the problem will reduce to that of finding, for given conditions, a suitable function f.

However, we shall first obtain the density distribution associated with the mode.

It is to be expected that the density $\sigma(r)$ will be large wherever the slope of $g(r)$ is large (see fig.4). Indeed, the density is given by

$$\sigma = \sigma_o + \delta\sigma = \sigma_o / D \tag{48}$$

where $\sigma_o$ is the unperturbed density, given by (7), and

$$D = \begin{vmatrix} \frac{\partial x}{\partial a_x} & \frac{\partial x}{\partial a_y} \\ \frac{\partial y}{\partial a_x} & \frac{\partial y}{\partial a_y} \end{vmatrix} \tag{49}$$

In writing (48) we have assumed that the distribution of the tips of the $\vec{a}$-vectors is as that of the unperturbed particles (and hence the Jacobian between them is unity).

D can easily be calculated; for the more general case (47) we obtain

$$\sigma = \sigma_o \frac{1}{1 + \omega^2 f' \cos(\omega t + \theta a)}, \text{ i.e.,} \tag{50}$$

$$\delta\sigma = \sigma_o \frac{-\omega^2 f' \cos(\omega t + \theta a)}{1 + \omega^2 f' \cos(\omega t + \theta a)} \tag{50a}$$

($f'$ being the derivative $df/da$); for the particular mode (45), we obtain

$$\sigma = \sigma o \frac{1}{1 - \alpha k \sin ka \cos(\omega t + \theta a)} \tag{51}$$

$$\delta\sigma = \sigma o \frac{\alpha k \sin ka \cos(\omega t + \theta a)}{1 - \alpha k \sin ka \cos(\omega + \theta a)} \tag{51a}$$

In both (50) and (51) we have to think of a as expressed in terms of x and g(x), through (41). When $\alpha k \ll 1$, (51) can be expanded to first order of $\alpha k$; $\delta\sigma$ then describes a sinusoidal perturbation of the density which is obtainable also from the linear analysis (e.g. ref.(   )). The case of interest here is $\alpha k \lesssim 1$. When $\alpha k \approx 1$, (51) describes "spikes" of large density concentration (see fig.5) which occur at certain radial distances (approximately at $\frac{\pi}{2k}$, $\frac{3\pi}{2k}$, $\frac{5\pi}{2k}$ ...) and at azimuthal angles which fulfill $\cos(\omega t + \theta a) = 1$. Assuming $\alpha$ to be sufficiently small, $\theta a$ is close enough to $\theta$ (see fig.3) to describe the azimuthal position of the spike by $\theta = \omega t$, i.e., there is <u>one</u> spike on each of the circles $r = \frac{\pi}{2k}, \frac{3\pi}{2k}$, ... and this spike rotates (clockwise) with angular velocity $-\omega$, as seen from the frame fixed in the disk (i.e., the stationary observer sees the spike rotating (clockwise) with angular velocity $-\frac{\omega}{2}$). As seen in fig.5, the spikes become less and less sharp the further they are from the center of the disk; eventually they will degenerate into an ordinary sinusoidal perturbation.

## II.4. Typical features of dynamical nonlineararities

In this section we shall illustrate some typical circumstances which arise when the effect of the perturbed densities on the motions of the particles is taken into account. This effect results in what was called, in section 1, a "dynamical" nonlinearity, whereas a "kinematical" nonlinearity (which is the only linearity present in a one-dimensional plasma) is the result of the bunching up of perturbed particles which move under the gravitational influence of the unperturbed density.

To obtain an understanding of principle, we shall assume that the initial perturbation of the particles was limited to a sufficiently thin ring $r_1 < r < r_2$, so that only one of the spikes (the sharpest one, say) discussed towards the end of the preceding section, will be present. We also assume $ka \approx 1$ so that the spike is sufficiently narrow to be considered as a mass-point Q, of total mass $\mu$, say.

In the frame moving with the disk, both the spike Q and a typical particle $P^o$ are seen to rotate clockwise with angular velocity $\omega$, as long as $P^o$ is not yet accelerated by Q. Q rotates about the center 0 of the disk, $P^o$ rotates about the tip 0' of its (as yet unperturbed) $\vec{a}^o$-vector (see fig.6). If one chooses both rotations to be in phase, it is easy to see from fig.6 that the lines $\overline{Q_1P_1^o}$ , $\overline{Q_2P_2^o}$, ... through corresponding positions all pass through a fixed point S and that

$$\overline{Q(t)P^o(t)} \; : \; \overline{P^o(t)\,S} = c \tag{52}$$

where $dc/dt = 0$.

Therefore, the gravitational attraction

$$F_Q = G\mu \frac{\overrightarrow{P^oQ}}{PQ^3}$$

exerted by the spike on $P^o$ (per unit mass) can be written

$$F_Q = G\mu c^2 \frac{\overrightarrow{SP^o}}{SP^{o3}} \tag{53}$$

In other words, $F_Q$ can be replaced by a repulsive force coming from S, again with an inverse-square law and with an effective "charge" $g = G\mu c^2$ (while the "charge" at P is assumed to be one).

If the two rotations are not in phase, essentially the same result can be obtained, with a somewhat more complicated geometry.

To find the perturbing effect of $F_Q$ on the $P^o$-motion, we decompose the vector equation (53) into a component in the direction $\overrightarrow{SP^o}$ and one perpendicular to it. We denote the true (i.e., perturbed) position of the particle by P, and

$$\left.\begin{aligned} \overrightarrow{SP^o} &\equiv \vec{s}^{\,o}, \qquad \overrightarrow{P^oP} \equiv \overrightarrow{\delta s} \\ \overrightarrow{SP} &\equiv \vec{s} \equiv \vec{s}^{\,o} + \overrightarrow{\delta s} \end{aligned}\right\} \tag{54}$$

and from the geometry of fig.6 (see the broken lines) we conclude that

$$\overrightarrow{O'S} = \frac{\rho - \alpha}{\alpha}\,\vec{a} = c\vec{a} \tag{55}$$

($\rho$ is the distance of the spike from O, while $\alpha$ - according to (42) - is the distance of $P^o$ from O').

If $\vec{x}^{\,o}$ denotes the vector $\overrightarrow{OP^o}$,

$$\vec{x}^{\,o} = (1 + e)\,\vec{a} + \vec{s}_o \tag{56}$$

Hence the equation of motion of $P^o$, (36c), can be written in terms of $\vec{s}_o$

$$\frac{d^2\vec{s}_o}{dt^2} + \omega^2\vec{s}_o = -\,\omega^2\,(1+c)\,\vec{a} \tag{57}$$

To this equation we now have to add the force $F_Q$. Taking $F_Q$ to zeroth order in $\delta s$, one obtains the equation of variation (with $q=Gpc^2 > 0$)

$$\frac{d^2\,\delta\vec{s}}{dt^2} + \omega^2\,\delta\vec{s} = q\,\frac{\vec{s}^{\,o}(t)}{|s^o|^3} \tag{57}$$

To this equation we now have to add the force $F_Q$. Taking $F_Q$ to zeroth order in $\delta s$, one obtains the equation of variation (with $q = Gpc^2 > 0$)

$$\frac{d^2\delta\vec{s}}{dt^2} + \omega^2\,\delta\vec{s} = q\,\frac{\vec{s}^{\,o}(t)}{|s^o|^3} \tag{58}$$

and to first order (expanding $(\vec{s}^{\,o} + \delta\vec{s})/|s^o + \delta s|^3$)

$$\frac{d^2\delta\vec{s}}{dt^2} + \left(\omega^2 - \frac{q}{|s^o|^3}\right)\delta\vec{s} + 3q\,\frac{(\vec{s}^{\,o}\cdot\delta\vec{s})}{|s^o|^5}\,\vec{s}^{\,o} = q\,\frac{s^o(t)}{|s^o|^3} \tag{59}$$

Equation (58) is simple to solve: The component of (58) in the direction $\vec{s}^{\,o}$ is of the form

$$\delta\ddot{s}_{11} + \omega^2\delta s_{11} = f(t) \tag{60}$$

whose solution is (with arbitrary constants $A,\beta$)

$$\delta s_{11} = \frac{1}{\omega}\int_0^t d\tau \sin\omega\,(t-\tau)f(\tau) + A\cos(\omega t + \beta) \tag{61}$$

and the component in the direction perpendicular to $\vec{s}^{\,o}$ is simply

$$\delta s_{\perp}\ B\cos(\omega t + \gamma)$$

with arbitrary constants $B,\gamma$. More interesting for the typical nonlinear phenomena of the disk is the next order - equation (59): The important component of (59) is the one in the direction of $\vec{s}^{\,o}$, it si (dropping the$_{11}$ for convenience)

$$\delta\ddot{s} + \left(\omega^2 + 2q\,\frac{1}{|s^o|^3}\right)\delta s = f(t) \tag{62}$$

The crucial point about this equation is that the eigenfrequency-squared, $\omega^2$, has been supplemented by a positive term which, through $s^o(t)$, is periodic (with frequency $\omega$). Thus, (62) is of Hill's type, and the delineation of regions of stability and instability of its family of solutions depends sensitively on the value of g (see, e.g., (      )). This means that while there may be many particles of the disk whose circular perturbation motion (as seen from the rotating disk) will be merely perturbed somewhat further by the spike into a nearby "apple-shape" (fig.6), there may be others (those nearest to the spike) which will be thrown out of orbit altogether. In terms of a Eulerian representation, this instability - which contains features which are above and beyond those predicted by a linearized analysis in the Eulerian representation - can again be obtained using the general approach of section 3. One eliminates the Lagrangian constants of motion between the solution and the equation of motion, and one obtains the nonlinear waves in the density by calculating a suitable Jacobian. The problem of treating dynamical nonlinearities is hereby formulated, at least for the case of sharp spikes (which is the case of most interest). The solution of this problem is a chapter in its own right and will be pursued in another paper, with C. Cuvaj.

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## Appendix : The possibility of spiral solutions

To see that nonlinear solutions of the type (42) may describe also spiral waves, we need only re-include the Lagrangian constant $\beta$ which was dropped from equation (38). For $\beta$ we may choose any function $\beta(\theta_a)$, since $\theta_a$ (see fig.3) is a Lagrangian constant. In particular, we may choose

$$\beta(\theta_a) = - m\,\theta_a = - m\,\theta - m\,\theta''$$

where m is an integer and, as in section 3, $\theta''$ can be assumed to be small compared to $\theta$. This gives a wave in the field strength which is of the form

$$g(\vec{x},t) = - \omega^2\alpha \cos\left[k\left(r^2 + \frac{g^2}{\omega^4} + \frac{2rg}{\omega^2}\cos(\omega t + \theta)\right)^{\frac{1}{2}} - m\,\theta - m\,\theta''\right]$$

For sufficiently small $\alpha$, this is of the form

$$g = - \omega^2\,\alpha\,\cos\,(k\,r - m\,\theta)$$

which is clearly of the spiral type. In the more exact form of g, the appearance of the term $m\,\theta$ can be seen to lead to a similarly spiral behaviour, although more complex, as long as $\theta''$ is not comparable with $\theta$ (in which case the picture given by fig.4 breaks down anyway, since then the manyvaluedness mentioned in connection with (45) sets in).

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