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TRACKING SYSTEMS, THEIR MATHEMATICAL MODELS AND THEIR ERRORS

PART II: LEAST SQUARES TREATMENT

by W. D. Kahn and F. O. Vonbun

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Greenbelt, Md.

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ABSTRACT

This paper describes the development of mathematical models which reflect the effects of tracking system measurement errors (i.e., noise and bias) and tracking system location uncertainties on RMS errors in spacecraft position and velocity. Through the application of the method of weighted least squares, a generalized mathematical model is developed which permits simulation of tracking the spacecraft with several tracking stations in the simultaneous or nonsimultaneous tracking mode. In Part I only simultaneous tracking was considered.

Mathematical models are also developed for: (1) effects of a priori knowledge on the RMS errors in spacecraft position and velocity; (2) the ellipsoid of errors in spacecraft position and velocity; and (3) target body impact parameters, the errors in impact position and impact time.

The last two models are especially important because they provide the techniques to use when describing the geometry for the RMS errors in spacecraft position and velocity at discrete points along the trajectory and at any targeting or impact point. Finally, a numerical example is given which serves to illustrate the effects of known error sources on the RMS errors in spacecraft position and velocity.

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INTRODUCTION

The rapid increase in the number of analyses of space missions during the past few years and in the increasing complexity of the missions has intensified the need for more accurate determination of the position and velocity of spacecraft. A realistic estimate of the errors in any such determination should include errors resulting from a variety of sources. These are the errors associated with the basic measurements, the uncertainty of the station locations, the basic physical constants, and the mathematical models used to describe the orbit. In addition, random errors, which are errors subject to statistical analysis, must be distinguished from bias errors, which are errors that stay constant or change slowly during the measurement process.

This report presents a detailed error analysis utilizing a weighted least squares approach, and shows how all of the types of errors mentioned above can be combined for different tracking systems. This is important, from the practical point of view, since on each space mission many tracking systems and stations are being utilized to determine the spacecraft trajectory. In order to design and evaluate a tracking system properly, all significant error sources mentioned must be considered.

Most of the formulas are taken from Part I of this paper (Reference 1) keeping changes at a minimum. Many of the equations given in Part I are included for the reader's convenience. Particular attention is paid to the bias errors in the measurements and in the locations of the stations. This paper shows that the bias errors are predominant when only short tracking times can be utilized for the trajectory determination. To cite a practical example, when a spacecraft is inserted into a parking orbit, as in the case of the Apollo Missions, only a tracking ship located in the Atlantic Ocean can "see" the insertion and then only for 2 to 4 minutes (Reference 2). This time is too short to evaluate bias errors and to correct for them properly. Evaluation of some of these bias errors is possible when more than one station is utilized for the trajectory determination and when time is less important. This is the case for multiple earth orbits, transfer orbits to the moon and planets, and orbits with long periods (days rather than hours).

Errors, for proper interpretation, are treated in a geometric sense; that is, the error ellipsoids for position and velocity are calculated. The principal axes of these ellipsoids are determined and their projection along a space trajectory is treated in three dimensional form. The hyperellipsoid treatment (6-dimensional) has been avoided since it lacks geometric interpretation, whereas the error ellipsoids of the position and velocity can easily be visualized and transformed into impact miss parameters at the moon or planets.

ANALYSIS

Part I of this paper was primarily concerned with the development of the error equations for the fundamental spacecraft tracking systems. These systems are: (1) Radar Systems which measure range, azimuth, and elevation; (2) Interferometer Systems which measure direction cosine and direction cosine rates; (3) Ranging Systems which measure range only; and (4) Range Rate Systems which measure Doppler only. These are called fundamental systems because they can be combined to handle all known present and future tracking systems. As an example, consider the USBS,* which is presently used in the Apollo program. This system measures range, range rate, and x-y angles, which are readily converted to azimuth and elevation (Reference 3). This system can be presented as a combination of a radar system and a range rate system. Previously the determination of RMS errors in spacecraft position and velocity was made by assuming k different tracking stations making one measurement each at the discrete points along the trajectory which were visible to each of the k tracking stations. In brief, simultaneous observations were necessary to determine the spacecraft position and velocity errors without utilizing the orbit itself (Reference 4). This method is important when extremely accurate spacecraft position and velocity are to be obtained. SECOR† is a typical example of a system which does not make use of the orbit and thus is not subject to the errors introduced in the equations of motion.

For the analysis presented here, error sources are of secondary importance and major emphasis is given to the errors themselves which influence the trajectory determination. Although the errors and their sources are interrelated, a separation is assumed in order to reduce the size of Part II. The position and velocity errors are determined by combining the measurements, n_k, made by each of the k tracking stations during the time the spacecraft is "radio-visible" from the station. "Radio-visibility" means that the station position vector intersects the spacecraft antenna at sufficiently close range to provide a signal strength within given limits; e.g., 3 db, 10 db, etc. (Reference 5). The solution for these errors is obtained by the weighted least squares method, combining all of the measurements for the stations viewing the spacecraft either simultaneously or nonsimultaneously. The basic requirement here is that the number of measurements has to exceed the number of unknown parameters to be determined. In most cases these unknown parameters are the six orbital parameters or the position and velocity vectors. Since only the errors in the state vector (6×1 vector composed of the spacecraft position and velocity components)

*USBS stands for Unified S-Band System which is a system combining range and range rate measurements with voice and telemetry and television.

†SECOR is the Sequential Electronic Correlation Ranging System.

are solved for, a spacecraft need not be visible to each of the k tracking stations at the same time. Obviously there are many more observations than unknowns to be solved for.

Random as well as bias errors are treated in this paper. In its colloquial use, the word "random" is applied to any method of choice which lacks aim or purpose. In the true sense, a measurement subject to random errors must be repeated many times in order to ascertain the most probable value and to assure that any future measurement lies within a specified interval of confidence. This is to some extent in conflict with what is being done in spacecraft tracking, since no quantity considered can be measured more than once because the spacecraft is in motion. In most practical cases, however, the relative changes in the magnitude of these quantities are small during the measurement interval so that proven techniques for error analysis can be applied.

Bias or systematic errors are in general those which stay constant from measurement to measurement or fluctuate in periods which are larger than necessary to make the measurements to determine a quantity within given limits. Throughout Part II, bias errors are not solved for and thus add to the spacecraft total position and velocity errors. However, bias errors in the measurements and tracking station locations are considered in the overall analysis along with the tracking system's uncertainties. For single station solutions, bias errors play a larger role than does measurement noise. As an example, if only measurement noise is used in the analysis, the errors in the state vector tend to approach zero; this situation is obviously meaningless in the practical sense. Incorporation of the bias error effects, which cannot be avoided, will in practice wipe out this phenomenon automatically and add considerably to the errors in the state vector (Reference 6). The error equations in this paper are derived from the variational equations given in Part I.

Since Part II treats all random measurements in a least squares sense, the letters α, β, γ, and δ (representing the type of tracking systems in Part I) are no longer applicable in the same way. The reason is that in a practical sense each system makes a different number of measurements. To be as consistent as possible, these letters are now used as subscripts indicating the type of tracking systems used in the analysis; that is

α designates radar,

β designates interferometer,

γ designates ranging system, and

δ designates range rate system.

The total number of measurements made by each of these systems are designated as follows:

$$n_\alpha,\ n_\beta,\ n_\gamma,\ \text{and}\ n_\delta \quad .$$

A. Definition of Symbols

The following symbols will be used throughout this paper.

ϕ - Geodetic latitude of the tracking station.

λ - Geodetic longitude of the tracking station.

h - Height above geoid of the tracking station.

$a_\oplus$ - Equatorial radius of spheroid used to represent the Earth, (For Hayford Spheroid $a_\oplus$ = 6378.388 km).

e_1^2 - Square of eccentricity for spheroid, (for Hayford Spheroid e_1^2 = .0067226700223).

θ_{G_0} - Greenwich sidereal time at 0^h Universal Time (U.T.), obtained from the various almanacs.

θ_G - Greenwich sidereal time at U.T. of observation.

t - U.T. of observation.

α - Azimuth of the object being tracked.

ϵ - Elevation of the object being tracked.

r - Slant range of the object being tracked.

a_s - Semi-major axis of satellite orbit.

λ' - Right ascension of satellite.

δ - Declination of satellite.

e_s - Eccentricity of satellite orbit.

i_s - Inclination of satellite orbit.

Ω_s - Longitude of ascending node.

ω_s - Argument of perigee.

E_s - Eccentric anomaly.

θ_s - True anomaly.

P_s - Period of revolution of satellite.

N - Radius of curvature along prime vertical.

ρ_s - Magnitude of the radius vector to the satellite in the inertial coordinate system.

h_s - Satellite altitude above the Earth or other central body.

$|\vec{v}|$ - Magnitude of velocity vector of the satellite.

γ_s - Flight path angle.

B. Error Analysis for Spacecraft Position and Velocity

Mathematical Models

The variational equations corresponding to the fundamental tracking systems are presented in this section. These equations differ from those derived in Part I in that they reflect the measurement variation with respect to the bias in the measurement. For a single measurement in the inertial coordinate system these equations are as follows:*

1. Radar Systems (Measuring r, α, ϵ)

$$\mathbf{D}_{(3i_\alpha \times 3)}\, \delta\mathbf{K}_{(3\times 1)} = \mathbf{L}_{(3i_\alpha \times 3)}\, \delta\mathbf{X}_{(3\times 1)} - \left(\mathbf{J}^T\mathbf{R}\right)_{(3i_\alpha \times 3)}\, \delta\mathbf{S}_{(3\times 1)} + \mathbf{D}_{(3i_\alpha \times 3)}\, \delta\boldsymbol{\beta}_{(3\times 1)}\ , \tag{B-1}$$

where

$$\delta\boldsymbol{\beta} = \text{Bias in Measurement}$$

and

$$i_\alpha = 1,\ 2,\ \ldots,\ n_\alpha\ .$$

2. Angle and Angular Rate System (Interferometer—Measuring α, ϵ, $\dot{\alpha}$, $\dot{\epsilon}$)

$$\mathbf{D}^0_{a_{(2i_\beta \times 3)}}\, \delta\mathbf{K}_{(3\times 1)} = \mathbf{L}^0_{a_{(2i_\beta \times 3)}}\, \delta\mathbf{X}_{(3\times 1)} - (\mathbf{F}_a\, \mathbf{J}^T\mathbf{R})_{(2i_\beta \times 3)}\, \delta\mathbf{S}_{(3\times 1)} + \mathbf{D}^0_{a_{(2i_\beta \times 3)}}\, \delta\boldsymbol{\beta} \tag{B-2}$$

and

$$\mathbf{D}^0_{a_{(2i_\beta \times 3)}}\, \delta\dot{\mathbf{K}}_{(3\times 1)} = \mathbf{M}^0_{a_{(2i_\beta \times 3)}}\, \delta\mathbf{X}_{(3\times 1)} + \mathbf{L}^0_{a_{(2i_\beta \times 3)}}\, \delta\dot{\mathbf{X}}_{(3\times 1)} - \left(\mathbf{F}_a\, \mathbf{V}\, \mathbf{J}^T\mathbf{R}\right)_{(2i_\beta \times 3)}\, \delta\mathbf{S}_{(3\times 1)} + \mathbf{D}^0_{a_{(2i_\beta \times 3)}}\, \delta\dot{\boldsymbol{\beta}}_{(3\times 1)}, \tag{B-3}$$

where

$$i_\beta = 1,\ 2,\ \ldots,\ n_\beta\ .$$

3. Ranging Systems (Measuring r only)

$$\mathbf{D}^0_{r_{(i_\gamma \times 3)}}\, \delta\mathbf{K}_{(3\times 1)} = \mathbf{L}^0_{r_{(i_\gamma \times 3)}}\, \delta\mathbf{X}_{(3\times 1)} - (\mathbf{F}_r\, \mathbf{J}^T\mathbf{R})_{(i_\gamma \times 3)}\, \delta\mathbf{S}_{(3\times 1)} + \mathbf{D}^0_{r_{(i_\gamma \times 3)}}\, \delta\boldsymbol{\beta}_{(3\times 1)}\ , \tag{B-4}$$

where

$$i_\gamma = 1,\ 2,\ 3,\ \ldots,\ n_\gamma\ .$$

4. Range Rate Systems (Measuring $\dot{r}$ only)

$$\mathbf{D}^0_{r_{(i_\delta \times 3)}}\, \delta\dot{\mathbf{K}}_{(3\times 1)} = \mathbf{M}^0_{r_{(i_\delta \times 3)}}\, \delta\mathbf{X}_{(3\times 1)} + \mathbf{L}^0_{r_{(i_\delta \times 3)}}\, \delta\dot{\mathbf{X}}_{(3\times 1)} - (\mathbf{F}_r\, \mathbf{V}\, \mathbf{J}^T\mathbf{R})_{(i_\delta \times 3)}\, \delta\mathbf{S}_{(3\times 1)} + \mathbf{D}^0_{r_{(i_\delta \times 3)}}\, \delta\dot{\boldsymbol{\beta}}\ , \tag{B-5}$$

*All pertinent matrices used in the equations discussed in this paper are fully defined in Appendix a.

where

$$i_\delta = 1, 2, \ldots, n_\delta \quad .$$

Error Equations in Combined Form

Present day tracking systems can be presented as combinations of the so-called basic systems which are used in this paper. The mathematical models of the error equations for such systems can therefore be formed by writing in partitioned matrix form combinations of Equations (B-1) through (B-5). A solution for the RMS error in the state vector (position and velocity) can then be made by applying the method of weighted least squares. Listed below are well known combinations of advanced types of present day tracking systems. In all instances, they represent single measurements.

1. Range and Range Rate System (Measuring r, $\dot{r}$, α, ϵ, or equivalent)

This system (for example, the USBS and the Goddard Range and Range Rate System) is a combination of the models shown in Equations (B-1) through (B-5). The variational equation in partitioned matrix form is:

$$\underset{(4\times 1)}{\begin{bmatrix} (\mathbf{D}\,\delta\mathbf{K})_{(3\times 1)} \\ \hline (\mathbf{D}_r^0\,\delta\dot{\mathbf{K}})_{(1\times 1)} \end{bmatrix}} = \underset{(4\times 6)}{\left[\begin{array}{c:c} \mathbf{L}_{(3\times 3)} & \mathbf{0}_{(3\times 3)} \\ \hline \mathbf{M}^0_{r(1\times 3)} & \mathbf{L}^0_{r(1\times 3)} \end{array}\right]} \underset{(6\times 1)}{\begin{bmatrix} \delta\mathbf{X}_{(3\times 1)} \\ \hline \delta\dot{\mathbf{X}}_{(3\times 1)} \end{bmatrix}} + \underset{(4\times 3)}{\begin{bmatrix} (-\mathbf{J}^T\mathbf{R})_{(3\times 3)} \\ \hline (-\mathbf{F}_r\,\mathbf{V}\,\mathbf{J}^T\mathbf{R})_{(1\times 3)} \end{bmatrix}} \delta\mathbf{S}_{(3\times 1)} + \underset{(4\times 6)}{\left[\begin{array}{c:c} \mathbf{D}_{(3\times 3)} & \mathbf{0}_{(1\times 3)} \\ \hline \mathbf{0}_{(1\times 3)} & \mathbf{D}^0_{r(1\times 3)} \end{array}\right]} \underset{(6\times 1)}{\begin{bmatrix} \delta\boldsymbol{\beta}_{(3\times 1)} \\ \hline \delta\dot{\boldsymbol{\beta}}_{(3\times 1)} \end{bmatrix}} \cdot \quad \text{(B-6)}$$

In essence, Equation (B-6) is applicable for other combinations of systems, such as (1) a radar system and a Doppler system and (2) systems measuring range only or range rate only with an interferometer system measuring α and ϵ or their equivalent.

2. Range Rate - Interferometer System (Measuring $\dot{r}$, α, ϵ, or equivalent)

This combination of systems can be mathematically represented by combining Equation (B-2) and (B-5) as follows:

$$\underset{(3\times 1)}{\begin{bmatrix} (\mathbf{D}_a^0\,\delta\mathbf{K})_{(2\times 1)} \\ \hline (\mathbf{D}_r^0\,\delta\dot{\mathbf{K}})_{(1\times 1)} \end{bmatrix}} = \underset{(3\times 6)}{\left[\begin{array}{c:c} \mathbf{L}^0_{a(2\times 3)} & \mathbf{0}_{(2\times 3)} \\ \hline \mathbf{M}^0_{r(1\times 3)} & \mathbf{L}^0_{r(1\times 3)} \end{array}\right]} \underset{(6\times 1)}{\begin{bmatrix} \delta\mathbf{X}_{(3\times 1)} \\ \hline \delta\dot{\mathbf{X}}_{(3\times 1)} \end{bmatrix}} + \underset{(3\times 3)}{\begin{bmatrix} -(\mathbf{F}_a\,\mathbf{J}^T\mathbf{R})_{(2\times 3)} \\ \hline -(\mathbf{F}_r\,\mathbf{V}\,\mathbf{J}^T\mathbf{R})_{(1\times 3)} \end{bmatrix}} \delta\mathbf{S}_{(3\times 1)} + \underset{(3\times 6)}{\left[\begin{array}{c:c} \mathbf{D}^0_{a(2\times 3)} & \mathbf{0}_{(2\times 3)} \\ \hline \mathbf{0}_{(1\times 3)} & \mathbf{D}^0_{r(1\times 3)} \end{array}\right]} \underset{(6\times 1)}{\begin{bmatrix} \delta\boldsymbol{\beta}_{(3\times 1)} \\ \hline \delta\dot{\boldsymbol{\beta}}_{(3\times 1)} \end{bmatrix}} \cdot \quad \text{(B-7)}$$

3. The Angle and Angular Rate System or Interferometer (Measuring α, ϵ, $\dot{\alpha}$, and $\dot{\epsilon}$ or equivalent)

Matrix Equations (B-2) and (B-3) are combined:

$$\begin{bmatrix} (\mathbf{D}_a^0\,\delta\mathbf{K})_{(2\times1)} \\ \hdashline (\mathbf{D}_a^0\,\delta\dot{\mathbf{K}})_{(2\times1)} \end{bmatrix}_{(4\times1)} = \begin{bmatrix} \mathbf{L}^0_{a(2\times3)} & \mathbf{0}_{(2\times3)} \\ \mathbf{M}^0_{a(2\times3)} & \mathbf{L}^0_{a(2\times3)} \end{bmatrix}_{(4\times6)} \begin{bmatrix} \delta\mathbf{X}_{(3\times1)} \\ \hdashline \delta\dot{\mathbf{X}}_{(3\times1)} \end{bmatrix}_{(6\times1)} + \begin{bmatrix} -(\mathbf{F}_a\mathbf{J}^T\mathbf{R})_{(2\times3)} \\ \hdashline -(\mathbf{F}_a\mathbf{V}\mathbf{J}^T\mathbf{R})_{(2\times3)} \end{bmatrix}_{(4\times3)} \delta\mathbf{S}_{(3\times1)} + \begin{bmatrix} \mathbf{D}^0_{a(2\times3)} & \mathbf{0}_{(2\times3)} \\ \mathbf{0}_{e(2\times3)} & \mathbf{D}^0_{a(2\times3)} \end{bmatrix}_{(4\times6)} \begin{bmatrix} \delta\boldsymbol{\beta}_{(3\times1)} \\ \hdashline \delta\dot{\boldsymbol{\beta}}_{(3\times1)} \end{bmatrix}_{(6\times1)} . \quad \text{(B-8)}$$

Formulation of the Variance-Covariance Matrix for the Spacecraft's State Vector

The fundamental problem considered here is that of estimating the RMS errors in the state vector when uncertainties in the tracking system measurements, measurement biases, and bias errors in station locations are known. It should be pointed out that station location errors (except for ships and aircraft) are to be considered as constant (or bias errors) for orbit determination, since the station location does not change as the spacecraft transits a tracking station. For illustrative purposes, the mathematical model for the range and range rate system is used to determine the errors in the state vector resulting from utilizing such a tracking system. In order to facilitate future matrix manipulations, Equation (B-6) is rewritten in abbreviated form as follows:

$$\widetilde{\mathbf{Y}}_{(4\times1)} = \mathbf{A}_{(4\times6)}\,\widetilde{\mathbf{X}}_{(6\times1)} + \mathbf{B}_{(4\times3)}\,\widetilde{\mathbf{S}}_{(3\times1)} + \mathbf{C}_{(4\times6)}\,\widetilde{\boldsymbol{\beta}}_{(6\times1)}* . \quad \text{(B-9)}$$

where

$$\widetilde{\mathbf{Y}}_{(4\times1)} \equiv \begin{bmatrix} (\mathbf{D}\,\delta\mathbf{K})_{(3\times1)} \\ \hdashline (\mathbf{D}_r^0\,\delta\dot{\mathbf{K}})_{(1\times1)} \end{bmatrix}_{(4\times1)} ,$$

$$\mathbf{A}_{(4\times6)} \equiv \begin{bmatrix} \mathbf{L}_{(3\times3)} & \mathbf{0}_{(3\times3)} \\ \mathbf{M}^0_{r(1\times3)} & \mathbf{L}^0_{r(1\times3)} \end{bmatrix}_{(4\times6)} ,$$

$$\widetilde{\mathbf{X}}_{(6\times1)} \equiv \begin{bmatrix} \delta\mathbf{X}_{(3\times1)} \\ \hdashline \delta\dot{\mathbf{X}}_{(3\times1)} \end{bmatrix}_{(6\times1)} ,$$

*Matrix A is not to be confused with the generalized Matrix A given in Part I.

$$B_{(4\times3)} \equiv \begin{bmatrix} (-J^T R)_{(3\times3)} \\ \hdashline (-F_r V J^T R)_{(1\times3)} \end{bmatrix}_{(4\times3)} ,$$

$$\tilde{S}_{(3\times1)} \equiv (\delta S)_{(3\times1)} = \begin{bmatrix} \delta s_1 \\ \delta s_2 \\ \delta s_3 \end{bmatrix}_{(3\times1)} ,$$

$$C_{(4\times6)} \equiv \begin{bmatrix} D_{(3\times3)} & \vdots & 0_{(1\times3)} \\ \hdashline 0_{(1\times3)} & \vdots & D^0_{r(1\times3)} \end{bmatrix}_{(4\times6)} ,$$

and

$$\tilde{\beta}_{(6\times1)} \equiv \begin{bmatrix} (\delta\beta)_{(3\times1)} \\ \hdashline (\delta\dot{\beta})_{(3\times1)} \end{bmatrix}_{(6\times1)} .$$

Equation (B-9) is a linearized equation (terms of a Taylor expansion terminated with the first order partial derivatives) which states that the deviations in the measurements represented by the column matrix $\tilde{Y}_{(4\times1)}$ are dependent on the deviations in spacecraft position and velocity represented by the column matrix $\tilde{X}_{(6\times1)}$, the deviations in the location of the instrument making the measurement represented by the column matrix $\tilde{S}_{(3\times1)}$, and the bias errors represented by the column matrix $\tilde{\beta}_{(6\times1)}$. A solution of the matrix Equation (B-9) will now be made utilizing the method of weighted least squares. Before doing so, several preliminary conditions must be met; these are:

1. Proper evaluations in spacecraft position and velocity over the interval (t_0, t).

2. Proper weighting of the measurements in accordance with known accuracies associated with these measurements. This means giving greatest weight to the most accurate measurement and the least weight to the least accurate measurement.

To fulfill condition 1, it is necessary to find a set of variables which are independent of time. Since, in Equation (B-9), the variation in the state vector $\tilde{X}_{(6\times1)}$ is time dependent, this equation in its present form cannot be solved in the least squares sense. In order to overcome this, $\tilde{X}_{(6\times1)}$ has to be expressed as a linear combination of another set of variables. In general, this can be represented by the following equation:

$$\tilde{X}_{(6\times1)} = \left[\frac{\partial X(t)}{\partial X(t_0)}\right]_{(6\times6)} \tilde{X}_{0_{(6\times1)}} = [\Phi(t, t_0)]_{(6\times6)} \tilde{X}_{0_{(6\times1)}} , \qquad \text{(B-10)*}$$

where $\Phi(t, t_0)_{(6\times6)}$ is the "state transition matrix" which linearly transforms the state of the vector $\tilde{X}_{(6\times1)}$. This transformation is valid provided that linearity is preserved in the interval (t_0, t).

The elements of the state transition matrix are the first order partial derivatives of the state vector components at time t with respect to time t_0. For a precision trajectory these partial derivatives are determined by numerical integration of the equations of motion.

A precision trajectory is often approximated with a patched conic trajectory (Reference section entitled "Target Body Impact Parameters"). Using a patched conic trajectory for error analysis permits expressing the state transition matrix elements in closed form. This approach significantly reduces computation time on large digital computers and usually provides a good estimate to the actual state transition matrix.

In this paper the elements of the state transition matrix are expressed in closed form using the Keplerian elements as the bridge between the state at time t and at time t_0. The state transition matrix using Keplerian elements is derived as follows:

$$\left.\begin{aligned} \tilde{X}_{(6\times1)} &= P(t)_{(6\times6)} \Lambda_{(6\times1)} \\ \tilde{X}_{0_{(6\times1)}} &= P(t_0)_{(6\times6)} \Lambda_{(6\times1)} \end{aligned}\right\} , \qquad \text{(B-11)}$$

where

$\tilde{X}_{(6\times1)}$ ≡ Variation in state evaluated at time t,

$\tilde{X}_{0_{(6\times1)}}$ ≡ Variation in state evaluated at time t_0,

$$P(t)_{(6\times6)} \equiv \left[\left\{\frac{\partial(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)}{\partial(\Omega_s, i_s, \omega_s, a_s, e_s, \tau_s)}\right\}_t\right]_{(6\times6)} ,$$

*Dr. S. F. Schmidt of Philco Corporation - Western Development Laboratories pointed out to the authors the need to introduce the state transition matrix into Equation (B-9) in order that the unknown variables comprising the vector $\tilde{X}_{(6\times1)}$ can be evaluated in the least squares sense.

and

$$\Lambda_{(6\times 1)} \equiv \begin{bmatrix} \delta\Omega_s \\ \delta i_s \\ \delta\omega_s \\ \delta a_s \\ \delta e_s \\ \delta\tau_s \end{bmatrix}_{(6\times 1)} , \text{ Variation in the Keplerian Elements.}$$

As can be seen, the matrix $P(t)_{(6\times 6)}$ has to be the Jacobian matrix transforming the variation in the orbital elements into the variation in the state vector. The matrix $P(t)_{(6\times 6)}$ is fully derived in Appendix a of this paper. From Equations (B-11), the following result is obtained:

$$\tilde{X}_{(6\times 1)} = [P(t)\, P^{-1}\,(t_0)]_{(6\times 6)}\, \tilde{X}_{0_{(6\times 1)}} = \left(PP_0^{-1}\right)_{(6\times 6)}\, \tilde{X}_{0_{(6\times 1)}} = [\Phi(t,\, t_0)]_{(6\times 6)}\, \tilde{X}_{0_{(6\times 1)}} . \qquad \text{(B-12)}$$

The matrix product $[PP_0^{-1}]_{(6\times 6)}$ represents the state transition matrix* in the interval $(t_0,\, t)$. It is now possible to solve for the errors in the state vector in the least squares sense by introducing (B-12) into (B-9), resulting in:

$$\tilde{Y}_{(4\times 1)} = \Gamma_{(4\times 6)}\, \tilde{X}_{0_{(6\times 1)}} + B_{(4\times 3)}\, \tilde{S}_{(3\times 1)} + C_{(4\times 4)}\, \tilde{\beta}_{(4\times 1)} , \qquad \text{(B-13)}$$

where

$$\Gamma_{(4\times 6)} \equiv (APP_0^{-1})_{(4\times 6)} .$$

The unbiased linear estimate of the unknown vector $\tilde{X}_{0_{(6\times 1)}}$ is obtained from an abbreviated form of Equation (B-13):

$$\tilde{Y}_{(4\times 1)} = \Gamma_{(4\times 6)}\, \tilde{X}_{0_{(6\times 1)}} . \qquad \text{(B-14)}$$

Equation (B-14) represents the linear terms of a Taylor's expansion about the deviations which comprise the column matrix $\tilde{X}_{0_{(6\times 1)}}$ (higher order terms are neglected, see Part I, Appendix b). This equation has to be modified in order to reflect the omission of higher order terms of the Taylor's expansion as well as the effect of the bias errors and station location uncertainties. The

*W. H.Goodyear of IBM Federal Systems Division has derived the state transition matrix in terms of Cartesian coordinates. This form of the transition matrix bypasses numerical difficulties which arise whenever the spacecraft's trajectory is non-elliptic; i.e., parabolic or hyperbolic. For a description and derivation of the state transition in Cartesian form see Reference 7.

modified expression for $\tilde{Y}_{(4\times 1)}$ then is:

$$\tilde{Y}_{(4\times 1)} = \Gamma_{(4\times 6)}\ \tilde{X}_{0_{(6\times 1)}} + \tilde{\varepsilon}_{(4\times 1)} . \tag{B-15}$$

Matrix $\tilde{\varepsilon}_{(4\times 1)}$ is a column matrix of the residuals resulting from the linearization process.

The matrix Equation (B-15) represents a single observation made by the range and range rate system. In that equation, six components of the variation in the state vector are estimated in the least squares sense. It is therefore necessary for the number of measurements, n, to be much larger than the number of unknowns. In terms of Equation (B-15), a "single measurement" means a measurement of one range, one azimuth, one elevation, and one range rate. Therefore, n measurements of such a system actually means $4n$ single observations. Therefore, Equation (B-15) is rewritten:

$$\tilde{Y}_{(4n\times 1)} = \Gamma_{(4n\times 6)}\ \tilde{X}_{0_{(6\times 1)}} + \tilde{\varepsilon}_{(4n\times 1)} , \tag{B-16}$$

where

$$4n > 6 .$$

Equation (B-16) represents the general equation of a range and range rate tracking system making n measurements (this is the same as a radar and a Doppler). All errors in the system are attributed to random noise. Extending Equation (B-16) to two tracking systems necessitates the following modification:

$$\underbrace{\begin{bmatrix} \tilde{Y}_{1_{(4n_1\times 1)}} \\ \hdashline \tilde{Y}_{2_{(4n_2\times 1)}} \end{bmatrix}}_{4(n_1+n_2)\times 1} = \underbrace{\begin{bmatrix} \Gamma_{1_{(4n_1\times 6)}} \\ \hdashline \Gamma_{2_{(4n_2\times 6)}} \end{bmatrix}}_{4(n_1+n_2)\times 6} \tilde{X}_{0_{(6\times 1)}} + \underbrace{\begin{bmatrix} \tilde{\varepsilon}_{1_{(4n_1\times 1)}} \\ \hdashline \tilde{\varepsilon}_{2_{(4n_2\times 1)}} \end{bmatrix}}_{4(n_1+n_2)\times 1} . \tag{B-17}$$

The general variational equation for i tracking stations, $i > 2$, is obtained by expanding the system shown for two stations by Equation (B-17) as follows:

$$\underbrace{\begin{bmatrix} \tilde{Y}_{1_{(4n_1\times 1)}} \\ \hdashline \tilde{Y}_{2_{(4n_2\times 1)}} \\ \hdashline \vdots \\ \tilde{Y}_{i_{(4n_i\times 1)}} \end{bmatrix}}_{4(n_1+n_2+\cdots+n_i)\times 1} = \underbrace{\begin{bmatrix} \Gamma_{1_{(4n_1\times 6)}} \\ \hdashline \Gamma_{2_{(4n_2\times 6)}} \\ \hdashline \vdots \\ \Gamma_{i_{(4n_i\times 6)}} \end{bmatrix}}_{4(n_1+n_2+\cdots+n_i)\times 6} \tilde{X}_{0_{(6\times 1)}} + \underbrace{\begin{bmatrix} \tilde{\varepsilon}_{1_{(4n_1\times 1)}} \\ \hdashline \tilde{\varepsilon}_{2_{(4n_2\times 1)}} \\ \hdashline \vdots \\ \tilde{\varepsilon}_{i_{(4n_i\times 1)}} \end{bmatrix}}_{4(n_1+n_2+\cdots+n_i)\times 1} . \tag{B-18}$$

Or, abbreviated form,

$$\widetilde{K}_{(4\ell\times 1)} = G_{(4\ell\times 6)}\,\widetilde{X}_{0_{(6\times 1)}} + \widetilde{\varepsilon}_{(4\ell\times 1)}\ , \tag{B-19}$$

where

$$\ell \equiv \sum_{k=1}^{i} n_k\,,\quad i \geq 1$$

and

$$\ell >>> 6\ .$$

The Gaussian condition for solving a system of equations is that:

$$\phi = \left(\widetilde{\varepsilon}^T\,\widetilde{W}^{-1}\,\widetilde{\varepsilon}\right)_{(1\times 1)} = \text{Min}^* \tag{B-20}$$

$$= (\widetilde{K}^T - \widetilde{X}_0^T\,G^T)_{(1\times 4\ell)}\,\widetilde{W}^{-1}_{(4\ell\times 4\ell)}\,(\widetilde{K} - G\,\widetilde{X}_0)_{(4\ell\times 1)} = \text{Min}\ , \tag{B-21}$$

where $W^{-1}_{(4\ell\times 4\ell)}$ is the covariance matrix associated with the measurements. This matrix weights measurements in accordance with their relative accuracy, so that the greatest weight is given to those measurements having the smallest uncertainties associated with them.

The condition that $\phi =$ minimum can be obtained by differentiating Equation (B-21) with respect to $\widetilde{X}_{0_{(6\times 1)}}$, that is:

$$\left[\frac{\partial\phi}{\partial\widetilde{X}_0}\right]_{(6\times 1)} = -\,2\left(G^T\,\widetilde{W}^{-1}\,\widetilde{K}\right)_{(6\times 1)} + 2(G^T\,\widetilde{W}^{-1}\,G)_{(6\times 6)}\,\widetilde{X}_{0_{(6\times 1)}} = 0_{(6\times 1)}\ . \tag{B-22}$$

The minimum condition also requires that $[\partial^2\phi/\partial\widetilde{X}_0^2]_{(6\times 6)}$ be positive definite.

From (B-22),

$$\left[\frac{\partial^2\phi}{\partial\widetilde{X}^2}\right]_{(6\times 6)} = (G^T\,\widetilde{W}^{-1}\,G)_{(6\times 6)}\ .$$

*See Reference 1, Equation (B-6) Page 29.

If $(G^T \widetilde{W}^{-1} G)_{(6\times6)}$ is positive definite, (eigenvalues are all positive) then ϕ is a minimum. From (B-22) the estimate to the column matrix $\widetilde{X}_0$ is:

$$\widetilde{X}_{0_{(6\times1)}} = (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times6)} (G^T \widetilde{W}^{-1} \widetilde{K})_{(6\times1)} . \tag{B-23}$$

Let a matrix $\widetilde{N}$ be defined to represent only the random noise in the measurements. Then if

$$\widetilde{K}_{(4\ell\times1)} = \widetilde{N}_{(4\ell\times1)}$$

and

$$E(\widetilde{K}\widetilde{K}^T)_{(4\ell\times4\ell)} = E(\widetilde{N}\widetilde{N}^T)_{(4\ell\times4\ell)} = \widetilde{W}_{(4\ell\times4\ell)} , \tag{B-24}*$$

measurements from different tracking stations are assumed to be statistically independent (uncorrelated). The following expansion of matrix $\widetilde{W}_{(4\ell\times4\ell)}$ reflects this assumption by having null matrices off the main diagonal.

$$\widetilde{W}_{(4\ell\times4\ell)} \equiv \begin{bmatrix} W_{1_{(4n_1\times4n_1)}} & 0 & 0 & \cdots 0 \\ 0 & W_{2_{(4n_2\times4n_2)}} & 0 & \cdots\cdot \\ 0 & 0 & W_{3_{(4n_3\times4n_3)}} & \cdots\cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & W_{i_{(4n_i\times4n_i)}} \end{bmatrix}_{(4\ell\times4\ell)} .$$

The covariance matrix in terms of spacecraft position and velocity is derived by using Equations (B-23) and (B-24) as follows:

$$E(\widetilde{X}_0 \widetilde{X}_0^T)_{(6\times6)} = (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times6)} G^T_{(6\times4\ell)} \widetilde{W}^{-1}_{(4\ell\times4\ell)} E(\widetilde{K}\widetilde{K}^T)_{(4\ell\times4\ell)} \widetilde{W}^{-1}_{(4\ell\times4\ell)} G_{(4\ell\times6)} (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times6)}$$

*E( ): The expectation operator (Reference 8).

$$= (G^T W^{-1} G)^{-1}_{(6\times 6)} \, G^T_{(6\times 4\ell)} \, \widetilde{W}^{-1}_{(4\ell\times 4\ell)} \, E(\widetilde{N}\widetilde{N}^T)_{(4\ell\times 4\ell)} \, \widetilde{W}^{-1}_{(4\ell\times 4\ell)} \, G_{(4\ell\times 6)} \, (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)}$$

$$= (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)} \, G^T_{(6\times 4\ell)} \, \widetilde{W}^{-1}_{(4\ell\times 4\ell)} \, \widetilde{W}_{(4\ell\times 4\ell)} \, W^{-1}_{(4\ell\times 4\ell)} \, G_{(4\ell\times 6)} \, (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)}$$

$$= (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)} \, (G^T \widetilde{W}^{-1} G)_{(6\times 6)} \, (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)}$$

$$= (G^T \widetilde{W}^{-1} G)^{-1}_{(6\times 6)} \quad . \tag{B-25}$$

In terms of i tracking stations, Equation 25 when written in expanded form using equation (B-18) is:

$$E(\widetilde{X}_0 \widetilde{X}_0^T)_{(6\times 6)} = \left[\sum_{k=1}^{i} \sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} \Gamma_k)_{j_k} \right]^{-1}_{(6\times 6)} , \tag{B-26}$$

where

$$n_k > 6 \quad ,$$

$$k \geq 1 \quad .$$

and

$$[\mathbf{W}_k]_{(4\times4)} \equiv \left\{ \begin{bmatrix} \eta_r^2 & 0 & 0 & 0 \\ 0 & r^2 \cos^2 \epsilon\ \eta_\alpha^2 & 0 & 0 \\ 0 & 0 & r^2\ \eta_\epsilon^2 & 0 \\ 0 & 0 & 0 & \eta_{\dot r}^2 \end{bmatrix}_k \right\}_{(4\times4)} .$$

To obtain the biased estimate for the range and range rate system, let the error in the measurement reflect the effect of measurement noise, measurement bias, and station location uncertainties as follows:

$$\tilde{\mathbf{K}}_{(4\ell\times1)} = \tilde{\mathbf{N}}_{(4\ell\times1)} + \tilde{\mathbf{B}}_{(4\ell\times3i)}\tilde{\boldsymbol{\mathcal{S}}}_{(3i\times1)} + \tilde{\mathbf{C}}_{(4\ell\times4i)}\tilde{\mathbf{b}}_{(4i\times1)} , \qquad \text{(B-27)}$$

where

$$\tilde{\mathbf{B}}_{(4\ell\times3)} \equiv \begin{bmatrix} \mathbf{B}_{1\,(4n_1\times3)} \\ \hdashline \mathbf{B}_{2\,(4n_2\times3)} \\ \hdashline \vdots \\ \hdashline \mathbf{B}_{i\,(4n_i\times3)} \end{bmatrix}_{(4\ell\times3i)} ; \quad \tilde{\mathbf{C}} \equiv \begin{bmatrix} \mathbf{C}_{1\,(4n_1\times4)} \\ \hdashline \mathbf{C}_{2\,(4n_2\times4)} \\ \hdashline \vdots \\ \hdashline \mathbf{C}_{i\,(4n_i\times4)} \end{bmatrix}_{(4\ell\times4i)}$$

and

$$\tilde{\boldsymbol{\mathcal{S}}}_{(3i\times1)} \equiv \begin{bmatrix} \tilde{\mathbf{S}}_{1\,(3\times1)} \\ \hdashline \tilde{\mathbf{S}}_{2\,(3\times1)} \\ \hdashline \vdots \\ \hdashline \tilde{\mathbf{S}}_{i\,(3\times1)} \end{bmatrix}_{(3i\times1)} ; \quad \tilde{\mathbf{b}}_{(4i\times1)} \equiv \begin{bmatrix} \tilde{\beta}_{1\,(4\times1)} \\ \hdashline \tilde{\beta}_{2\,(4\times1)} \\ \hdashline \vdots \\ \hdashline \tilde{\beta}_{i\,(4\times1)} \end{bmatrix}_{(4i\times1)} .$$

By substituting Equation (B-27) in (B-23), the biased estimate to the variations in the state vector is:

$$\tilde{X}_{0_{(6\times 1)}} = (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \; \{G^T \tilde{W}^{-1} (\tilde{N} + \tilde{B}\tilde{S} + \tilde{C}\tilde{b})\}_{(6\times 1)} \; . \qquad \text{(B-28)}$$

The covariance matrix in the state vector for the biased estimate then is:*

$$E(\tilde{X}_0 \tilde{X}_0^T)_{(6\times 6)} = (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \, G^T_{(6\times 4\ell)} \, \tilde{W}^{-1}_{(4\ell\times 4\ell)} \, \{E(\tilde{N}\tilde{N}^T) + \tilde{B}\,E(\tilde{S}\tilde{S}^T)\,\tilde{B}^T + \tilde{C}E(\tilde{b}\tilde{b}^T)\,\tilde{C}^T\}_{(4\ell\times 4\ell)} \, \tilde{W}^{-1}_{(4\ell\times 4\ell)} \, G_{(4\ell\times 6)} \, (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \; .$$

Let it be assumed that:

$$E(\tilde{N}\tilde{S}^T) = E(\tilde{N}\tilde{b}^T) = E(\tilde{S}\tilde{b}^T) = 0$$

then

$$\begin{aligned} E(\tilde{X}_0 \tilde{X}_0^T)_{(6\times 6)} &= (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \, G^T_{(6\times 4\ell)} \, \tilde{W}^{-1}_{(4\ell\times 4\ell)} \, \{\tilde{W} + \tilde{B}E(\tilde{S}\tilde{S}^T)\,\tilde{B}^T + \tilde{C}\,E(\tilde{b}\tilde{b}^T)\,\tilde{C}^T\}_{(4\ell\times 4\ell)} \, \tilde{W}^{-1}_{(4\ell\times 4\ell)} G_{(4\ell\times 6)} \, (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \\ &= (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} + (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \, (G^T \tilde{W}^{-1} \tilde{B})_{(6\times 3i)} \, E(\tilde{S}\tilde{S}^T)_{(3i\times 3i)} \, (\tilde{B}^T \tilde{W}^{-1} G)_{(3i\times 6i)} (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \\ &\quad + (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \, (G^T \tilde{W}^{-1} \tilde{C})_{(6\times 4i)} \, E(\tilde{b}\tilde{b}^T)_{(4i\times 4i)} \, (\tilde{C}^T \tilde{W}^{-1} G)_{(4i\times 6)} \, (G^T \tilde{W}^{-1} G)^{-1}_{(6\times 6)} \end{aligned} \qquad \text{(B-29)}$$

or, in expanded form,

$$\begin{aligned} E(\tilde{X}_0 \tilde{X}_0^T)_{(6\times 6)} = {} & \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k W_k^{-1} \Gamma_k)_{j_k} \right] \right\}^{-1} + \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\}^{-1} \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k W_k^{-1} B_k)_{j_k} \right] \overline{W}_k \left[\sum_{j_k=1}^{n_k} (B_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\} \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\}^{-1} \\ & + \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\}^{-1} \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} C_k)_{j_k} \right] W_{\overline{\beta}_k} \left[\sum_{j_k=1}^{n_k} (C_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\} \left\{ \sum_{k=1}^{i} \left[\sum_{j_k=1}^{n_k} (\Gamma_k^T W_k^{-1} \Gamma_k)_{j_k} \right] \right\}^{-1} , \end{aligned} \qquad \text{(B-30)}$$

*This proof was suggested by W. Goodyear, IBM Federal Systems Division.

where

$$[\overline{W}_k]_{(3\times3)} \equiv [E(\tilde{S}\tilde{S}^T)_k]_{(3\times3)} = \left\{\begin{bmatrix} \eta_{s_1}^2 & 0 & 0 \\ 0 & \eta_{s_2}^2 & 0 \\ 0 & 0 & \eta_{s_3}^2 \end{bmatrix}_k\right\}_{(3\times3)}$$

and

$$(W_{\tilde{\beta}_k})_{(4\times4)} \equiv [E(\tilde{\beta}\tilde{\beta}^T)_k]_{(4\times4)} = \left\{\begin{bmatrix} \eta_r^2 & 0 & 0 & 0 \\ 0 & \eta_\alpha^2 & 0 & 0 \\ 0 & 0 & \eta_\epsilon^2 & 0 \\ 0 & 0 & 0 & \eta_{\dot{r}}^2 \end{bmatrix}_k\right\}_{(4\times4)},$$

where

$$k \equiv \text{index for the } k^{th} \text{ tracking station.}$$

The errors in the state vector as obtained from Equation (B-29) or (B-30) are those which correspond to a least squares solution in which the effects of the bias errors were included but not solved for. This has the effect of obtaining a nonoptimal least squares estimate of the errors in the state vector. To obtain the optimum least squares estimate of the errors in the state vector requires solving for corrections to the tracking station location as well as for corrections to the measurements. The optimal least squares estimator is now to be derived for a single range and range rate tracking system. For n measurements Equation (B-13) reads:

$$\tilde{Y}_{(4n\times1)} = \Gamma_{(4n\times6)}\,\tilde{X}_{0_{(6\times1)}} + B_{(4n\times3)}\,\tilde{S}_{(3\times1)} + C_{(4n\times4)}\,\tilde{\beta}_{(4\times1)} \; . \tag{B-31}$$

Assuming that the deviations in station location and in the measurements remain constant necessitates introducing two equations of constraint, which are stated as follows:

$$\left.\begin{aligned} \tilde{S}_{0_{(3\times1)}} &= \tilde{S}_{(3\times1)} && \text{(For Deviations in Station Location)} \\ (C\tilde{\beta}_0)_{(4n\times1)} &= C_{(4n\times4)}\,\tilde{\beta}_{(4\times1)} && \text{(For Deviations in the Measurements)} \end{aligned}\right\} \; . \tag{B-32}$$

From a combination of Equation (B-31) with (B-32) the following matrix equation results:

$$\left[\begin{array}{c} \tilde{Y}_{(4n\times 1)} \\ \hdashline \tilde{S}_{0_{(3\times 1)}} \\ \hdashline (C\tilde{\beta}_0)_{4n\times 1} \end{array}\right]_{(8n+3)\times 1} = \left[\begin{array}{c:c:c} \Gamma_{(4n\times 6)} & B_{(4n\times 3)} & C_{(4n\times 4)} \\ \hdashline 0_{(3\times 6)} & I_{(3\times 3)} & 0_{(3\times 4)} \\ \hdashline 0_{(4n\times 6)} & 0_{(4n\times 3)} & C_{(4n\times 4)} \end{array}\right]_{(8n+3)\times 13} \left[\begin{array}{c} \tilde{X}_{0_{(6\times 1)}} \\ \hdashline \tilde{S}_{(3\times 1)} \\ \hdashline \tilde{\beta}_{(4\times 1)} \end{array}\right]_{(13\times 1)} \tag{B-33}$$

or, in abbreviated form,

$$\tilde{L}_{(8n+3)\times 1} = \mathcal{G}_{(8n+3)\times 13}\,\tilde{T}_{0_{(13\times 1)}} \quad . \tag{B-34}$$

The omission of higher order terms resulting from the linearization process necessitates restating Equation (B-34) as follows:

$$\tilde{L}_{(8n+3)\times 1} = \mathcal{G}_{(8n+3)\times 13}\,\tilde{T}_{0_{(13\times 1)}} + \bar{\epsilon}_{(4n+3)\times 1} \quad . \tag{B-35}$$

As described in Equations (B-20) through (B-23), a best estimate for the unknown column matrix $\tilde{T}$ is given by the following:

$$\tilde{T}_{0_{(13\times 1)}} = (\mathcal{G}^T \hat{W}^{-1} \mathcal{G})^{-1}_{(13\times 13)} (G^T \hat{W}^{-1} \tilde{L})_{(13\times 1)} \quad , \tag{B-36}$$

where

$$\hat{W} \equiv E(\tilde{L}_0 \tilde{L}_0^T)_{(8n+3)\times(8n+3)} = \left[\begin{array}{c:c:c} E(\tilde{Y}\tilde{Y}^T)_{(4n\times 4n)} & 0 & 0 \\ \hdashline 0 & E(\tilde{S}_0 \tilde{S}_0^T)_{(3\times 3)} & 0 \\ \hdashline 0 & 0 & \{C\,E(\tilde{\beta}\tilde{\beta}_0^T)\,C^T\}_{(4n\times 4n)} \end{array}\right]_{(8n+3)\times(8n+3)}$$

and

$$\begin{cases} E(\tilde{Y}\tilde{Y}^T)_{(4n\times 4n)} = W_{(4n\times 4n)} \\ E(\tilde{S}_0 \tilde{S}_0^T)_{(3\times 3)} = \overline{W}_{(3\times 3)} \\ E(\tilde{\beta}_0 \tilde{\beta}_0^T)_{(4n\times 4n)} = [W_{\overline{\beta}}]_{(4n\times 4n)} \end{cases} \quad .$$

It then follows that the optimal estimate for the errors in the components of the column matrix $\tilde{T}$ is:

$$E(\tilde{T}_0\,\tilde{T}_0^T)_{(13\times 13)} = (\mathcal{G}^T\,\hat{W}^{-1}\,\mathcal{G})^{-1}_{(13\times 13)} \quad , \tag{B-37}$$

where

$$E(\tilde{T}_0\,\tilde{T}_0^T)_{(13\times 13)} \equiv \left[\begin{array}{c:c:c} E(\tilde{X}_0\,\tilde{X}_0^T) & E(\tilde{X}_0\,\tilde{S}^T) & E(\tilde{X}_0\,\tilde{\beta}^T) \\ \hdashline E(\tilde{S}\tilde{X}_0^T) & E(\tilde{S}\tilde{S}^T) & E(\tilde{S}\tilde{\beta}^T) \\ \hdashline E(\tilde{\beta}\tilde{X}_0^T) & E(\tilde{\beta}\,\tilde{S}^T) & E(\tilde{\beta}\,\tilde{\beta}^T) \end{array}\right]_{(13\times 13)} .$$

and

$$(\mathcal{G}^T\,\hat{W}^{-1}\,\mathcal{G})^{-1}_{(13\times 13)} \equiv \left[\begin{array}{c:c:c} \sum_{j=1}^{n} (\Gamma^T\,W^{-1}\,\Gamma)_j & \sum_{j=1}^{n} (\Gamma^T\,W^{-1}\,B)_j & \sum_{j=1}^{n} (\Gamma^T\,W^{-1}\,C)_j \\ \hdashline \sum_{j=1}^{n} (B^T\,W^{-1}\,\Gamma)_j & \left\{\sum_{j=1}^{n} (B^T\,W^{-1}\,B)_j + \overline{W}^{-1}\right\} & \sum_{j=1}^{n} (B^T\,W^{-1}\,C)_j \\ \hdashline \sum_{j=1}^{n} (C^T\,W^{-1}\,\Gamma)_j & \sum_{j=1}^{n} (C^T\,W^{-1}\,B)_j & \left\{\sum_{j=1}^{n} (C^T\,W^{-1}\,C)_j + W_{\overline{\beta}}\right\} \end{array}\right]^{-1}_{(13\times 13)} .$$

By inverting the matrix on the right hand side of Equation (B-37), the optimal estimate of the co-variance matrix for the state vector $E(\tilde{X}_0\tilde{X}_0^T)$ is:

$$E(\tilde{X}_0\,\tilde{X}_0^T)_{(6\times 6)} = \left[A_{11} - A_{13}\,A_{33}^{-1}\,A_{13}^T - (A_{12} - A_{13}\,A_{33}^{-1}\,A_{23}^T)\,(A_{22} - A_{23}\,A_{33}^{-1}\,A_{23}^T)^{-1}\,(A_{12}^T - A_{23}\,A_{33}^{-1}\,A_{13}^T)\right]^{-1}_{(6\times 6)}, \tag{B-38}$$

where

$$A_{11} \equiv \left[\sum_{j=1}^{n} (\Gamma^T\,W^{-1}\,\Gamma)_j\right]_{(6\times 6)} \quad ,$$

$$A_{12} \equiv \left[\sum_{j=1}^{n} (\Gamma^T W^{-1} B)_j\right]_{(6\times 3)} ,$$

$$A_{13} \equiv \left[\sum_{j=1}^{n} (\Gamma^T W^{-1} C)_j\right]_{(6\times 4)} ,$$

$$A_{21} = A_{12}^T ,$$

$$A_{22} \equiv \left\{\sum_{j=1}^{n} (B^T W^{-1} B)_j + \overline{W}^{-1}\right\}_{(3\times 3)} ,$$

$$A_{23} \equiv \left\{\sum_{j=1}^{n} (B^T W^{-1} C)_j\right\}_{(3\times 4)} ,$$

$$A_{31} = A_{13}^T ,$$

$$A_{32} = A_{23}^T ,$$

$$A_{33} \equiv \left\{\sum_{j=1}^{n} (C^T W^{-1} C)_j + W_{\overline{\beta}}\right\}_{(4\times 4)} ,$$

and

$$C_{(4\times 4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -r\cos\epsilon & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{(4\times 4)} .$$

The nonoptimal estimate of the errors in the state vector for a single tracking station obtained by setting index i of Equation (B-30) to one is:

$$E(X_0 X_0^T)_{(6\times 6)} = \left\{A_{11}^{-1} + A_{11}^{-1} A_{12} \overline{W} A_{12}^T A_{11}^{-1} + A_{11}^{-1} A_{13} W_{\overline{\beta}} A_{13}^T A_{11}^{-1}\right\}_{(6\times 6)} . \qquad \text{(B-39)}$$

A numerical evaluation of Equations (B-38) and (B-39) will show the estimate for the errors in the state vector to be low when Equation (B-38) is used and high when Equation (B-39) is used. The most reasonable estimate of the errors in the state vector probably is obtained from a combination of the non-optimal estimator with the optimal estimator. This can be accomplished by solving for some but not all of the bias errors in the least squares sense.

Effects of a Priori Knowledge on the Errors in the State Vector

As the powered flight portion of a trajectory is terminated and the free flight portion is initiated, some knowledge about the RMS errors in spacecraft position and velocity at the transition point can be assumed. Such knowledge is defined here to be a priori knowledge for the free-flight trajectory; that is to say, the following RMS errors are assumed to be known at the time the spacecraft is injected into orbit.

η_h - RMS error in spacecraft height above the earth,

η_{v_s} - RMS error in speed,

η_{γ_s} - RMS error in flight path angle,

η_a - RMS error in injection azimuth,

$\eta_{\lambda'}$ - RMS error in right ascension (equivalent to RMS error in longitude), and

η_δ - RMS error in declination (equivalent to RMS error in latitude).

In order to include the effects of these RMS errors in the RMS error in spacecraft position and velocity, the following procedure is used. The transformation relating $[\delta h, \delta v_s, \delta\gamma_s, \delta a_i, \delta\lambda', \delta(\delta)]$ with $[\delta x_1, \delta x_2, \delta x_3, \delta\dot{x}_1, \delta\dot{x}_2, \delta\dot{x}_3]$ is given by the following matrix equation:

$$\underset{(6\times1)}{\tilde{\mathbf{X}}(t)} = \underset{(6\times6)}{\psi(t)} \ \underset{(6\times1)}{\bar{\bar{\beta}}(t)} , \qquad \text{(B-40)}$$

where

$$\tilde{\mathbf{X}}_{(6\times1)} \equiv \begin{vmatrix} \delta\mathbf{X}_{(3\times1)} \\ \hdashline \delta\dot{\mathbf{X}}_{(3\times1)} \end{vmatrix}_{(6\times1)} ,$$

$$\underset{(6\times6)}{\psi(t)} \equiv \left[\left\{\frac{\partial(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)}{\partial(h, v_s, \gamma_s, a_i, \lambda', \delta)}\right\}_t\right]_{(6\times6)} ,$$

and

$$\bar{\bar{\beta}}_{(6\times 1)} \equiv \begin{bmatrix} \delta h \\ \delta v_s \\ \delta \gamma_s \\ \delta \alpha_i \\ \delta \lambda' \\ \delta(\delta) \end{bmatrix}_{(6\times 1)} .$$

The time that a spacecraft is inserted into orbit corresponds to the epoch time τ. Equation (B-40) evaluated at time τ reads:

$$\tilde{X}(\tau)_{(6\times 1)} = \Psi(\tau)_{(6\times 6)} \; \bar{\bar{\beta}}(\tau)_{(6\times 1)} . \tag{B-41}$$

By using the result from Equation (B-12), Equation (B-41) can be restated as follows:

$$\begin{aligned} \tilde{X}(\tau)_{(6\times 1)} &= [P(\tau)\, P_0^{-1}]_{(6\times 6)} \tilde{X}_{0_{(6\times 1)}} = \Psi(\tau)_{(6\times 6)} \bar{\bar{\beta}}(\tau)_{(6\times 1)} , \\ \bar{\bar{\beta}}(\tau)_{(6\times 1)} &= \bar{\bar{\beta}}_{\tau_{(6\times 1)}} = \Psi^{-1}_{\tau_{(6\times 6)}} \; [P_\tau\, P_0^{-1}]_{(6\times 6)} \tilde{X}_{0_{(6\times 1)}} , \end{aligned} \tag{B-42}$$

or

$$\bar{\bar{\beta}}_{\tau_{(6\times 1)}} = \left\{ \Psi^{-1}_{\tau_{(6\times 6)}} \; [\Phi(\tau,\, t_0)]_{(6\times 6)} \right\} \tilde{X}_{0_{(6\times 1)}} .$$

Equation (B-42) is combined with Equation (B-19) as follows:

$$\begin{bmatrix} \tilde{K} \\ \hline \bar{\bar{\beta}}_\tau \end{bmatrix}_{(4\ell+6)\times 1} = \begin{bmatrix} G_{(4\ell\times 6)} \\ \hline \{\Psi^{-1}\, \Phi(\tau,\, t_0)\}_{(6\times 1)} \end{bmatrix}_{(4\ell+6)\times 6} \tilde{X}_{0_{(6\times 1)}} + \begin{bmatrix} \tilde{\varepsilon}_{(4\ell\times 1)} \\ \hline 0_{(6\times 1)} \end{bmatrix}_{(4\ell+6)\times 1} . \tag{B-43}$$

The covariance matrix reflecting the effect of a priori knowledge of the state vector components is derived in accordance with Equation (B-25) as

$$E(\tilde{X}_0 \tilde{X}_0^T)_{(6\times 6)} = \left[(G^T \tilde{W}^{-1} G)_{(6\times 6)} + \Phi^T_{(6\times 6)} \{\Psi_\tau E(\bar{\bar{\beta}}_\tau \bar{\bar{\beta}}_\tau^T) \Psi_\tau^T\}^{-1}_{(6\times 6)} \Phi_{(6\times 6)} \right]^{-1}_{(6\times 6)} , \tag{B-44}$$

where

$$\Phi_{(6\times6)} = \Phi(\tau, t_0) = [\mathbf{P}(\tau)\,\mathbf{P}^{-1}(t_0)]_{(6\times6)} .$$

For the nonoptimal least squares estimator, the influence of a priori knowledge introduces a modification of Equation (B-39) as follows:

$$E(\tilde{\mathbf{X}}_0 \tilde{\mathbf{X}}_0^T)_{(6\times6)} = [\mathbf{B}_{11}^{-1} + \mathbf{B}_{11}^{-1} \mathbf{A}_{12} \overline{\mathbf{W}} \mathbf{A}_{12}^T \mathbf{B}_{11}^{-1} + \mathbf{B}_{11}^{-1} \mathbf{A}_{13} \mathbf{W}_{\overline{\beta}} \mathbf{A}_{13}^T \mathbf{B}_{11}^{-1}] , \tag{B-45}$$

where

$$\mathbf{B}_{11_{(6\times6)}} \equiv \left[\mathbf{A}_{11} + \Phi^T \{\Psi_\tau\, E(\bar{\bar{\beta}}_\tau \bar{\bar{\beta}}_\tau^T)\, \Psi_\tau^T\}^{-1} \Phi\right]_{(6\times6)} .$$

Similarly, using matrix $[\mathbf{B}_{11}]_{(6\times6)}$ in place of matrix $[\mathbf{A}_{11}]_{(6\times6)}$ in Equation (B-38) will incorporate the influence of a priori knowledge into the optimal least squares estimator. To incorporate the effects of a priori knowledge into the covariance matrix of spacecraft position and velocity, it is sometimes presumed that the RMS errors are uncorrelated; that is,

$$E(\bar{\bar{\beta}}\bar{\bar{\beta}}^T)_{(6\times6)} = \text{Trace } E(\bar{\bar{\beta}} \cdot \bar{\bar{\beta}}^T)_{(6\times6)} = \begin{bmatrix} \eta_h^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_{v_s}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_{\gamma_s}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{\alpha_i}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{\lambda'}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_\delta^2 \end{bmatrix}_{(6\times6)} . \tag{B-46}$$

By neglecting the covariance elements in the covariance matrix, a conservative estimate for the error volume is always obtained (References 9 and 10). Whenever the covariances in the variables (h, v_s, γ_s, α_i, λ', δ) are available, they are to be included in the covariance matrix. This means that $E(\bar{\bar{\beta}}\bar{\bar{\beta}}^T)_{(6\times6)}$ is nondiagonal and symmetric; or, in other words:

$$\eta_{\bar{\bar{\beta}}_i \bar{\bar{\beta}}_j} = \rho_{ij}\, \eta_{\bar{\bar{\beta}}_i}\, \eta_{\bar{\bar{\beta}}_j} , \tag{B-47}$$

where

$$\rho_{ij} \equiv \text{Correlation Coefficient}$$

$$\begin{cases} -1 < \rho_{ij} < 1 & \text{for} \quad i \neq j \\ \rho_{ij} = 1 & \text{for} \quad i = j \end{cases}$$

$$\eta_{\bar{\beta}_i \bar{\beta}_j} \equiv \text{Covariance}.$$

The effect of a priori knowledge is very significant whenever the amount of tracking data is limited. As the amount of tracking data is increased, the influence of the a priori knowledge diminishes to a point of insignificance.

The Ellipsoid of Errors in Spacecraft Position and Velocity

The normal distribution function for the vector (Reference 11)

$$\tilde{X}_{(6\times1)} = \begin{bmatrix} \delta X_{(3\times1)} \\ \hdashline \delta \dot{X}_{(3\times1)} \end{bmatrix}_{(6\times1)} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta \dot{x}_1 \\ \delta \dot{x}_2 \\ \delta \dot{x}_3 \end{bmatrix}_{(6\times1)}$$

is given by

$$f(\delta x_1, \delta x_2, \ldots \delta \dot{x}_3) = \frac{1}{(2\pi)^3 \, |\Delta|^{1/2}} \exp -\frac{1}{2} (\tilde{X}^T \Delta^{-1} \tilde{X})_{(1\times1)}, \tag{B-48}$$

where

$$\Delta_{(6\times6)} \equiv E(\tilde{X}\tilde{X}^T)_{(6\times6)}$$ (The variance-covariance matrix in spacecraft position and velocity)

and

$$|\Delta| \equiv \text{Determinant of matrix } \Delta.$$

The normal distribution function (B-48) is constant for points of the hyperellipsoid $(\tilde{X}^T \Delta^{-1} \tilde{X})_{(1\times 1)} = k$ where k is a constant; that is to say, the probability level that a random point $(\delta x_1,\ \delta x_2,\ \cdots\ \delta \dot{x}_3)$ will fall inside the hyperellipsoid is dictated by the value assigned to the constant k. It is more logical to consider the error ellipsoids in position and velocity separately, primarily because greater physical significance can be associated with such surfaces rather than with the hyperellipsoid. In so doing let

$$\delta X_{(3\times 1)} \equiv (X - \bar{X})_{(3\times 1)}$$

$$\bar{\Delta}_{(3\times 3)} \equiv E(\delta X\, \delta X^T)_{(3\times 3)} \text{ (Covariance matrix in spacecraft position),}$$

$$\delta \dot{X}_{(3\times 1)} \equiv (\dot{X} - \bar{\dot{X}})_{(3\times 1)}\ ,$$

in which

$$\hat{\Delta}_{(3\times 3)} \equiv E(\delta \dot{X}\, \delta \dot{X}^T)_{(3\times 3)} \text{ (Covariance matrix in spacecraft velocity),}$$

$$\bar{X} \equiv \text{Mean value of } X\ ,$$

$$\bar{\dot{X}} \equiv \text{Mean value of } \dot{X}\ ,$$

$$|\bar{\Delta}| \equiv \text{Determinant of matrix } \bar{\Delta}\ ;$$

and

$$|\hat{\Delta}| \equiv \text{Determinant of matrix } \hat{\Delta}\ .$$

The normal distribution functions corresponding to position and velocity vectorial components are as follows:

$$\bar{f}(\delta x_1,\ \delta x_2,\ \delta x_3) = \frac{1}{(2\pi)^{3/2}\ |\bar{\Delta}|^{1/2}} \exp - 1/2\ (\delta X^T\ \bar{\Delta}^{-1}\ \delta X)_{(1\times 1)}$$

and

$$\hat{f}(\delta\dot{x}_1, \delta\dot{x}_2, \delta\dot{x}_3) = \frac{1}{(2\pi)^{3/2} \, |\hat{\Delta}|^{1/2}} \exp - \left[\frac{1}{2} (\delta\dot{X}^T \hat{\Delta}^{-1} \delta\dot{X})_{(1\times1)}\right]. \quad \text{(B-49)}$$

Each of the normal distribution functions (B-49) is constant for points on the ellipsoids:

$$(\delta X^T \bar{\Delta}^{-1} \delta X)_{(1\times1)} = \bar{k}_1 \quad \text{(B-50)}$$

and

$$(\delta\dot{X}^T \hat{\Delta}^{-1} \delta\dot{X})_{(1\times1)} = \hat{k}_1 ,$$

where $\bar{k}_1$ and $\hat{k}_1$ are constants. Therefore, the probability that a random point $(\delta x_1, \delta x_2, \delta x_3)$ will fall inside the error ellipsoid for position is (Reference 11):

$$p = \sqrt{\frac{2}{\pi}} \int_0^{\bar{k}_1} t^2 e^{-t^2/2} \, dt , \quad \text{(B-51)}$$

so that when $\bar{k}_1 = 1.5382$, $p = .5$. The corresponding ellipsoid is called the 50 percent error ellipsoid. The same probability level is associated with the velocity error ellipsoid.

The probability levels for different values of $\bar{k}_1$, as generated by Equation (B-51), are given by the following table (Reference 11):

$\bar{k}_1$	p
1.101	.25
1.538	.50
2.027	.75
2.500	.90
2.795	.95
3.368	.99

To simplify the mathematical analysis of the shape and orientation of the error ellipsoids (B-50), a technique known as "Transformation to the principal axis" is used. Described mathematically, the equation of an ellipsoid falls into the general category of the "quadratic form." It can then be stated that there exist linear transformations

$$\delta X_{(3\times1)} = T_{(3\times3)} \, \xi_{(3\times1)} \quad \text{(B-52)}$$

which reduce the quadratic form

$$(\delta \mathbf{X}^T \bar{\Delta}^{-1} \delta \mathbf{X})_{(1\times 1)} = \bar{k}_1 \tag{B-53}$$

into a linear combination of squares:

$$[\xi^T \tilde{\mathbf{D}} \xi]_{(1\times 1)} = \bar{k}_1 \ , \tag{B-54}$$

where

$$\tilde{\mathbf{D}}_{(3\times 3)} \equiv (\mathbf{T}^T \bar{\Delta}^{-1} \mathbf{T})_{(3\times 3)} \quad .$$

Matrix $\tilde{\mathbf{D}}_{(3\times 3)}$ is diagonal; that is, the elements of the matrix have the following properties:

$$\tilde{d}_{ij} = \begin{cases} 0 & i \neq j \\ \tilde{d}_{ii} & i = j \end{cases} \quad .$$

Elements $\tilde{d}_{ii}$ are often referred to as the eigenvalues of the quadratic equation and the columns of matrix $\mathbf{T}_{(3\times 3)}$ in (B-52) as the eigenvectors. Thus the orientation of the error ellipsoid can be established with reference to a designated coordinate system.

To show the transformation to the principal axis in more detail, let Equation (B-53) be re-written as follows:

$$\sum_{j=1}^{3} \sum_{i=1}^{3} c_{ij} \, \delta \mathbf{x}_i \, \delta \mathbf{x}_j = \bar{k}_1 \ , \tag{B-55}$$

where

$$[c_{ij}]_{(3\times 3)} = \bar{\Delta}^{-1}_{(3\times 3)}$$

and

$$c_{ij} = c_{ji} \quad .$$

It remains to reduce Equation (B-55) to the form

$$\sum_{i=1}^{3} \tilde{d}_{ii}\, \xi_i^2 = \bar{k}_1 \ , \tag{B-56}$$

where

$$[\tilde{d}_{ii}]_{(3\times3)} = (T^T \bar{\Delta}^{-1} T)_{(3\times3)} \ .$$

To do so, let it be assumed that element $c_{12}(= c_{21})$ is the largest off-diagonal element of matrix $\bar{\Delta}^{-1}$, which is to be eliminated by rotation of axes through an angle θ in the $x_1 x_2$-plane. The transformation equations to be used are:

$$\left.\begin{aligned} \delta x_1 &= x_1 - \bar{x}_1 = \xi_1 \cos\theta - \xi_2 \sin\theta \\ \delta x_2 &= x_2 - \bar{x}_2 = \xi_1 \sin\theta + \xi_2 \cos\theta \\ \delta x_3 &= x_3 - \bar{x}_3 = \xi_3 \end{aligned}\right\} \ . \tag{B-57}$$

Substituting Equation (B-57) into (B-55), the coefficient of $\xi_1 \xi_2$ is:

$$2\cos\theta \sin\theta\,(c_{11} - c_{22}) + 2c_{12}\,(\cos^2\theta - \sin^2\theta) \ . \tag{B-58}$$

Eliminating the term (i.e., by setting the coefficient of $\xi_1 \xi_2$ equal to zero),

$$\tan 2\theta = \frac{2c_{12}}{(c_{22} - c_{11})} \tag{B-59}$$

then defines the rotation through the angle θ.

Thus if element c_{12} is the largest off-diagonal element in matrix $\bar{\Delta}^{-1}$, the linear transformation given by Equation (B-52) will reduce Equation (B-55) to the form given by Equation (B-56). For this example matrix T is of the form:

$$T_{(3\times3)} \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)} \ .$$

If, on the other hand, Equation (B-59) is:

$$\tan 2\theta = \frac{2c_{13}}{(c_{33} - c_{11})} ,$$

then the corresponding Matrix T would be defined in the following manner:

$$T_{(3\times3)} \equiv \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}_{(3\times3)} .$$

In general, a symmetric matrix can be diagonalized by a linear transformation such as given by (B-52), and the matrix $T_{(3\times3)}$ of this transformation equation always will be diagonal.

In practice the diagonalization process of a symmetric matrix is accomplished by iteration; that is,

$$\left.\begin{aligned} \tilde{D}_{1(3\times3)} &= (T_1^T \, \Delta^{-1} \, T_1)_{(3\times3)} \\ \tilde{D}_{2(3\times3)} &= (T_2^T \, \tilde{D}_1 \, T_2)_{(3\times3)} \\ &\vdots \\ \tilde{D}_{n(3\times3)} &= (T_n \, \tilde{D}_{n-1} \, T_n)_{(3\times3)} \end{aligned}\right\} . \tag{B-60}$$

For each iteration the largest off-diagonal element can be any $\tilde{d}_{ij}$ $(i \neq j)$. Again it is to be mentioned that the elements $\tilde{d}_{ij}$ $(i = j)$ in matrix $\tilde{D}$ are the eigenvalues and the columns of matrix $(T = T_1 T_2 \ldots T_n)_{(3\times3)}$ are the components of the eigenvectors.

Target Body Impact Parameters — The Errors in Impact Position and Impact Time

In most cases, the impact of a target body by a spacecraft is best described by the errors in impact position and impact time. A set of parameters developed by W. Kizner (Reference 12), which is based on a technique applied in scattering theory in atomic physics, is used to describe these impact errors.

To properly derive the planetary impact parameters, it is first necessary to describe the "patched conic trajectory" concept. Although a precision trajectory for a spacecraft is computed by either direct integration of the equations of motion (such as Cowell's method) or by a variational scheme (such as Encke's method), these precision trajectories can be approximated by a patched conic trajectory. The term "conic" arises from the fact that the solution to the two-body problem in celestial mechanics is a conic section, such as a circle, ellipse, parabola, or hyperbola. The type of conic section is dependent on the energy in the system relative to the central body. A conic trajectory is always referenced to a particular central force field. When a conic approximation to an actual trajectory (initially referenced to the central force field of body 1) reaches the region of dominance of the central force field of body 2, it is referenced to body 2's central force field. The effect of switching the field of reference is that the characteristic of the conic section is altered. In the case of a patched conic trajectory from the earth to the moon, the trajectory usually is an ellipse with respect to the earth's central force field. Upon entering the region of dominance of the moon's central force field, the trajectory is referenced to the moon. A conic section characterized by a hyperbola results. The process of combining the two conics in order to approximate an integrated trajectory is known as "patching conics."

The concept of the patched conic trajectory makes it possible to develop a set of parameters which describes the "miss" geometry at a target body such as the moon or planet. Figure 1 depicts the geometry as a spacecraft approaches a target body. The target plane, given by the unit vectors $\vec{R}^\circ$ and $\vec{T}^\circ$, contains the impact vector $\vec{B}$. This plane is perpendicular to the asymptote of the approach hyperbola, which is the conic resulting from switching from the earth's central force field to the force field of the target body. A unit vector $\vec{S}^\circ$ is therefore defined as that unit vector

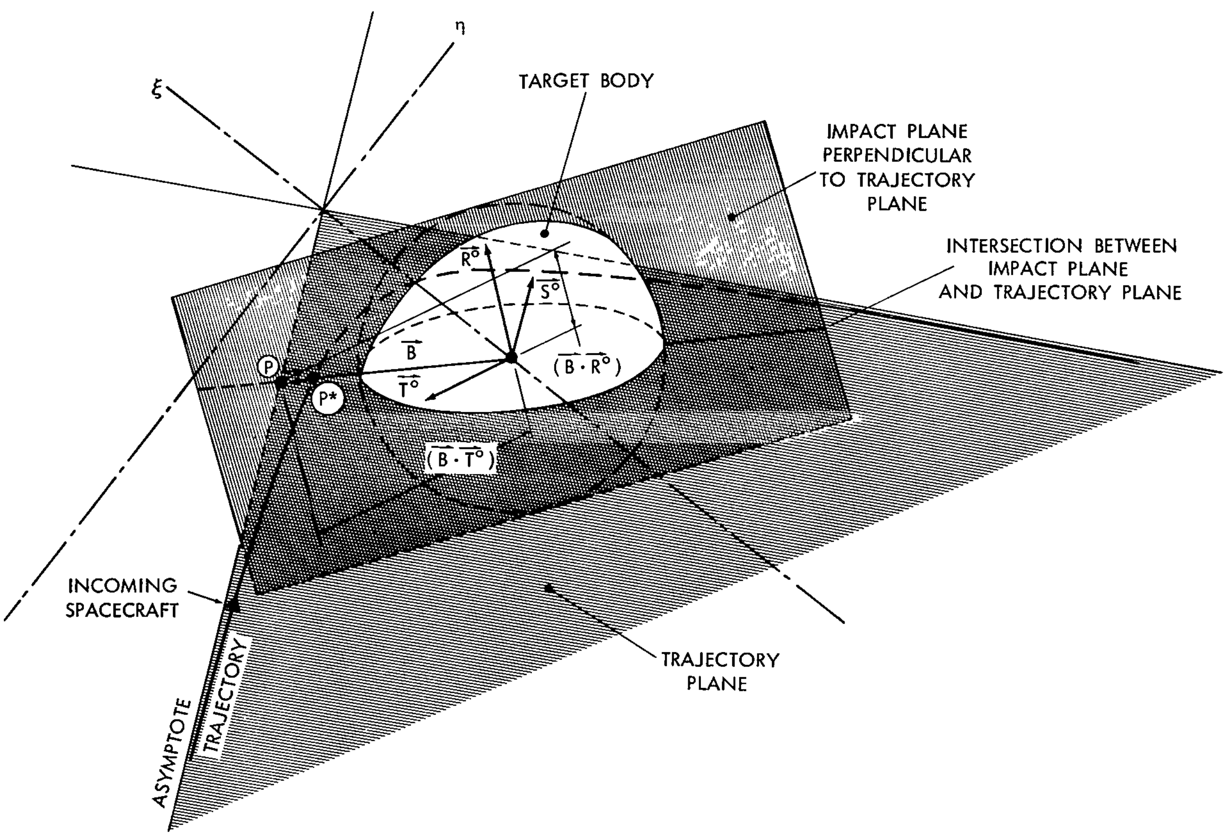

Figure 1—Target body impact parameters.

directed along the incoming asymptote which is normal to the impact plane. The vector $\vec{T}^\circ$ represents an arbitrary vector normal to $\vec{S}^\circ$ (usually assumed parallel to some fundamental plane such as the ecliptic, celestial equator, etc.). The vector $\vec{R}^\circ$ is normal to both $\vec{S}^\circ$ and $\vec{T}^\circ$; i.e., $\vec{R}^\circ = (\vec{S}^\circ \times \vec{T}^\circ)$. The miss vector $\vec{B}$ is that vector which is perpendicular from the center of the target body to the asymptote of the approach hyperbola.

Figure 2 depicts in greater detail the "impact plane" and the miss vector $\vec{B}$. In actuality the vector $\vec{B}$ is not the real "miss" vector but that vector which is directed from the center of the target planet to the point of penetration Ⓟ (Figures 1 and 2) of the trajectory hyperbola and the "impact plane." The real impact point is the trajectory penetration point (P*) (Figures 1, 2, and 3). From the "miss" vector $\vec{B}$ projection onto the unit vectors $\vec{T}^\circ$ and $\vec{R}^\circ$, the direction of the "miss" with respect to the penetration point Ⓟ is determined. A more accurate solution for describing the direction of "miss" is with respect to the actual penetration point (P*).

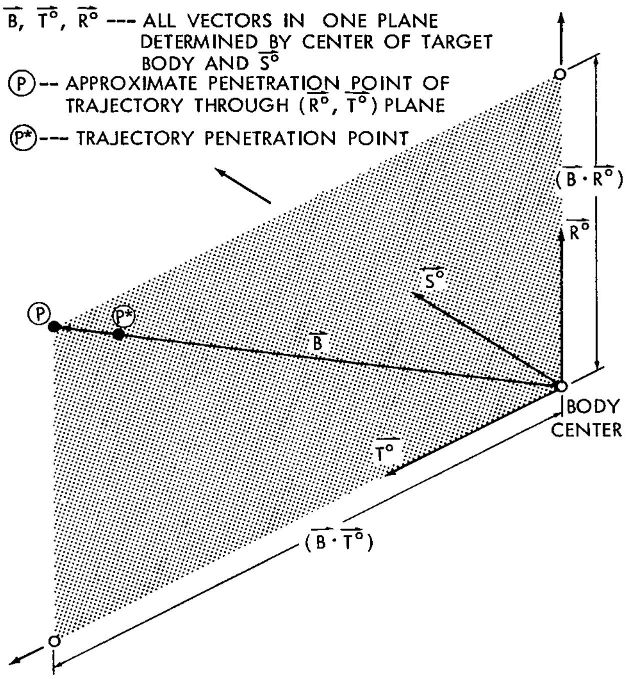

Figure 2—Target body miss components, impact plane $(\vec{B}\cdot\vec{R}^\circ)$ and $(\vec{B}\cdot\vec{T}^\circ)$.

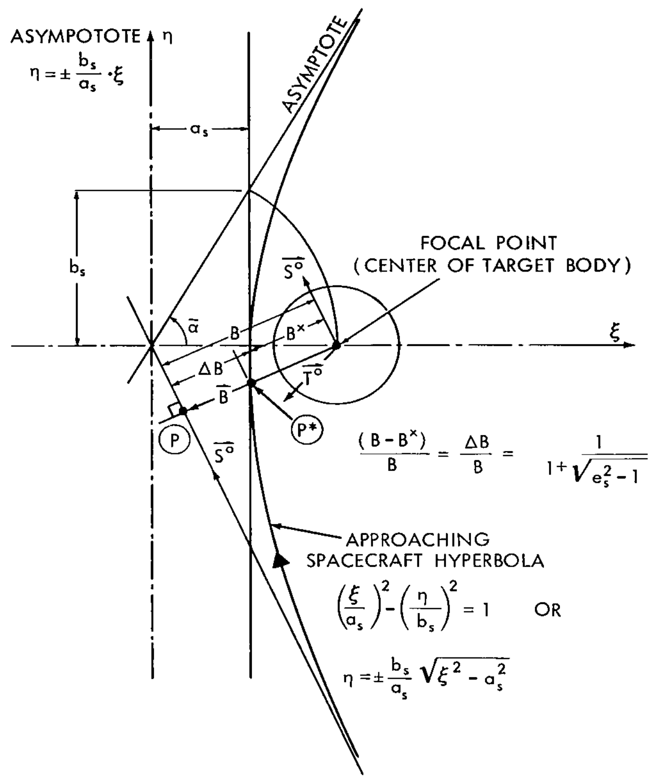

Figure 3—Target body-approach hyperbola.

The necessary mathematical derivation for these parameters is based on the description of the "miss" parameters. Therefore, let the spacecraft's position and velocity vectors be defined within the target body's region of dominance. The coordinate reference frame in which these two vectors are referenced is chosen for convenience (such as target body — equatorial, target body — ecliptic, etc.). The conic trajectory with reference to the target body is a hyperbola. In order to derive the "miss" parameters using Figure 3, it is necessary to define the following vectors:

$$\vec{\rho}_s^{\,\circ} = \frac{\vec{\rho}_s}{|\vec{\rho}_s|} \qquad \text{(unit position vector of spacecraft)} ,$$

$$\vec{v}_s^{\,\circ} = \frac{\vec{v}_s}{|\vec{v}_s|} \qquad \text{(unit velocity vector of spacecraft)} , \tag{B-61}$$

and

$$\vec{n}^{\circ} = \frac{\vec{\rho}_s \times \vec{v}_s}{|\vec{\rho}_s \times \vec{v}_s|} \qquad \text{(unit momentum vector directed along trajectory rotational axis and perpendicular to trajectory plane).}$$

A unit vector directed to the point of closest approach from the target body is defined as:

$$\vec{p}^{\circ} = \cos\theta_s \, \vec{\rho}_s^{\,\circ} - \sin\theta_s (\vec{n}^{\circ} \times \vec{\rho}_s^{\,\circ}) \quad , \tag{B-62}$$

where

$$|\vec{\rho}_s| = \frac{a_s(e_s^2 - 1)}{1 + e_s \cos\theta_s} \qquad \text{(polar form for equation of a hyperbola)} ,$$

$$\{|\vec{v}_s|\}^2 = \mu_{T.B.} \left(\frac{2}{|\vec{\rho}_s|} + \frac{1}{a_s}\right) \qquad \text{(energy integral for hyperbolic orbits)} ,$$

θ_s = true anomaly or polar angle ,

a_s = semi-major axis ,

e_s = eccentricity ,

and

$\mu_{T.B.}$ = gravitational constant of target body.

A unit vector perpendicular to $\vec{p}^{\circ}$ is:

$$\vec{q}^{\circ} = \sin\theta_s \, \vec{\rho}_s^{\,\circ} + \cos\theta_s \, (\vec{n}^{\circ} \times \vec{\rho}_s^{\,\circ}) . \tag{B-63}$$

The unit vectors $\vec{p}^{\circ}$ and $\vec{q}^{\circ}$ are directed along the positive ξ and η axes of Figure 3. Consequently the unit vector in the direction of the incoming asymptote is:

$$\vec{S}^{\circ} = \cos\tilde{\alpha} \, \vec{p}^{\circ} + \sin\tilde{\alpha} \, \vec{q}^{\circ} \quad , \tag{B-64}$$

where

$$\cos\tilde{\alpha} = \frac{a_s}{(a_s^2 + b_s^2)^{1/2}} = \frac{1}{e_s} \quad ,$$

$$\sin\tilde{\alpha} = \frac{b_s}{(a_s^2 + b_s^2)^{1/2}} = \left(1 - \frac{1}{e_s^2}\right)^{1/2} \quad ,$$

and

$$b_s = a_s\,(e_s^2 - 1)^{1/2} \quad .$$

By definition, the "miss" vector $\vec{B}$ is normal to $\vec{S}°$:

$$\vec{B} = |\vec{B}|\ \sin\tilde{\alpha}\,\vec{p}° - |\vec{B}|\ \cos\tilde{\alpha}\,\vec{q}° \quad , \tag{B-65}$$

where

$$|\vec{B}| = \{a_s + a_s\,(e_s - 1)\}\ \sin\tilde{\alpha}$$

$$= a_s\,(e_s^2 - 1)^{1/2} \quad .$$

The miss vector with respect to the actual penetration point P^* is:

$$\vec{B}^* = |\vec{B}^*|\ \sin\tilde{\alpha}\,\vec{p}° - |\vec{B}^*|\ \cos\tilde{\alpha}\,\vec{q}° \quad , \tag{B-66}$$

where

$$|\vec{B}^*| = \frac{a_s\,(e_s^2 - 1)}{1 + e_s\ \sin\tilde{\alpha}} \quad .$$

Let the arbitrary unit vector $\vec{T}°$, which lies parallel to a chosen fundamental plane (i.e., equatorial plane, ecliptic plane, etc.) be defined as perpendicular to $\vec{S}°$. A unit vector $\vec{R}°$ then is defined as normal to $\vec{T}°$ and $\vec{S}°$, i.e.,

$$\vec{R}° = (\vec{S}° \times \vec{T}°) \quad . \tag{B-67}$$

The definition of $\vec{R}^\circ$ implies that the vector $\vec{R}^\circ$ is normal to the fundamental plane; thus its components are {0, 0, 1}. Therefore it is possible to write the components of vector $\vec{T}^\circ$ as follows:

$$\vec{T}^\circ = -\frac{S_2}{(S_1^2 + S_2^2)^{1/2}} \vec{i} + \frac{S_1}{(S_1^2 + S_2^2)^{1/2}} \vec{j} \quad , \tag{B-68}$$

where

$$\vec{i} = \{1, 0, 0\}, \quad \vec{j} = \{0, 1, 0\}.$$

Using Equation (B-65) or (B-66) in combination with Equations (B-67) and (B-68), it is possible to resolve the "miss" vector $\vec{B}$ or $\vec{B}^*$ into its components and thus obtain a measure of how much in-plane and out-of-plane the "miss" of the target point was (referenced to some chosen fundamental plane). That is,

$$(\vec{B} \cdot \vec{T}^\circ) = \text{in-plane miss distance}$$

and

$$(\vec{B} \cdot \vec{R}^\circ) = \text{out-of-plane miss distance} .$$

Transformation of the covariance matrix in spacecraft position and velocity (at the time of impact with a target body) into a covariance matrix in terms of the "miss" parameters furnishes more significant information than can be derived from the RMS errors in the state vector's components at impact. To make this transformation, a third variable is defined. This variable can either be energy, usually defined as C_3, or impact time defined at t_0, where

$$C_3 = \left(\{|\vec{v}_s|\}^2 - 2\frac{\mu_{T.B.}}{|\vec{\rho}_s|}\right) . \tag{B-69}$$

The following transformations are used to express the covariance matrix in terms of the "miss" or "impact" parameters:

$$\delta(\vec{B} \cdot \vec{T}^\circ)_{(1\times1)} = \left(\frac{\partial(\vec{B} \cdot \vec{T}^\circ)}{\partial \mathbf{X}_0}\right)_{(1\times6)} \delta\mathbf{X}_{0_{(6\times1)}} ,$$

$$\delta(\vec{B} \cdot \vec{R}^\circ)_{(1\times1)} = \left(\frac{\partial(\vec{B} \cdot \vec{R}^\circ)}{\partial \mathbf{X}_0}\right)_{(1\times6)} \delta\mathbf{X}_{0_{(6\times1)}} ,$$

$$\begin{cases} \delta C_{3_{(1\times1)}} = \left(\dfrac{\partial C_3}{\partial \mathbf{X}_0}\right)_{(1\times6)} \delta \mathbf{X}_{0_{(6\times1)}} \\ \text{or} \\ \delta t_{0_{(1\times1)}} = \left(\dfrac{\partial t_0}{\partial \mathbf{X}_0}\right)_{(1\times6)} \delta \mathbf{X}_{0_{(6\times1)}} \end{cases} , \qquad \text{(B-70)}$$

and

$$\delta \mathbf{X}_{0_{(6\times1)}} \equiv \tilde{\mathbf{X}}_{0_{(6\times1)}} .$$

If impact time is chosen as the third variable, the covariance matrix in terms of the "miss" parameters is expressed as follows:*

$$E(M^* M^{*T})_{(3\times3)} = \left(\frac{\partial M^*}{\partial \mathbf{X}_0}\right)_{(3\times6)} E(\tilde{\mathbf{X}}_0 \tilde{\mathbf{X}}_0^T)_{(6\times6)} \left(\frac{\partial M^*}{\partial \mathbf{X}_0}\right)^T_{(6\times3)} , \qquad \text{(B-71)}$$

where

$$M^*_{(3\times1)} \equiv \begin{bmatrix} \delta(\vec{B}\cdot\vec{T}^\circ) \\ \delta(\vec{B}\cdot\vec{R}^\circ) \\ \delta t_0 \end{bmatrix}_{(3\times1)}$$

and

$$\left(\frac{\partial M^*}{\partial \mathbf{X}_0}\right)_{(3\times6)} \equiv \begin{bmatrix} \dfrac{\partial(\vec{B}\cdot\vec{T}^\circ)}{\partial \mathbf{X}_0} \\ \\ \dfrac{\partial(\vec{B}\cdot\vec{R}^\circ)}{\partial \mathbf{X}_0} \\ \\ \dfrac{\partial t_0}{\partial \mathbf{X}_0} \end{bmatrix}_{(3\times6)} .$$

*The partial derivatives in matrix $(\partial M^*/\partial \mathbf{X}_0)_{(3\times6)}$ are in practice evaluated numerically and not in closed analytic form due to their complexity. Numerical evaluation of these partial derivatives is also more efficient on high speed digital computers.

C. Discussion and Interpretation of Results

The mathematical theory described in the previous sections was used in developing the "Erran Error Propagation Computer Program (Reference 13). This computer program was principally used for studying the effects of errors inherent in tracking data on the spacecraft's trajectory or orbit. This program has also been utilized for the optimization of tracking schedules used during different phases of a space mission.

Figures 4 and 5 show how the RMS errors in spacecraft position and velocity behave during the early phase of a transfer trajectory to the moon. The effects of different error sources are compared. On each graph, curve A shows the RMS errors in spacecraft position and velocity due to measurement noise in the tracking data. Curve B shows the effects of measurement noise and measurement bias, and Curve C shows the effects of measurement noise, measurement bias, and station location uncertainties.

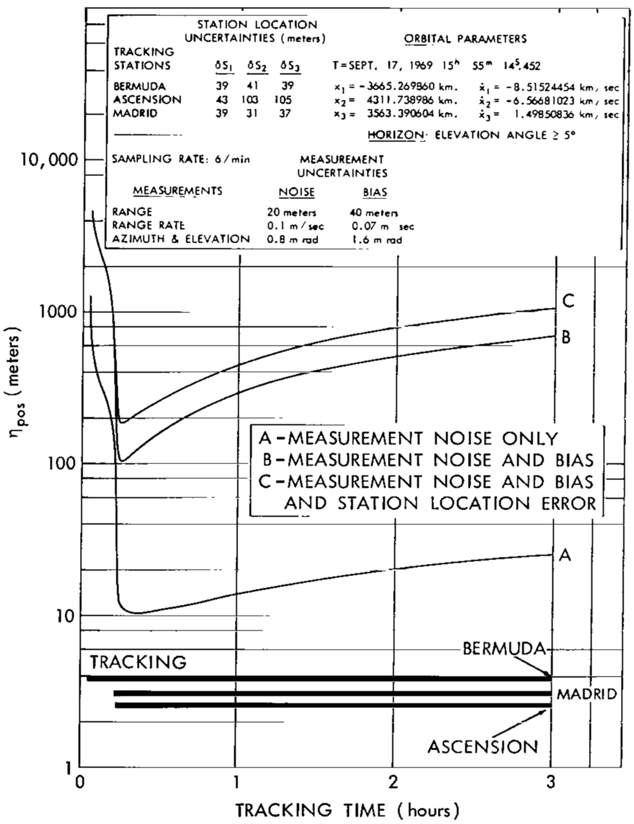

Figure 4—Position errors for the Apollo lunar transfer trajectory.

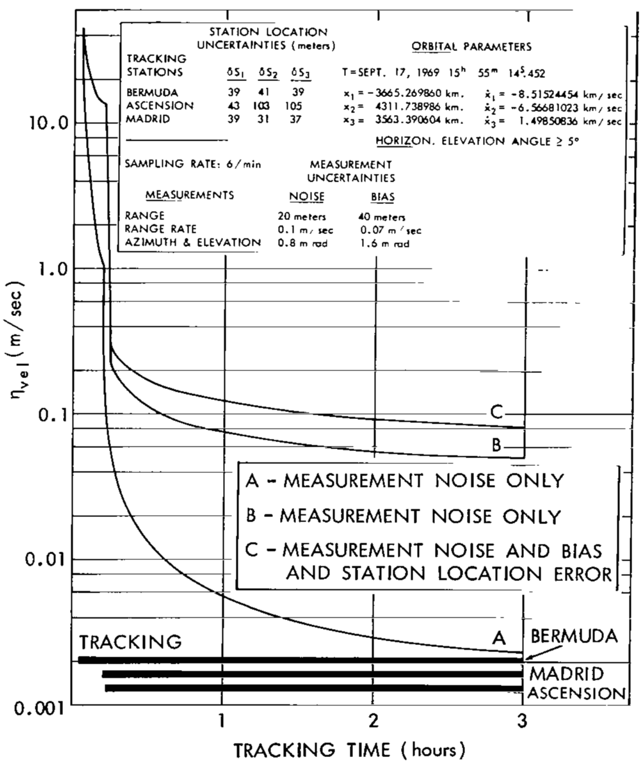

Figure 5—Velocity errors for Apollo lunar transfer trajectory.

Because no a priori information is presumed about the state, the RMS errors in the state vector may be considered infinite. As soon as sufficient tracking data become available (a minimum of six independent measurements are required to define the components of the state vector), these RMS errors become finite and may be evaluated.

As increasing amounts of tracking data are utilized, the spacecraft's position and velocity errors are found to behave differently. Indeed, the RMS error in spacecraft position tends to reach an initial minimum rapidly and then increases monotonically, whereas the velocity continues to decrease monotonically. Several factors contribute to this difference in behavior:

1. As the spacecraft's distance from earth increases, the spacecraft - tracker geometry is weakened, resulting in an increase in the RMS errors in spacecraft position.

2. The motion of a spacecraft along a lunar transfer trajectory is directed away from perigee towards apogee, resulting in a decrease in velocity (since maximum velocity is at perigee and minimum at apogee). The effect of decreasing velocity coupled with increasing distance from the earth causes a corresponding decrease in the RMS error in spacecraft velocity and a corresponding increase in the RMS error in spacecraft position.

3. The RMS errors, when evaluated at discrete points along the spacecraft's trajectory, are influenced by local anomalies such as the strong dependence on where along the trajectory these errors are evaluated, the number of tracking stations which track the spacecraft, etc. If the RMS errors in spacecraft position and velocity were evaluated at some fixed targeting point such as arrival at the moon's region of dominance, then RMS errors in both position and velocity would tend to decrease as increasing amounts of tracking data become available. The needs of individual missions determine whether to evaluate the RMS errors at discrete points or at a fixed targeting point. For the Apollo mission, the instantaneous knowledge of the state (simulated by evaluating the RMS errors at discrete points) is required for updating the onboard computer. Likewise, for this mission, knowledge of the RMS errors evaluated at some targeting point (the lunar region of dominance) provides a basis for any corrective space maneuvers which may be required to minimize the errors at the targeting point. For the error analysis study described here, the RMS errors in spacecraft position and velocity are evaluated only at discrete points along the trajectory.

Figures 4 and 5 show that the measurement bias errors are the principal contributors to the RMS errors in spacecraft position and velocity. The uncertainty in tracking station location becomes less significant as the spacecraft's distance from the earth increases. For the error analysis study described, the bias errors are not solved for but their effects are included; this results in a conservative estimate of the RMS errors in the state vector. Solving for some of the bias errors would provide a more optimistic estimate. However, to insure that a safety factor is associated with the results presented in the error analysis, it is more prudent not to solve for any of the bias errors.

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Appendix a

Fundamental Matrices in Expanded Form as Used in This Report

A listing of matrices used is presented here from Part I for completeness and convenience.

$$\delta \mathbf{K}_{(3\times1)} = \delta \boldsymbol{\beta}_{(3\times1)} \equiv \begin{bmatrix} \delta r \\ \delta \alpha \\ \delta \epsilon \end{bmatrix}_{(3\times1)} , \qquad \text{(a-1)}$$

$$\delta \dot{\mathbf{K}}_{(3\times1)} = \delta \dot{\boldsymbol{\beta}}_{(3\times1)} \equiv \begin{bmatrix} \delta \dot{r} \\ \delta \dot{\alpha} \\ \delta \dot{\epsilon} \end{bmatrix}_{(3\times1)} , \qquad \text{(a-2)}$$

$$\delta \mathbf{X}_{(3\times1)} \equiv \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}_{(3\times1)} , \qquad \text{(a-3)}$$

$$\delta \dot{\mathbf{X}}_{(3\times1)} \equiv \begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \\ \delta \dot{x}_3 \end{bmatrix}_{(3\times1)} , \qquad \text{(a-4)}$$

$$[\mathbf{R}_2(\epsilon)]_{(3\times3)} \equiv \begin{bmatrix} \cos\epsilon & 0 & \sin\epsilon \\ 0 & 1 & 0 \\ -\sin\epsilon & 0 & \cos\epsilon \end{bmatrix}_{(3\times3)} , \qquad \text{(a-5)}$$

$$[\mathbf{R}_3(\pi/2 - \alpha)]_{(3\times3)} \equiv \begin{bmatrix} \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)} , \qquad \text{(a-6)}$$

$$\mathbf{J}^{T}_{(3\times3)} \equiv \left[\mathbf{R}_2\,(-\epsilon)\,\mathbf{R}_3\,(\pi/2 - \alpha) \right]_{(3\times3)} , \qquad \text{(a-7)}$$

$$\mathbf{D}_{(3\times 3)} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -r\cos\epsilon & 0 \\ 0 & 0 & r \end{bmatrix}_{(3\times 3)} \quad , \tag{a-8}$$

$$\mathbf{D}^0_{r_{(1\times 3)}} \equiv \begin{bmatrix} 1, & 0, & 0 \end{bmatrix}_{(1\times 3)} \quad , \tag{a-9}$$

$$\mathbf{D}^0_{a_{(2\times 3)}} \equiv \begin{bmatrix} 0 & -r\cos\epsilon & 0 \\ 0 & 0 & r \end{bmatrix}_{(2\times 3)} \quad , \tag{a-10}$$

and

$$\mathbf{V}_{(3\times 3)} \equiv \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ -v_{12} & v_{22} & v_{23} \\ -v_{13} & -v_{23} & v_{33} \end{bmatrix}_{(3\times 3)} \quad ; \tag{a-11}$$

where

$$v_{11} = 0 \ ,$$

$$v_{12} \equiv -\dot{\alpha}\cos\epsilon = -\frac{1}{r}(\dot{z}_1\cos\alpha - \dot{z}_2\sin\alpha) \ ,$$

$$v_{13} \equiv \dot{\epsilon} = -\frac{1}{r}(\dot{z}_1\sin\epsilon\sin\alpha + \dot{z}_2\sin\epsilon\cos\alpha - \dot{z}_3\cos\epsilon) \quad ,$$

$$v_{22} \equiv \left(\dot{\epsilon}\tan\epsilon - \frac{\dot{r}}{r}\right) = \frac{-\sec\epsilon}{r}(\dot{z}_1\sin\alpha + \dot{z}_2\cos\alpha) \quad ,$$

$$v_{23} \equiv -\dot{\alpha}\sin\epsilon = \frac{-\tan\epsilon}{r}(\dot{z}_1\cos\alpha - \dot{z}_2\sin\alpha) \quad ,$$

$$v_{33} \equiv \frac{-\dot{r}}{r} = \frac{-1}{r}(\dot{z}_1\cos\epsilon\sin\alpha + \dot{z}_2\cos\epsilon\cos\alpha + \dot{z}_3\sin\epsilon) \ ,$$

$$v_{12} = -v_{21} \ ,$$

$$v_{13} = - v_{31} ,$$

and

$$v_{23} = - v_{32} .$$

Continuing the listing of matrices,

$$[\mathbf{R}_1 (\pi/2 - \varphi)]_{(3\times3)} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & \cos\varphi \\ 0 & -\cos\varphi & \sin\varphi \end{bmatrix}_{(3\times3)} , \tag{a-12}$$

$$[\mathbf{R}_3 (\pi/2 + \lambda)]_{(3\times3)} \equiv \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)} , \tag{a-13}$$

$$\mathbf{R}_{(3\times3)} \equiv \left[\mathbf{R}_1 (\pi/2 - \varphi)\, \mathbf{R}_3 (\pi/2 + \lambda)\right]_{(3\times3)} , \tag{a-14}$$

$$[\mathbf{R}_3(\theta_G)]_{(3\times3)} \equiv \begin{bmatrix} \cos\theta_G & \sin\theta_G & 0 \\ -\sin\theta_G & \cos\theta_G & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)} , \tag{a-15}$$

and

$$[\hat{\mathbf{R}}_3(\theta_G)]_{(3\times3)} \equiv \left(\frac{d\theta_G}{dt}\right) \left[\frac{d}{d\theta_G} \left\{\mathbf{R}_3 (\theta_G)\right\} \right]_{(3\times3)} , \tag{a-16}$$

where

$$\left\{ \begin{aligned} & \frac{d\theta_G}{dt} = 7.29211585 \times 10^{-5} \text{ rad./sec.} \\ & \frac{d}{d\theta_G}\left\{\mathbf{R}_3 (\theta_G)\right\}_{(3\times3)} \equiv \begin{bmatrix} -\sin\theta_G & \cos\theta_G & 0 \\ -\cos\theta_G & -\sin\theta_G & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(3\times3)} \end{aligned} \right.$$

Continuing the listing of matrices,

$$\mathbf{L}_{(3\times3)} \equiv \left[\mathbf{J}^T \mathbf{R}\mathbf{R}_3(\theta_G)\right]_{(3\times3)}, \tag{a-17}$$

$$\mathbf{M}_{(3\times3)} \equiv \left[\mathbf{V}\mathbf{J}^T \mathbf{R}\mathbf{R}_3(\theta_G) + \frac{d\theta_G}{dt}\left\{\mathbf{J}^T\mathbf{R}\left[\frac{d}{d\theta_G}\mathbf{R}_3(\theta_G)\right]\right\}\right]_{(3\times3)}, \tag{a-18}$$

$$\mathbf{F}_{r_{(1\times3)}} \equiv \left[1, \quad 0, \quad 0\right]_{(1\times3)}, \tag{a-19}$$

$$\mathbf{F}_{a_{(2\times3)}} \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(2\times3)}, \tag{a-20}$$

$$\mathbf{L}^0_{r_{(1\times3)}} \equiv \left[\mathbf{F}_r\,\mathbf{L}\right]_{(1\times3)}, \tag{a-21}$$

$$\mathbf{L}^0_{a_{(2\times3)}} \equiv \left[\mathbf{F}_a\,\mathbf{L}\right]_{(2\times3)}, \tag{a-22}$$

$$\mathbf{M}^0_{r_{(1\times3)}} \equiv \left[\mathbf{F}_r\,\mathbf{M}\right]_{(1\times3)}, \tag{a-23}$$

$$\mathbf{M}^0_{a_{(2\times3)}} \equiv \left[\mathbf{F}_a\,\mathbf{M}\right]_{(2\times3)}, \tag{a-24}$$

$$[\mathbf{R}_3(-\Omega_s)]_{(3\times3)} \equiv \begin{bmatrix} \cos\Omega_s & -\sin\Omega_s & 0 \\ \sin\Omega_s & \cos\Omega_s & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)}, \tag{a-25}$$

$$[\mathbf{R}_1(-i_s)]_{(3\times3)} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i_s & -\sin i_s \\ 0 & \sin i_s & \cos i_s \end{bmatrix}_{(3\times3)}, \tag{a-26}$$

$$[\mathbf{R}_3(-\omega_s)]_{(3\times3)} \equiv \begin{bmatrix} \cos\omega_s & -\sin\omega_s & 0 \\ \sin\omega_s & \cos\omega_s & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3\times3)}, \tag{a-27}$$

$$\overline{\mathbf{R}}_{(3\times3)} \equiv \left[\mathbf{R}_3(-\Omega_s)\,\mathbf{R}_1(-i_s)\,\mathbf{R}_3\,(-\omega_s)\right]_{(3\times3)} \quad , \tag{a-28}$$

$$\left[\frac{\partial\overline{\mathbf{R}}}{\partial\Omega_s}\right]_{(3\times3)} \equiv \left[\overline{\mathbf{R}}_{\Omega_s}\right]_{(3\times3)} \equiv \left\{\begin{bmatrix} -\sin\Omega_s & -\cos\Omega_s & 0 \\ \cos\Omega_s & -\sin\Omega_s & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{R}_1\,(-i_s)\,\mathbf{R}_3\,(-\omega_s)\right\}_{(3\times3)} \quad , \tag{a-29}$$

$$\left[\frac{\partial\overline{\mathbf{R}}}{\partial i_s}\right]_{(3\times3)} \equiv \left[\overline{\mathbf{R}}_{i_s}\right]_{(3\times3)} \equiv \left\{\mathbf{R}_3\,(-\Omega_s)\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin i_s & -\cos i_s \\ 0 & \cos i_s & -\sin i_s \end{bmatrix}\mathbf{R}_3\,(-\omega_s)\right\}_{(3\times3)} \quad , \tag{a-30}$$

$$\left[\frac{\partial\overline{\mathbf{R}}}{\partial\omega_s}\right]_{(3\times3)} \equiv \left[\overline{\mathbf{R}}_{\omega_s}\right]_{(3\times3)} \equiv \left\{\mathbf{R}_3\,(-\Omega_s)\,\mathbf{R}_1\,(-i_s)\begin{bmatrix} -\sin\omega_s & -\cos\omega_s & 0 \\ \cos\omega_s & -\sin\omega_s & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}_{(3\times3)} \quad , \tag{a-31}$$

$$\mathbf{U}_{(3\times1)} \equiv \begin{bmatrix} a_s\,(\cos E_s - e_s) \\ a_s\,(1-e_s^2)^{1/2} \quad \sin E_s \\ 0 \end{bmatrix}_{(3\times1)} \quad , \tag{a-32}$$

and

$$\dot{\mathbf{U}}_{(3\times1)} \equiv \begin{bmatrix} \dfrac{-n_s\,a_s\sin E_s}{(1-e_s\cos E_s)} \\ \dfrac{n_s\,a_s\,(1-e_s^2)^{1/2}\cos E_s}{(1-e_s\cos E_s)} \\ 0 \end{bmatrix}_{(3\times1)} \quad ; \tag{a-33}$$

where

$$n_s = (\mu^{1/2}\,a_s^{-3/2}) \quad .$$

Continuing the listing of matrices,

$$\hat{N}_{(3\times 3)} \equiv \left[\bar{R}_{\Omega_s} \dot{U} \,\vdots\, \bar{R}_{i_s} \dot{U} \,\vdots\, \bar{R}_{\omega_s} \dot{U} \right]_{(3\times 3)} , \tag{a-34}$$

$$\bar{N}_{(3\times 3)} \equiv \left[\bar{R}_{\Omega_s} U \,\vdots\, \bar{R}_{i_s} U \,\vdots\, \bar{R}_{\omega_s} U \right]_{(3\times 3)} , \tag{a-35}$$

and

$$\bar{M}_{(3\times 3)} \equiv \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} & \bar{m}_{13} \\ \bar{m}_{21} & \bar{m}_{22} & \bar{m}_{23} \\ \bar{m}_{31} & \bar{m}_{32} & \bar{m}_{33} \end{bmatrix}_{(3\times 3)} , \tag{a-36}$$

where

$$\bar{m}_{11} \equiv \left[(\cos E_s - e_s) + \frac{3 n_s (t - \tau) \sin E_s}{2 (1 - e_s \cos E_s)} \right] ,$$

$$\bar{m}_{12} \equiv - a_s \left[1 + \frac{\sin^2 E_s}{(1 - e_s \cos E_s)} \right] ,$$

$$\bar{m}_{13} \equiv \frac{a_s n_s \sin E_s}{(1 - e_s \cos E_s)} ,$$

$$\bar{m}_{21} \equiv \left[(1 - e_s^2)^{1/2} \sin E_s - \frac{3 n_s (t - \tau) (1 - e_s^2)^{1/2} \cos E_s}{2 (1 - e_s \cos E_s)} \right] ,$$

$$\bar{m}_{22} = \frac{a_s \sin E_s (\cos E_s - e_s)}{(1 - e_s^2)^{1/2} (1 - e_s \cos E_s)} ,$$

$$\bar{m}_{23} \equiv \frac{-a_s n_s (1 - e_s^2)^{1/2} \cos E_s}{(1 - e_s \cos E_s)} ,$$

and

$$\bar{m}_{31} = \bar{m}_{32} = \bar{m}_{33} = 0 .$$

Continuing the listing of matrices,

$$\hat{M}_{(3\times 3)} \equiv \begin{bmatrix} \hat{m}_{11} & \hat{m}_{12} & \hat{m}_{13} \\ \hat{m}_{21} & \hat{m}_{22} & \hat{m}_{23} \\ \hat{m}_{31} & \hat{m}_{32} & \hat{m}_{33} \end{bmatrix}_{(3\times 3)} , \tag{a-37}$$

where

$$\hat{m}_{11} \equiv \frac{n_s}{(1 - e_s \cos E_s)} \left[\sin E_s + \frac{3 n_s (t - \tau)(\cos E_s - e_s)}{(1 - e_s \cos E_s)^2} \right] ,$$

$$\hat{m}_{12} \equiv \frac{-n_s a_s \sin E_s}{(1 - e_s \cos E_s)^2} \left[\cos E_s + \frac{(\cos E_s - e_s)}{(1 - e_s \cos E_s)} \right] ,$$

$$\hat{m}_{13} \equiv \frac{n_s^2 a_s (\cos E_s - e_s)}{(1 - e_s \cos E_s)^3} ,$$

$$\hat{m}_{21} \equiv \frac{-n_s (1 - e_s^2)^{1/2}}{2(1 - e_s \cos E_s)} \left[\cos E_s - \frac{3 n_s (t - \tau) \sin E_s}{(1 - e_s \cos E_s)^2} \right] ,$$

$$\hat{m}_{22} \equiv \frac{n_s a_s}{(1 - e_s \cos E_s)^2} \left[\frac{\cos E_s (\cos E_s - e_s)}{(1 - e_s^2)^{1/2}} - \frac{(1 - e_s^2)^{1/2} \sin^2 E_s}{(1 - e_s \cos E_s)} \right] ,$$

$$\hat{m}_{23} \equiv \frac{n_s^2 a_s (1 - e_s^2)^{1/2} \sin E_s}{(1 - e_s \cos E_s)^3} ,$$

and

$$\hat{m}_{31} \equiv \hat{m}_{32} \equiv \hat{m}_{33} = 0 .$$

Continuing the listing of matrices,

$$[\mathbf{P}(t)]_{(6\times 6)} \equiv \left[\begin{array}{c|c} \bar{\mathbf{N}}_{(3\times 3)} & (\bar{\mathbf{R}}\,\bar{\mathbf{M}})_{(3\times 3)} \\ \hline \hat{\mathbf{N}}_{(3\times 3)} & (\bar{\mathbf{R}}\,\hat{\mathbf{M}})_{(3\times 3)} \end{array} \right]_{(6\times 6)} , \tag{a-38}$$

and

$$[\psi(t)]_{(6\times 6)} \equiv \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} & \psi_{26} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} & \psi_{36} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} & \psi_{45} & \psi_{46} \\ \psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55} & \psi_{56} \\ \psi_{61} & \psi_{62} & \psi_{63} & \psi_{64} & \psi_{65} & \psi_{66} \end{bmatrix}_{(6\times 6)} , \tag{a-39}$$

where

$$\psi_{11} = \cos\delta\cos\lambda' ,$$

$$\psi_{12} = \psi_{13} = \psi_{14} = 0 ,$$

$$\psi_{15} = - r_s \cos\delta\sin\lambda' ,$$

$$\psi_{16} = - r_s \sin\delta\cos\lambda' ,$$

$$\psi_{21} = \cos\delta\sin\lambda' ,$$

$$\psi_{22} = \psi_{23} = \psi_{24} = 0 ,$$

$$\psi_{25} = r_s \cos\delta\cos\lambda' ,$$

$$\psi_{26} = - r_s \sin\delta\sin\lambda' ,$$

$$\psi_{31} = \sin\delta ,$$

$$\psi_{32} = \psi_{33} = \psi_{34} = \psi_{35} = 0 ,$$

$$\psi_{36} = r_s \cos\delta ,$$

$$\psi_{41} = 0 ,$$

$$\psi_{42} = [\sin\gamma_s\cos\delta\cos\lambda' - \cos\gamma_s\sin\alpha_i\sin\lambda' - \cos\gamma_s\cos\alpha_i\sin\delta\cos\lambda'] ,$$

$$\psi_{43} = v_s[\cos\gamma_s\cos\delta\cos\lambda' + \sin\gamma_s\sin\alpha_i\sin\lambda' + \sin\gamma_s\cos\alpha_i\sin\delta\cos\lambda'] ,$$

$$\psi_{44} = v_s[-\cos\gamma_s\cos\alpha_i\sin\lambda' + \cos\gamma_s\sin\alpha_i\sin\delta\cos\lambda'] ,$$

$$\psi_{45} = v_s[-\sin\gamma_s\cos\delta\sin\lambda' - \cos\gamma_s\sin\alpha_i\cos\lambda' + \cos\gamma_s\cos\alpha_i\sin\delta\sin\lambda'] ,$$

$$\psi_{46} = v_s[-\sin\gamma_s\sin\delta\cos\lambda' - \cos\gamma_s\cos\alpha_i\cos\delta\cos\lambda'] ,$$

$$\psi_{51} = 0 ,$$

$$\psi_{52} = [\sin\gamma_s\cos\delta\sin\lambda' + \cos\gamma_s\sin\alpha_i\cos\lambda' - \cos\gamma_s\cos\alpha_i\sin\delta\sin\lambda'] ,$$

$$\psi_{53} = v_s[\cos\gamma_s\cos\delta\sin\lambda' - \sin\gamma_s\sin\alpha_i\cos\lambda' + \sin\gamma_s\cos\alpha_i\sin\delta\sin\lambda'] ,$$

$$\psi_{54} = v_s[\cos\gamma_s\cos\alpha_i\cos\lambda' + \cos\gamma_s\sin\alpha_i\sin\delta\sin\lambda'] ,$$

$$\psi_{55} = v_s[\sin\gamma_s\cos\delta\cos\lambda' - \cos\gamma_s\sin\alpha_i\sin\lambda' - \cos\gamma_s\cos\alpha_i\sin\delta\cos\lambda'] ,$$

$$\psi_{56} = v_s[-\sin\gamma_s\sin\delta\sin\lambda' - \cos\gamma_s\cos\alpha_i\cos\delta\sin\lambda'] ,$$

$$\psi_{61} = 0 ,$$

$$\psi_{62} = [\sin\gamma_s\sin\delta + \cos\gamma_s\cos\alpha_i\cos\delta] ,$$

$$\psi_{63} = v_s[\cos\gamma_s \sin\delta - \sin\gamma_s \cos\alpha_i \cos\delta] ,$$

$$\psi_{64} = -v_s \cos\gamma_s \sin\alpha_i \cos\delta ,$$

$$\psi_{65} = 0 ,$$

and

$$\psi_{66} = v_s[\sin\gamma_s \cos\delta - \cos\gamma_s \cos\alpha_i \sin\delta] .$$

Appendix b

Coordinate Systems and Their Transformations

This appendix lists the equations which relate the inertial coordinate system to the local coordinate system. All the transformations considered here, previously given in Part I, are renumbered to be consistent with the order of presentation of the coordinate systems in Part II.

1. The inertial coordinate system is a right handed Cartesian coordinate system defined as: (Figure 1, Part 1)

$$\mathbf{X}_{(3\times 1)} \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{(3\times 1)}, \tag{b-1}$$

where

Origin is: Earth's center of mass

x_1 -axis: Directed towards vernal equinox.

x_2 - axis: Normal to both x_1 and x_3-axes.

x_3 -axis: Directed along earth's axis of rotation.

2. The local coordinate system is a right handed Cartesian coordinate system defined as: (Figure 1 and 3, Part I)

$$\mathbf{Z}_{(3\times 1)} \equiv \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}_{(3\times 1)} \tag{b-2}$$

where

Origin is: At the observer

z_1-axis: Directed towards local east.

z_2-axis: Directed towards local north.

z_3-axis: Directed along normal to local horizon plane.

3. The coordinates of the observer. (Figure 4, Part I)

$$S_{(3\times1)} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} (N + h) \cos \varphi \cos \lambda \\ (N + h) \cos \varphi \sin \lambda \\ [N(1 - e_1^2) + h] \sin \varphi \end{bmatrix}_{(3\times1)} \tag{b-3}$$

Origin: The center of the reference ellipsoid

and

$$N = a_\oplus/(1 - e_1^2 \sin^2 \varphi)^{1/2}$$

4. Transformation equations relating the local coordinate system with the inertial coordinate system (Figure 1, Part I)

$$Z_{(3\times1)} = R_{(3\times3)} R_3(\theta_G)_{(3\times3)} X_{(3\times1)} - R_{(3\times3)} S_{(3\times1)} \quad , \tag{b-4}$$

where

$$R_{(3\times3)} \equiv [R_1 (\pi/2 - \varphi) R_3 (\pi/2 + \lambda)]_{(3\times3)}$$

(see Appendix a for the expanded form of matrices $R_{(3\times3)}$ and $R_3(\theta_G)$).

$$\theta_G \equiv \theta_{G_0} + \left(\frac{d\theta_G}{dt}\right) t \quad ,$$

$$0 \le \theta_G \le 2\pi \quad ,$$

and

$\theta_G \equiv$ Greenwich sidereal time

$\theta_{G_0} \equiv$ Greenwich sidereal time at 0^h U.T.

$$\left(\frac{d\theta_G}{dt}\right) = 15^\circ\!.04106864/\text{hr}$$

or

$$= 7.29211585 \times 10^{-5} \text{ radians/sec.}$$

t : Universal time of observation.

5. The transformation from the local coordinate system into the coordinates of the spacecraft in its orbit is obtained by combining equation (c-20) with (b-4):

$$Z_{(3\times1)} = \underset{(3\times3)}{[R R_3(\theta_G) \bar{R}]} U_{(3\times1)} - R_{(3\times3)} S_{(3\times1)} \quad , \tag{b-5}$$

where

$$\overline{\mathbf{R}}_{(3\times3)} \equiv [\mathbf{R}_3(-\Omega)\ \mathbf{R}_1(-i)\ \mathbf{R}_3(-\omega)]_{(3\times3)}$$

and

$$\mathbf{U}_{(3\times1)} \equiv \begin{bmatrix} a_s(\cos E_s - e_s) \\ a_s(1 - e_s^2)^{1/2} \sin E_s \\ 0 \end{bmatrix}_{(3\times1)} .$$

Appendix c

Transformation of Position and Velocity Vectors into Keplerian Orbital Elements

This appendix gives the transformation of the position and velocity vectors from the inertial coordinate system into the Keplerian orbital elements. Before doing so the transformation of the following spacecraft orbital injection parameters into the inertial coordinate system will be given.

For the purposes of this paper, injection orbital parameters are defined as follows:

ρ_s — Radial distance of spacecraft from center of earth.

v_s — Magnitude of velocity vector.

γ_s — Flight path angle.

α_i — Injection azimuth.

λ' — Right ascension of spacecraft at time of injection into orbit.

δ — Declination of spacecraft at time of injection into orbit.

τ — Epoch time (time spacecraft is inserted into orbit).

The transformation of the injection orbital parameters into the position and velocity vectorial components in the inertial coordinate system are:

$$\vec{\rho}_s = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} \rho_s \cos\delta\cos\lambda' \\ \rho_s \cos\delta\sin\lambda' \\ \rho_s \sin\delta \end{bmatrix}_{(3\times1)} \tag{c-1}$$

$$\vec{v}_s = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}_{(3\times1)} = \begin{bmatrix} v_s\{\sin\gamma_s\cos\delta\cos\lambda' - \cos\gamma_s\sin\alpha_i\sin\lambda' - \cos\gamma_s\cos\alpha_i\sin\delta\cos\lambda'\} \\ v_s\{\sin\gamma_s\cos\delta\sin\lambda' + \cos\gamma_s\sin\alpha_i\cos\lambda' - \cos\gamma_s\cos\alpha_i\sin\delta\sin\lambda'\} \\ v_s\{\sin\gamma_s\sin\delta + \cos\gamma_s\cos\alpha_i\cos\delta\} \end{bmatrix}_{(3\times1)} \tag{c-2}$$

From the position and velocity vectors given by (c-1) and (c-2), the Keplerian orbital elements are derived. The geometry of the conditions at the time of insertion into orbit is shown in Figure c-1.

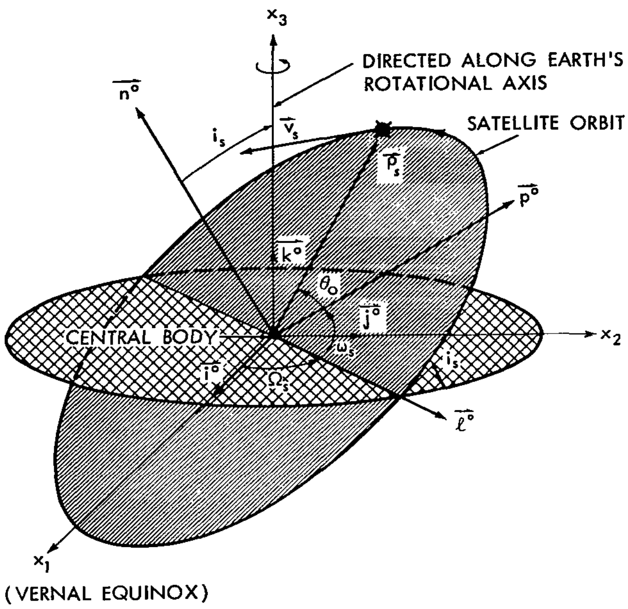

Figure C-1—Geometry of satellite orbit in the inertial reference frame.

For the determination of the orbit orientation with respect to the inertial coordinate system, the following vectors are required:

$$\vec{\rho}_s^{\,\circ} = \frac{\vec{\rho}_s}{|\vec{\rho}_s|} \tag{c-3}$$

$$\vec{v}_s^{\,\circ} = \frac{\vec{v}_s}{|\vec{v}_s|} \tag{c-4}$$

$$\vec{n}^\circ = \frac{\vec{\rho}_s \times \vec{v}_s}{|\vec{\rho}_s \times \vec{v}_s|} \tag{c-5}$$

$$\vec{p}^\circ = \cos\theta_0\, \vec{\rho}_s^{\,\circ} - \sin\theta_0 (\vec{n}^\circ \times \vec{\rho}_s^{\,\circ}) \tag{c-6}$$

$$\vec{q}^\circ = \sin\theta_0\, \vec{\rho}_s + \cos\theta_0 (\vec{n}^\circ \times \vec{\rho}_s^{\,\circ}) \tag{c-7}$$

$$\vec{\ell}^\circ = \frac{\vec{n}^\circ \times \vec{k}^\circ}{|\vec{n}^\circ \times \vec{k}^\circ|} \tag{c-8}$$

From the energy integral the semi-major axis is determined:

$$a_s = \left(\frac{2}{|\vec{\rho}_s|} - \frac{\{|\vec{v}_s|\}^2}{\mu} \right)^{-1} . \tag{c-9}$$

The eccentricity is given by:

$$e_s = \left(1 - \frac{\{|\vec{\rho}_s \times \vec{v}_s|\}^2}{\mu a_s} \right)^{1/2} . \tag{c-10}$$

The true anomaly is determined from the polar form of the equation of an ellipse.

$$\left.\begin{aligned} \cos\theta_0 &= \frac{1}{e_s} \left[\frac{a_s(1-e_s^2)}{|\vec{\rho}_s|} - 1 \right] \\ \sin\theta_0 &= \frac{1}{e_s} \left\{ \left| \left(\frac{a_s(1-e_s^2)}{\mu} \right)^{1/2} \right| (\vec{\rho}_s^{\,\circ} \cdot \vec{v}_s) \right\} \end{aligned}\right\} , \tag{c-11}$$

where $\theta_0 = \theta_s(\tau)$. From the true anomaly the transformation into the eccentric anomaly at epoch is given by:

$$E_s^\circ = 2\tan^{-1}\left\{\left(\frac{1-e_s}{1+e_s}\right)^{1/2}\left(\frac{1-\cos\theta_0}{1+\cos\theta_0}\right)^{1/2}\right\} \quad . \tag{c-12}$$

The mean anomaly at epoch is then derived from Kepler's equation:

$$M_0 = (E_s^\circ - e_s \sin E_s^\circ) \quad . \tag{c-13}$$

To evaluate the eccentric anomaly at times other than at epoch, let

$$M_s = M_0 + (\mu^{1/2} a_s^{-3/2})(t_1 - \tau) \quad , \tag{c-14}$$

where $(t_1 - \tau)$ is to be expressed in seconds of time.

M_0 is in radians.

τ is epoch time.

Then:

$$\left.\begin{aligned} E_s^{(0)} &= M_s \\ E_s^{(1)} &= M_s + e_s \sin E_s^{(0)} \\ E_s^{(2)} &= M_s + e_s \sin E_s^{(1)} \\ &\cdot \\ &\cdot \\ &\cdot \\ E_s^{(n)} &= M_s + e_s \sin E_s^{(n-1)} \end{aligned}\right\} \quad . \tag{c-15}$$

That value of E_s is then chosen for which the following convergence criterion is met,

$$\{E^{(n)} - (M_s + e_s \sin E_s^{(n-1)})\} < 10^{-8} \quad . \tag{c-16}$$

The orbital plane orientation remains invariant whenever a Keplerian-type orbit is presumed. Using the fundamental vectors given by (c-3) to (c-8), the following orientation parameters are obtained.

Inclination of orbit to equatorial plane:

$$\left.\begin{aligned} i_s &= \cos^{-1}(\vec{n}^\circ \cdot \vec{k}^\circ) \\ 0 \le i_s &\le \pi/2 \quad . \end{aligned}\right. \tag{c-17}$$

Argument of perigee:

$$\sin \omega_s = \frac{\vec{\rho}_s^\circ \cdot \vec{k}^\circ}{\sin i_s} \cos \theta_0 - \sin \theta_0 \, (\vec{\ell}^\circ \cdot \vec{\rho}_s^\circ)$$

$$\left.\begin{aligned} \cos \omega_s &= (\vec{\ell}^\circ \cdot \vec{p}^\circ) \\ 0 \le \omega_s &\le 2\pi \quad . \end{aligned}\right. \tag{c-18}$$

The right ascension of the ascending node:

$$\left.\begin{aligned} \sin \Omega_s &= \frac{(\vec{\ell}^\circ \cdot \vec{\rho}_s^\circ) - (\vec{\ell}^\circ \cdot \vec{i}^\circ)(\vec{\rho}_s^\circ \cdot \vec{i}^\circ)}{(\vec{\rho}_s^\circ \cdot \vec{j}^\circ)} \\ \cos \Omega_s &= (\vec{\ell}^\circ \cdot \vec{i}^\circ) \\ 0 \le \Omega_s &\le 2\pi \end{aligned}\right\} \quad . \tag{c-19}$$

Once the position and velocity vectors at epoch have been transformed into the orbital elements, it is possible to generate an ephemeris of the satellite's motion in the Keplerian sense using the following transformation equations.

The position vector at any time t is given by:

$$\mathbf{X}_{(3\times1)} = \{\mathbf{R}_3(-\Omega_s)\,\mathbf{R}_1(-i_s)\,\mathbf{R}_3(-\omega_s)\}_{(3\times3)}\,\mathrm{U}_{(3\times1)} \ , \tag{c-20}$$

where

$$\vec{\rho}_s(t) = \mathbf{X}_{(3\times1)} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}_{(3\times1)} ,$$

$$U_{(3\times1)} = \begin{bmatrix} a_s(\cos E_s - e_s) \\ a_s(1 - e_s^2)^{1/2} \sin E_s \\ 0 \end{bmatrix} \quad ,$$

$$E_s = M_s + e_s \sin E_s; \quad M_s = M_0 + (\mu^{1/2}\, a_s^{-3/2})\,(t - \tau) \quad ,$$

and the velocity is given by

$$\dot{X}_{(3\times1)} = \{R_3(-\Omega_s)\, R_1(-i_s)\, R_3(-\omega_s)\}_{(3\times3)}\, \dot{U}_{(3\times1)} \quad , \tag{c-21}$$

where

$$\vec{v}_s(t) = \dot{X}_{(3\times1)} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}_{(3\times1)} \quad ,$$

$$\dot{U}_{(3\times1)} = \begin{bmatrix} \dfrac{-n_s a_s \sin E_s}{(1 - e_s \cos E_s)} \\ \\ \dfrac{n_s a_s (1 - e_s^2)^{1/2} \cos E_s}{(1 - e_s \cos E_s)} \\ \\ 0 \end{bmatrix}_{(3\times1)} \quad ,$$

$$n_s = \mu^{1/2}\, a_s^{-3/2}; \quad n_s \equiv \text{Mean motion} \quad ,$$

and

$$\mu_{(\text{earth})} = 3.986032 \times 10^5 \ \text{km}^3/\text{sec}^2 \ .$$

By assuming no perturbations, a Keplerian orbit adequately describes the spacecraft's motion around the earth. Utilizing Equations (c-20) and (c-21) the position and velocity can be calculated for any instant of time.

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