NASA TECHNICAL NOTE

NASA TN D-3768

NASA TN D-3768

c. 1

AN INCREMENTAL PROCEDURE FOR SOLUTION OF NONLINEAR PROBLEMS WITH APPLICATIONS TO PLATES AND SHELLS

by Manuel Stein

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1966

NASA TN D-3768

AN INCREMENTAL PROCEDURE FOR SOLUTION OF NONLINEAR PROBLEMS WITH APPLICATIONS TO PLATES AND SHELLS

By Manuel Stein

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $2.00

AN INCREMENTAL PROCEDURE FOR SOLUTION OF NONLINEAR PROBLEMS WITH APPLICATIONS TO PLATES AND SHELLS

By Manuel Stein
Langley Research Center

SUMMARY

An incremental step-by-step procedure is described for obtaining solutions to nonlinear boundary value problems. This procedure is especially suited for computer solution of problems cast in numerical form; it is used in this paper to obtain numerical solutions for the clamped plate with uniform lateral load, for the stretched simply supported plate with a central concentrated load, and for a spherical cap with internal pressure and an outward acting central concentrated load. There is good agreement between the present results and exact results which have been obtained previously.

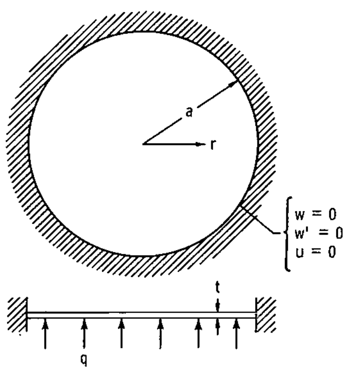
INTRODUCTION

Linear equations are often inadequate for use in the analysis of space vehicle and aircraft plate and shell structures. Some design problems involve lateral deflections many times the plate or shell thickness such that the neutral surface strains depend nonlinearly on lateral deflection. Other problems may involve nonlinear material properties such as those due to temperature rise or to plasticity. The design of such structures may be concerned primarily with strength or with deformation. Useful estimates of the deformations can usually be found from rough approximate solutions of the nonlinear equations. To obtain meaningful stress distributions, however, much more accurate solutions of the nonlinear equations are required.

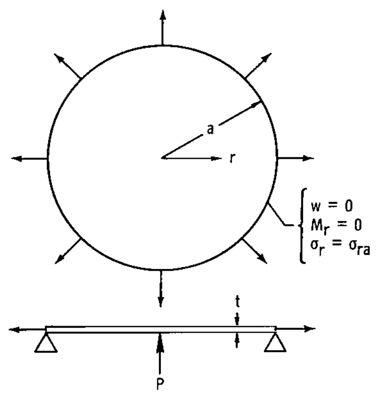
Many methods are available for solving nonlinear problems. (See ref. 1.) Analytical solutions of the differential equations of the problem are sometimes possible through special devices or inverse procedures. Direct methods include iteration, Newton's method, and methods which make use of trial and error. For problems involving a small parameter, a regular perturbation method may be employed; for problems involving a boundary layer, methods of asymptotic integration may be employed. Weighted residual techniques are also available which include the energy method, the Galerkin method, collocation, and the method of least squares. These available methods (for a fairly complete survey with examples, see ref. 1) have certain limitations and often are applicable to rather restricted types of problems.

The purpose of this paper is to present a procedure which would have relatively general applicability to boundary value problems. In particular, the idea of incrementing a control parameter is used in such a way that a solution may be obtained for a broad spectrum of practical values for this parameter.

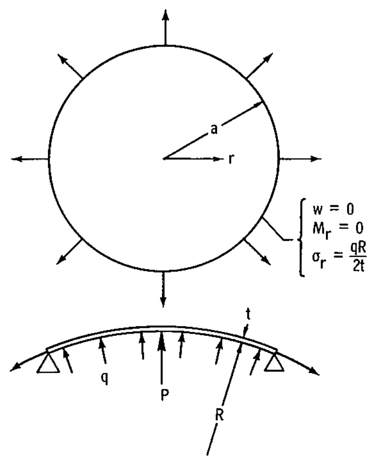
By using this procedure, solutions are obtained for the deformations and stresses in the following axisymmetric problems: a clamped circular plate under uniform load (sketch 1), a simply supported circular plate with a central concentrated load and uniform edge forces (sketch 2), and a simply supported spherical cap with central concentrated load and with internal pressure and corresponding edge forces (sketch 3). The results obtained for these problems are in good agreement with "exact" theoretical results and with the end results given by membrane theory. Although all the examples in the present paper are for axisymmetric problems, the method basically is not limited to such problems and can also be applied to problems of asymmetric loading and deformation.



Sketch 1



Sketch 2



Sketch 3

SYMBOLS

a	radius of plate or chord radius of shell
D	plate stiffness, $\dfrac{Et^3}{12\left(1 - \mu^2\right)}$
$\bar{D}$	nondimensional stiffness parameter, $\dfrac{D\left(1 - \mu^2\right)}{Eta^2Q^{2/3}}$
E	Young's modulus of material
H	shell rise, $\dfrac{a^2}{2R}$

M_r	radial plate bending moment
S	number of stations considered
P	concentrated lateral load
q	uniform lateral load
Q	load parameter, $\dfrac{P\left(1 - \mu^2\right)}{\pi Eta}$ or $\dfrac{qa\left(1 - \mu^2\right)}{Et}$
r	radial distance
R	radius of spherical shell
t	thickness of plate or shell
T	nondimensional edge radial stress, $\dfrac{\sigma_{ra}\left(1 - \mu^2\right)}{EQ^{2/3}}$ or $\dfrac{qR\left(1 - \mu^2\right)}{2EtQ^{2/3}}$
u	radial displacement
w	deflection
w_o	central deflection
z_o	function defining the spherical shell considered
β	slope, w'
λ	curvature parameter, $\lambda^4 = \dfrac{12a^4}{R^2t^2}$
μ	Poisson's ratio of material
σ_r	radial stress
σ_{ro}	central radial stress
σ_{ra}	edge radial stress $\left(\text{for spherical shell, } \sigma_{ra} = \dfrac{qR}{2t}\right)$

Primes indicate differentiation with respect to independent variable.

DESCRIPTION OF INCREMENTAL PROCEDURE

The procedure makes use of a one-term Galerkin solution to obtain approximate results and a subsequent linearization to improve these approximate results.

The problems examined herein can be written in the general form

$$AL(w) + N(w) = l \tag{1}$$

where L is a linear differential operator, N is a nonlinear operator, A is the control parameter (or a measure of several parameters but not necessarily large or small), and l is a loading term. The choice of the coordinate system and the details of the method of solution depend on the particular problem solved; here, let $w = w(x)$. It is presumed that a solution is available at each step which satisfies this equation and the boundary conditions for a given value of A, for example, $A^{(n-1)}$. Call this solution $f^{(n-1)}$. For a neighboring value of $A = A^{(n)}$, let w be approximated by

$$w_{approx}^{(n)} = c^{(n)}f^{(n-1)} \tag{2}$$

Using equation (2) as a one-term expansion function in the approximate Galerkin solution of equation (1) leads to the following equation which determines the constant $c^{(n)}$ when $f^{(n-1)}$ is known:

$$\int f^{(n-1)}\left[c^{(n)}A^{(n)}L\left(f^{(n-1)}\right) + N\left(c^{(n)}f^{(n-1)}\right)\right] dx = \int f^{(n-1)}l\ dx \tag{3}$$

(See refs. 2 and 3 for a discussion of the Galerkin method.) Note that equation (3), with limits on the integrals specifying the boundaries of interest, is a nonlinear algebraic equation in $c^{(n)}$ (of same degree as the differential eq. (1)). It is important that the proper root of the nonlinear algebraic equation for $c^{(n)}$ is chosen as dictated by the physical problem.

Now set

$$w^{(n)} = c^{(n)}\left(f^{(n-1)} + F^{(n)}\right) = c^{(n)}f^{(n)} \tag{4}$$

With $F^{(n)}$ assumed to be sufficiently small compared with $f^{(n-1)}$, a linearized form of equation (1) may be written as

$$c^{(n)}A^{(n)}L\left(F^{(n)}\right) + \overline{N}\left(F^{(n)},f^{(n-1)},c^{(n)}\right) = l - c^{(n)}A^{(n)}L\left(f^{(n-1)}\right) - N\left(c^{(n)}f^{(n-1)}\right) \tag{5}$$

where $\overline{N}$ is the part of $N\left[c^{(n)}\left(f^{(n-1)} + F^{(n)}\right)\right]$ which contains $F^{(n)}$ linearly. Equation (5) and the boundary conditions determine $F^{(n)}$ when $c^{(n)}$ and $f^{(n-1)}$ are known. With equation (4) defining $f^{(n)}$, the procedure can be repeated for the next value of $A = A^{(n+1)}$. The present method uses equations (2) to (5) to advance from a solution available at one value of A to a solution at any desired value of A. For each value of A – that is, for each step – a check must be made to insure that the linearization assumption is fulfilled (this validates the solution) and, if not, the solution for this step must be discarded and the small increments in A must be reduced in magnitude. Thus, the present procedure combines the incrementing of the control parameter A with the one-term Galerkin method to effect accurate solution over the complete range of A. At each step the solution is validated a posteriori.

For problems of the form of equation (1) the starting point might be the solution of the linear equation obtained from equation (1) by neglecting the nonlinear term. Call this solution $f^{(0)}$. Then, for a large value of $A = A^{(1)}$ (an appropriate $A^{(1)}$ may be found by trial and error) the procedure begins with equations (2), (3), (4), and (5) in which $n = 1$ for this first step. In the next and all ensuing steps, A is decreased in small increments to $A^{(2)}$, to $A^{(3)}$, and so on to $A^{(j)}$, where $A^{(j)}$ is the A of interest, or until A approaches zero, if it is desired to cover the complete range. The procedure just described is especially suited for high-speed digital computer solution of problems cast in numerical form.

APPLICATIONS

In order to illustrate the use of the incremental procedure, solutions are presented for two problems. The first problem is the determination of the stresses and deformations of a clamped circular plate under uniform lateral loading. This problem was chosen because "exact" solutions existed for part of the nonlinear range and for the membrane case. The second problem considers a simply supported spherical cap subjected to internal pressure with corresponding edge forces and a central concentrated outward acting lateral load. For zero curvature this problem reduces to that of the flat plate with in-plane tension and a concentrated central load. Existing solutions are available for this problem only for the end point of the flat membrane.

In the first problem the critical behavior is near the edge, whereas in the second problem the critical behavior is near the center. Both problems were solved numerically on the digital computer by replacing derivatives by differences and integrals by sums. The conversion of the equations into numerical form is described in the appendix.

Clamped Circular Plate With Uniform Lateral Load

Sketch 1 illustrates the configuration and the loading considered. The large deflection (nonlinear) differential equations to be solved (see, for example, ref. 4 or 5) are

$$\left.\begin{aligned} D\left(\beta'' + \frac{1}{r}\beta' - \frac{1}{r^2}\beta\right) - \frac{Et}{1-\mu^2}\beta\left(u' + \frac{\mu}{r}u + \frac{1}{2}\beta^2\right) &= -\frac{qr}{2} \\ u'' + \frac{1}{r}u' - \frac{1}{r^2}u + \beta\left(\beta' + \frac{1-\mu}{2r}\beta\right) &= 0 \end{aligned}\right\} \tag{6}$$

where the dependent variables are the slope $\beta = w'$ and the radial displacement u and the primes indicate differentiation with respect to the independent variable r.

The boundary and continuity conditions considered are

$$\beta(0) = \beta(a) = u(0) = u(a) = 0 \tag{7}$$

and, if it is required to find the deflection w, there is the additional condition $w(a) = 0$.

The numerical equivalent of the method of solution described in the previous section is to be used. Since the region of an expected rapid change of slope is in the neighborhood of the edge, the transformation $x = \frac{r^2}{a^2}$ is made to permit equally spaced stations in the numerical procedure but more closely spaced stations near the edge in the physical problem. It is also convenient (but not necessary) to introduce new dependent variables B and U according to

$$\left.\begin{aligned}\beta &= \frac{a}{r} B\left(\frac{r^2}{a^2}\right) = \frac{1}{\sqrt{x}} B(x) \\ u &= \frac{a^2}{r} U\left(\frac{r^2}{a^2}\right) = \frac{a}{\sqrt{x}} U(x)\end{aligned}\right\} \tag{8}$$

With these changes equations (6) become

$$\left.\begin{aligned}&4DxB'' - \frac{Eta^2}{1 - \mu^2} B\left(2U' - \frac{1 - \mu}{x} U + \frac{1}{2x} B^2\right) = -q \frac{a^3 x}{2} \\ &4xU'' + B\left(2B' - \frac{1 + \mu}{2x} B\right) = 0\end{aligned}\right\} \tag{9}$$

where the primes now indicate differentiation with respect to x, and the boundary conditions in terms of the new variables are

$$B(0) = B(1) = U(0) = U(1) = 0$$

From the second of equations (9) and the boundary conditions of U,

$$U = \frac{1 - \mu}{8} x \int_x^1 \frac{1}{x^2} B^2 dx + \frac{1 + \mu}{8}\left(x \int_x^1 \frac{1}{x} B^2 dx - \int_0^x \frac{1}{x} B^2 dx\right) \tag{10}$$

so that the first of equations (9) becomes

$$4DxB'' - \frac{Eta^2}{1 - \mu^2} B\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} B^2 dx + \frac{1}{x} \int_0^x \frac{1}{x} B^2 dx\right) + \frac{(1 + \mu)^2}{8} \int_0^1 \frac{1}{x} B^2 dx\right] = -\frac{qa^3 x}{2} \tag{11}$$

Let an approximate solution be given by

$$B = cf \tag{12}$$

where f is a known function of x which satisfies the boundary conditions and c is a constant that is to be determined by the Galerkin equation. The differential equation (11), after substitution for B from equation (12), becomes

$$4Dcxf'' - \frac{Eta^2c^3}{1 - \mu^2} f\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} f^2dx + \frac{1}{x}\int_0^x \frac{1}{x} f^2dx\right) + \frac{(1 + \mu)^2}{8}\int_0^1 \frac{1}{x} f^2dx\right] = -\frac{qa^3x}{2} \tag{13}$$

To get equation (11) in dimensionless form, divide by qa^3; thus,

$$4\bar{D}\bar{c}xf'' - \bar{c}^3f\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} f^2dx + \frac{1}{x}\int_0^x \frac{1}{x} f^2dx\right) + \frac{(1 + \mu)^2}{8}\int_0^1 \frac{1}{x} f^2dx\right] = -\frac{x}{2} \tag{14}$$

where

$$\bar{D} = \frac{D\left(1 - \mu^2\right)}{Eta^2Q^{2/3}} = \frac{t^2}{12a^2Q^{2/3}}$$

$$\bar{c} = \frac{c}{Q^{1/3}}$$

$$Q = \frac{qa\left(1 - \mu^2\right)}{Et}$$

The following Galerkin equation is used to determine $\bar{c}$:

$$\int_0^1 f\left\{4\bar{D}\bar{c}xf'' - \bar{c}^3f\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} f^2dx + \frac{1}{x}\int_0^x \frac{1}{x} f^2dx\right) + \frac{(1 + \mu)^2}{8}\int_0^1 \frac{1}{x} f^2dx\right]\right\}dx = -\frac{1}{2}\int_0^1 fx\,dx \tag{15}$$

This equation corresponds to equation (3) if $A = \bar{D}$. Now with c known let

$$B = c(f + F) \tag{16}$$

with F a function assumed to be sufficiently small, so that squares and higher powers of F can be neglected, compared with a known function f. Consequently, a linear differential equation may be written for F:

$$4\bar{D}\bar{c}xF'' - \bar{c}^3\left\{\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} f^2dx + \frac{1}{x}\int_0^x \frac{1}{x} f^2dx\right) + \frac{(1 + \mu)^2}{8}\int_0^1 \frac{1}{x} f^2dx\right]F + f\left[\frac{1 - \mu^2}{4}\left(\int_x^1 \frac{1}{x^2} fF\,dx + \frac{1}{x}\int_0^x \frac{1}{x} fF\,dx\right)\right.\right.$$
$$\left.\left. + \frac{(1 + \mu)^2}{4}\int_0^1 \frac{1}{x} fF\,dx\right]\right\} = -\frac{x}{2} - 4\bar{D}\bar{c}xf'' + \bar{c}^3f\left[\frac{1 - \mu^2}{8}\left(\int_x^1 \frac{1}{x^2} f^2dx + \frac{1}{x}\int_0^x \frac{1}{x} f^2dx\right) + \frac{(1 + \mu)^2}{8}\int_0^1 \frac{1}{x} f^2dx\right] \tag{17}$$

Equation (17) corresponds to equation (5). The numerical procedure used on the digital computer to solve these problems is described in the appendix.

Results for deflections and midplane stresses at the center of the plate are plotted as a function of $\overline{D} = \frac{D(1 - \mu^2)}{Eta^2Q^{2/3}}$ in figures 1 and 2. There is very good agreement between the present numerical results and the exact solution by power series of Way (ref. 6) using the same equations as used in the present paper. The present results extend the solution of Way to thinner plates or higher loads (smaller $\overline{D}$) and, in the limit as $D \rightarrow 0$, approach the exact membrane solution of Hencky (ref. 7).

For comparison, the one-term Galerkin solution of the nonlinear equilibrium equations using the function obtained from linear theory was determined; it gives only a fair approximation for the complete range of $\overline{D}$ as shown in figures 1 and 2.

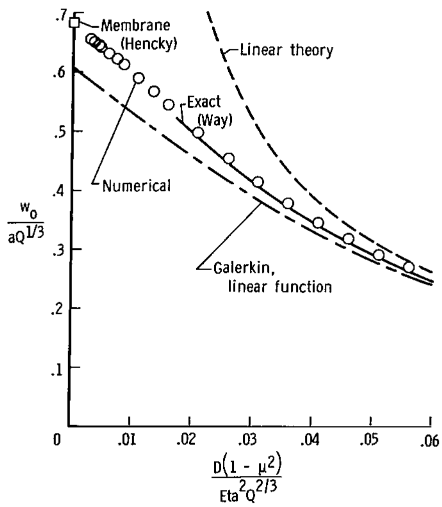


Figure 1.- Deflection at the center of a uniformly loaded clamped circular plate. $Q = \frac{qa(1 - \mu^2)}{Et}$.

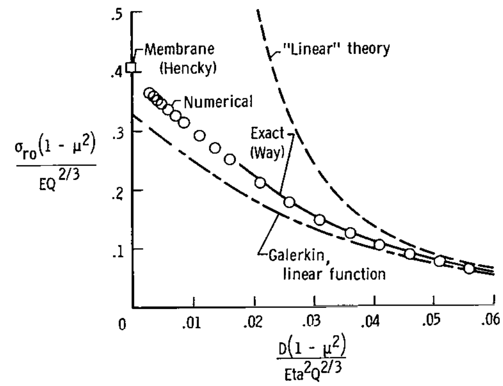


Figure 2.- Radial midplane stress at the center of a uniformly loaded clamped circular plate. $Q = \frac{qa(1 - \mu^2)}{Et}$.

In figures 3 and 4 radial distributions of deflection and in-plane stresses are plotted for three different values of $\overline{D}$. The deflection distribution for the highest value of $\overline{D}$ as obtained from linear plate theory is also plotted. These results indicate that the deflections according to linear theory are everywhere larger than those given by nonlinear theory. No unusual changes in the shapes of the curves occur as $\overline{D}$ changes. Of course, in the limit as $\overline{D}$ approaches zero – that is, for the pure membrane – the boundary conditions change from clamped to supported.

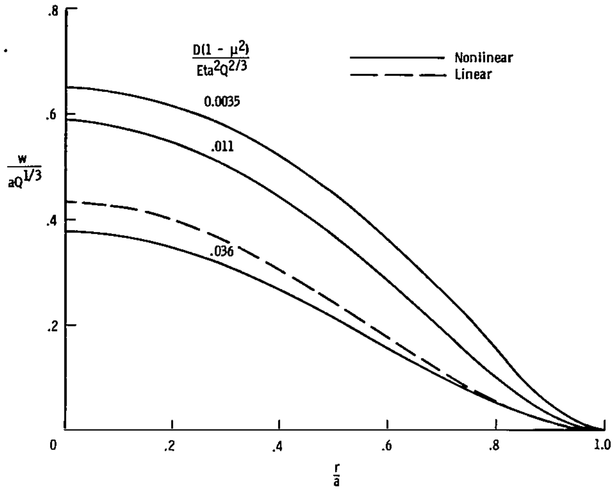


Figure 3.- Deflection distribution of a uniformly loaded clamped circular plate. $Q = \frac{qa(1 - \mu^2)}{Et}$.

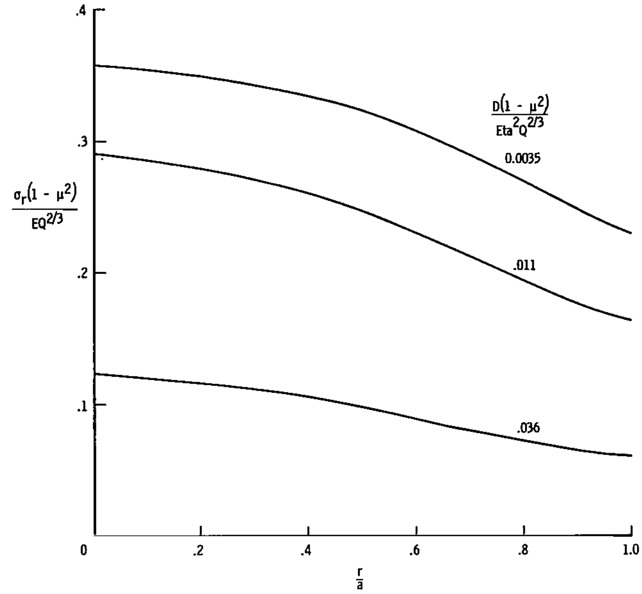


Figure 4.- Radial midplane stress distribution of a uniformly loaded clamped circular plate. $Q = \frac{qa(1 - \mu^2)}{Et}$.

Shallow Spherical Shell

The problem of a simply supported spherical cap with central concentrated load and with internal pressure and corresponding edge forces is illustrated in sketch 3. For zero curvature the problem reduces to that of a simply supported circular plate with a central concentrated load and uniform edge forces (sketch 2). The large deflection (nonlinear) differential equations to be solved (see ref. 5, for example) are

$$\left.\begin{aligned} &D\left(\beta'' + \frac{1}{r}\beta' - \frac{1}{r^2}\beta\right) - \frac{Et}{1-\mu^2}\left(\beta + z_o'\right)\left(u' + \frac{\mu}{r}u + z_o'\beta + \frac{1}{2}\beta^2\right) = \frac{qr}{2} + \frac{P}{2\pi r} \\ &u'' + \frac{1}{r}u' - \frac{1}{r^2}u + z_o'\beta' + \beta\left[\beta' + z_o'' + \frac{1-\mu}{2r}\left(\beta + 2z_o'\right)\right] = 0 \end{aligned}\right\} \tag{18}$$

where

$$z_o = H\left(1 - \frac{r^2}{a^2}\right)$$

H being the shell rise $\left(H = \frac{a^2}{2R}\right)$ and R the radius of curvature, and again the dependent variables are the slope $\beta = w'$ and the radial displacement u. Primes indicate differentiation with respect to the independent variable r. The boundary and continuity conditions are

$$\left.\begin{aligned}
&\beta(0) = u(0) = 0 \\
&D\left(\beta' + \frac{\mu}{r}\beta\right)_{r=a} = 0 \\
&\frac{E}{1-\mu^2}\left(u' + \frac{\mu}{r}u + z_o'\beta + \frac{1}{2}\beta^2\right)_{r=a} = \sigma_{ra}
\end{aligned}\right\} \tag{19}$$

with the additional condition on deflection at the edge

$$w(a) = 0$$

For convenience a new dependent variable U is introduced to get homogeneous boundary conditions:

$$\frac{u}{a} = \frac{\sigma_{ra}(1-\mu)}{E}\frac{r}{a} + U \tag{20}$$

The region of rapid change of slope for this problem is expected to be near the center; therefore, it is expedient to change the independent variable to y where

$$y^2 = \frac{r}{a}$$

With these changes equations (18) become

$$\left.\begin{aligned}
&D\left(\frac{1}{4}\beta'' + \frac{1}{4y}\beta' - \frac{1}{y^2}\beta\right) - \frac{Eta^2}{1-\mu^2}y\left(\beta - \frac{2H}{a}y^2\right)\left[\frac{\sigma_{ra}\left(1-\mu^2\right)}{E}y + \frac{1}{2}U'\right. \\
&\left. + \frac{\mu}{y}U - \frac{2H}{a}y^3\beta + \frac{1}{2}y\beta^2\right] = \frac{qa^3}{2}y^4 + \frac{Pa}{2\pi} \\
&\frac{1}{4}U'' + \frac{1}{4y}U' - \frac{1}{y^2}U - \frac{H}{a}y^3\beta' + y\beta\left[\frac{1}{2}\beta' - \frac{2H}{a}y + \frac{1-\mu}{2y}\left(\beta - \frac{4H}{a}y^2\right)\right] = 0
\end{aligned}\right\} \tag{21}$$

where the primes now indicate differentiation with respect to y. Poisson's ratio μ is set equal to one-third and both sides of equations (21) are multiplied through by y to give

$$\left.\begin{aligned}
&D\left[\frac{1}{4}\left(y\beta'\right)' - \frac{1}{y}\beta\right] - \sigma_{ra}ta^2y^3\left(\beta - \frac{2H}{a}y^2\right) \\
&- \frac{Eta^2}{1-\mu^2}y^2\left(\beta - \frac{2H}{a}y^2\right)\left[\frac{1}{2y^{2/3}}\left(y^{2/3}U\right)' - \frac{2H}{a}y^3\beta + \frac{y}{2}\beta^2\right] = \frac{qa^3}{2}y^5 + \frac{Pa}{2\pi}y \\
&\frac{1}{4}\left(yU'\right)' - \frac{1}{y}U - \frac{H}{a}y^{2/3}\left(y^{10/3}\beta\right)' + \frac{1}{4}y^{2/3}\left(y^{4/3}\beta^2\right)' = 0
\end{aligned}\right\} \tag{22}$$

If the edge stress is identified as the stress due to the pressure forces, then $\sigma_{ra} = \frac{qR}{2t}$, and since $\frac{H}{a} = \frac{a}{2R}$, there is a cancellation with the q term on the right-hand side of the first of equations (22). The boundary conditions in terms of the new variable are

$$\left.\begin{aligned} &\beta(0) = U(0) = 0 \\ &\left(y^{2/3}\beta\right)'_{y=1} = 0 \\ &\left[\frac{1}{2y^{2/3}}\left(y^{2/3}U\right)' - \frac{2H}{a} y^3\beta + \frac{y}{2}\beta^2\right]_{y=1} = 0 \end{aligned}\right\} \tag{23}$$

Because of the different transformation used in this application compared with that used in the clamped plate application, it is not convenient to integrate the second of equations (22) directly. With f and g known functions of y and c_1 and c_2 constants that are to be determined later by the Galerkin equations, let

$$\left.\begin{aligned} \beta &= c_1 f \\ U &= c_2 g \end{aligned}\right\} \tag{24}$$

Upon substitution of equations (24) into equations (22), the following equations result:

$$\left.\begin{aligned} &Dc_1\left[\frac{1}{4}(yf')' - \frac{1}{y}\right]f - \frac{qa^2R}{2} c_1 y^3 f \\ &- \frac{Eta^2}{1-\mu^2} y^2\left(c_1 f - \frac{a}{R} y^2\right)\left[\frac{c_2}{2y^{2/3}}\left(y^{2/3}g\right)' - \frac{a}{r} c_1 y^3 f + \frac{y}{2} c_1{}^2 f^2\right] = \frac{Pay}{2\pi} \\ &c_2\left[\frac{1}{4}(yg')' - \frac{1}{y} g\right] - \frac{a}{4R} c_1 y^{2/3}\left(y^{10/3} f\right)' - \frac{c_1{}^2}{4} y^{2/3}\left(y^{4/3} f^2\right)' = 0 \end{aligned}\right\} \tag{25}$$

To get in dimensionless form, divide by Pa/π; thus,

$$\left.\begin{aligned} &\bar{D}\bar{c}_1\left[\frac{1}{4}(yf')' - \frac{1}{y} f\right] - T\bar{c}_1 y^3 f - \left(\bar{c}_1 f - \bar{D}^{1/2}\lambda^2 y^2\right)\left[\frac{\bar{c}_2}{2} y^{4/3}\left(y^{2/3}g\right)'\right. \\ &\left. - \bar{D}^{1/2}\lambda^2\bar{c}_1 y^5 f + \frac{\bar{c}_1{}^2}{2} y^3 f^2\right] = \frac{y}{2} \\ &\bar{c}_2\left[\frac{1}{4}(yg')' - \frac{1}{y} g\right] - \frac{1}{2}\bar{D}^{1/2}\lambda^2\bar{c}_1 y^{2/3}\left(y^{10/3} f\right)' - \frac{\bar{c}_1{}^2}{4} y^{2/3}\left(y^{4/3} f^2\right)' = 0 \end{aligned}\right\} \tag{26}$$

where

$$\bar{D} = \frac{D(1 - \mu^2)}{Eta^2Q^{2/3}} = \frac{t^2}{12a^2Q^{2/3}}$$

$$T = \frac{qR(1 - \mu^2)}{2EtQ^{2/3}} = \frac{\sigma_{ra}(1 - \mu^2)}{EQ^{2/3}}$$

$$\lambda^4 = 12\frac{a^4}{R^2t^2}$$

$$Q = \frac{P(1 - \mu^2)}{\pi Eta}$$

$$\bar{c}_1 = \frac{c_1}{Q^{1/3}}$$

$$\bar{c}_2 = \frac{c_2}{Q^{2/3}}$$

The following Galerkin equations may be used to determine c_1 and c_2:

$$\left.\begin{aligned}
&\int_0^1 f\left\{\bar{D}\bar{c}_1\left[\frac{1}{4}(yf')' - \frac{1}{y}f\right] - T\bar{c}_1y^3f - \left(\bar{c}_1f - \bar{D}^{1/2}\lambda^2y^2\right)\left[\frac{\bar{c}_2}{2}y^{4/3}\left(y^{2/3}g\right)' - \bar{D}^{1/2}\lambda^2\bar{c}_1y^5f\right.\right.\\
&\left.\left.+\frac{\bar{c}_1^{\,2}}{2}y^3f^2\right]\right\}dy = \frac{1}{2}\int_0^1 fy\,dy\\
&\int_0^1 g\left\{\bar{c}_2\left[\frac{1}{4}(yg')' - \frac{1}{y}g\right] - \frac{1}{2}\bar{D}^{1/2}\lambda^2\bar{c}_1y^{2/3}\left(y^{10/3}f\right)' - \frac{\bar{c}_1^{\,2}}{4}y^{2/3}\left(y^{4/3}f^2\right)'\right\}dy = 0
\end{aligned}\right\} \tag{27}$$

Equations (27) are treated in a manner similar to that for equation (3) with $A = \bar{D}$.

Let

$$\left.\begin{aligned}
\beta &= c_1(f + F)\\
U &= c_2(g + G)
\end{aligned}\right\} \tag{28}$$

with functions F and G assumed to be small on the average compared with functions f and g, respectively, and with c_1 and c_2 known constants. With the aforementioned assumptions, the following linear differential equations can be written for F and G:

$$
\left.
\begin{aligned}
&\bar{D}\bar{c}_1\left[\frac{1}{4}(yF')' - \frac{1}{y}F\right] + \frac{1}{2}\bar{D}^{1/2}\lambda^2\bar{c}_2y^{10/3}\left(y^{2/3}G\right)' - T\bar{c}_1y^3F - \bar{D}\bar{c}_1\lambda^4y^7F + 3\bar{D}^{1/2}\lambda^2\bar{c}_1{}^2y^5fF \\
&- \frac{\bar{c}_1\bar{c}_2}{2}y^{4/3}f\left(y^{2/3}G\right)' - \left[\frac{\bar{c}_1\bar{c}_2}{2}y^{4/3}\left(y^{2/3}g\right)' + \frac{3}{2}\bar{c}_1{}^3y^3f^2\right]F \\
&\qquad = \frac{1}{2}y - \bar{D}\bar{c}_1\left[\frac{1}{4}(yf')' - \frac{1}{y}f\right] + T\bar{c}_1y^3f + \left(\bar{c}_1f - \bar{D}^{1/2}\lambda^2y^2\right)\left[\frac{\bar{c}_2}{2}y^{4/3}\left(y^{2/3}g\right)' - \bar{D}^{1/2}\lambda^2\bar{c}_1y^5f + \frac{\bar{c}_1{}^2}{2}y^3f^2\right] \\
&\bar{c}_2\left[\frac{1}{4}(yG')' - \frac{1}{y}G\right] - \frac{1}{2}\bar{D}^{1/2}\lambda^2\bar{c}_1y^{2/3}\left(y^{10/3}F\right)' - \frac{\bar{c}_1{}^2}{2}y^{2/3}\left(y^{4/3}fF\right)' \\
&\qquad = -\bar{c}_2\left[\frac{1}{4}(yg')' - \frac{1}{y}g\right] + \frac{1}{2}\bar{D}^{1/2}\lambda^2\bar{c}_1y^{2/3}\left(y^{10/3}f\right)' + \frac{\bar{c}_1{}^2}{4}y^{2/3}\left(y^{4/3}f^2\right)'
\end{aligned}
\right\} \tag{29}
$$

Equations (29) correspond to equation (5).

The numerical procedure used on the digital computer to solve this problem is described in the appendix. Results for deflections and the midplane stresses at the center are plotted as a function of $\bar{D} = \frac{D\left(1 - \mu^2\right)}{Eta^2Q^{2/3}}$ for values of $T = \frac{\sigma_{ra}\left(1 - \mu^2\right)}{EQ^{2/3}}$ between 0 and 1 for a flat plate in figures 5 and 6. The center deflection for $T = 0$ and $T = 1$ for a flat plate is compared with corresponding results for spherical caps in figure 7. Similar comparisons for center midplane stress are shown in figure 8. Similar comparisons of deflection distribution and radial stress distribution are shown in figures 9, 10, 11, and 12.

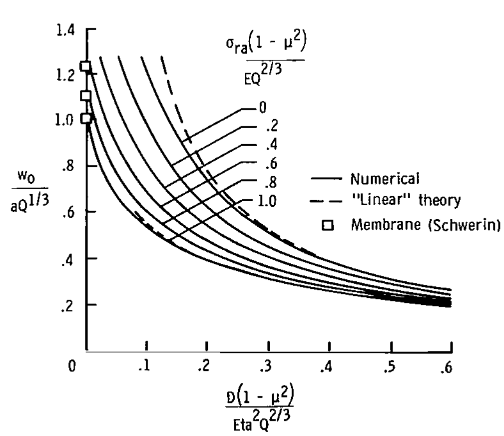


Figure 5.- Deflection under a concentrated lateral load of a simply supported circular plate with edge stress σ_{ra}. $Q = \frac{P(1 - \mu^2)}{\pi Eta}$.

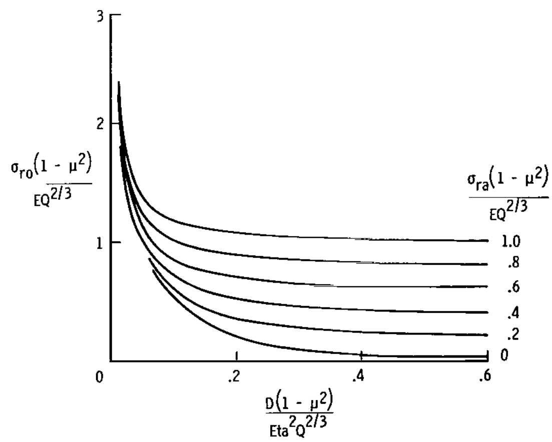


Figure 6.- Radial midplane stress under a concentrated lateral load of a simply supported circular plate with edge stress σ_{ra}. $Q = \frac{P(1 - \mu^2)}{\pi Eta}$.

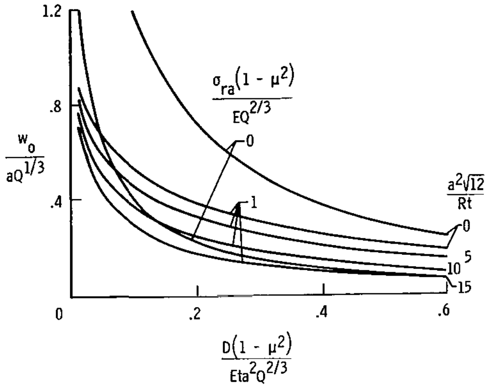


Figure 7.- Deflection under a concentrated load for a circular plate and spherical caps. $Q = \frac{P(1 - \mu^2)}{\pi Eta}$.

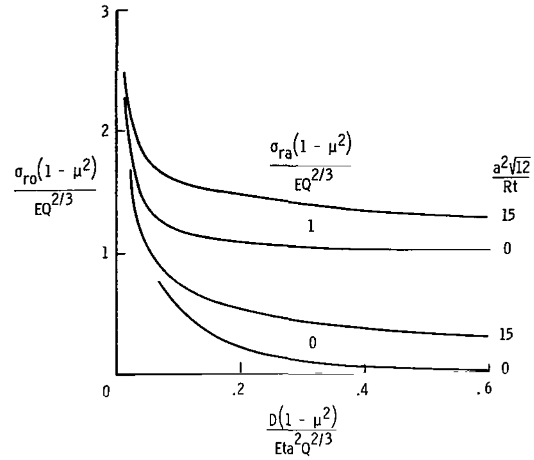


Figure 8.- Radial midplane stress under a concentrated load for a circular plate and a spherical cap. $Q = \frac{P(1 - \mu^2)}{\pi Eta}$.

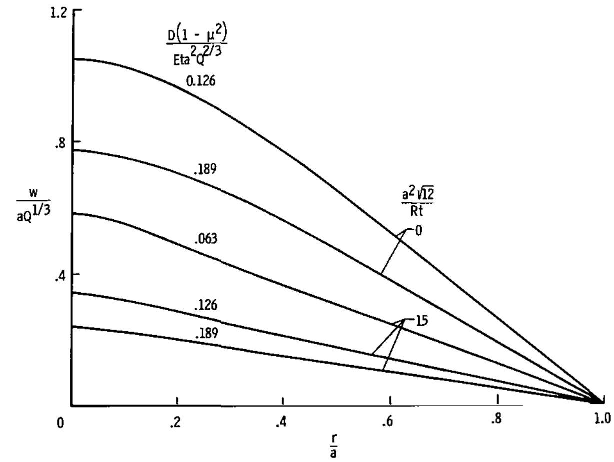


Figure 9.- Deflection distribution for a circular plate and a spherical cap with prescribed edge stress. $\frac{\sigma_{ra}(1 - \mu^2)}{EQ^{2/3}} = 0.$

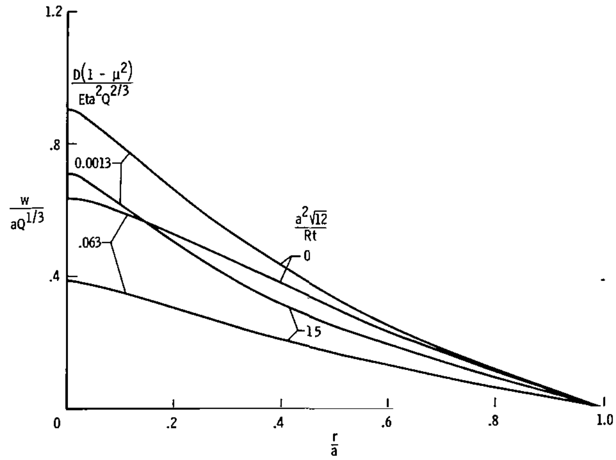


Figure 10.- Deflection distribution for a circular plate and a spherical cap with prescribed edge stress. $\frac{\sigma_{ra}(1 - \mu^2)}{EQ^{2/3}} = 1.$

The results show the trend toward membrane behavior as the thickness decreases or the load increases (as $\overline{D} \rightarrow 0$). In figure 8 the flat-plate deflections for $T = 0.6$, 0.8, and 1.0 approach the nonlinear membrane values obtained from Schwerin's results as $\overline{D} \rightarrow 0$ (ref. 8). For values of T lower than 0.5, membrane theory gives infinite deflections; the stress at the edge is not enough to prevent this unlimited deflection (the membrane is not in static equilibrium). Also, in figure 10 the deflection distribution curves for $\overline{D} = 0.0013$ (and $T = 1$) illustrate the present results when membrane behavior predominates for both the flat plate and the spherical cap.

Figures 11 and 12 show that, although the flat plate and spherical cap have roughly the same midplane radial stresses under the concentrated load, away from the concentrated load the radial stress in the flat plate decreases and the radial stress in the spherical cap first increases appreciably and then decreases.

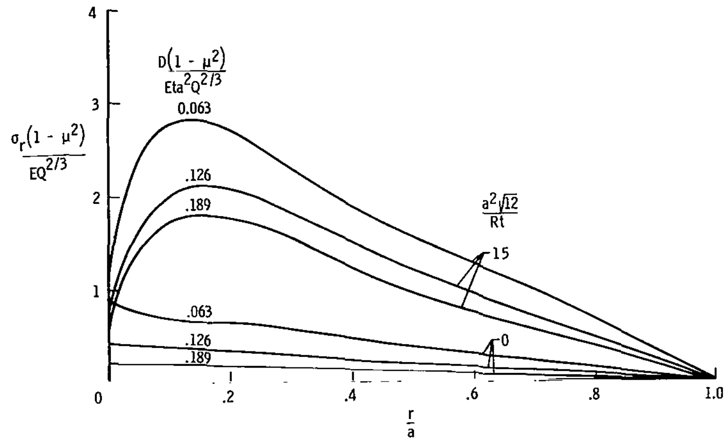


Figure 11.- Radial midplane stress distribution for a circular plate and a spherical cap with zero edge stress. $\frac{\sigma_{ra}(1 - \mu^2)}{EQ^{2/3}} = 0.$

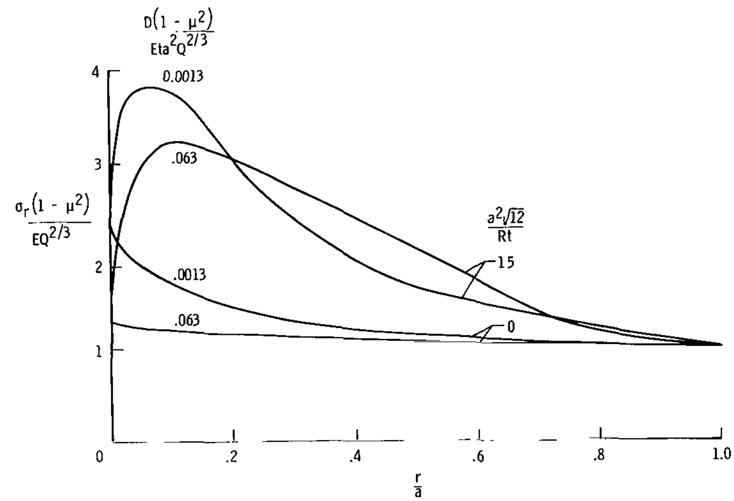


Figure 12.- Radial midplane stress distribution for a circular plate and a spherical cap with prescribed edge stress. $\frac{\sigma_{ra}(1 - \mu^2)}{EQ^{2/3}} = 1.$

CONCLUDING REMARKS

A procedure for the solution of nonlinear problems has been presented which involves step-by-step incrementing of a control parameter in such a way that solutions may be obtained for a wide spectrum of practical values of the parameter. The procedure has been applied to the solution of several axisymmetric problems in plates and shells having moderately large deformations. In these problems the control parameter is the flexural stiffness. Results are obtained for the complete spectrum of parameter values, from the linear plate or shell solution to the nonlinear membrane solution. These solutions provide an evaluation of the suggested procedure in three ways. First, a posteriori examination of the nonlinear terms at each step of the procedure shows that they are sufficiently small to justify linearization and indicates the appropriate size of increments in the control parameter. Second, close agreement is obtained with previously known linear and nonlinear plate and shell solutions. Third, the plate and shell solutions approach closely known nonlinear membrane results for the limiting case of the flexural stiffness approaching zero.

It is believed that this procedure is fast, accurate, more flexible, and more widely applicable (with appropriate modification) than many of the methods of solution of nonlinear boundary value problems now in existence.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., July 11, 1966,
124-08-06-01-23.

APPENDIX

GOVERNING EQUATIONS IN NUMERICAL FORM

The equations of the section entitled "Applications" are herein cast in numerical form in order to proceed with their solution on the digital computer. Stations are measured out radially from the center of the plate or shell in terms of x or y, and these stations are taken to be equally spaced and are numbered from $s = 0$ at the center to $s = S$ at the edge. The following (central) difference approximation for the first derivative is used:

$$\left(\frac{df}{dx}\right)_{s+\frac{1}{2}} = \frac{f_{s+1} - f_s}{\epsilon}$$

where s indicates the station and ϵ the distance between stations. With the differential relations replaced by such differences, the differential equations are then defined for each station. Integrals are approximated by finite sums according to the trapezoidal rule. Of course, the equations written for the stations near the center and near the edge must be modified to account for continuity and boundary conditions. Thus the linear differential equations are replaced by a set of simultaneous linear algebraic equations in terms of the unknown displacements at the various stations.

Clamped Circular Plate With Uniform Lateral Load

Equations (15) and (17) are now expressed in numerical form. The algebraic relation determining $\bar{c}$ corresponding to equation (15) can be written in matrix form as

$$\lceil f \rfloor \left(4\bar{D}\bar{c} \lceil x \rfloor \lceil d \rfloor - \bar{c}^3 \lceil f \rfloor \lceil M \rfloor\right) \left| f \right| = \frac{1}{2} \lceil f \rfloor \left| x \right| \qquad \text{(A1)}$$

where

$$\lceil x \rfloor = \frac{1}{S}\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \cdots & \\ & & & S-1 \end{bmatrix}$$

$$\lceil d \rfloor = S^2\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & & \cdots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}$$

APPENDIX

$$
[M] = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \cdots & \\ & & & M_{S-1} \end{bmatrix}
$$

$$
[f] = \begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & \cdots & \\ & & & f_{S-1} \end{bmatrix}
$$

and vertical lines $| \; |$ indicate column vectors and the half brackets indicate a row matrix – that is,

$$
|f| = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{S-1} \end{bmatrix}
$$

$$
|x| = \frac{1}{S} \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ S-1 \end{bmatrix}
$$

$$
\lfloor f \rfloor = \lfloor f_1 \quad f_2 \quad \cdots \quad f_{S-1} \rfloor
$$

Also,

$$
|M| = [G]|f|
$$

and

$$
[G] = \left\{ \frac{1-\mu^2}{4} \left([i] \left[\frac{1}{x^2}\right] + \left[\frac{1}{x}\right] [i]^T \left[\frac{1}{x}\right] \right) + \frac{(1+\mu)^2}{4} [1] \left[\frac{1}{x}\right] \right\} [f]
$$

where

$$
[i] = \frac{1}{S} \begin{bmatrix} 1/2 & 1 & \cdots & & 1 \\ & & \cdots & & \\ & & & 1/2 & 1 \\ & & & & 1/2 \end{bmatrix}
$$

$$\left[1\right] = \frac{1}{S}\begin{bmatrix} 1 & . & . & . & 1 \\ & . & . & . & \\ 1 & . & . & . & 1 \end{bmatrix}$$

$$\left[\frac{1}{x}\right] = S\begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & \cdots & \\ & & & \frac{1}{S-1} \end{bmatrix}$$

$$\left[\frac{1}{x^2}\right] = S^2\begin{bmatrix} 1 & & & \\ & 1/4 & & \\ & & \cdots & \\ & & & \frac{1}{(S-1)^2} \end{bmatrix}$$

and the superscript T indicates the transpose.

The set of simultaneous algebraic equations replacing equation (17) can be written in the matrix form as

$$\left\{4\bar{D}\bar{c}\left[x\right]\left[d\right] - \bar{c}^3\left(\left[M\right] + \left[f\right]\left[G\right]\right)\right\}\left|F\right| = -\frac{1}{2}\left|x\right| - \left(4\bar{D}\bar{c}\left[x\right]\left[d\right] - \bar{c}^3\left[f\right]\left[M\right]\right)\left|f\right| \tag{A2}$$

To obtain a solution for a particular case, it is necessary to specify Poisson's ratio μ, number of stations S, and initial values of $|f|$, $\bar{c}$, and $\bar{D}$. Results are presented for $\mu = 0.3$ and $S = 12$. (It was found that accurate results were obtained for $S = 12$.) The initial values of $|f|$ were obtained from the linear (continuous) solution so that

$$f_s = \frac{s}{S} - \frac{s^2}{S^2} \qquad (s = 0, 1, 2, \ldots, S)$$

Trial and error suggested a starting value of $\bar{D}$ of 0.056; then from a Galerkin solution using this linear expression, $\bar{c} = 1.032$. Insert these initial values in equation (A2) and solve for $|F|$. If the elements of $|F|$ are small compared with $|f|$, this may be taken to be a valid solution so that all the quantities of interest are determined at this value of $\bar{D}$. For this problem the criterion used to satisfy the linearization assumption was to require that the sum of the magnitudes of the components of $|F|$ was less than 4 percent of the corresponding sum of those of $|f|$. From a study of the Galerkin solution a change in $\bar{D}$ of 0.005 was chosen so that, with the new value of $\bar{D}$ ($\bar{D} = 0.051$) and $|f|$

(i.e., the sum of the old $|f|$ and the $|F|$ as just found), a new value of $\bar{c}$ is determined from equation (A1). With these new values of $\bar{c}$, $\bar{D}$, and $|f|$, a new $|F|$ is determined from equation (A2). This procedure is repeated until results for $\bar{c}$, $|f|$, and $|F|$ are obtained for the $\bar{D}$ value of interest.

When each new $|F|$ is determined, it should be checked to see whether it is small compared with the $|f|$ used to obtain it. If it is not small, the change in $\bar{D}$ should be decreased from the previous step and this step should be recalculated.

The quantities of interest which may be desired at each value of $\bar{D}$ may be calculated from the numerical equivalents of equations (8), (10), and (12) and from $\beta = w'$. In particular, the deflection and the midplane stresses at the center of the clamped circular plate were calculated by using Simpson's rule for integration. The center deflection is

$$\frac{w_o}{aQ^{1/3}} = \frac{\bar{c}}{2S}\left[\Sigma\right|\left[\frac{1}{x}\right]|f| \tag{A3}$$

The radial midplane stress is given by

$$\sigma_r = \frac{E}{1 - \mu^2}\left(u' + \frac{\mu}{r} u + \frac{1}{2}\beta^2\right)$$

so that the stress at the center is

$$\frac{\sigma_{ro}(1 - \mu^2)}{EQ^{2/3}} = \frac{\bar{c}^2}{S}\left\{\frac{1 - \mu^2}{8}\left[\Sigma\right|\left[\frac{1}{x^2}\right]|f^2| + \frac{(1 + \mu)^2}{8}\left[\Sigma^*\right|\left[\frac{1}{x}\right]|f^2|\right\} \tag{A4}$$

where

$$\left[\Sigma\right| = \frac{1}{3}\left[6, \frac{3}{2}, 4, 2, 4, 2, \ldots, 4, 2, 4\right]$$

$$\left[\Sigma^*\right| = \frac{1}{3}\left[4, 2, 4, 2, \ldots, 4, 2, 4\right]$$

Center deflection and midplane stress as given by equations (A3) and (A4) are plotted as a function of $\bar{D}$ in figures 1 and 2. Figures 3 and 4 present deflection and stress distributions obtained for particular values of $\bar{D}$.

APPENDIX

Shallow Spherical Shell

Equations (27) and (29) are now expressed in numerical form. The algebraic relations determining $\bar{c}_1$ and $\bar{c}_2$ corresponding to equations (27) can be written in matrix form as

$$\left.\begin{aligned} \lceil f \rfloor \left(\left[A_{11}\right] \left| f \right| + \left[A_{12}\right] \left| g \right| \right) &= \frac{1}{2} \lceil f \rfloor \left| y \right| \\ \lceil g \rfloor \left(\left[A_{21}\right] \left| f \right| + \left[A_{22}\right] \left| g \right| \right) &= 0 \end{aligned}\right\} \tag{A5}$$

where

$$\left[A_{11}\right] = \bar{D}\bar{c}_1 \left\{ \frac{S}{4} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & & & \\ & 3/2 & & \\ & & \ddots & \\ & & & S+\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & -1 & 1 \\ & & & \left(\frac{S-1}{S+1}\right)^{2/3} & 1 \end{bmatrix} - S \begin{bmatrix} 1 & & & & \\ & 1/2 & & & \\ & & 1/3 & & \\ & & & \ddots & \\ & & & & \frac{1}{S} \end{bmatrix} \right\}$$

$$- \frac{T\bar{c}_1}{S^3} \begin{bmatrix} 1 & & & & \\ & (2)^3 & & & \\ & & (3)^3 & & \\ & & & \ddots & \\ & & & & (S)^3 \end{bmatrix} - \frac{Dc_1\lambda^4}{S^7} \begin{bmatrix} 1 & & & & \\ & (2)^7 & & & \\ & & (3)^7 & & \\ & & & \ddots & \\ & & & & (S)^7 \end{bmatrix} + \frac{3\bar{D}^{1/2}\lambda^2\bar{c}_1^{\,2}}{2S^5} \begin{bmatrix} 1 & & & & \\ & (2)^5 & & & \\ & & (3)^5 & & \\ & & & \ddots & \\ & & & & (S)^5 \end{bmatrix} \begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & \ddots & \\ & & & f_S \end{bmatrix}$$

$$- \frac{\bar{c}_1\bar{c}_2}{2S} \begin{bmatrix} 1 & & & & & \\ & (2)^{4/3} & & & & \\ & & (3)^{4/3} & & & \\ & & & \ddots & & \\ & & & & (S-1)^{4/3} & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_S \end{bmatrix} - \frac{\bar{c}_1^{\,3}}{2S^3} \begin{bmatrix} 1 & & & & & \\ & (2)^3 & & & & \\ & & (3)^3 & & & \\ & & & \ddots & & \\ & & & & (S-1)^3 & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} f_1^{\,2} & & & \\ & f_2^{\,2} & & \\ & & \ddots & \\ & & & f_S^{\,2} \end{bmatrix}$$

$$\left[A_{12}\right] = \frac{\bar{D}^{1/2}\lambda^2\bar{c}_2}{2S^3} \begin{bmatrix} 1 & & & & \\ & (2)^{10/3} & & & \\ & & (3)^{10/3} & & \\ & & & \ddots & \\ & & & & (S)^{10/3} \end{bmatrix} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \ddots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix}$$

$$- \frac{\bar{c}_1\bar{c}_2}{2S} \begin{bmatrix} 1 & & & & & \\ & (2)^{4/3} & & & & \\ & & (3)^{4/3} & & & \\ & & & \ddots & & \\ & & & & (S-1)^{4/3} & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & \ddots & \\ & & & f_S \end{bmatrix} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \ddots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix}$$

APPENDIX

$$
\left[A_{21}\right] = -\frac{\bar{D}^{1/2}\lambda^{2}\bar{c}_{1}}{2S^{3}}
\begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \cdots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix}
\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \\ & & & & -1 \quad (1-Y) \end{bmatrix}
\begin{bmatrix} 1 & & & & \\ & (2)^{10/3} & & & \\ & & (3)^{10/3} & & \\ & & & \cdots & \\ & & & & (S)^{10/3} \end{bmatrix}
$$

$$
+\frac{\bar{c}_{1}^{2}}{4S}
\begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \cdots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix}
\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \\ & & & & -1 \quad (1-Y) \end{bmatrix}
\begin{bmatrix} f_{1} & & & \\ & f_{2} & & \\ & & \cdots & \\ & & & f_{S} \end{bmatrix}
\begin{bmatrix} 1 & & & & \\ & (2)^{4/3} & & & \\ & & (3)^{4/3} & & \\ & & & \cdots & \\ & & & & (S)^{4/3} \end{bmatrix}
$$

$$
\left[A_{22}\right] = \bar{c}_{2}\left\{\frac{S}{4}
\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \cdots & \\ & & & S \end{bmatrix}
\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \\ & & & & (X-1) \end{bmatrix}
- N
\begin{bmatrix} 2 & & & & \\ & 2/3 & & & \\ & & 2/5 & & \\ & & & \cdots & \\ & & & & \frac{1}{S-\frac{1}{2}} \end{bmatrix}
\right\}
$$

with $Y = \dfrac{S^{4/3}}{\left(S-\frac{1}{2}\right)^{2/3}\left(S+\frac{1}{2}\right)^{2/3}}$ and $X = \left(\dfrac{S-\frac{1}{2}}{S+\frac{1}{2}}\right)^{2/3}$. Also,

$$
|f| = \begin{bmatrix} f_{1} \\ f_{2} \\ \cdot \\ \cdot \\ \cdot \\ f_{S} \end{bmatrix}
$$

$$
|g| = \begin{bmatrix} g_{1/2} \\ g_{3/2} \\ \cdot \\ \cdot \\ \cdot \\ g_{S-\frac{1}{2}} \end{bmatrix}
$$

APPENDIX

$$
\left|y\right| = \frac{1}{S}\begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ S \end{bmatrix} .
$$

and the M's required for the diagonal matrix appearing in $\left[A_{11}\right]$ are given by

$$
\begin{bmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ \cdot \\ M_S \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \cdots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix} \begin{bmatrix} g_{1/2} \\ g_{3/2} \\ \cdot \\ \cdot \\ \cdot \\ g_{S-\frac{1}{2}} \end{bmatrix}
$$

The set of simultaneous linear algebraic equations replacing equations (29) is

$$
\left.\begin{aligned} \left[A_{11}{}^*\right]\left|F\right| + \left[A_{12}\right]\left|G\right| &= \left|R_1\right| \\ \left[A_{21}{}^*\right]\left|F\right| + \left[A_{22}\right]\left|G\right| &= \left|R_2\right| \end{aligned}\right\} \tag{A6}
$$

where

$$
\left[A_{11}{}^*\right] = \left[A_{11}\right] + \frac{3\bar{D}^{1/2}\lambda^2\bar{c}_1{}^2}{2S^5}\begin{bmatrix} 1 & & & & \\ & (2)^5 & & & \\ & & (3)^5 & & \\ & & & \cdots & \\ & & & & (S)^5 \end{bmatrix}\begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & \cdots & \\ & & & f_S \end{bmatrix} - \frac{\bar{c}_1{}^3}{S^3}\begin{bmatrix} 1 & & & & & \\ & (2)^3 & & & & \\ & & (3)^3 & & & \\ & & & \cdots & & \\ & & & & (S-1)^3 & \\ & & & & & 0 \end{bmatrix}\begin{bmatrix} f_1{}^2 & & & \\ & f_2{}^2 & & \\ & & \cdots & \\ & & & f_S{}^2 \end{bmatrix}
$$

$$
\left[A_{21}{}^*\right] = \left[A_{21}\right] + \frac{\bar{c}_1{}^2}{4S}\begin{bmatrix} (1/2)^{2/3} & & & \\ & (3/2)^{2/3} & & \\ & & \cdots & \\ & & & \left(S-\frac{1}{2}\right)^{2/3} \end{bmatrix}\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & -1 & 1 & \\ & & & -1 & (1+Y) \end{bmatrix}\begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & \cdots & \\ & & & f_S \end{bmatrix}\begin{bmatrix} 1 & & & & \\ & (2)^{4/3} & & & \\ & & (3)^{4/3} & & \\ & & & \cdots & \\ & & & & (S)^{4/3} \end{bmatrix}
$$

and

$$
\left|R_1\right| = \frac{1}{2}\left|y\right| - \left[A_{11}\right]\left|f\right| - \left[A_{12}\right]\left|g\right|
$$

$$
\left|R_2\right| = -\left[A_{21}\right]\left|f\right| - \left[A_{22}\right]\left|g\right|
$$

and

$$|F| = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_S \end{bmatrix}$$

$$|G| = \begin{bmatrix} G_{1/2} \\ G_{3/2} \\ \cdot \\ \cdot \\ \cdot \\ G_{S-\frac{1}{2}} \end{bmatrix}$$

To obtain a solution for a particular case, it is necessary to specify the number of stations S, the tension at the edge T corresponding to the pressure, the curvature parameter λ, and an initial value of $\overline{D}$. The initial value of $\overline{D}$ chosen should be in the range where linear theory is expected to be valid. The initial values of f and g are found from the following relations:

$$\left.\begin{aligned} \left[\overline{A}_{11}\right]|f| + \left[\overline{A}_{12}\right]|g| &= \frac{1}{2S}\begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ S \end{bmatrix} \\ \left[\overline{A}_{21}\right]|f| + \left[A_{22}\right]|g| &= \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \end{aligned}\right\} \tag{A7}$$

where $\left[\overline{A}_{11}\right]$ is $\left[A_{11}\right]$ less the last three terms (the terms nonlinear in $\bar{c}_1$ and $\bar{c}_2$). $\left[\overline{A}_{12}\right]$ is only the first term in $\left[A_{12}\right]$, and $\left[\overline{A}_{21}\right]$ is only the first term in $\left[A_{21}\right]$. Thus, the initial values of $|f|$ and $|g|$ are the numerical solutions to basic equations (26) with nonlinear terms omitted. Use these initial values with the initial value of $\overline{D}$ in equations (A5) and solve for $\bar{c}_1$ and $\bar{c}_2$.

Equations (A6) may be solved for $|F|$ and $|G|$ for these initial values of $\bar{c}_1$, $\bar{c}_2$, $\overline{D}$, $|f|$, and $|g|$. If elements in the $|F|$ and $|G|$ are small compared with the largest elements in $|f|$ and $|g|$, then the solution for the initial value of $\overline{D}$ is complete. If not, start again in equations (A5) with a larger value of $\bar{D}$ until the solution for the initial $\bar{D}$ is complete. Then smaller values of $\bar{D}$ can be considered.

A new (smaller) $\bar{D}$ for the next step is obtained by subtracting a prescribed increment from the previous $\bar{D}$. In equations (A5) use the new value of $\bar{D}$ and the new values for $|f|$ and $|g|$ equal to the previous value of $|f|$ plus $|F|$ and the previous value of $|g|$ plus $|G|$, respectively, and solve for new $\bar{c}_1$ and $\bar{c}_2$. To obtain $\bar{c}_1$ and $\bar{c}_2$, it will be necessary to solve a cubic equation and to determine the proper root. Use the new values of $\bar{D}$, $\bar{c}_1$, and $\bar{c}_2$ and the new $|f|$ and $|g|$ in equations (A6) and solve for new $|F|$ and $|G|$. If the elements of $|F|$ and $|G|$ are of permissible size, the solution is now complete for the new $\bar{D}$. With each incremental decrease in $\bar{D}$, this cycle is repeated until a solution is obtained for the $\bar{D}$ desired.

The quantities of interest which may be desired at each value of $\bar{D}$ may be calculated as in the clamped plate problem. In particular, the deflection and the midplane stresses at the center of the shallow spherical shell were calculated by using Simpson's rule for integration. The center deflection is, therefore,

$$\frac{w_o}{aQ^{1/3}} = \frac{2\bar{c}_1}{3S^2}\left\lfloor 4,\ 2,\ 4,\ 2,\ .\ .\ .,\ 4,\ 2,\ 4,\ 1 \right\rceil \begin{bmatrix} f_1 \\ 2f_2 \\ \cdot \\ \cdot \\ \cdot \\ Sf_S \end{bmatrix} \tag{A8}$$

The equation determining the midplane radial stress is

$$\sigma_r = \frac{E}{1 - \mu^2}\left(u' + \frac{\mu}{r}\,u + z_o{}'\beta + \frac{1}{2}\,\beta^2\right)$$

so that at the center σ_r is given by

$$\frac{\sigma_{ro}\left(1 - \mu^2\right)}{EQ^{2/3}} = T + \bar{c}_2 S^2\left\{\left(\frac{3}{2}\right)^{2/3} g_{3/2} - \left(\frac{1}{2}\right)^{2/3} g_{1/2} - \frac{1}{2(2)^{5/3}}\left[\left(\frac{5}{2}\right)^{2/3} g_{5/2} - \left(\frac{3}{2}\right)^{2/3} g_{3/2}\right]\right\} \tag{A9}$$

In figures 5 and 6, these quantities are plotted as a function of $\bar{D}$ for values of T $(0 \leqq T \leqq 1)$ for a flat plate. The center deflections for $T = 0$ and $T = 1$ for a flat plate are compared with corresponding results for a spherical cap in figure 7. Similar comparisons for center midplane stress are shown in figure 8. Comparisons of the deflection distribution for $T = 0$ and $T = 1$ for a flat plate and the corresponding spherical shell are shown in figures 9 and 10, respectively. Similar comparisons of the radial stress distribution are given in figures 11 and 12.

REFERENCES

1. Ames, W. F.: Nonlinear Partial Differential Equations in Engineering. Academic Press, 1965.

2. Kantorovich, L. V.; and Krylov, V. I. (Curtis D. Benster, trans.): Approximate Methods of Higher Analysis. Interscience Publ., Inc., 1958.

3. Duncan, W. J.: Galerkin's Method in Mechanics and Differential Equations. R. & M. No. 1798, British A.R.C., 1937.

4. Timoshenko, S.; and Woinowsky-Krieger, S.: Theory of Plates and Shells. Second ed., McGraw-Hill Book Co., Inc., 1959, p. 402.

5. Reissner, Eric: Rotationally Symmetric Problems in the Theory of Thin Elastic Shells. Proceedings of the Third U.S. National Congress of Applied Mechanics, Am. Soc. Mech. Engrs., c.1958, pp. 51-69.

6. Way, Stewart: Bending of Circular Plates With Large Deflection. Trans. ASME, vol. 56, no. 8, Aug. 1934, pp. 627-636.

7. Hencky, H.: Über den Spannungszustand in kreisrunden Platten mit verschwindender Biegungssteifigkeit. Z. f. Math. u. Phys., vol. 63, 1915, p. 311.

8. Schwerin, E.: Über Spannungen und Formänderungen kreisringförmiger Membranen. Z. Tech. Physik., Bd. 10, Nr. 12, 1929, pp. 651-659.

NASA-Langley, 1966 L-3846

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546