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OPTIMAL CONTROL OF A FLEXIBLE LAUNCH VEHICLE

By: Edmund G. Rynaski, Richard F Whitbeck, Walter W. Wierwille

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FOREWORD

The research documented in this report was performed for the Aero-Astrodynamics Laboratory of the George C. Marshall Space Flight Center, Huntsville, Alabama by the Flight Research Department of the Cornell Aeronautical Laboratory, Inc., of Buffalo, New York. This study was done under NASA Contract NAS8-20067 and was administered for MSFC by Mr. James C. Blair. The work was performed primarily by Mr. Edmund Rynaski (the principal investigator), Dr. Richard Whitbeck, and Dr. Walter Wierwille.

Important suggestions and guidance were provided by Mr. James C. Blair, Mr. Judson A. Lovingood, and Mr. Jerome R. Redus of the Aero-Astrodynamics Laboratory and Mr. W. R. Deazley and Mr. J. M. Schuler of CAL. Mr. Joseph Francis of CAL analyzed the vehicle equations of motion to determine appropriate design parameters for use in the control system studies. Grateful acknowledgement is made to Mr. C. Mesiah for his computing services and Mrs. J. Martino, Miss D. Kantorski and Mrs. J. Cornell for their report preparation skills.

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ABSTRACT

Linear optimal control techniques are used as a synthesis tool to conceptually design control systems for a large, highly flexible launch vehicle. Quadratic performance indices are specified, some of which include a drift minimum model, that yield a realizable optimal control law providing adequate damping of two structural modes as well as satisfactory closed-loop speed of response. These designs are shown to be relatively insensitive to bending mode shapes and slopes. A theory of optimal control is developed for systems possessing parameters whose values are known only on a sample space and an optimal compensation network is designed for a launch vehicle with an uncertain first bending mode slope. In addition, a theory is developed for a design procedure that limits the feedback gains from one or more state variables. An experiment is described that obtains optimal solutions for systems having one or more feedback gains arbitrarily set to preselected values.

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LIST OF SYMBOLS

Vectors

x	state vector whose components define the variables of the first-order set of equations of motion of the plant.
y	the output vector; defined by a transformation on x, y = Hx. Also, the vector whose components define the variables in the equations of booster motion.
z	an alternate state vector, whose components define measurable sensor outputs
u	the control vector; the control input to a plant
u_o	the optimal control vector; the input motion that forces the plant to respond optimally
λ	the adjoint or costate vector; the undetermined multiplier of the Euler-Lagrange equations.
γ	fictitious state vector associated with the model in the performance index; $\dot{\gamma} = L\gamma$
$x(s)$	the Laplace transformed state vector
$y(s)$	the Laplace transformed output vector
$z(s)$	the Laplace transformed vector of measurable quantities. (When used with the Wiener-Hopf equations, $z(s)$ indicates a rational polynomial which is analytic in the left-half plane.)
$u(s)$	the Laplace transformed control input vector
$u_o(s)$	the Laplace transformed optimal control vector
$\lambda(s)$	the Laplace transformed adjoint state vector

Matrices

Matrices are generally denoted by upper case letters. Elements of matrices or 1 x 1 matrices are denoted by lower case letters.

F	n x n matrix; matrix of constants that define the interactions among the state variables of the plant
G	n x p input matrix; matrix of constants defining the effect of a control motion on the rate of change of the state vector

H	r x n matrix transformation on x that defines the output, $y = Hx$.
A, B	n x n matrix of constants defining a measurable state vector; $z = Ay$ or $z = By$
L	the model system matrix; matrix of constants defining the interactions among the model state variables
Q	r x r matrix of constants that weight the output in the performance index
R	p x p matrix of constants weighting the control in the performance index
P	n x n symmetrical positive definite Riccati equation matrix
K	p x n matrix of feedback gains

Determinants and Scalars

$D(s)$	characteristic polynomial of the open-loop launch vehicle, $D(s) = \mid Is - F \mid$
$\Delta(s)$	characteristic polynomial of the optimal closed-loop system, $\Delta(s) = \mid Is - F + GK \mid$
V	the performance index, $2V = \int_0^\infty (y'Qy + u'Ru)\,dt$
$W(s)$	transfer function of the fixed elements of the plant
$\overline{W}$	$\overline{W} = W(-s)$
$W_c(s)$	compensation network
$\mathcal{R}(s)$	Laplace transformation of system excitation
$\mathcal{L}$	The Lagrangian, $\mathcal{L} = \frac{1}{2}(y'Qy + u'Ru) + \lambda'(-\dot{x} + Fx + Gu)$

Notation and Abbreviations

[ ]	denotes a matrix or a vector
$\mid\ \mid$	denotes a determinant
$[\ ]'$	transpose of a vector or matrix
$[\ ]^{-1}$	inverse of a matrix
$[\ ]^{adj}$	adjugate of a matrix, defined by $[A]^{-1} = [A]^{adj} / \mid A \mid$

$[\ ]^{\dagger}$	the generalized inverse of a matrix
$[\ ]_{ij}$	matrix element appearing in the i^{th} row and the j^{th} column
$[\ ]_{i}$	the i^{th} component of a vector
$x(0)$	initial conditions of a vector

Launch Vehicle Symbols

		(units)
ϕ	attitude angle	rad
α	angle of attack	rad
β	control deflection angle	rad
η_i	generalized displacement of the i^{th} mode	m
x_i	body station	m
ℓ_{cg}	distance from engine gimbal to vehicle cg	m
N'	aerodynamic force	kg
I_y	pitch plane moment of inertia about cg	kg-m-sec^2
F	total thrust of the vehicle booster	kg
X	drag force along body axis	kg
m	total mass of the vehicle	kg-sec^2/m
I	inertia of the gimballed engine	kg-m-sec^2
V	velocity	m/sec
R'	thrust of control engines	kg
M_i	generalized mass for the i^{th} mode	kg-sec^2/m
Q_i	generalized force for the i^{th} mode	kg
ω_i	natural frequency of the i^{th} mode	rad/sec
ζ_i	damping ratio of the i^{th} mode	---
$Y_{i(x)}$	normalized displacement of i^{th} mode at station x	---
$Y'_{i(x)}$	normalized slope of the i^{th} mode at station x	1/m

SECTION 1

INTRODUCTION AND SUMMARY

The design of control systems for launch vehicles must account for the effects of structural flexibility and slosh. These effects present more and more significant complications as launch vehicles increase in size and slenderness. The approach of removing flexibility effects by filtering is no longer sufficient, and active control of structural modes is becoming a necessity. Thus, a broader demand is placed on the control system designer.

The use of linear optimal control theory to meet this demand is the subject of this report. The theory has been developed to the point where it offers a systematic approach to complex problems of automatic control system design. The aim of the study was to show that this theory can be used as a practical and effective tool for the design of control systems for flexible launch vehicles.

Linear optimal control theory provides a synthesis procedure for linear systems which determines in any particular case a unique closed-loop characteristic polynomial. A unique linear combination of the state variables is fed back to the control inputs of the plant in accordance with this determination. The theory recognizes the desirability of keeping errors in system outputs small, while at the same time using amplitudes of control motions that are no larger than necessary. This dual objective is expressed in terms of minimizing a performance index, which includes weighted measures of the output motions and the control input motions. The approach can be explained in terms of the problem of returning a system to equilibrium from a disturbance state that is represented by a set of initial conditions. The following quadratic performance index is used

$$2V = \min_{u} \int_0^\infty (y'Qy + u'Ru)\,dt$$

In this expression, y is a vector representing the deviation of the system state from the equilibrium state; u is a vector representing the control input quantities; and Q and R are weighting matrices determining the relative importance attached to minimizing the variations of y and u respectively. The variables y and u are related as follows. The linear constant coefficient equations of motion of the vehicle are put in the form

$$\dot{x} = Fx + Gu$$

where x is a vector defining the state of the system, and F and G are matrices. The output quantities of interest are determined by the equation

$$y = Hx$$

The problem is to find u_0, the time history of u which minimizes the performance index. Then the theory shows that it is possible to generate u_0 from linear combinations of the state variables x. This means that it is possible to find a feedback matrix K such that optimal performance will be obtained if we use the control law

$$u_0 = -Kx.$$

A family of optimal solutions can be found, depending on the choices of the weighting matrices Q and $\mathcal{R}$. A major advantage of this approach is that the theory guarantees that all members of this family will be stable. It is found also that the optimal system configurations generally have smooth, well-behaved responses. The matter of choosing among the optimal solutions is a subject of discussion in the body of this report.

The unique problems associated with the control of a large, flexible launch vehicle make the optimal control approach attractive. To date, most of the launch vehicle control system design techniques have used filtering of one kind or another, such as notch filters, to attenuate the bending mode signals in the feedback control loop. This procedure has the effect of gain- or phase-stabilizing the poles associated with the bending modes. Although the poles are not destabilized, the damping ratios of the bending mode are frequently not much improved over their open-loop values. Wind or gust disturbances can still excite the lightly damped modes of the structure.

As launch vehicles become larger and more slender, the principal bending mode frequencies become lower, approaching the desired closed-loop natural frequency of the poles originally associated with the rigid-body motions. Under these circumstances, it is no longer practical to filter or attenuate the bending modes. Instead, it becomes necessary to exert positive control over the bending motions of the vehicle.

Optimal control techniques yield a positive, direct design procedure for controlling the bending motions of a large, flexible launch vehicle, while at the same time accomplishing the task of controlling the overall rigid-body motions of the vehicle. As noted above, a linear optimal control system is guaranteed stable despite the complexities introduced by the structural modes. More than the assurance of stability is gained by the use of linear optimal control techniques. Inherent in the quadratic nature of the performance index is the tendency to penalize large motions of the error and the control, yet place little penalty on small errors or deviations. In his book, S. S. L. Chang (Reference 4) demonstrates that for the same speed of response, an optimal design requires lower amplitudes of control motion than does a conventionally designed system.

In addition to these advantages, the demands of the quadratic performance index require that closed-loop poles of the system approximate a Butterworth filter distribution. The significance of this is that the optimal designs provide substantial increases in the damping ratios of the bending modes. The elastic motions of the vehicle are blended with the rigid-body motions in such a way that the net response of the vehicle tends to be smooth and well-behaved to either a command input or other excitations.

The application of linear optimal design techniques can be divided into two parts:

1) The selection of the quadratic performance index. It can be shown that the selection of the performance index can be interpreted as a selection of the closed-loop poles of the system in a logical, satisfactory manner.

2) Determination of the feedback control law that will yield the desired closed-loop pole configuration.

The optimal design theory approaches a dynamic control problem in a reverse sense from that of conventional design procedures. In the optimal approach, the closed-loop poles are selected first and the control configuration is then determined. In a conventional design procedure, the closed-loop control configuration is first selected, and then the closed-loop poles that result from this selection are obtained.

Summary

The work presented in this report was motivated by the knowledge that use of a quadratic performance index can satisfy many of the control system criteria important to the elastic launch vehicle problem. In addition, it was felt that the analysis tools developed in Reference 2 would enable a designer to systematically specify a performance index that would yield a satisfactory control system design.

This report describes two efficient techniques for optimal launch vehicle control system analysis. But, as might be expected, the problems associated with the design are primarily ones of synthesis, not analysis. The control law must be expressed in terms of measurable quantities, with feedback gain magnitudes that do not exceed limits set by previous experience in launch vehicle control system designs. The problem is further complicated by the fact that the elastic properties of large launch vehicles are not known to a high degree of accuracy, requiring a design that is insensitive to parameter variations of the elastic characteristics of the vehicle.

The work described in this report systematically investigates the problem areas of elastic launch vehicle control system synthesis outlined above. The following section briefly outlines linear optimal control theory in order to acquaint the reader with the techniques that are subsequently used in later sections of the report. More complete descriptions of the theory are given in References 2, 3, and 4.

The third section describes a method for selecting quadratic performance indices in a way that will yield satisfactory closed-loop dynamics. The aim is to provide a suitably fast closed-loop system response and at the same time increase the damping ratio of the bending modes. The control law is synthesized in terms of measurable quantities.

In the fourth section, a drift minimum model is used in conjunction with optimal control techniques to specify a control system that approximates the drift minimum criterion in an error squared sense and within the allowable control motions of a single control variable. Direct control law synthesis procedures are demonstrated and the optimal control is computed for several sets of state variables. A brief investigation was made of the sensitivity of the optimal system to variations of the bending mode slopes and shapes and preliminary results indicated that the optimal system was remarkably insensitive to changes in these parameters. The state vector was approximated in a least squared sense by three measurements of the system dynamics. Finally, it is shown that higher order structural modes which are not accounted for in the optimal control design may be destabilized. There are several practical approaches to the solution of this problem.

These results led to a renewed interest in a study of sensitivity and parameter variations, considered in Section 5 of this report. This section describes the theory and practice of optimal design of systems subject to parameters whose values are known only as random variables described on a sample space. A method of optimal compensation is developed in the frequency domain and several examples are used to expose the basic analytical difficulties which are evolved. The resultant theory is then applied to the design of a compensating network required for a flexible booster when the value for the slope of the first bending mode is uncertain.

Section 6 describes a study in which an attempt was made to develop a design procedure that limits the feedback gains of one or more state variables yet still satisfies a quadratic performance index criterion. A theory is developed for this difficult problem and the results show that a direct application of the theory requires the solution of a complex set of nonlinear algebraic equations. An analog computer program was run to obtain optimal solutions experimentally for systems having one or more feedback gains arbitrarily set to preselected values. The approach was to start with an optimal solution obtained without gain constraints. Then those gains that were beyond allowable limits were gradually reduced, while systematically adjusting the other gains to keep the performance index as low as possible. The cases tried indicated that this procedure is workable and that satisfactory closed-loop system designs may be obtained with this approach.

The report ends with a section on conclusions and recommendations. The linear optimal control theory is shown to be applicable to the design of launch vehicle control systems. Practical problems of system design were considered in the study, such as choice of sensors and their locations, sensitivity to parameter uncertainty, and limitations on usable gains. In the situations studied, these problems were found to be entirely manageable. It was felt that designs of optimal control systems were obtained which met practical requirements and which could be mechanized. A systematic approach was developed for dealing with uncertainties in the values of system parameters. A method was also found for imposing constraints on some of the feedback gains and finding optimal values for the system gains within these constraints.

In summary, the study reported here has shown that linear optimal control techniques can be used as effective tools in the design of control systems for flexible launch vehicles.

SECTION 2
A BRIEF OUTLINE OF LINEAR OPTIMAL CONTROL THEORY

The optimal control problem can be briefly stated as follows: starting with some initial condition of the plant, the problem is to find a time history of control motion that forces the plant to dynamically respond in a manner that minimizes an integral function of the control vector and the state vector. If the plant is linear and the integral contains quadratic scalar functions of the state variables and control variables, the control motion can be generated as a linear combination of the state variables, and the result is a linear feedback control law. This closed-loop system then responds optimally to any initial condition of the plant. If the plant is describable by a set of constant-coefficient linear differential equations of motion and the integral is taken to infinity, the optimal feedback control law is composed of a set of constant feedback gains from the state variables of the plant.

For this problem, the differential equations of motion are assumed to have constant coefficients and so can be written as a matrix set of first-order equations of the form

$$\dot{x} = Fx + Gu \qquad y = Hx \tag{2-1}$$

where x = the variables of the differential equations of motion

u = the control vector

y = the output; a transformed set on the state whose motions are to be minimized

F = an n x n matrix of constants describing the coupling among state variables in the equations of motion

G = an n x p matrix of constants describing the effect of control inputs on the equations of motion.

A control motion u_0 is to be found that minimizes the quadratic performance index

$$2V = \int_0^\infty (y'Qy + u'Ru)\,dt \tag{2-2}$$

where Q = an r x r positive definite symmetric matrix whose elements weight the contributions of each output in the integral

R = a p x p positive definite symmetric matrix whose elements weight the contributions of each control motion in the integral.

There are several ways to obtain the optimal control motion u_0 that minimizes the performance index. In the time domain, the calculus of variations may be used to obtain a solution (see References 11 and 2). If this approach is taken, the Euler-Lagrange partial differential equations of motion must be satisfied.

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0 \qquad (2\text{-}3)$$

where $\mathcal{L}$ is the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(y'Qy + u'Ru) + \lambda'(-\dot{x} + Fx + Gu)$$

and λ is an undetermined vector called the adjoint state vector or the costate.

The Euler-Lagrange equations, together with the original equations of motion are

$$\begin{aligned} \dot{x} - Fx - Gu &= 0 && \text{(a)} \\ Ru_0 + G'\lambda &= 0 && \text{(b)} \\ H'QHx + \dot{\lambda} + F'\lambda &= 0 && \text{(c)} \end{aligned} \qquad (2\text{-}4)$$

From Equation 2-4b the optimal control law is $u_0 = -R^{-1}G'\lambda$.

If the substitution $\lambda = Px$, $\dot{\lambda} = P\dot{x}$ is made, Equations 2-4 can be reduced to the single quadratic Riccati equation in P which was described by Kalman (Reference 3) and others.

$$0 = PF + F'P - PGR^{-1}G'P + H'QH \qquad (2\text{-}5)$$

The feedback gains are determined from the positive definite symmetric solution of the Riccati equation.

Now consider the Laplace transform of Equations 2-4:

$$\begin{bmatrix} Is-F & -G & 0 \\ 0 & R & G' \\ -H'QH & 0 & -Is-F' \end{bmatrix} \begin{bmatrix} x(s) \\ u_0(s) \\ \lambda(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ 0 \\ -\lambda(0) \end{bmatrix} \qquad (2\text{-}6)$$

The determinant of Equation 2-6, when set to zero, yields an expression for the closed-loop roots of the optimal system with state variable x and "adjoint" system with adjoint variable λ. The roots of this characteristic equation can be determined from the root square locus expression

$$\left| I + R^{-1}G'\left[-Is - F'\right]^{-1}H'QH\left[Is - F\right]^{-1}G \right| = 0 \qquad (2\text{-}7)$$

which can be derived from the determinant of Equation 2-6 (References 2 and 5), or from a frequency domain formulation of the problem (References 2, 9). $H[Is-F]^{-1}G$ is a matrix of transfer functions $y_i/u_j\ (s)$ relating the outputs of the system to the inputs and $G'[-Is-F']^{-1}H'$ is the transpose of the transfer function matrix with s replaced by -s.

The linear optimal control problem can also be solved entirely in the frequency domain. Parseval's theorem is applied to the performance index, Equation 2-2:

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (y_* Q y + u_* R u)\, ds \tag{2-8}$$

where

$$\begin{aligned} y = y(s) &= H[Is-F]^{-1}Gu + H[Is-F]^{-1}x(0) \\ &= Wu + Bx(0) \\ y_* = y'(-s) &= u_* W_* + x(0) B_* \\ u_* &= u'(-s) \\ W_* &= G'[-Is-F']^{-1}H' \end{aligned}$$

It is desired to take a variation to find the control that minimizes 2-8. To do this, substitute for y and y_* in Equation 2-8 and let

$$u = u_0 + \lambda u_1$$

It will be found that the requirement for minimization (References 2 and 9) is that

$$J_C = \frac{\lambda}{2\pi j}\int_{-j\infty}^{j\infty} u_*\left[W_* Q W u_0 + W_* Q B x(0) + R u_0\right] ds = 0$$

or

$$\left[R + W_* Q W\right] u_0 + W_* Q B x(0) = z(s) \tag{2-9}$$

where $z(s)$ must be analytic in the left-half plane.* Equation 2-9 is a matrix equation of the Wiener-Hopf type that must be satisfied to obtain the optimal control vector. The solution to Equation 2-9 is given by

$$u_0 = -Y^{-1}\left[Y_*^{-1} J\right]_+ \tag{2-10}$$

where

$$Y Y_* = \left[R + W_* Q W\right]$$

and $Y_*^{-1}\left[W_* Q B x(0)\right]$ has been decomposed by a partial fraction expansion into the sum

$$Y_*^{-1}\left[W_* Q B x(0)\right] = \left[Y_*^{-1} J\right]_+ + \left[Y_*^{-1} J\right]_- \tag{2-11}$$

---

*Appendix B describes this requirement when the open-loop plant is nonminimum phase.

It can be seen that

$$|R + W_* Q W| = 0 \tag{2-12}$$

yields the same expression for the root square locus as Equation 2-7.

A complete method of analysis and synthesis is therefore obtained. The selection of a performance index that yields a satisfactory closed-loop response is possible through the use of the root square locus expression. The feedback control law can be obtained by solving the Riccati equation or by satisfying the matrix Wiener-Hopf equation. References 2 and 5 describe the solution through the use of the Riccati equation, while References 2 and 9 describe the proper technique for solving linear optimal control problems using the matrix Wiener-Hopf equation.

When the plant has only one control input, the synthesis problem can be solved in a direct, straightforward manner. The plant is described by the set of first-order linear equations of motion

$$\dot{x} = Fx + Gu$$

and the optimal feedback control law is of the form

$$u_0 = -Kx \tag{2-13}$$

The equations of motion of the optimal regulator are given by

$$\dot{x} = (F - GK)x \tag{2-14}$$

whose closed-loop characteristic polynomial is

$$\Delta(s) = |Is - F + GK| \tag{2-15}$$

The closed-loop characteristic polynomial can be reconstructed for any values of weighting in the performance index from the root square locus plot. Equating powers of s of the characteristic polynomial obtained from the root square locus plot to the scalar expression of Equation 2-15 yields a linear algebraic set of equations in the k_i feedback gains. This set of linear algebraic equations can be simply solved by machine in a straightforward manner. Examples of optimal analysis using the root square locus plot and synthesis using the method described above are given in the next two sections. In addition to the examples in this report, Reference 2 contains numerous examples designed to acquaint the engineer with the required mathematics.

SECTION 3
SELECTION OF THE PERFORMANCE INDEX

Introduction

The selection of the parameters of the performance index is not automatic. The variable or variables that are included in the performance index will have an optimal response that tends towards the response of a Butterworth filter as these variables are weighted more and more heavily with respect to the control. The implications of this statement are significant. It means, for instance, that if the performance index contained only a bending mode variable, say η_1 , then η_1 would tend to respond as a second-order system with a damping ratio of approximately $\zeta = 0.7$. The responses of the other variables, however, are almost certain to be unacceptable. The approach, then, will be to include a measure of both a rigid-body variable and bending mode variables in the performance index. This approach will permit a rapid and stable response of the rigid-body contribution to the dynamics and acceptable damping of the bending mode contribution to the closed-loop response.

One Bending Mode

Although it is clear that a measure of both the rigid-body mode and the body bending modes will be included in the performance index description, the mix of the two variables should be investigated. Thus the attitude angle and first bending mode variable may be included in the performance index in the following ways:

$$2V = \min_{\beta_c} \int_0^\infty \left[q(\phi_R + k\eta_1)^2 + r\beta_c^2 \right] dt \qquad (3\text{-}1)$$

or

$$2V = \min_{\beta_c} \int_0^\infty \left[q_1 \phi_R^2 + q_2 \eta_1^2 + r\beta_c^2 \right] dt \qquad (3\text{-}2)$$

The analysis of the first performance index, Equation 3-1, is a relatively simple matter, involving only a single root square locus plot with the parameter q/r . A preliminary analysis of a conventional root locus plot of the function

$$k \frac{\phi_R}{\beta_c}(s) + \frac{\eta_1}{\beta_c}(s) = 0 \qquad (3\text{-}3)$$

will yield the zeros of the function that will appear in the root square locus plot. In general, the roots of Equation 3-3 are chosen to obtain an adequate separation between the poles and zeros of the function of Equation 3-3. This will guarantee that adequate damping of the bending mode will be achievable for some value of the performance index parameter q/r.

The analysis of the performance index of Equation 3-2 involves two root square locus plots instead of one conventional root locus plot and one root square locus. As before, the first locus serves to define the zeros for the second root square locus plot. The root square locus expression for the performance index of Equation 3-2 is given by

$$1 + \frac{q_1}{r}\frac{\phi_R}{\beta_c}(s)\frac{\phi_R}{\beta_c}(-s)\left[1 + \frac{q_2}{q_1}\frac{\frac{\eta_1}{\beta_c}(s)\frac{\eta_1}{\beta_c}(-s)}{\frac{\phi_R}{\beta_c}(s)\frac{\phi_R}{\beta_c}(-s)}\right] = 0 \qquad (3\text{-}4)$$

where the term in brackets

$$1 + \frac{q_2}{q_1}\frac{\frac{\eta_1}{\beta_c}(s)\frac{\eta_1}{\beta_c}(-s)}{\frac{\phi_R}{\beta_c}(s)\frac{\phi_R}{\beta_c}(-s)} = 0 \qquad (3\text{-}5)$$

defines the zeros, or terminal points for the locus of the closed-loop poles of the optimal system. Because of the quadratic nature of Equation 3-5, the locus of the zeros of Equation 3-5 will generally have a higher damping ratio for any value of the parameter q_2 / q_1 than will the zeros defined by Equation 3-3 for any value of the parameter k.

To illustrate this process of the selection of the terminal points of the root square locus process, it was decided to use the performance index of Equation 3-1 for the analysis of an optimal system that contains only one bending mode and the performance index of Equation 3-2 for the two bending mode analysis and synthesis problem of this section of the report.

Single Bending Mode Analysis

The elastic booster equations of motion including a single bending mode are (see Appendix A):

$$\begin{aligned} &\ddot{\phi}_R - .0733\alpha + 0.45\beta = 0 \\ &-\dot{\phi}_R + .0405\,\phi_R + \dot{\alpha} + .01067\alpha + .0211\beta = 0 \\ &-5.453\alpha + \ddot{\eta}_1 + .0232\dot{\eta}_1 + 5.37\eta_1 - 15.83\beta = 0 \\ &\dot{\beta} + 17.9\beta = 17.9\beta_c \end{aligned} \qquad (3\text{-}6)$$

or, in matrix form

$$\begin{bmatrix} \dot{\phi}_R \\ \ddot{\phi}_R \\ \dot{\alpha} \\ \dot{\eta}_1 \\ \ddot{\eta}_1 \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0733 & 0 & 0 & -.45 \\ -.0405 & 1.0 & -.0107 & 0 & 0 & -.0211 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 5.453 & -5.37 & -.0232 & 15.83 \\ 0 & 0 & 0 & 0 & 0 & -17.9 \end{bmatrix} \begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha \\ \eta_1 \\ \dot{\eta}_1 \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 17.9 \end{bmatrix} \beta_c \qquad (3\text{-}7)$$

The design will involve the analysis of the performance index

$$2V = \min_{\beta_c} \int_0^{\infty} \left[q(\eta_1 + k\,\phi_R)^2 + r\beta_c{}^2 \right] dt \tag{3-8}$$

It has been shown in Section 2 that the closed-loop roots of the optimal system and adjoint are given by the root square locus expression

$$\left| I + R^{-1} G' \left[-Is - F' \right]^{-1} H' Q H \left[Is - F \right]^{-1} G \right| = 0 \tag{3-9}$$

where, for this example

$$H = \begin{bmatrix} k & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} q \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} \frac{1}{r} \end{bmatrix}$$

$$H \left[Is - F \right]^{-1} G = \left[\frac{\eta_1}{\beta_c}(s) + k\,\frac{\phi_R}{\beta_c}(s) \right] \qquad G'\left[-Is - F' \right]^{-1} H' = \left[\frac{\eta_1}{\beta_c}(-s) + k\,\frac{\phi_R}{\beta_c}(-s) \right]$$

A preliminary root locus plot of $\frac{\eta_1}{\beta_c}(s) + k\,\frac{\phi_R}{\beta_c}(s) = 0$ was performed to select a suitable value for k. Since this locus defines the zeros of the root square locus plot, the selection of k was made on the basis that the roots of the closed-loop optimal system would be such to yield a sufficiently rapid booster response and increase the damping ratio of the bending mode. A value of k = 30 produced the desired result. This locus is not shown because its construction is straightforward and relatively simple.

Substituting the transfer functions from Appendix A,

$$P(s) = \frac{\eta_1}{\beta_c}(s) + 30\,\frac{\phi_R}{\beta_c}(s)$$

$$= \frac{9.18\left(1 - \frac{s}{.0408}\right)\left(1 + \frac{s}{.4985}\right)\left(1 - \frac{s}{.454}\right) + 30(-2.14)\left(1 + \frac{s}{.0141}\right)\left[1 + \frac{2(.005)}{2.32}s + \left(\frac{s}{2.32}\right)^2\right]}{\left(1 - \frac{s}{.0418}\right)\left(1 - \frac{s}{.242}\right)\left(1 + \frac{s}{.294}\right)\left(1 + \frac{s}{17.9}\right)\left[1 + \frac{2(.005)}{2.317}s + \left(\frac{s}{2.317}\right)^2\right]}$$

$$= \frac{55.02\left(1 + \frac{s}{.0115}\right)\left(1 - \frac{s}{2.29}\right)\left(1 + \frac{s}{2.24}\right)}{\left(1 - \frac{s}{.04175}\right)\left(1 - \frac{s}{.242}\right)\left(1 + \frac{s}{.294}\right)\left(1 + \frac{s}{17.9}\right)\left[1 + \frac{2(.005)}{2.317}s + \left(\frac{s}{2.317}\right)^2\right]} \tag{3-10}$$

Substituting P(s) and P(-s) into the root square locus expression (Equation 3-9) yields

$$0 = 1 + \frac{q}{r}\;\frac{3027\left(1 \pm \frac{s}{.0115}\right)\left(1 \pm \frac{s}{2.29}\right)\left(1 \pm \frac{s}{2.24}\right)}{\left(1 \pm \frac{s}{.0418}\right)\left(1 \pm \frac{s}{.242}\right)\left(1 \pm \frac{s}{.294}\right)\left(1 \pm \frac{s}{17.9}\right)\left[1 \pm \frac{2(.005)}{2.317}s + \left(\frac{s}{2.317}\right)^2\right]} \tag{3-11}$$

One half of the solutions to Equation 3-11 are plotted in Figure 3.1. The stable, optimal solutions are shown and the unstable, adjoint system solutions are omitted for simplicity and clarity. The parameter of the locus is q/r, the ratio of the weighting of the state variables and the control variables in the performance index. The locus then shows the closed-loop poles of the optimal system as a function of the ratio of the weighting of the two parts of the performance index. For purposes of illustration, a value of q/r = 0.01 was chosen as a candidate for a weighting of the performance index factors yielding satisfactory closed-loop optimal system dynamic characteristics.

From the root square locus plot, a value of q/r = 0.01 yields the closed-loop characteristic polynomial

$$\Delta(s) = (s+17.9)(s+.0117)\left[s^2+2(.81)(1.13)s+(1.13)^2\right]\left[s^2+2(.12)(2.42)s+(2.42)^2\right] \qquad (3\text{-}12)$$

Equation 3-12 shows that the damping ratio of the first bending mode has been increased from an open-loop value of $\zeta = .005$ to $\zeta = .12$ in the closed-loop design. At the same time, the rigid-body contribution to the response contains a predominant pair of poles at $\omega_n = 1.13$ rad/sec, $\zeta = 0.81$. The attitude response of the vehicle will therefore be smooth, with little overshoot.

The optimal control law that yields the closed-loop characteristic equation 3-11 and satisfies the performance index

$$2V = \min_{\beta_c} \int_0^\infty \left[0.01(30\phi_R + \eta_1)^2 + \beta_c^{\,2}\right] dt \qquad (3\text{-}13)$$

can be obtained by solving the Wiener-Hopf equation, 2-9, or the matrix Riccati equation, 2-5, or by a direct method that is described in the next section. For this problem, the feedback control law is found to be

$$\beta_c = 3.00\,\phi_R + 4.98\,\dot{\phi}_R + 4.45\alpha - .0827\eta_1 - .0231\dot{\eta}_1 - 0.135\beta \qquad (3\text{-}14)$$

The magnitudes of the feedback gains are generally within the guidelines outlined by the Marshall Space Flight Center of NASA, and can be mechanized in terms of measurable quantities. However, it is clear that the quantities contained in the optimal control law of Equation 3-14 do not generally represent measurable quantities. It would be more realistic to express the feedback control law in terms of the outputs of sensors located on the vehicle. Because this particular problem, which includes only one bending mode, has been presented for illustrative purposes only, the feedback control law mechanization aspects are not presented. The remainder of Section 3 and all of Section 4, which includes a description of the vehicle containing two bending modes, does consider control law synthesis problems in detail.

Selection of Performance Index for Two Bending Modes

The addition of a second bending mode to the description of the launch vehicle adds complexity to the problem because now a measure of the second

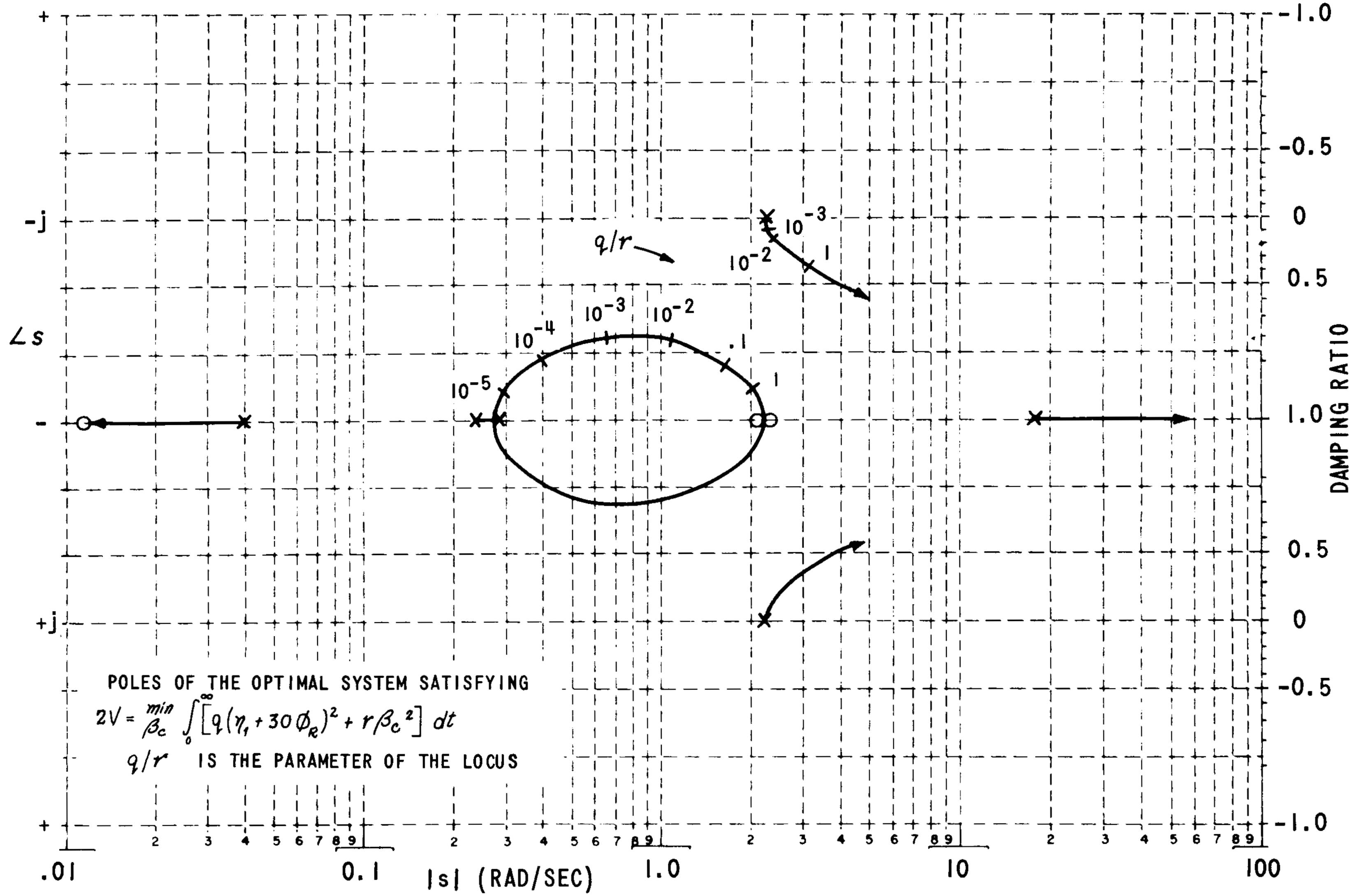


Figure 3.1 Locus of Poles of the Optimal System

mode dynamics must be included in the performance index. The equations of motion for this problem are (see Appendix A)

$$\begin{bmatrix} \dot{\phi}_R \\ \ddot{\phi}_R \\ \dot{\alpha} \\ \dot{\eta}_1 \\ \ddot{\eta}_1 \\ \dot{\eta}_2 \\ \ddot{\eta}_2 \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0733 & 0 & 0 & 0 & 0 & -.45 \\ -.0405 & 1 & -.0107 & 0 & 0 & 0 & 0 & -.0211 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5.45 & -5.37 & -.0563 & 0 & 0 & 15.83 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2.36 & 0 & 0 & -31.8 & -.0564 & 22.77 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -17.9 \end{bmatrix} \begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha \\ \eta_1 \\ \dot{\eta}_1 \\ \eta_2 \\ \dot{\eta}_2 \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 17.9 \end{bmatrix} \beta_c \qquad (3\text{-}15)$$

A performance index is chosen of the form

$$2V = \min_{\beta_c} \int_0^{\infty} \left(q_1 \phi_R^2 + q_3 \eta_1^{\,2} + q_2 \eta_2^{\,2} + r\beta_c^{\,2} \right) dt \qquad (3\text{-}16)$$

requiring the selection of three weighting parameters of the performance index. The analysis of this problem can be carried out using the root square locus expression

$$\left| I + R^{-1} G' \left[-Is - F' \right]^{-1} H' Q H \left[Is - F \right]^{-1} G \right| = 0 \qquad (3\text{-}17)$$

where for this problem

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_3 & 0 \\ 0 & 0 & q_2 \end{bmatrix} \qquad R^{-1} = \frac{1}{r} \qquad (3\text{-}18)$$

The expression $H \left[Is - F \right]^{-1} G$ is a column matrix of transfer functions of those variables appearing in the performance index.

$$H\left[Is - F\right]^{-1} G = \begin{bmatrix} \dfrac{\phi_R}{\beta_c}(s) \\ \dfrac{\eta_1}{\beta_c}(s) \\ \dfrac{\eta_2}{\beta_c}(s) \end{bmatrix} \quad \text{and} \quad G'\left[-Is - F'\right]^{-1} H' = \begin{bmatrix} \dfrac{\phi_R}{\beta_c}(-s) & \dfrac{\eta_1}{\beta_c}(-s) & \dfrac{\eta_2}{\beta_c}(-s) \end{bmatrix} \qquad (3\text{-}19)$$

Substituting into Equation 3-17,

$$-1 = \frac{1}{r} \begin{bmatrix} \frac{\phi_R}{\beta_c}(-s) & \frac{\eta_1}{\beta_c}(-s) & \frac{\eta_2}{\beta_c}(-s) \end{bmatrix} \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_3 & 0 \\ 0 & 0 & q_2 \end{bmatrix} \begin{bmatrix} \frac{\phi_R}{\beta_c}(s) \\ \frac{\eta_1}{\beta_c}(s) \\ \frac{\eta_2}{\beta_c}(s) \end{bmatrix} \tag{3-20}$$

or

$$0 = 1 + \frac{q_1}{r} \frac{\phi_R}{\beta_c}(s) \frac{\phi_R}{\beta_c}(-s) + \frac{q_2}{r} \frac{\eta_2}{\beta_c}(s) \frac{\eta_2}{\beta_c}(-s) + \frac{q_3}{r} \frac{\eta_1}{\beta_c}(s) \frac{\eta_1}{\beta_c}(-s) \tag{3-21}$$

Equation 3-21 can be written

$$0 = 1 + \frac{q_1}{r} \frac{\phi_R}{\beta_c}(s) \frac{\phi_R}{\beta_c}(-s) \left\{ 1 + \frac{q_2}{q_1} \frac{\frac{\eta_2}{\beta_c}(s) \frac{\eta_2}{\beta_c}(-s)}{\frac{\phi_R}{\beta_c}(s) \frac{\phi_R}{\beta_c}(-s)} \left[1 + \frac{q_3}{q_2} \frac{\frac{\eta_1}{\beta_c}(s) \frac{\eta_1}{\beta_c}(-s)}{\frac{\eta_2}{\beta_c}(s) \frac{\eta_2}{\beta_c}(-s)} \right] \right\} \tag{3-22}$$

Equation 3-21 shows that the closed-loop poles of the optimal system and its adjoint are a function of three parameters q_1/r, q_2/r, and q_3/r. Equation 3-22 shows that three separate root square loci are involved in the analysis of the closed-loop system and the selection of the values of q_1/r, q_2/r, and q_3/r.

Defining

$$\frac{\eta_1}{\beta_c}(s) \frac{\eta_1}{\beta_c}(-s) = K_{\eta_1}^2 \frac{N_{\eta_1} \bar{N}_{\eta_1}}{D\bar{D}}$$

$$\frac{\eta_2}{\beta_c}(s) \frac{\eta_2}{\beta_c}(-s) = K_{\eta_2}^2 \frac{N_{\eta_2} \bar{N}_{\eta_2}}{D\bar{D}} \tag{3-23}$$

$$\frac{\phi_R}{\beta_c}(s) \frac{\phi_R}{\beta_c}(-s) = K_{\phi}^2 \frac{N_{\phi} \bar{N}_{\phi}}{D\bar{D}}$$

then the first of the three root square loci can be written

$$0 = 1 + \frac{q_3}{q_2} \frac{K_{\eta_1}^2}{K_{\eta_2}^2} \frac{N_{\eta_1} \bar{N}_{\eta_1}}{N_{\eta_2} \bar{N}_{\eta_2}} \tag{3-24}$$

Substituting the numerical values for the booster transfer functions, Equation 3-24 becomes

$$0 = 1 + \frac{q_3}{q_2} \frac{(9.18)^2}{(1.17)^2} \frac{\left(1 \pm \frac{s}{.0408}\right)\left(1 \pm \frac{s}{.454}\right)\left(1 \pm \frac{s}{.499}\right)\left[1 \pm \frac{2(.005)}{5.639} s + \left(\frac{s}{5.639}\right)^2\right]}{\left(1 \pm \frac{s}{.0412}\right)\left(1 \pm \frac{s}{.319}\right)\left(1 \pm \frac{s}{.369}\right)\left[1 \pm \frac{2(.005)}{2.317} s + \left(\frac{s}{2.317}\right)^2\right]} \tag{3-25}$$

One half of the root square locus of Equation 3-25 is plotted in Figure 3.2. The stable, optimal system part of the locus is shown, and the unstable, adjoint part is omitted. The locus in itself is difficult to interpret precisely, but a few general comments can be made. This locus and the one to follow will define the zeros, or terminal locations for the closed-loop poles of the optimal system. Therefore, in general it can be said that the aim of the locus of Figure 3.2 will be to select roots that have an adequate damping ratio. This will guarantee that both the first and second bending modes will have adequate damping ratios in the closed-loop design. The locus does indicate that if good first mode damping is to be achieved, q_3/q_2 should not have too small a value, while too large a value for q_3/q_2 will result in an insufficient second mode damping ratio. A selection of q_3/q_2 in the range $.2 \leq q_3/q_2 \leq 2$ will likely serve the purpose of obtaining good damping ratios for both the first and second modes.

The value $q_3/q_2 = 0.5$ was chosen as a logical value, yielding a maximum damping ratio for the roots of Figure 3.2. With a value for q_3/q_2 chosen, the second root square locus expression to be formulated is

$$0 = 1 + \frac{q_2}{q_1} \frac{\left[K_{\eta_2}^2 + \frac{q_3}{q_2} K_{\eta_1}^2\right]}{K_\phi^2} = \frac{(\text{Roots from Figure 3.2 with } q_3/q_2 = .05)}{N_\phi \bar{N}_\phi} \quad (3\text{-}26)$$

Substituting the appropriate dynamic quantities from the transfer functions and from the first root square locus expression, the second root square locus expression becomes

$$0 = 1 + 9.52 \frac{q_2}{q_1} \frac{\left[1 \pm \frac{2(.39)}{3.92} s + \left(\frac{s}{3.92}\right)^2\right]\left[1 \pm \frac{2(.98)}{.47} s + \left(\frac{s}{.47}\right)^2\right]}{\left(1 \pm \frac{s}{.014}\right)\left[1 \pm \frac{2(.005)}{2.317} s + \left(\frac{s}{2.317}\right)^2\right]\left[1 \pm \frac{2(.005)}{5.639} s + \left(\frac{s}{5.639}\right)^2\right]} \quad (3\text{-}27)$$

As in Figure 3.2, one half of the root square locus expression of Equation 3-27 is plotted in Figure 3.3. This locus defines the end points or zeros of the root square locus for the closed-loop poles of the optimal system. An inspection of the plot shows that a value of $q_2/q_1 = 4$ allows for good damping of both the first and second bending modes.

The third root square locus expression can then be formulated as

$$0 = 1 + \frac{q_1}{r} \frac{\left[K_\phi^2 + \frac{q_2}{q_1} K_{\eta_2}^2 + \frac{q_3}{q_2} K_{\eta_1}^2\right] (\text{Roots from Figure 3.3 with } q_2/q_1 = 4)}{D\bar{D}} \quad (3\text{-}28)$$

where D is the open-loop characteristic polynomial of the elastic vehicle and $\bar{D}$ is the open-loop characteristic polynomial with s replaced by -s.

Substituting the proper expressions from the second root square locus plot and from the transfer functions yields

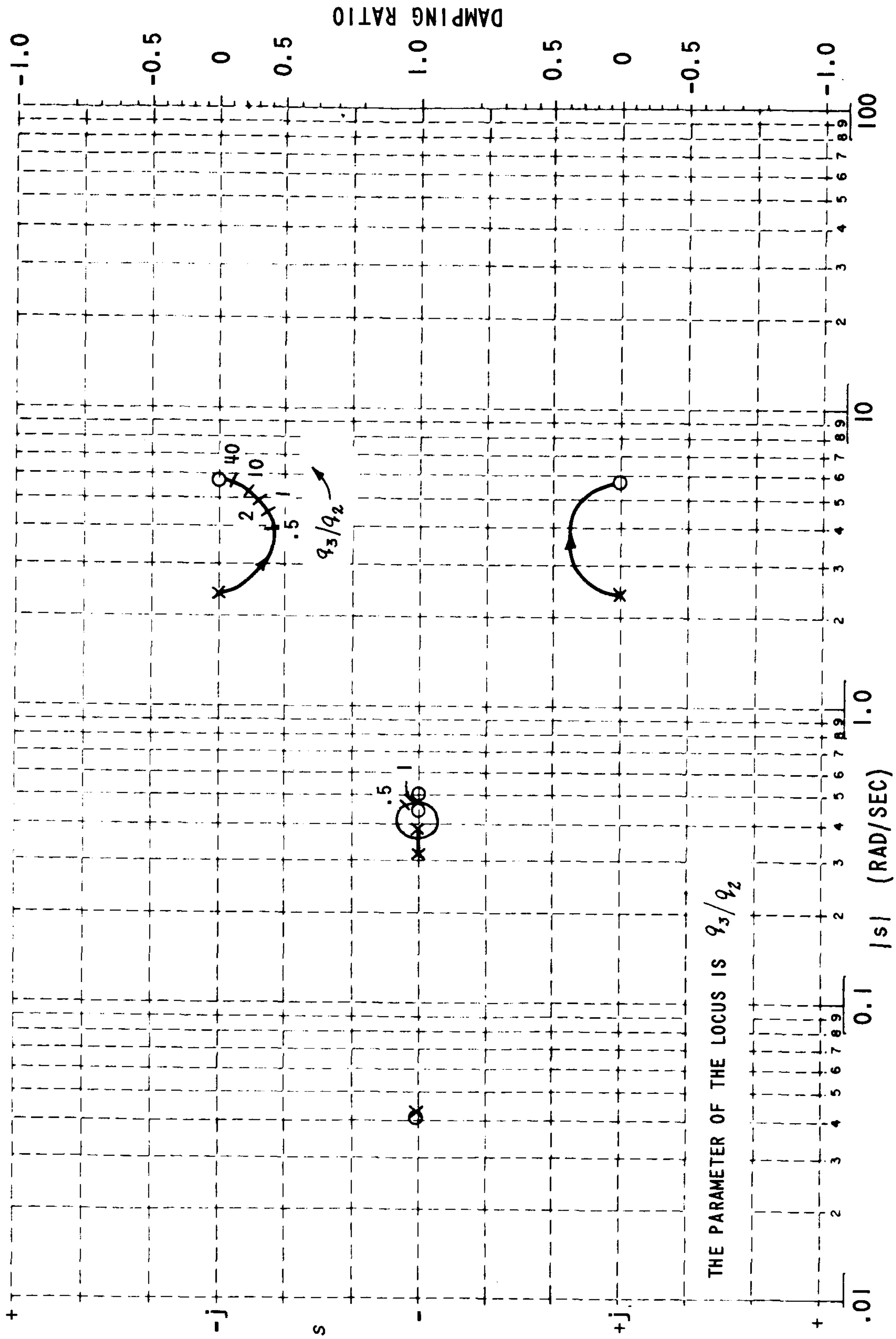


Figure 3.2 Root Square Locus Plot of Equation 3-25

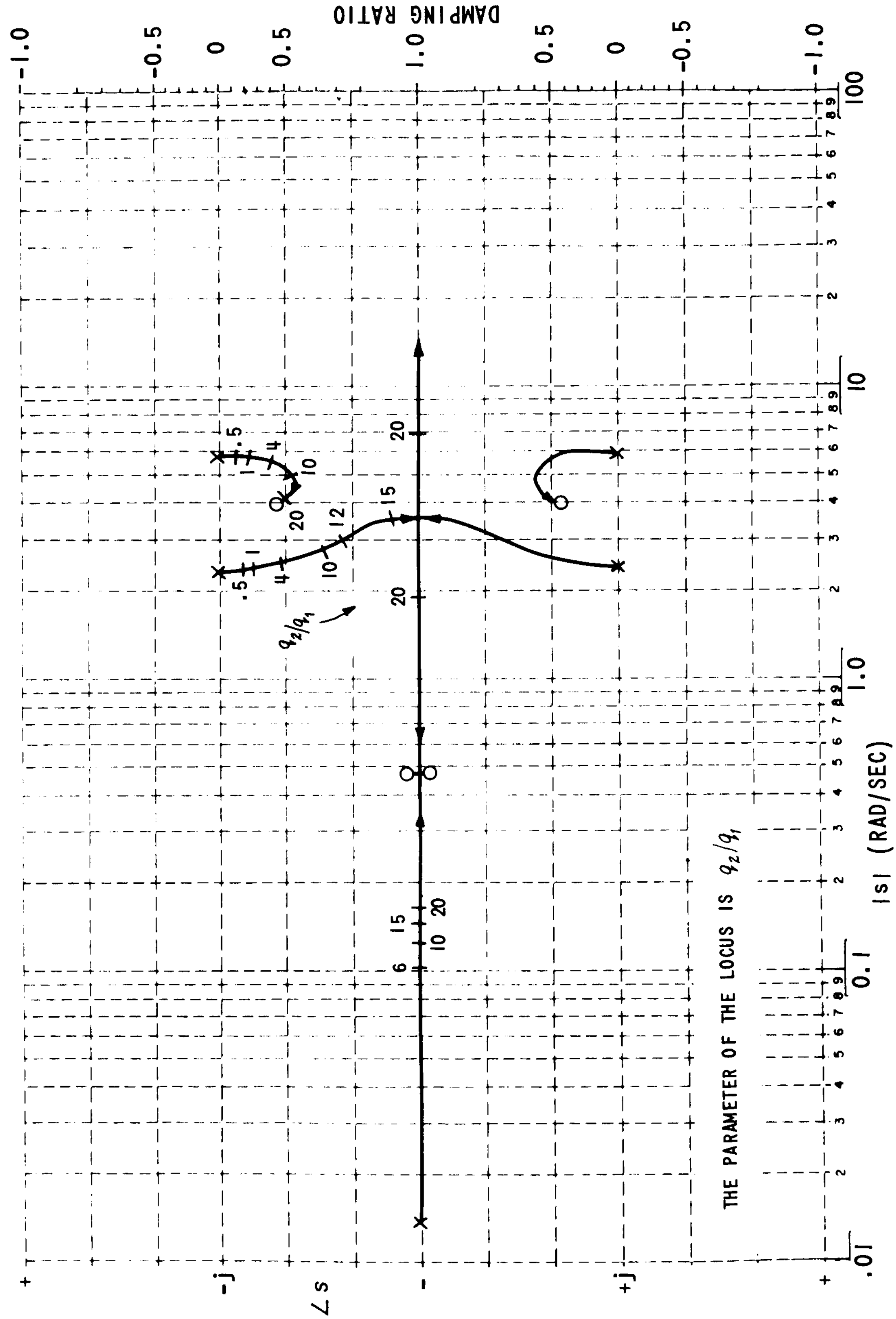


Figure 3.3 Root Square Locus Plot of Equation 3-27

$$0 = 1 + 52.2\,\frac{q_1}{r}\,\frac{\left(1 \pm \frac{s}{.0825}\right)\left[1 \pm \frac{2(.48)}{2.47}s + \left(\frac{s}{2.47}\right)^2\right]\left[1 \pm \frac{2(.38)}{5.6}s + \left(\frac{s}{5.6}\right)^2\right]}{\left(1 \pm \frac{s}{17.9}\right)\left(1 \pm \frac{s}{.294}\right)\left(1 \pm \frac{s}{.242}\right)\left(1 \pm \frac{s}{.0418}\right)\left[1 \pm \frac{2(.005)}{2.317}s + \left(\frac{s}{2.317}\right)^2\right]\left[1 \pm \frac{2(.005)}{5.64}s + \left(\frac{s}{5.64}\right)^2\right]} \tag{3-29}$$

The root square locus plot of the poles of the optimum system is shown in Figure 3.4. As in the previous plots, only the stable, optimal half of the locus is shown and the adjoint system is omitted. The locus shows that the objective of the design procedure has been achieved. The open-loop poles of the booster originating from the rigid-body dynamics become stable, well damped and increase in natural frequency as the parameter of the locus q_1/r is increased. In addition, the damping ratios of the poles originally associated with the bending modes increase significantly. The net result is a logical design procedure that yields a desirable and satisfactory pattern of closed-loop poles.

Optimal Control Law

It is a relatively simple matter, once the root square locus plot has been obtained, to calculate the optimal control law that will yield the closed-loop polynomial selected from the root locus plot. The original open-loop equations of motion are written

$$\dot{x} = Fx + Gu$$

and it is desired to find the optimal feedback control law

$$u_o = -Kx$$

so that the closed-loop optimal system becomes

$$\dot{x} = (F - GK)x$$

The closed-loop characteristic equation is given by

$$\left| Is - F + GK \right| = 0 \tag{3-30}$$

Equating the determinant of Equation 3-30 to the closed-loop polynomial obtained from the root square locus plot yields a set of linear algebraic equations from which the feedback gains can be easily computed.

As an example, a value of $q_1/r = 50$ was chosen to yield a closed-loop system whose speed of response would be satisfactory. From the root square locus plot of the poles of the closed-loop optimal system (Figure 3.4), the closed-loop polynomial is given by

$$\Delta(s) = (s + .0825)(s + 17.91)\left[s^2 + 2(.76)(1.28)s + (1.28)^2\right]\left[s^2 + 2(.17)(2.4)s + (2.4)^2\right]\left[s^2 + 2(.05)(5.65)s + (5.65)^2\right] \tag{3-31}$$

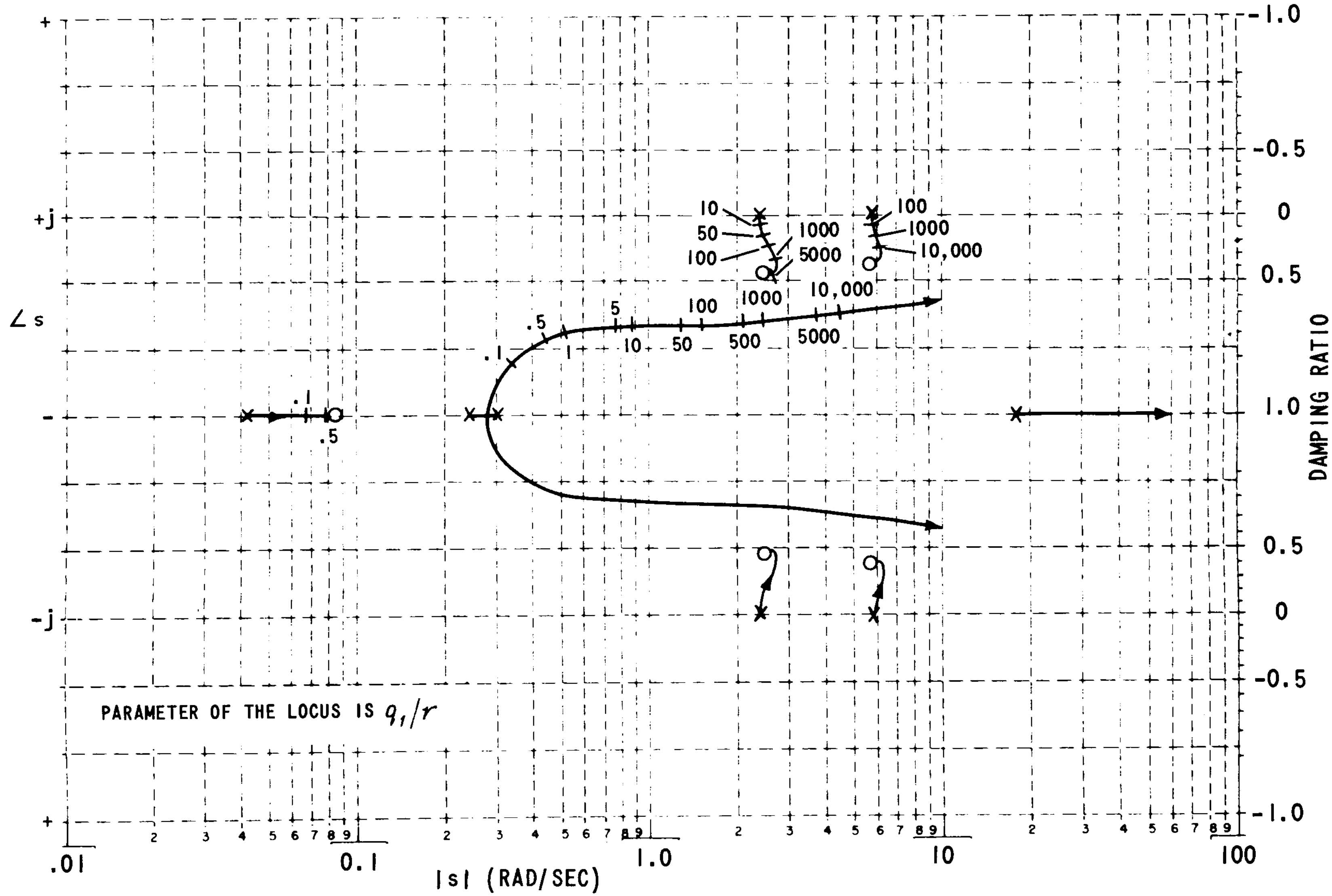


Figure 3.4 Locus of the Poles of the Closed-Loop Optimal System

For this problem, the determinant $\Delta(s) = |Is - F + GK|$ or

$$\Delta(s) = \begin{vmatrix} s & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & -.0733 & 0 & 0 & 0 & 0 & .45 \\ .0405 & -1.0 & s+.0107 & 0 & 0 & 0 & 0 & .0211 \\ 0 & 0 & 0 & s & -1.0 & 0 & 0 & 0 \\ 0 & 0 & -5.453 & 5.37 & s+.0232 & 0 & 0 & -15.83 \\ 0 & 0 & 0 & 0 & 0 & s & -1.0 & 0 \\ 0 & 0 & -2.36 & 0 & 0 & 31.8 & s+.0564 & -22.77 \\ 17.9k_1 & 17.9k_2 & 17.9k_3 & 17.9k_4 & 17.9k_5 & 17.9k_6 & 17.9k_7 & s+17.9+17.9k_8 \end{vmatrix} \tag{3-32}$$

Equating powers of s of Equations 3-31 and 3-32 and solving for K yields

$$K = \left[-9.104,\ -5.993,\ 4.467,\ .1020,\ .03163,\ .03513,\ .02264,\ .1860\right] \tag{3-33}$$

and the optimum control law becomes

$$\beta_c = +9.104\,\phi_R + 5.993\,\dot{\phi}_R - 4.467\alpha - .1020\,\eta_1 - .03163\,\dot{\eta}_1 - .03513\,\eta_2 - .0226\,\dot{\eta}_2 - .186\,\beta \tag{3-34}$$

The control law of Equation 3-34 is not realizable because the variables of Equation 3-34 are not directly measurable. It is necessary to define measurable quantities and express the control law in terms of these quantities. For purposes of illustration, the following set of state variables was chosen

$$z' = \left[\phi_{x_1},\ \dot{\phi}_{x_1},\ a_{z_{x_1}},\ \dot{\phi}_{x_2},\ a_{z_{x_2}},\ \dot{\phi}_{x_3},\ a_{z_{x_3}},\ \beta\right] \tag{3-35}$$

where ϕ_{x_i}, $\dot{\phi}_{x_i}$, and $a_{z_{x_i}}$ represent the quantities measured by a position gyro, a rate gyro and an accelerometer at station x_i on the launch vehicle body. Choosing the body stations x_1 = 41.5 meters, x_2 = 86.0 meters, and x_3 = 122.5 meters, the transformation $z = Ax$ becomes, for this example

$$\begin{bmatrix} \phi_{x_1} \\ \dot{\phi}_{x_1} \\ a_{z_{x_1}} \\ \dot{\phi}_{x_2} \\ a_{z_{x_2}} \\ \dot{\phi}_{x_3} \\ a_{z_{x_3}} \\ \beta \end{bmatrix} = \begin{bmatrix} 1.0 & 0 & 0 & -.028 & 0 & -.0085 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & -.028 & 0 & -.0085 & 0 \\ 0 & 0 & 2.105 & 1.9037 & -.00905 & 16.786 & .0313 & -7.881 \\ 0 & 1.0 & 0 & 0 & .0250 & 0 & .060 & 0 \\ 0 & 0 & 7.441 & 2.453 & .01624 & -35.38 & -.05866 & 3.550 \\ 0 & 1.0 & 0 & 0 & .1230 & 0 & -.143 & 0 \\ 0 & 0 & 23.37 & -14.32 & -.0473 & -8.526 & -.01861 & 14.287 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha \\ \eta_1 \\ \dot{\eta}_1 \\ \eta_2 \\ \dot{\eta}_2 \\ \beta \end{bmatrix} \qquad (3\text{-}36)$$

and the control law becomes

$$\beta_c = -Kx = -KA^{-1}z$$

$$\beta_c = \underbrace{9.104\phi_{x_1} + 4.226\dot{\phi}_{x_1} - .4202a_{z_{x_1}}}_{\text{sensors located at station 41.5m}} + \underbrace{1.317\dot{\phi}_{x_2} - .177a_{z_{x_2}}}_{\text{sensors located at station 86.0m}} + \underbrace{.453\dot{\phi}_{x_3} - .0969a_{z_{x_3}}}_{\text{sensors located at station 122.5m}} - .1493\beta \qquad (3\text{-}37)$$

The control law of Equation 3-37 represents a solution to the two bending mode problem. By locating three instrument packages at three different locations along the body of the launch vehicle, every state variable of the system is measurable. The feedback gains are low enough that the control law may be mechanized without difficulty, resulting in a design that may be practical for a launch vehicle whose significant elastic properties can be described by two bending modes.

The transient responses of the optimal system, defined by the control law of either Equation 3-34 or 3-37, are shown in Figures 3.5 through 3.7. The characteristic polynomial of the closed-loop system is given by Equation 3-31, which indicates that the first mode damping has been increased from $\zeta = .005$ open loop to approximately $\zeta = .17$ for the closed-loop design, while the second mode damping ratio has been increased by a factor of 10, from $\zeta = .005$ to approximately $\zeta = .05$. The transient responses to a unit step command input clearly show the increases in the first and second mode damping ratios. In addition, the rigid-body contribution to the closed-loop response, dominated by a pair of poles located at $\omega_n = 1.28$ rad/sec and $\zeta = 0.76$, contribute to the smoothness of the vehicle's response. The control motions of Figure 3.7 show that the gimbal deflections are reasonable and well within the frequency capability of the actuator.

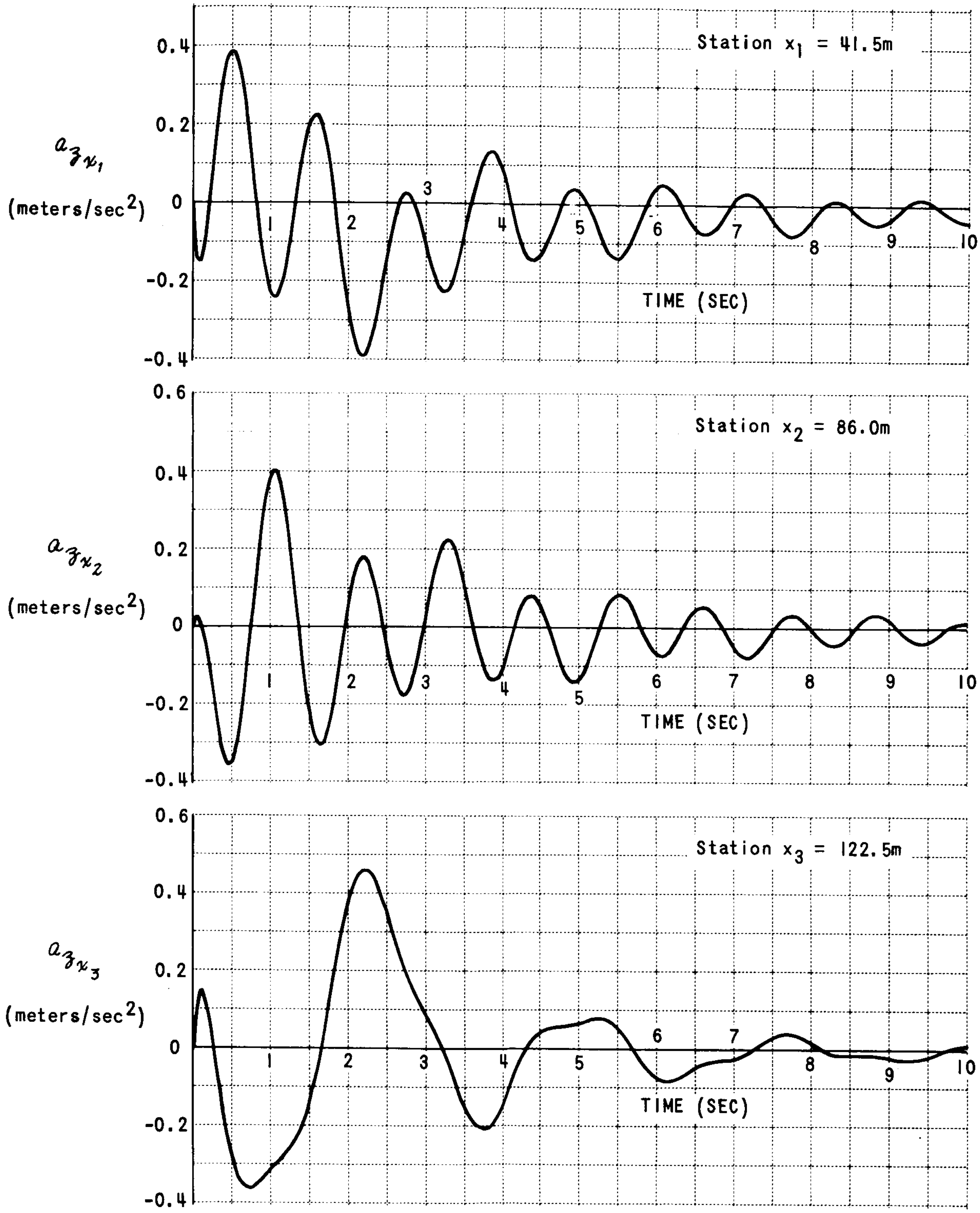


Figure 3.5 Normal Acceleration Response at Three Different Body Stations

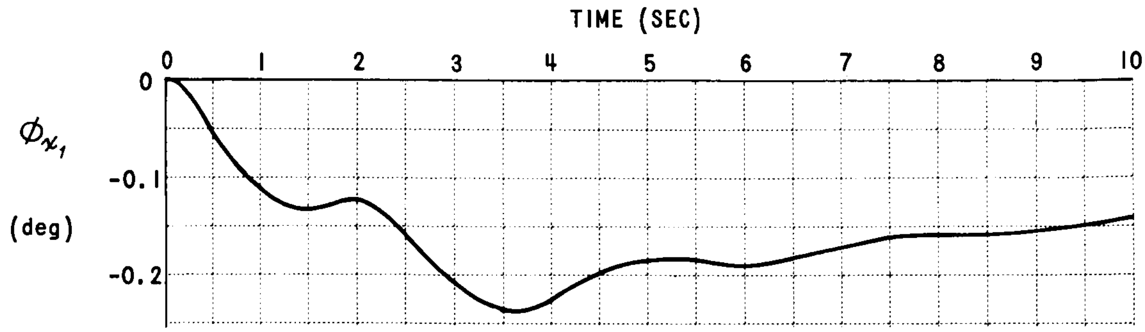


Figure 3.6 Attitude Response at Center of Gravity

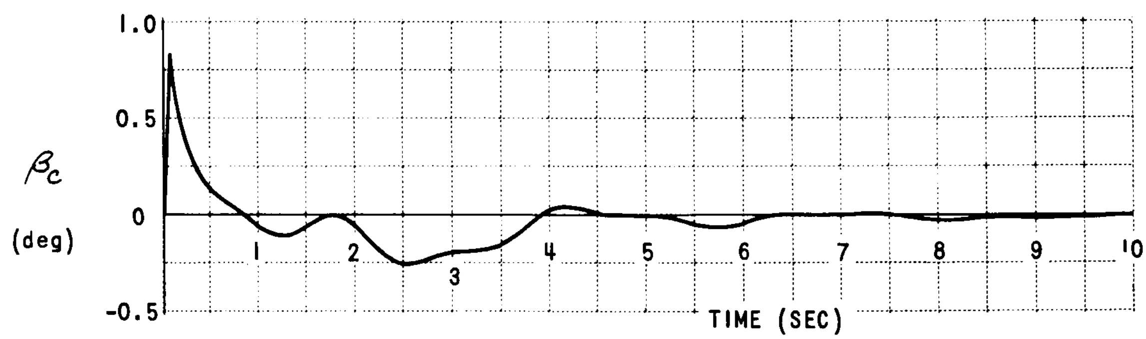


Figure 3.7 The Optimal Control Motion

SECTION 4

MODEL ORIENTED DESIGN STUDY

Introduction

This section describes the use of a model in the linear optimal control design procedure. The model defines a set of equations that describe the motions of an ideal launch vehicle. Optimal control is then used to select a feedback control system configuration that minimizes the differences between the motions of the launch vehicle and the model in an integral error squared sense and control squared sense. Minimizing the motions in this manner guarantees a stable solution and also yields an error response that is smooth and well behaved, resulting in control motions that do not require violent deflections if the model is smooth. A well behaved model is desirable for the launch vehicle, so it is expected that the control motion will be smooth for this application.

The following theoretical development and examples show that the use of a model is a logical design aid, resulting in a more direct and simple analysis procedure than the procedure presented in Section 3. The optimal aspect of the design tends to satisfy some of the design objectives, such as stability and smoothness of response, while the model aspect satisfies other requirements such as drift minimum characteristics and speed of response of the closed-loop system.

A model of a desirably responding system is formulated as (Ref. 3)

$$\dot{\gamma} = L\gamma \tag{4-1}$$

where L defines the matrix of constants describing the coupling among variables in the equations of motion of the model.

A control motion u_o is to be found that minimizes the quadratic performance index

$$2V = \int_0^\infty \left[(\dot{y}-Ly)'Q(\dot{y}-Ly)+u'Ru\right]dt \tag{4-2}$$

where

Q = an rxr positive definite symmetric matrix whose elements weight the contribution of each error in the performance index.

R = a pxp positive definite symmetric matrix whose elements weight the contribution of each control motion in the integral.

The error portion of the performance index, the $(\dot{y}-Ly)$ term, is a vector that vanishes when the closed-loop optimal system behaves exactly like the model. In general, the feedback cannot exactly match the booster to the model for several reasons. In general, the L and F matrices are of different dimension, and a limited number of controllers is available for feedback purposes.

Although it will be impossible to exactly match the booster and the model, the error can be made as small as practical by weighting the error portion of the performance index heavily with respect to the control. It will be shown that approximating the model response through the use of a quadratic performance index results in not only an optimal, but an acceptable feedback control system design.

The problem of minimizing Equation 4-2 subject to the differential constraint of the equations of motion is again a straightforward problem in the calculus of variations. The Lagrangian of this problem can be defined as:

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\left[(\dot{y}-Ly)'Q(\dot{y}-Ly)+u'Ru\right]+\lambda'(-\dot{x}+Fx+Gu) \\ &= \tfrac{1}{2}\left[\dot{x}'H'QH\dot{x}-\dot{x}'H'QLHx-x'H'L'QH\dot{x}\right. \\ &\quad \left.+\,x'L'H'QLHx+u'Ru\right]+\lambda'(-\dot{x}+Fx+Gu) \end{aligned} \tag{4-3}$$

where λ is an n x 1 vector called the adjoint state vector or the costate.

The minimization of the quadratic performance index requires that the Euler-Lagrange equations be satisfied

$$\frac{\partial\mathcal{L}}{\partial x}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)=0 \qquad \frac{\partial\mathcal{L}}{\partial u}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{u}}\right)=0 \tag{4-4}$$

Using the Lagrangian of Equation 4-3, the Euler-Lagrange equations are

$$\dot{\lambda}+F'\lambda-H'QH\ddot{x}+H'QLH\dot{x}-H'L'QH\dot{x}+H'L'QLHx=0 \qquad \text{(a)}$$

$$Ru_o+G'\lambda=0 \qquad \text{(b)} \tag{4-5}$$

The optimal control law from Equation 4-5b is $u_o=-R^{-1}G'\lambda$ and the optimal closed-loop system becomes

$$\dot{x}-Fx+GR^{-1}G'\lambda=0 \tag{4-6}$$

The exact solution for λ as a function of x has been obtained by Kalman (Ref. 3) and Tyler (Ref 6). For the case of the single control input the feedback control law can be more quickly and easily obtained by developing a root square locus expression, which, when plotted, spectral factors the poles of the closed-loop optimal and adjoint systems. The closed-loop characteristic polynomial of the optimal system can then be reconstructed from the root square locus plot, and the feedback gains are then easily obtained. The root square locus expression is developed below.

In Laplace transform form, Equations 4-5a and 4-6 become

$$\begin{bmatrix} [Is-F] & GR^{-1}G' \\ -H'[-Is-L']Q[Is-L]H & [-Is-F'] \end{bmatrix}\begin{bmatrix} x(s) \\ \lambda(s)\end{bmatrix}=\begin{bmatrix} x(0) \\ -\lambda(0)+H'QH[Is\,x(0)+\dot{x}(0)] \\ \quad -H'[QL+L'Q]Hx(0)\end{bmatrix} \tag{4-7}$$

The determinant of the left hand side of Equation 4-7, when set to zero, defines the characteristic equation of the optimal system and its adjoint. It can be shown (Ref. 2) that this determinant may be written

$$\begin{vmatrix} [Is-F] & GR^{-1}G' \\ -H'[-Is-L']Q[Is-L]H & [-Is-F'] \end{vmatrix} = R\left|Is-F\right|\left|-Is-F'\right|\left|I+R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q[Is-L]H[Is-F]^{-1}G\right| = 0 \tag{4-8}$$

if $[Is-F]$ and $[-Is-F']$ are square, non-singular matrices. But the determinants $|Is-F|$ and $|-Is-F'|$ are the characteristic polynomials of the open-loop system and its adjoint, which vanish only at their root locations. The closed-loop roots of the optimal system and adjoint are therefore obtained from the expression

$$\left|I + R^{-1}G'[-Is-F']^{-1}H'[-Is-L']Q\,[Is-L]H[Is-F]^{-1}G\right| = 0 \tag{4-9}$$

The term $H[Is-F]^{-1}G$ is a matrix of transfer functions $y_i/u_j\,(s)$ relating the outputs of the system to the inputs and $G'[-Is-F']^{-1}H'$ is the transpose of the transfer function matrix with s replaced by -s. The locus of roots obtained from Equation 4-9 defines a root square locus expression (Ref. 2 and 5). Using this expression one can investigate the closed-loop characteristic polynomial of the optimal system as a function of the weighting parameters of the performance index of Equation 4-2. Because the root square locus is a p x p expression, while the original determinant of Equation 4-8 is of dimension 2n x 2n, a considerable simplification has been made for purposes of analysis. In particular, because the booster has a single controller (the gimballed rocket engines), the root square locus reduces to a scalar expression.

Basic Limitations of Modeling

The use of a "mathematical model" to define a control system criterion has limitations and clearly cannot be used without some conception of the implications of its use. In general, it is mathematically and physically impossible to exactly match a system to a model. Generally speaking, there are two reasons for this:

1. The mathematical model is usually of an order or dimension that differs from that of the actual system and often differs from the mathematical description of the actual system.

2. In general, the number of controllers does not equal the number of degrees of freedom of motion of the system. It is basic that the number of controllers must equal the number of degrees of freedom the plant possesses to have complete control of the dynamics of the plant.

To illustrate the two principles stated above, consider the following simple examples:

1. Assume a second-order, single controller plant is describable by the first-order set of equations

$$\dot{x} = Fx + Gu$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} u \tag{4-10}$$

Assume that it is desirable to use feedback control of the form

$$u = -K x \tag{4-11}$$

in an attempt to match the plant to a model described by the matrix

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \tag{4-12}$$

Substituting Equation 4-11 into 4-10 yields, in the closed loop,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} - \begin{bmatrix} g_1 k_1 & g_1 k_2 \\ g_2 k_1 & g_2 k_2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4-13}$$

or

$$\dot{x} = \left[F - G K \right] x$$

It is clear, by an examination of 4-13 that only one row of L, either $[l_{21} \quad l_{22}]$ or $[l_{11} \quad l_{12}]$ can be matched by the two feedback gains k_1 and k_2.

The best that one can do is to specify L and F as companion matrices, i.e.,

$$L = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m \omega_m \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta_n \omega_n \end{bmatrix}$$

Then feedback control can be used to match the natural frequency and damping ratio of the plant and the model. This approach has been incorporated into the design philosophy of the flexible launch vehicle.

2. Assume that the plant is describable by the third-order set of equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} u$$

and the desired "model" is described by the same 2 x 2 matrix

$$L = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

It can then be deduced, using the principles from example 1 that it is possible to match only one row of any 2 x 2 submatrix of F to L. If F and L are specified as companion matrices, then the two poles of the model may be exactly obtained and the third pole "arbitrarily" selected.

The approach to flexible launch vehicle control system design incorporates both of the principles described above with one additional, and perhaps important, factor. Referring to example 2, one may ask the following:

> Instead of exactly matching two of the model poles and "arbitrarily" selecting the third pole, what distribution of the <u>three</u> (closed-loop) system poles approximates the two model poles in a minimum quadratic "error" squared and "control" squared sense?

The reason for specifying the closed-loop poles in this sense is logical. It eliminates the arbitrariness of the selection of the third pole. For if the third pole were too "large", the control motions would be rapid and high feedback gains would probably be required. If the third pole were chosen too "small", it would contribute significantly to the closed-loop system response. Therefore, the quadratic type of design used in this report is a compromise design.

If, in the flexible launch vehicle design, the exact drift minimum property must be obtained, it would be a simple matter to precalculate the feedback control required to yield a pole at the origin, then select the remaining poles in a minimum quadratic "error" squared plus "control" squared sense. The penalty incurred is that greater or sharper control motions would be required than for the case when all the plant poles were chosen to satisfy the quadratic performance index criterion. Perhaps the penalty is significant; more likely the differences would be negligible.

Elastic Booster Application

Because the booster has a single control element, feedback control can alter only the characteristic polynomial of the vehicle and not the zeros of the transfer functions (unless other dynamic elements are deliberately added to the system). Therefore, the optimal control problem may be formulated to express the error in terms of a difference between the characteristic polynomial of the elastic booster and the characteristic polynomial of the model. To do this, assume that the F matrix associated with the booster is of dimension n x n and L is dimension $\ell \times \ell$. Define H of dimension ℓ x n

$$H = \begin{bmatrix} h_{11} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & h_{22} & & & & & & & \cdot \\ \cdot & 0 & \cdot & & & & & & \cdot \\ \cdot & & & \cdot & & & & & \cdot \\ \cdot & & & & \cdot & & & & \cdot \\ 0 & 0 & \cdots & 0 & h_{\ell\ell} & 0 & \cdots & & 0 \end{bmatrix} \tag{4-14}$$

where $h_{11} = h_{22} = \ldots = h_{\ell\ell} = 1$

The matrices F, L. and G are written

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & 1 & 0 \\ 0 & & & 0 & 1 \\ -b_1 & -b_2 & \cdots & -b_{n-1} & -b_n \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & 1 & 0 \\ & & & 0 & 1 \\ -d_1 & -d_2 & \cdots & -d_{\ell-1} & -d_\ell \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{4-15}$$

where

$$s^n + b_n s^{n-1} + b_{n-1} s^{n-2} + \ldots + b_2 s + b_1 = |Is - F| \qquad \text{(a)}$$

$$s^\ell + d_\ell s^{\ell-1} + d_{\ell-1} s^{\ell-2} + \ldots + d_2 s + d_1 = |Is - L| \qquad \text{(b)} \tag{4-16}$$

The Q matrix is defined as $Q = qI$, where I is an $\ell \times \ell$ identity matrix and $R = r$, a scalar.

When the quantities F, G, H, L, Q, and R are substituted into the root square locus expression of Equation 4-9 one obtains

$$-1 = \frac{q}{r} \frac{|Is-L||-Is-L'|}{|Is-F||-Is-F'|} \tag{4-17}$$

The numerator of Equation 4-17 contains the characteristic polynomial of the model and adjoint while the denominator of Equation 4-17 contains the characteristic polynomial of the booster and its adjoint. The locus of roots of Equation 4-17 will therefore originate at the launch vehicle open-loop poles and terminate at the model singularities. The parameter of the locus is proportional to q/r, the ratio of the weighting of the error to the control in the performance index. The net result of the use of Equation 4-17 is that the performance index has been formulated to minimize the difference between the characteristic polynomial of the booster and the model in the integral error and control squared sense.

Equations of Motion of the Elastic Booster

The equations of motion are taken from NASA "Model Vehicle No. 2" (Appendix A) which is representative of the post-Saturn vehicles under consideration by NASA. Assuming complete vehicle symmetry, the equations of motion in the x - z plane at t = 78 sec after launch (max q) are given below with certain simplifications. Two normal bending modes are included and the engine actuator is assumed to be described by a first-order differential equation

$$\begin{aligned}
&\ddot{\phi}_R - 0.0733\alpha + 0.45\beta = 0 && \text{pitching equation} \\
&-\dot{\phi}_R + 0.0405\phi_R + \dot{\alpha} + 0.01067\alpha + 0.211\beta = 0 && \text{heaving equation} \\
&\dot{\beta} + 17.9\beta - 17.9\beta_c = 0 && \text{actuator dynamics} \qquad (4\text{-}18) \\
&-5.453\alpha + \ddot{\eta}_1 + 0.02317\dot{\eta}_1 + 5.37\eta_1 - 15.63\beta = 0 && 1^{st} \text{ normal bending mode equation} \\
&-2.360\alpha + \ddot{\eta}_2 + 0.05642\dot{\eta}_2 + 31.80\eta_2 - 22.77\beta = 0 && 2^{nd} \text{ normal bending mode equation}
\end{aligned}$$

where ϕ_R = rigid-body attitude angle

α = angle of attack

β = control deflection angle

η_1 = first normal bending mode variable

η_2 = second normal bending mode variable

The F and G matrices are written in the first-order form of Equation 4-15

$$\dot{x} = Fx + Gu \qquad y = Ax$$

and the output matrix A is defined to be an 8 x 8 matrix of constants

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & & & & & a_{2n} \\ a_{31} & & & & & & \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & a_{(n-1)n} \\ a_{n1} & & \cdot & \cdot & \cdot & a_{n(n-1)} & a_{nn} \end{bmatrix} \qquad (4\text{-}19)$$

Then

$$\frac{y_i}{u} = \frac{a_{in}\, s^{n-1} + a_{in-1}\, s^{n-2} + \cdot \cdot \cdot\; a_{i1}}{s^n + b_n\, s^{n-1} + b_{n-1}\, s^{n-2} + \cdot \cdot \cdot\; b_{i1}}$$ is the transfer function

relating the y_i th variable of the equations of motion to the control input. In terms of the numbers of Equation 4-18, the first-order equations become:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -9.074 & 223.5 & -20.71 & -3010 & -193.7 & -665.5 & -38.71 & -17.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \beta_c \qquad (4\text{-}20)$$

$$y = Ax$$

$$\begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha \\ \eta_1 \\ \dot{\eta}_1 \\ \eta_2 \\ \dot{\eta}_2 \\ \beta \end{bmatrix} = \begin{bmatrix} -19.4 & -1376 & -12.6 & -299.4 & -0.755 & -8.055 & 0 & 0 \\ 0 & -19.4 & -1376 & -12.6 & -299.4 & -0.755 & -8.055 & 0 \\ 55.71 & -1375 & -60.62 & -299.8 & -14.33 & -8.085 & -.377 & 0 \\ 83.32 & -2057 & 29.74 & 8946 & 16.95 & 283.4 & 0 & 0 \\ 0 & 83.32 & -2057 & 29.74 & 8946 & 16.95 & 283.4 & 0 \\ 10.63 & -262.5 & 19.42 & 2140 & 12.9 & 407.6 & 0 & 0 \\ 0 & 10.63 & -262.5 & 19.42 & 2140 & 12.9 & 407.6 & 0 \\ 9.074 & -224 & 33.23 & 3008 & 26.66 & 664.1 & 1.616 & 17.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \qquad (4\text{-}21)$$

Model

The model to be used with this example will be defined by the third-order system

$$\begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1.4 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \qquad (4\text{-}22)$$

The model incorporates the concept of the drift minimum requirement, the condition that leads to the cancellation of the sum of all force components, such as gust inputs, perpendicular to the nominal flight plane. This cancellation of forces is equivalent to requiring a free integrator in the closed-loop characteristic polynomial. Therefore the model characteristic equation contains a root at the origin of the s plane

$$|Is - L| = s(s^2 + 1.4s + 1).$$

Analysis of the Optimal System

Substituting the characteristic polynomials of the open-loop booster and the model along with their adjoints in Equation 4-13, the root square locus expression, the result is

$$0 = 1 - \frac{q}{r} \frac{s^2(s^2 \pm 1.4s + 1)}{(s \pm 17.9)(s \pm .294)(s \pm .242)(s \pm .04175)\left[s^2 \pm 2(.005)2.32s + (2.32)^2\right]\left[s^2 + 2(.005)5.64s + (5.64)^2\right]} \tag{4-23}$$

The locus of roots of the optimal system is plotted in Figure 4.1. The adjoint, or right-half plane part of the locus is omitted for clarity. The locus shows that the open-loop roots of the booster originally associated with the rigid-body poles tend toward the model singularities, shown in the locus as zeros. The remaining poles, originally associated with the bending modes, tend to become distributed in a Butterworth filter pattern as the parameter of the locus, q/r, becomes large. For any value of the weighting of the error portion of the integral to the control, the accuracy of the approximation to the model in terms of the closed-loop root locations can be determined.

Optimal Control Law

The optimal control law is of the form $\beta_c = -Kx$ where K is a 1 x 8 matrix of feedback gains from the state variables of Equation 4-20 to the control inputs. The closed-loop characteristic polynomial is

$$\begin{aligned}\Delta(s) &= \left| Is - F + GK \right| \\ &= s^n + (b_n + k_n)s^{n-1} + (b_{n-1} + k_{n-1})s^{n-2} + \dots (b_2 + k_2)s + (b_1 + k_1) \\ &= s^8 + (17.99 + k_8)s^7 + (38.71 + k_7)s^6 + (665.5 + k_6)s^5 + (193.7 + k_5)s^4 \\ &\quad + (3010 + k_4)s^3 + (20.71 + k_3)s^2 + (-223.5 + k_2)s + (9.074 + k_1)\end{aligned} \tag{4-24}$$

The closed-loop characteristic polynomial can also be obtained directly from the root square locus plot of Figure 4.1 for any value of q/r. For instance, assuming that q/r = 100 will yield an optimal system that results in an acceptable approximation to the model, the closed-loop characteristic polynomial is

$$\begin{aligned}\Delta(s) &\cong (s + .0077)(s + 17.9)\left[s^2 + 2(.75)(.82)s + (.82)^2\right]\left[s^2 + 2(.46)(2.6)s + (2.6)^2\right]\left[s^2 + 2(.12)(6.2)s + (6.2)^2\right] \\ &= s^8 + 23.02s^7 + 145.87s^6 + 1135.8s^5 + 3372.5s^4 + 7885.6s^3 + 7157.4s^2 + 3107.8s + 24.11\end{aligned} \tag{4-25}$$

Equating coefficients of the powers of s in Equations 4-24 and 4-25 yields the feedback control law

$$\beta_c = -15.0x_1 - 3331.3x_2 - 7136.7x_3 - 4875.6x_4 - 3178.7x_5 - 470.3x_6 - 107.2x_7 - 5.03x_8 \tag{4-26}$$

Or, in terms of the variables of Equation 4-21, the control law

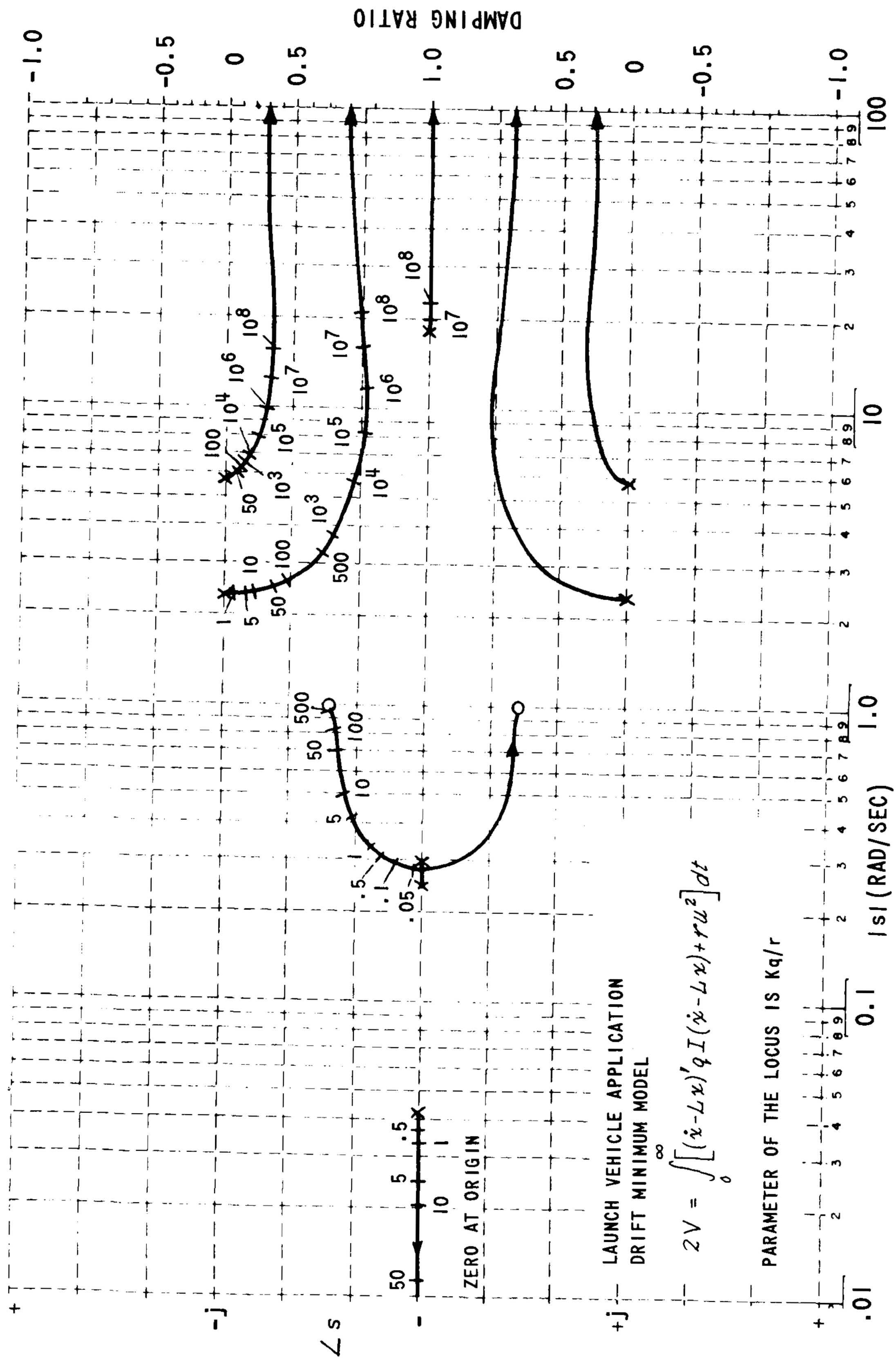


Figure 4.1 Locus of Poles of Optimal System (Drift Minimum Model)

becomes

$$\beta_0 = -K\psi = -KA^{-1}y$$
$$= 1.932\phi_R + 5.36\dot{\phi}_R + .893\alpha - .239\eta_1 - .162\dot{\eta}_1 - .456\eta_2 - .0421\dot{\eta}_2 - .281\beta \qquad (4\text{-}27)$$

With the exception of β the variables of Equation 4-27 are still not in terms of directly measurable quantities, i. e., in quantities that directly represent outputs of sensors located on the body of the vehicle. A set of measurable quantities can be chosen, such as the vector

$$z' = \left[\phi_{x_1}, \dot{\phi}_{x_1}, a_{z_{x_1}}, \Delta\phi_{x_1}, \Delta\phi_{x_2}, \Delta\dot{\phi}_{x_1}, \Delta\dot{\phi}_{x_2}, \beta\right] \qquad (4\text{-}28)$$

where ϕ_{x_1}, $\dot{\phi}_{x_1}$, and $a_{z_{x_1}}$, represent outputs of a position gyro, a rate gyro and a normal accelerometer located at station x_1 on the vehicle body and

$$\Delta\phi_1 = \phi_{x_1} - \phi_{x_2}$$
$$\Delta\phi_2 = \phi_{x_1} - \phi_{x_3}$$
$$\Delta\dot{\phi}_1 = \dot{\phi}_{x_1} - \dot{\phi}_{x_2}$$
$$\Delta\dot{\phi}_2 = \dot{\phi}_{x_1} - \dot{\phi}_{x_3}$$

A transformation $z = By$ can be formulated that defines the measurable quantities of Equation 4-28 in terms of the variables of Equation 4-27. With this transformation, the feedback control law becomes

$$\beta_0 = -KA^{-1}y = -KA^{-1}B^{-1}z \qquad (4\text{-}29)$$

If one chooses the vehicle body stations

x_1 = 41.5 meters
x_2 = 86 meters
x_3 = 122.5 meters

the feedback control law becomes

$$\beta_0 = 1.933\phi_{x_1} + 5.36\dot{\phi}_{x_1} + 0.424a_{z_{x_1}} + 72.98\Delta\phi_1 + 0.208\Delta\dot{\phi}_1 - 19.04\Delta\phi_2 + .0329\Delta\dot{\phi}_2 + 3.06\beta \qquad (4\text{-}30)$$

The differential gyro realization of the feedback control law for this particular case results in high feedback gains from several of the measured quantities, resulting in a feedback control law that may be difficult to mechanize. It would be more realistic to select another set of measurable quantities such as the set

$$z_1' = \left[(\phi_{x_1}, \dot{\phi}_{x_1}, a_{z_{x_1}})(\dot{\phi}_{x_2}, a_{z_{x_2}})(\dot{\phi}_{x_3}, a_{z_{x_3}})\beta\right] \qquad (4\text{-}31)$$

If this set is chosen, the control law becomes

$$\beta_o = \underbrace{1.933\,\phi_{x_1} + 5.48\,\dot{\phi}_{x_1} + .0486\,a_{z_{x_1}}}_{\substack{\text{sensors located at}\\ \text{station } x_1 = 41.5\text{m}}} \underbrace{-.0564\,\dot{\phi}_{x_2} + .0296\,a_{z_{x_2}}}_{\substack{\text{sensors located at}\\ \text{station } x_2 = 86.0\text{m}}} \underbrace{-.0592\,\dot{\phi}_{x_3} + .0244\,a_{z_{x_3}}}_{\substack{\text{sensors located at}\\ \text{station } x_3 = 122.5\text{m}}} - 0.35\,\beta_L \qquad (4\text{-}32)$$

where the units of β/ϕ and $\beta/\dot{\phi}$ are rad/rad and rad/(rad/sec) and the units of β/a_z are rad/(meter/sec^2). The control law of Equation 4-32 involves feedback gains more easily mechanized than those of Equation 4-30, but either control law will, of course, result in the same closed-loop characteristic polynomial and closed-loop transient response. It is clear that the realization of the optimal system is not unique, and many engineering factors enter into the choice of sensed quantities and sensor locations.

Equivalent Compensation Network

The optimal feedback control law may be synthesized as a filter or compensation network in the feedback path which shapes the signal of the output of one or a linear combination of sensors. There are, however, definite restrictions and qualifications that must be placed upon such a synthesis procedure. The two systems, one synthesized by feedback gains and the other synthesized through a filter network, will not respond exactly alike to initial conditions of the state. Therefore, one must be content, when using a feedback filter synthesis procedure, with a synthesis of the closed-loop poles of the optimal system rather than a complete time history equivalence.

A filter must satisfy certain characteristics of realizability that simply do not have to be considered when gains are fed back from a set of state variables. So the filter problem is more difficult than the straightforward feedback gain procedure of synthesis. The technique of specifying a filter is uncomplicated. Because the system has been assumed to be completely linear, the state variables are related to each other through transfer functions and therefore one state variable can be reconstructed from a measurement of a second state through a filter network (with the exception, of course, of the initial transients of the filter itself). There are limitations to this kind of reconstruction. The most important limitation is that the open-loop transfer function of the measured state variable must be minimum phase. Otherwise, the filter will have right-half plane poles, creating an unstable configuration for even the slightest variation of the launch vehicle or compensation network parameters. The second limitation to the reconstruction procedure is that the transfer function of the measured state should have a numerator polynomial of order n - 1, where n is the number of first-order equations of motion that describe the system's dynamic motions. This requirement will quarantee that the compensation network will contain a numerator polynomial that is not of higher order than the denominator polynomial and the filter will be realizable.

Each of the transfer functions $\frac{z_i}{\beta_c}(s)$ of the open-loop launch vehicle can be expressed as the ratio of two polynomials in s of the form

$$\frac{z_i}{\beta_c}(s) = \frac{N_i}{D}(s)$$

where

$N_i(s)$ = the numerator polynomial of the transfer function for the i^{th} state variable, and

$D(s)$ = the open-loop characteristic polynomial of the launch vehicle.

The transfer function of the z_i state variable is simply related to the transfer function of the z_j state variable through the expression

$$\frac{z_i}{\beta_c}(s) = \frac{N_j}{D}(s) \cdot \frac{N_i}{N_j}(s) \tag{4-33}$$

Using this relationship among the states, a feedback control law $\beta_o = K z$ can be reconstructed from a measurement of a single state, say z_1, in the following manner

$$\begin{aligned} \beta_o &= k_1 z_1 + k_2 z_2 + k_3 z_3 + \ldots + k_n z_n \\ &= k_1 z_1 + k_2 \frac{N_2}{N_1}(s) z_1 + k_3 \frac{N_3}{N_1}(s) z_1 + \ldots + k_n \frac{N_n}{N_1}(s) z_1 \\ &= \frac{z_1}{N_1}(s) \sum_{i=1}^{n} k_i N_i(s) \end{aligned} \tag{4-34}$$

The transfer function of the required compensation network is then given by

$$Y(s) = k_1 + \frac{1}{N_1(s)} \sum_{i=2}^{n} k_i N_i(s) \tag{4-35}$$

Suppose it is desirable to reconstruct the control law in the manner described above from a measurement of not more than three of the state variables of Equation 4-32. Consider the numerators of the transfer functions of these states which are given below:

$$
\begin{aligned}
N_{\phi(x_1)}(s) &= -19.4\,s^5 - 1.335\,s^4 - 5.70\,s^3 - 13.6\,s^2 - 1315\,s - 21.85 \\
N_{\dot\phi(x_1)}(s) &= -19.4\,s^6 - 1.335\,s^5 - 5.70\,s^4 - 13.6\,s^3 - 1315\,s^2 - 21.85\,s \\
N_{a_{z(x_1)}}(s) &= -141.1\,s^7 + 1.794\,s^6 + 2131\,s^5 + 164.3\,s^4 + 28{,}260\,s^3 - 33.70\,s^2 - 9450\,s + 382.8 \\
N_{\dot\phi(x_2)}(s) &= 23.49\,s^6 + .443\,s^5 + 52.65\,s^4 - 10.69\,s^3 - 1443\,s^2 - 16.68\,s \\
N_{a_{z(x_2)}}(s) &= 62.65\,s^7 - 16.46\,s^6 - 11{,}460\,s^5 - 411.9\,s^4 - 45{,}470\,s^3 - 966.9\,s^2 - 6774\,s + 274.6 \\
N_{\dot\phi(x_3)}(s) &= -31.48\,s^6 - .5146\,s^5 + 494.9\,s^4 - 11.72\,s^3 - 1591\,s^2 - 10.67\,s \\
N_{a_{z(x_3)}}(s) &= 255.7\,s^7 - 6.723\,s^6 + 1764\,s^5 - 784.3\,s^4 - 110{,}400\,s^3 - 1431\,s^2 - 3641\,s + 147.7 \\
N_{\beta}(s) &= 17.9\,s^7 + 1.616\,s^6 + 664.1\,s^5 + 26.66\,s^4 + 3008\,s^3 + 33.23\,s^2 - 224\,s + 9.074
\end{aligned}
\tag{4-36}
$$

We wish to choose as our measured quantity a state or combination of states whose transfer function contains a minimum phase numerator polynomial of not more than seventh order. None of the polynomials of Equation 4-36 qualify in themselves, but a combination of these polynomials will satisfy the measurability requirements. Several combinations will yield the desired characteristics, so as an example we can choose the sensed quantity to be $A = \beta - (\phi_{x_1} + \dot\phi_{x_1})$. The numerator of the transfer function is given by

$$N_A(s) = N_\beta(s) - \left[N_{\phi(x_1)}(s) + N_{\dot\phi(x_1)}(s)\right] \tag{4-37}$$

Substituting for $N_\beta(s)$, $N_{\phi(x_1)}(s)$ and $N_{\dot\phi(x_1)}(s)$ from Equations 4-36

$$N_A(s) = 17.9\,s^7 + 21.02\,s^6 + 685.\,s^5 + 598\,s^4 + 3590\,s^3 + 1362\,s^2 + 1113\,s + 30.9 \tag{4-38}$$

$N_A(s)$ is of seventh order and a Routh's criteria check of $N_A(s) = 0$ shows all the roots to have negative real parts.

Using this sensed quantity, there are still several ways the feedback control law may be synthesized. For instance the quantity $A = \beta - (\phi_{x_1} + \dot\phi_{x_1})$ may be the only quantity fed back to the gimbal actuator input, as shown in Figure 4.2.

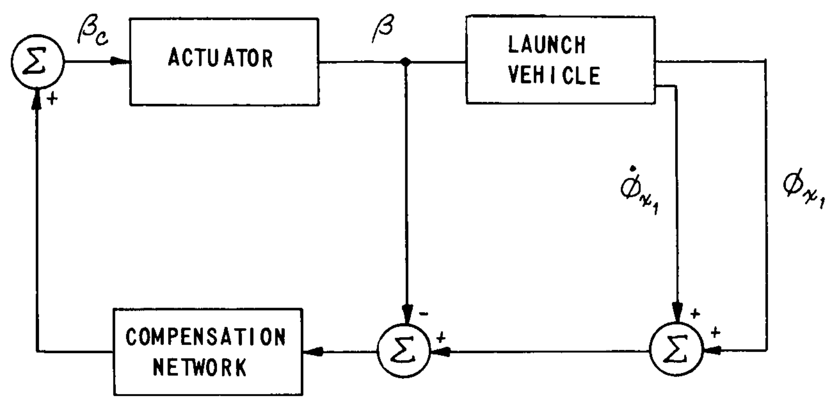


Figure 4.2 One Realization of the Control Law

or, as a second example, the attitude and attitude rate parts of the control law may be independently fed back, as in Figure 4.3 below:

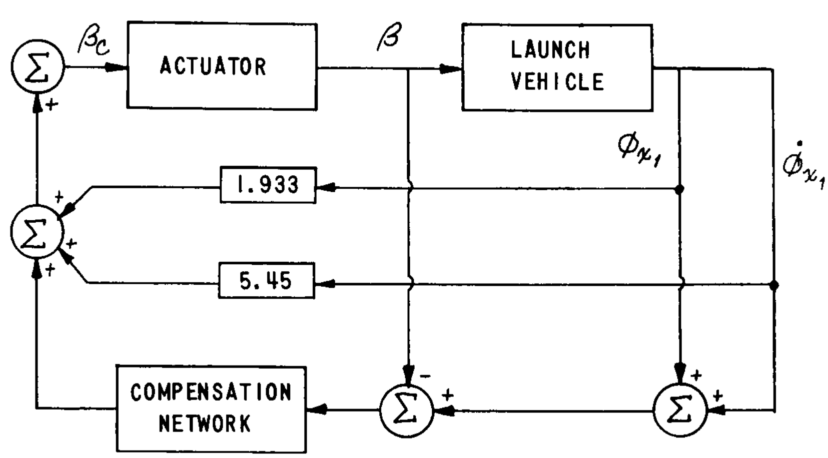


Figure 4.3 A Second Realization of the Control Law

Using the results of Equation 4-34, the transfer function of the compensation network of Figure 4.2, which realizes the control law of Equation 4-32 is given by Equation 4-39:

$$\sum_{i=1}^{8} \frac{k_i N_i(s)}{N_A(s)} = \frac{1}{N_A(s)} \Big[1.933 N_{\phi_{(x_1)}}(s) + 5.48 N_{\dot{\phi}_{(x_1)}}(s) + .0486 N_{a_{y(x_1)}}(s) - .0564 N_{\dot{\phi}_{(x_2)}}(s) - .0296 N_{a_{y(x_2)}}(s)$$

$$- .0592 N_{\dot{\phi}_{(x_3)}}(s) + .0244 N_{a_{y(x_3)}}(s) - 0.35 N_{\beta}(s) \Big]$$

$$= \frac{-5.30 s^7 - 106.8 s^6 - 469.2 s^5 - 3168.8 s^4 - 4880.3 s^3 - 7188.5 s^2 - 3327 s - 15.02}{17.9 s^7 + 21.02 s^6 + 685. s^5 + 598 s^4 + 3590 s^3 + 1362 s^2 + 1113 s + 30.9} \tag{4-39}$$

$$= \frac{-5.30(s+.0456)(s+.635)(s+16.82)\left[s^2+2(.163)(5.196)s+(5.196)^2\right]\left[s^2+2(.337)(1.468)s+(1.468)^2\right]}{17.9(s+.0287)\left[s^2+2(.0219)(5.64)s+(5.64)^2\right]\left[s^2+2(.114)(2.38)s+(2.38)^2\right]\left[s^2+2(.308)(.577)s+(.577)^2\right]}$$

The transfer function of the compensation network for the realization shown in Figure 4.3, on the other hand, is given by

$$\sum_{i=3}^{8} \frac{k_i N_i(s)}{N_A(s)} = \frac{1}{N_A(s)} \Big[.0486 N_{a_{y(x_1)}}(s) - .0564 N_{\dot{\phi}_{(x_2)}}(s) + .0296 N_{a_{y(x_2)}}(s) - .0592 N_{\dot{\phi}_{(x_3)}}(s) + .0244 N_{a_{y(x_3)}}(s) - 0.35 N_{\beta}(s) \Big]$$

$$= \frac{-5.30 s^7 - .587 s^6 - 424.4 s^5 - 46.2 s^4 - 3705.7 s^3 + 97.84 s^2 - 667.4 s + 27.18}{17.9 s^7 + 21.02 s^6 + 685 s^5 + 598 s^4 + 3590 s^3 + 1362 s^2 + 1113 s + 30.9} \tag{4-40}$$

$$= \frac{-5.30(s-.0415)\left[s^2+2(-.0012)(8.37)s+(8.37)^2\right]\left[s^2+2(.0256)(3.125)s+(3.125)^2\right]\left[s^2+2(.0145)(.4296)s+(.4296)^2\right]}{17.9(s+.0287)\left[s^2+2(.0219)(5.64)s+(5.64)^2\right]\left[s^2+2(.114)(2.38)s+(2.38)^2\right]\left[s^2+2(.308)(.577)s+(.577)^2\right]}$$

Although the filter transfer functions described in Equations 4-39 and 4-40 are of seventh order, they may be realized without great difficulty. If it is objectionable to have a seventh-order numerator polynomial, then it may be possible to synthesize Equation 4-30 instead of Equation 4-32 using the same sensed quantity $A = \beta - (\phi_{x_1} + \dot{\phi}_{x_1})$ but feeding back the actuator deflection part of the control law $3.06\,\beta$ separately (i.e., not reconstructing its effects by the compensation network). The resulting compensation network transfer function would be the ratio of a sixth- to a seventh-order polynomial.

So again it is found that the synthesis of the control law is not unique, and a search should be made of the control configuration that presents the fewest problems from a mechanization point of view.

The compensation network transfer functions of Equations 4-29 and 4-30 were synthesized from expressions of the exact optimal control law of Equation 4-32. If a compensation were to be synthesized for an actual vehicle, there are several steps that would be taken in the normal event of system design that are not shown in the preceding analysis. First, approximations to the control law of Equation 4-32 would be made. Perhaps some of the feedback gains can be eliminated entirely, a procedure that would alter the strict optimal character of the design, but may still yield a satisfactory and acceptable vehicle response. The second procedure would be to simplify the compensation network itself to yield a simpler design, still retaining an acceptable closed-loop vehicle response. It is felt that a fair number of simplifications can be made before the response becomes unacceptable (such as unstable bending modes). Therefore, the optimal control law can be used as a guide which defines a superior system. The aim is to realize a control law as close to the optimum as practical considerations will allow.

Multiple-Feedback, Multiple-Compensation Synthesis

Although it may be possible to synthesize the control either as feedback gains or as a single compensation network, neither of the two synthesis techniques are entirely satisfactory. The first technique requires a feedback path for each state variable used to describe the dynamic characteristics of the launch vehicle. It is clear that when many elastic modes of motion are to be controlled, the number of sensors required to realize the synthesis is prohibitive. In addition, there would be no way to compensate for the dynamics of the sensors themselves, which may present a problem if one is trying to control bending modes whose natural frequencies are in the neighborhood of the natural frequencies of the sensors themselves. Approximations to the control law, obtained by ignoring feedback paths or by using least squares approximations, may still require too many feedback paths or may result in closed-loop dynamics that are altered far beyond allowable tolerances.

The second technique, which requires that a compensation network be synthesized to shape the signal output of a single sensor, is an equally unsatisfactory approach to optimal control system synthesis. If the compensation network is synthesized in the feedforward path with unity feedback, the compensation network must contain zeros that cancel the unstable open-loop poles of the launch vehicle. Because pole-zero cancellation of unstable poles is physically impossible, the technique is worth not more than ephemeral consideration. No systematic approximations to the optimal will produce a mechanizable, stable result without at least an increase in the order of the system. A compensation network in the feedback path shaping the output of only one sensor has difficulties almost as insurmountable as the feedforward compensation scheme. If the transfer function associated with the measured output is non-minimum phase, the exact compensation network must contain right-half plane poles to cancel the right-half plane zeros of the transfer function of the sensed quantity. Therefore, in order to realize this type of compensation, one must either choose a sensed quantity whose transfer function is minimum phase or one must approximate the filter, either by deleting the unstable portion of the network or by approximating the unstable part with stable components, a procedure that may produce some difficulties. In any case, the order of the filter before approximations are made is equal to the number of state variables used to describe the launch vehicle dynamics. If a dozen elastic and slosh modes of motion are to be stabilized, the order of the required filter can be very high. Finally, the stability of the launch vehicle depends upon the reliability of sensing a single dynamic quantity, which may not be desirable.

A review of the difficulties associated with the synthesis procedures outlined above can lead one to the requirements for a desirable synthesis procedure:

1. The synthesis procedure should involve more than one sensor but fewer sensors than the number of states of the system.

2. The compensation networks should be in the feedback paths to avoid the pole-zero cancellation problems associated with feedforward compensation.

3. Each compensation network should be of order lower than n-1, where n = the order of the plant.

4. The synthesis procedure should be methodical, with systematic selection of the poles and zeros of the compensation networks.

5. The synthesis procedure should contain the ability to compensate for sensor dynamics.

6. The synthesis procedure should be amenable to rational, systematic methods of simplification. It should be possible to predict beforehand the consequences of a particular step in the simplification process.

7. The order of the characteristic polynomial of the closed-loop system should theoretically equal the order of the open-loop system, even after approximations and simplifications are made to the compensation networks.

8. The compensation networks should be chosen, if possible, such that the vehicle would not be unstable with the failure of any one feedback path.

The requirements dictate that the synthesis procedure should be a compromise between the feedback gain procedure, which avoids the pole-zero cancellation problem, and the single sensor synthesis technique, which eliminates the need for an excess number of feedback paths. Schematically, the block diagram of Figure 4.4 indicates the requirements:

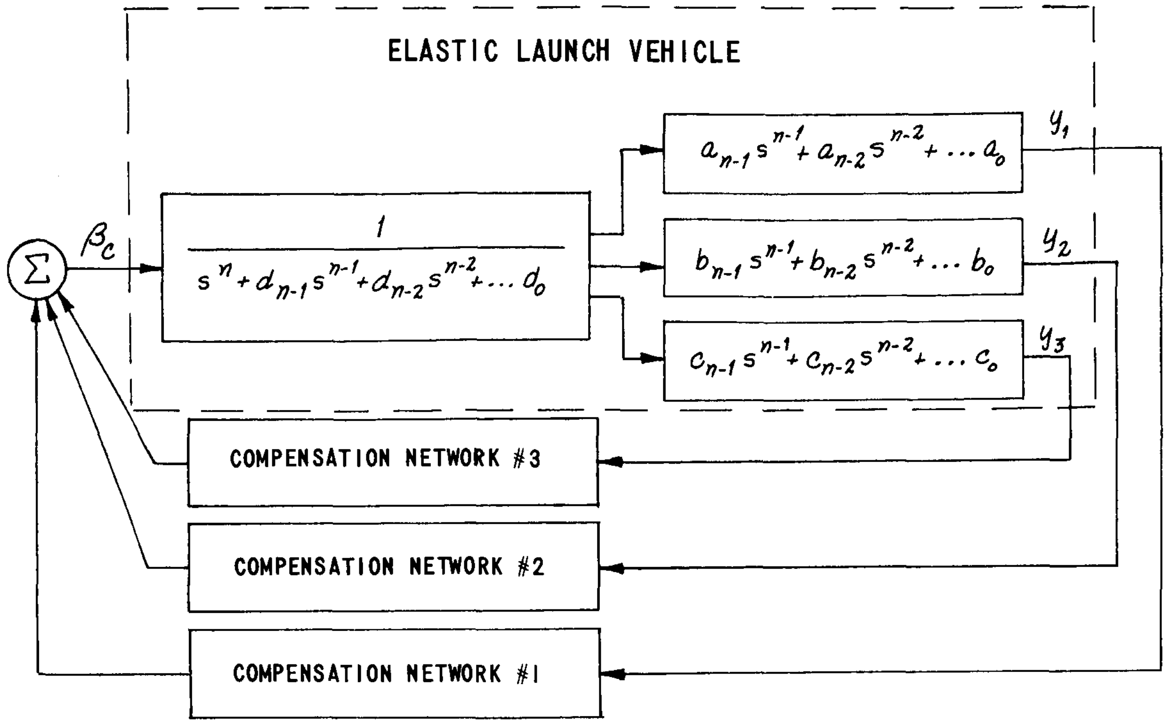


Figure 4.4 Block Diagram of Desirable Control Law

where the sum of the poles of the three compensation networks equals n - 3, where n is the order of the system.

The requirement that a systematic method be devised for the selection of the filter poles and the desire not to increase the order of the closed-loop system demands that the poles of the compensation networks be chosen from among the zeros of the $y_i/\beta_c\,(s)$ transfer functions.

The dynamical description of the elastic launch vehicle must be such that three of the states are y_1, y_2, y_3. The remaining states must be chosen such that each state is related to y_1, y_2, or y_3 by compensation network component parts. In other words, we wish to be able to describe the control law as

$$u = -\sum_{i=1}^{n} k_i y_i$$

$$\begin{aligned} u = &- \left[k_1 y_1 + k_2 y_1' + k_3 y_1'' + \dots k_{p+1} y_1^{p}\right] \\ &- \left[k_{p+2} y_2 + k_{p+3} y_2' + k_{p+4} y_2'' + \dots k_{p+q+2} y_2^{q}\right] \\ &- \left[k_{p+q+3} y_3 + k_{p+q+4} y_3' + k_{p+q+5} y_3'' + \dots k_n y_3^{r}\right], \quad p+q+r = n-3 \end{aligned} \tag{4-41}$$

where

$$\frac{y_1'}{y_1}(s) = \frac{c_1}{s+\alpha_1} \qquad \frac{y_1''}{y_1}(s) = \frac{c_2}{s+\alpha_2} \qquad etc.$$

$$\frac{y_2'}{y_2}(s) = \frac{d_1}{s+\beta_1} \qquad \frac{y_2''}{y_2}(s) = \frac{d_2}{s+\beta_2} \qquad etc.$$

$$\frac{y_3'}{y_3}(s) = \frac{e_1}{s+\gamma_1} \qquad \frac{y_3''}{y_3}(s) = \frac{e_2}{s+\gamma_2} \qquad etc.$$

and

$$a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_o = a_{n-1}\prod_{i=1}^{u}(s+\alpha_i) \qquad u \leq n-1$$

$$b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_o = b_{n-1}\prod_{j=1}^{v}(s+\beta_j) \qquad v \leq n-1$$

$$c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \ldots + c_o = c_{n-1}\prod_{k-1}^{w}(s+\gamma_k) \qquad w \leq n-1$$

The control law can then be specified in the form:

$$u(s) = -\left[k_1 + k_2\frac{y_1'}{y_1}(s) + k_3\frac{y_1''}{y_1}(s) + \ldots + k_{p+1}\frac{y_1^p}{y_1}(s)\right]y_1(s) \quad , p < u$$

$$-\left[k_{p+2} + k_{p+3}\frac{y_2'}{y_2}(s) + k_{p+4}\frac{y_2''}{y_2}(s) + \ldots + k_{p+q+2}\frac{y_2^q}{y_2}(s)\right]y_2(s) \quad , q < v$$

$$-\left[k_{p+q+3} + k_{p+q+4}\frac{y_3'}{y_3}(s) + k_{p+q+5}\frac{y_3''}{y_3}(s) + \ldots + k_n\frac{y_3^r}{y_3}(s)\right]y_3(s) \quad , r < w \qquad (4\text{-}42)$$

or

$$u(s) = -\left[k_1 + \frac{k_2 c_1}{s+\alpha_1} + \frac{k_3 c_2}{s+\alpha_2} + \ldots + \frac{k_{p+1}c_p}{s+\alpha_p}\right]y_1(s)$$

$$-\left[k_{p+2} + \frac{k_{p+3}d_1}{s+\beta_1} + \frac{k_{p+4}d_2}{s+\beta_2} + \ldots + \frac{k_{p+q+2}d_q}{s+\beta_q}\right]y_2(s)$$

$$-\left[k_{p+q+3} + \frac{k_{p+q+4}e_1}{s+\gamma_1} + \frac{k_{p+q+5}e_2}{s+\gamma_2} + \ldots + \frac{k_n e_r}{s+\gamma_r}\right]y_3(s) \qquad (4\text{-}43)$$

The control law of Equation 4-43 specifies three compensation networks shaping the outputs of the three sensed quantities y_1, y_2, and y_3. Because the order of the filters is $p < n - 1$, $q < n - 1$, $r < n - 1$; $p + q + r = n - 3$, the poles of the filters may be selectively chosen from the roots of $(s + \alpha_i)$, $(s + \beta_j)$, and $(s + \gamma_k)$, which have negative real parts, avoiding the pole-zero cancellation problems of single-sensor synthesis.

The control law of Equation 4-43 suggests a methodical sequence one may use in the process of simplifying or approximating the optimal control law. The contribution of each state variable to the total compensation network depends upon the magnitude of the feedback gains associated with the particular state. For instance, if in the compensation network

$$Y(s) = \left[k_1 + \frac{k_2 c_1}{s+\alpha_1} + \frac{k_3 c_2}{s+\alpha_2} + \dots + \frac{k_{p+1} c_p}{s+\alpha_p} \right] \tag{4-44}$$

the feedback gain k_2 is found to add negligibly to the closed-loop dynamics, it may be possible to ignore the gain, yielding the approximate network

$$Y_1(s) = \left[k_1 + \frac{k_3 c_2}{s+\alpha_2} + \dots + \frac{k_{p+1} c_p}{s+\alpha_p} \right] \tag{4-45}$$

resulting in a simpler, lower order compensation network.

Required Dynamical Description of the Launch Vehicle

In order to synthesize the system with three measured quantities and three compensation networks, it is probably necessary, and certainly expedient, to describe the system in the following form:

$$\dot{x} = Fx + Gu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & 1 & 0 \\ 0 & \cdot & \cdot & \cdots & 0 & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{n-2} & -d_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \beta_c \tag{4-46}$$

$$y = Ax$$

$$\begin{bmatrix} y_1 \\ y_1' \\ y_1'' \\ \vdots \\ y_2 \\ y_2' \\ y_2'' \\ \vdots \\ y_3 \\ y_3' \\ y_3'' \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ a_0' & a_1' & \cdots & a'_{n-3} & a'_{n-2} & 0 \\ a_0'' & a_1'' & \cdots & a''_{n-3} & a''_{n-2} & 0 \\ \vdots & \vdots & & & & \\ b_0 & b_1 & \cdots & b_{n-3} & b_{n-2} & b_{n-1} \\ b_0' & b_1' & \cdots & b'_{n-3} & b'_{n-2} & 0 \\ b_0'' & b_1'' & \cdots & b''_{n-3} & b''_{n-2} & 0 \\ & & & & & \\ c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} & c_{n-1} \\ c_0' & c_1' & \cdots & c'_{n-3} & c'_{n-2} & 0 \\ c_0'' & c_1'' & \cdots & c''_{n-3} & c''_{n-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{p+1} \\ x_{p+2} \\ x_{p+3} \\ \vdots \\ x_{p+q+1} \\ x_{p+q+2} \\ x_{p+q+3} \\ \vdots \\ x_n \end{matrix} \qquad (4\text{-}47)$$

The elements of the F matrix are obtained from the open-loop characteristic polynomial $D(s) = s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \ldots + d_0$. The elements of the A matrix are obtained as follows: The first row of A is obtained from the numerator of the $y_1/\beta_c\ (s)$ transfer function $a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0$. The second row of A is obtained from

$$\frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0}{s+\alpha_1} = a'_{n-2}s^{n-2} + a'_{n-3}s^{n-3} + \ldots + a_0' \qquad (4\text{-}48a)$$

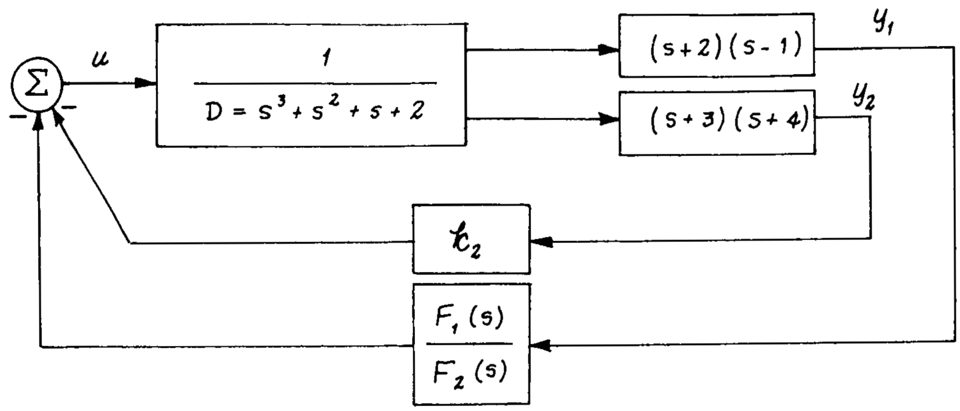
Similarly, the third row is obtained from

$$\frac{a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0}{s+\alpha_2} = a''_{n-2}s^{n-2} + a''_{n-3}s^{n-3} + \ldots + a_0'' \qquad (4\text{-}48b)$$

and all other rows of A are obtained in a similar manner.

Simple Example

Consider the plant represented by the block diagram



It is desired to synthesize the system as shown, with two feedback paths and with a realizable compensation network in the y_2 feedback path, such that the closed-loop characteristic polynomial is

$$\Delta(s) = s^3 + 2s^2 + 3s + 4.$$

From the transfer functions

$$\frac{y_1}{u}(s) = \frac{s^2 + s - 2}{s^3 + s^2 + s + 2} \qquad \frac{y_2}{u}(s) = \frac{s^2 + 7s + 12}{s^3 + s^2 + s + 2}$$

the state space description of the plant is obtained

$$\dot{x} = Fx + Gu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = Ax$$

$$\begin{bmatrix} y_1 \\ y_1' \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 12 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where the first row of A is obtained from numerator of the $y_1/u\ (s)$ transfer function $s^2 + s - 2$ and the second row is obtained from

$$\frac{s^2 + s - 2}{s + 2} = s - 1$$

while the third row is obtained from the numerator of the $y_2/u\ (s)$ transfer function $s^2 + 7s + 12$.

Solving for $u = -Kx$, we follow the procedure of Equation 4-24

$$\begin{aligned} -2 - k_1 &= -4 \qquad & k_1 &= 2 \\ -1 - k_2 &= -3 \qquad & k_2 &= 2 \\ -1 - k_3 &= -2 \qquad & k_3 &= 1 \end{aligned}$$

In terms of the variables y_1 , y_1' , and y_2 , the result is

$$u = -Kx = -KA^{-1}y$$

$$u = -\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 12 & 7 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_1' \\ y_2 \end{bmatrix} = -\frac{3}{4} y_1 + \frac{1}{2} y_1' - \frac{1}{4} y_2$$

Since $y_1'/y_1\ (s) = 1/s+2$, the control may be synthesized as

$$\begin{aligned} u(s) &= -\left[\frac{3}{4} - \frac{1/2}{s+2}\right] y_1(s) - \frac{1}{4} y_2(s) \\ &= -\left[\frac{\frac{3}{4}s + 1}{s+2}\ y_1(s) + \frac{1}{4}\ y_2(s)\right] \end{aligned}$$

which yields the desired closed-loop characteristic polynomial

$$\Delta(s) = s^3 + 2s^2 + 3s + 4.$$

Although the example is quite simple, it does illustrate the principles involved. The number of feedback paths is less than the number of states, yet greater than one. A compensation network has been systematically specified for the non-minimum phase variable y_1, yet the compensation network is realizable, containing no right-half plane poles. In addition, it would be relatively easy to investigate the consequences of omitting a feedback gain, but a simplification of this type would not increase the order of the closed-loop system.

The multiple-feedback, multiple-network synthesis procedure was not tried in the elastic launch vehicle problem, but it is clear that the techniques illustrated are directly applicable.

Response of the Optimal System

The closed-loop response of the vehicle, using the feedback control laws of Equation 4-32 or Equation 4-30, is shown in Figures 4.5, 4.6 and 4.7. Figure 4.5 shows the normal acceleration response at the three sensor locations indicated in Equation 4-32. The input command was a step, a severe excitation to such a system, but calculated to excite the bending motion and enhance the residues of the poles originally associated with the bending modes. The effect of the first bending mode shows clearly only at station $x_3 = 122.5$ meters. The second mode oscillations are clearly shown at all three locations. These oscillations are reasonably well damped (for a bending mode), the measured damping ratio of this mode closely matches the predicted $\zeta = .12$ from the root square locus plot. Figure 4.6 shows the attitude response of the vehicle at the c.g., $x = 41.5$ meters. The response is smooth, showing very little contribution from either bending mode, but this is to be expected because the mode slopes are small at this location. Figure 4.7 shows the control motions required to achieve the responses shown in the previous two figures. Except for the initial abrupt response, the control motion is reasonable, and shows, in general, that the system could probably have been speeded up somewhat before encountering an unacceptable control motion.

Parameter Variations

The bending characteristics of a large flexible launch vehicle are known only to a fair degree of accuracy. The normal bending mode shapes and slopes are subject to uncertainties depending upon the mode number and the vehicle body station. It is necessary, therefore, to calculate the effects of variations of the elastic properties of the vehicle on the closed-loop optimal system. A series of variations in these elastic properties was specified and their effect on the system, defined by the control law of Equation 4-30, was calculated. The transformation B relating the vector z to the vector y is defined by the equation shown at the top of page 53.

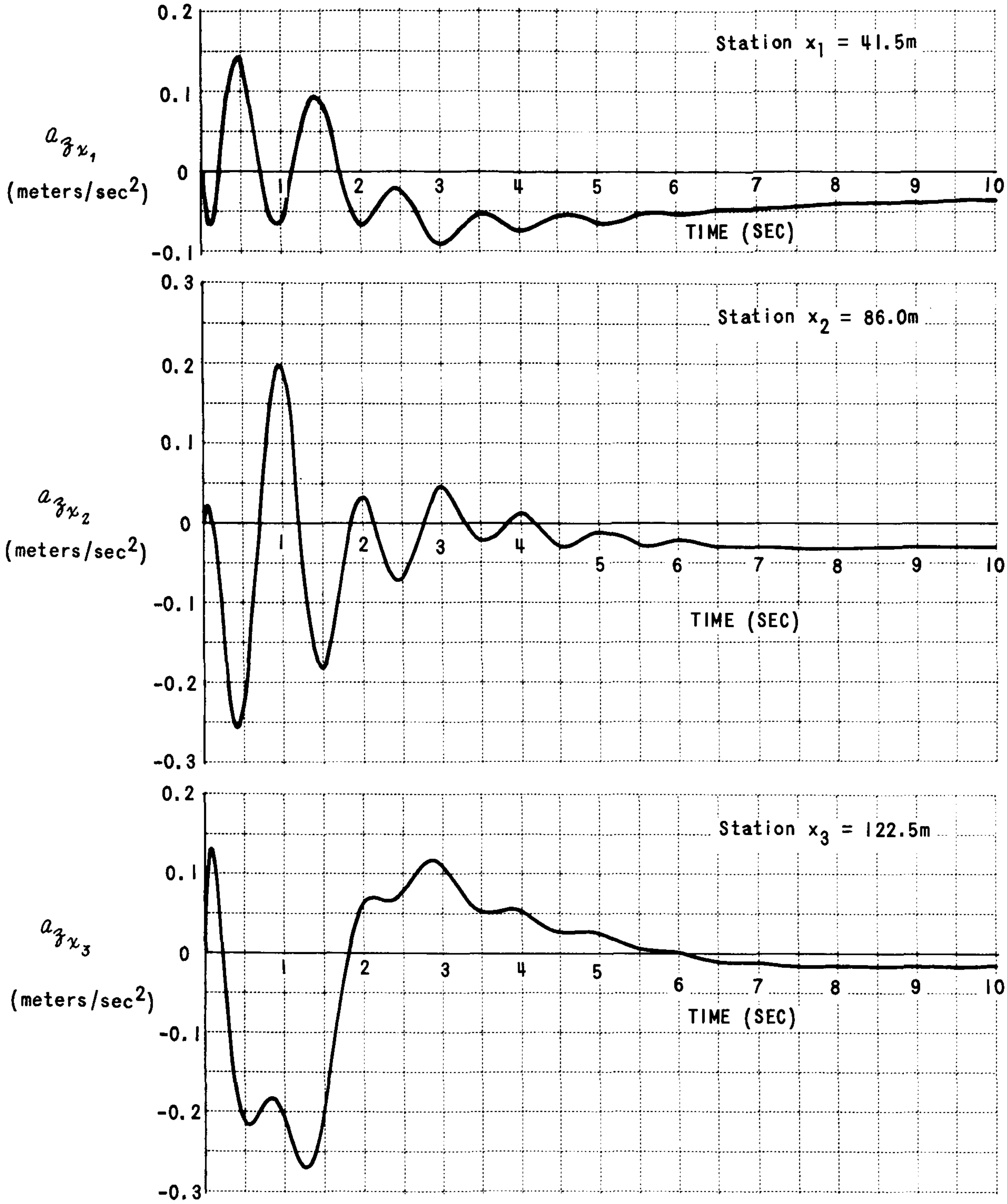


Figure 4.5 Normal Acceleration Response at Three Body Stations

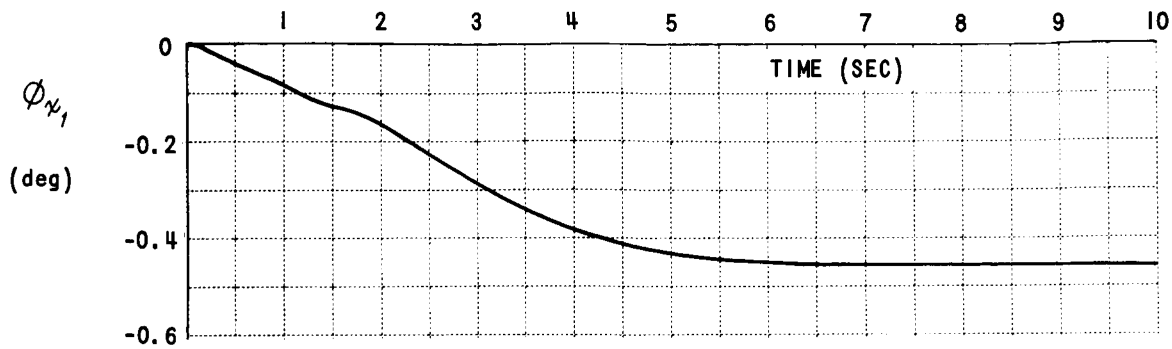


Figure 4.6 Attitude Response of Optimal System at Booster Center of Gravity

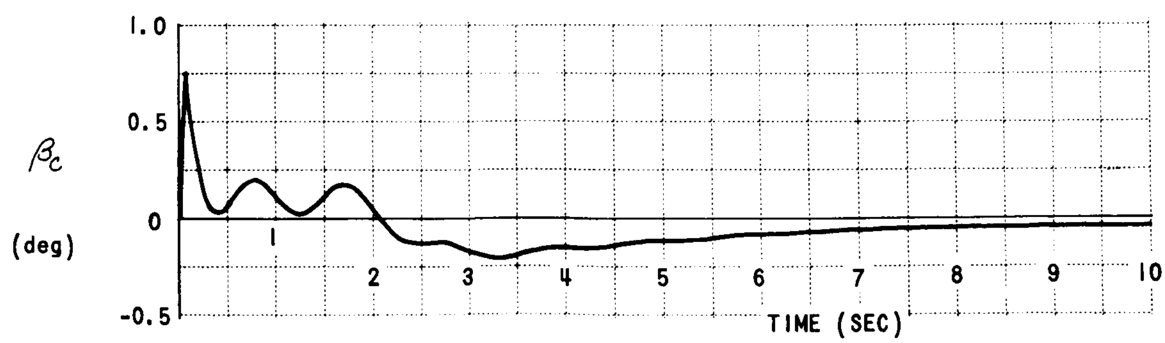


Figure 4.7 Time History of Optimal Control Motion

$$
\begin{bmatrix} \phi_{x_1} \\ \dot{\phi}_{x_1} \\ a_{z_{x_1}} \\ \Delta\phi_1 \\ \Delta\dot{\phi}_1 \\ \Delta\phi_2 \\ \Delta\dot{\phi}_2 \\ \beta \end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & -Y'_{1(x_1)} & 0 & -Y'_{2(x_1)} & 0 & 0 \\
0 & 1 & 0 & 0 & -Y'_{1(x_1)} & 0 & -Y'_{2(x_1)} & 0 \\
0 & 0 & \begin{matrix} 5.54 + 5.45\,Y_{1(x_1)} \\ + 2.36\,Y_{2(x_1)} \end{matrix} & \begin{matrix} -.78 - 5.37\,Y_{1(x_1)} \\ + 21.05\,Y'_{1(x_1)} \end{matrix} & -.0232\,Y_{1(x_1)} & \begin{matrix} 1.04 - 31.8\,Y_{2(x_2)} \\ + 21.05\,Y'_{2(x_2)} \end{matrix} & -.0564\,Y_{2(x_1)} & \begin{matrix} 10.93 + 15.8\,Y_{1(x_1)} \\ + 22.77\,Y_{2(x_1)} \end{matrix} \\
0 & 0 & 0 & -Y'_{1(x_1)} + Y'_{1(x_2)} & 0 & -Y'_{2(x_1)} + Y'_{2(x_2)} & 0 & 0 \\
0 & 0 & 0 & 0 & -Y'_{1(x_1)} + Y'_{1(x_2)} & 0 & 0 & 0 \\
0 & 0 & 0 & -Y'_{1(x_1)} + Y'_{1(x_3)} & 0 & -Y'_{2(x_1)} + Y'_{2(x_3)} & 0 & 0 \\
0 & 0 & 0 & 0 & -Y'_{1(x_1)} + Y'_{1(x_3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha_R \\ \eta_1 \\ \dot{\eta}_1 \\ \eta_2 \\ \dot{\eta}_2 \\ \beta_L \end{bmatrix}
$$

where

$Y_{j(x_i)}$ = normalized mode deflection of the j^{th} mode at the i^{th} body station

$Y'_{j(x_i)}$ = normalized mode slope of the j^{th} mode at the i^{th} body station

At body stations x_1 = 41.5 in., x_2 = 86.0 in. and x_3 = 122.5 in., the nominal bending mode and slope values and their selected variations from the nominal are given in Table 4.1 below:

TABLE 4.1

NOMINAL MODE CHARACTERISTICS AND EXPECTED RANGE OF VARIATIONS

Mode Characteristic	Nominal Value	Expected Range of Variation
$Y_{1(x_1)}$	-0.390	±15%
$Y'_{1(x_1)}$	+0.028	±15%
$Y_{2(x_1)}$	-0.555	±15%
$Y'_{2(x_1)}$	+0.0085	±15%
$Y'_{1(x_2)}$	-0.025	±25%
$Y'_{2(x_2)}$	-0.060	±35%
$Y'_{1(x_3)}$	-0.123	±25%
$Y'_{2(x_3)}$	+0.143	±25%

A brief program was conducted to investigate the variations of the closed-loop poles of the system with the control law fixed as defined by Equation 4-30. The results for a few of the parameter variations are tabulated in Table 4.2 below:

TABLE 4.2

VARIATIONS OF THE CLOSED-LOOP POLES

NOMINAL	ALL MODE PARAMETERS INCREASED (SEE TABLE 4.1)	ALL MODE PARAMETERS DECREASED (SEE TABLE 4.1)	$Y'_{2(x_3)}$ ONLY INCREASED	$Y'_{1(x_3)}, Y'_{2(x_3)}$ INCREASED
s = -.00770	-.00833	-.00790	-.00750	-.00820
s = -17.901	-17.901	-17.901	-17.901	-17.901
s = -.612 ± j.547	-.614 ± j.478	-.615 ± j.526	-.611 ± j.565	-.618 ± j.496
s = -1.200 ± j2.304	-1.187 ± j2.319	-1.197 ± j2.309	-1.200 ± j2.299	-1.193 ± j2.316
s = -.743 ± j6.155	-.744 ± j6.154	-.744 ± j6.155	-.744 ± j6.156	-.744 ± j6.1515

A casual examination of Table 4.2 will disclose that the optimal system is not particularly sensitive to variations of the bending mode parameters of the launch vehicle, even though the variations of the closed-loop roots tabulated in Table 4.2 represent the widest change of closed-loop dynamics discovered using the selected mode characteristics of Table 4.1. The variations were selected at random, and therefore an absolute conclusion concerning insensitivity cannot be made. However, all evidence points to a very insensitive design.

Control Law Approximation

A least squares fit of the control law of Equation 4-30 using only three of the eight state variables was calculated to determine if a direct approximation to the exact control law could result in an acceptable elastic booster control system design.

The requirement is that the outputs of three sensors located at station x_1 = 41.5 meters are to be used to approximate the vector

$$y' = \left[\phi_R, \dot{\phi}_R, \alpha, \eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \beta\right]$$

and therefore the optimal control law of Equation 4-30. The relationships defining the outputs of these three sensors are:

$$\begin{bmatrix} \phi_{x_1} \\ \dot{\phi}_{x_1} \\ a_{z_{x_1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -.028 & 0 & -.0085 & 0 & 0 \\ 0 & 1 & 0 & 0 & -.028 & 0 & -.0085 & 0 \\ 0 & 0 & 2.105 & 1.904 & .00905 & 16.786 & .0313 & -7.88 \end{bmatrix} \begin{bmatrix} \phi_R \\ \dot{\phi}_R \\ \alpha \\ \eta_1 \\ \dot{\eta}_1 \\ \eta_2 \\ \dot{\eta}_2 \\ \beta \end{bmatrix} \tag{4-49}$$

$$z_1 = B_1 y$$

where B_1 consists of the first three rows of the square matrix B defined by Equation 4-33.

The least squares fit is calculated by computing the generalized inverse of B_1 called B_1^+ (Ref. 7). The generalized inverse is obtained by solving for the matrices N and M such that

$$B_1' B_1 M = N B_1 B_1' = B_1' \tag{4-50}$$

then

$$B_1^+ = N B_1 M$$

The approximated feedback control law then becomes

$$\beta_0 = -K A^{-1} B_1^+ z_1 \tag{4-51}$$

For this particular problem, the least squares fit to the control law of Equation 4-30 is

$$\beta_0 = 1.9395\, \phi_{x_1} + 5.362\, \dot{\phi}_{x_1} - 0.01035\, a_{z_{x_1}} \tag{4-52}$$

The feedback control law of Equation 4-52 can be easily mechanized. The closed-loop characteristic polynomial obtained by using the approximate optimal control law of Equation 4-33 is:

$$\Delta(s) = (s + .0225)(s + .377)\left[s + .428 \pm j1.56\right]\left[s + 6.93 \pm j4.68\right]\left[s + .703 \pm j5.61\right]$$

Comparing this characteristic polynomial with the characteristic polynomial of Equation 4-21 shows that the approximation has preserved stability but the drift minimum property of the design has been compromised. It appears, however, that the technique shows promise for obtaining acceptable control system design approximations of the optimal system, although there is no reason to believe that all approximations of the type given above will result in stable closed-loop configurations.

Perhaps the best way to incorporate the idea of the generalized inverse into the control problem is to use the generalized inverse theory to adjust the zeros of the compensation networks of Equation 4-45 after the k_i having negligible effect on the closed loop dynamics have been deleted from the control law.

Addition of the Third Mode

A third bending mode was added to the definition of the launch vehicle equations of motion. The purpose of this addition was to obtain the effect on the third mode of a control system designed to increase the damping ratios of only the first two modes. The equations, which include the third mode, are given in Appendix A. The third mode has a natural frequency of ω = 9.18 rad/sec and a damping ratio ζ = .005. Using the control law of Equation 4-28, the closed-loop poles originating from the third mode were unstable, having a closed-loop natural frequency of ω_n = 7.885 and a damping ratio of ζ = -.206. This mode could, of course, be included in the system design and stabilized with a corresponding increase in the complexity of the control law.

It should be stated that the sensor positions selected for the control law of Equation 4-32 were chosen without regard to the third bending mode characteristics at the particular sensor locations. An obvious way of reducing the effect of the control law on higher modes is to judiciously locate the sensors to minimize the mode pickup. A less obvious way is to add a constraint to the problem that would have as an objective a design that would decrease the sensitivity of the third mode. This design would result in a filter to desensitize the third mode.

A few generalities can be made on the use of a model to specify a desirable response. The technique seems to be the straightforward, simplified way to design a closed-loop system, optimal or otherwise. The optimal aspect of the problem is secondary if the only requirement is to match the poles of the closed-loop system to the poles of a model. The optimal aspect of the selection of approximations to the model should be preserved for at least three reasons which are implied, but not proven, in this section. First, the optimal formulation of the problem considers control motion deflections. If the control motion amplitudes are too large (for a specific input) it is necessary only to weight the control portion of the performance index more heavily, resulting in lower amplitude control motions. No other design technique, particularly for complex systems, can specifically consider the control motions as part of the problem objectives. A second advantage of optimal control is also peculiar to the technique. It has been observed that the closed-loop poles rapidly approach the Butterworth configuration, i.e., the asymptotes of the locus. In general, the root square locus plot appears to approach its asymptotic values faster than for an ordinary root locus plot. Once the closed-loop poles approach the asymptotic values, the poles become relatively insensitive to changes of the coefficients of the original equations of motion. Since a large number of examples have shown that the root square locus approaches its asymptotes at a lower increment in frequency than an equivalent root locus, it can be safely stated that a linear optimal system is equivalently less sensitive than a conventionally designed system. Finally, the root square locus expression enables the designer to systematically select all of the poles of the system, particularly when the model and the plant are of different order.

SECTION 5

THE APPLICATION OF LINEAR OPTIMAL CONTROL TO SYSTEMS CONTAINING UNCERTAIN PARAMETERS

Introduction

Most of the synthesis techniques available to the designer depend upon the assumption that the plant parameters are well known. This assumption is not valid when one considers the design of a controller for a flexible booster. Variations in the plant parameters are, of course, expected to occur over the flight envelope of the booster, but quite aside from this effect, there is the problem that even at a specific flight condition (where one can usually assume the plant parameters are constant) the values of the various coefficients, particularly the flexible mode shapes and slopes, are not known exactly.

The bulk of the theory which treats this case has, in the past, considered the situation in which the statistical variations of the plant parameters take place over a period of time. The situation which is of more interest is the case where the plant parameters are relatively constant over time but can be considered as random variables described on a sample space. Some recent research in this area has approached the problem by incorporating some measure of the randomness or variability into the performance index prior to the minimization and specification of an optimal control law. In this way an expression for the optimal control is developed which reflects the fact that some of the plant parameters may not be at their nominal value.

One approach to this problem can be formulated by asking for the compensating networks which minimize the $E\{2V\}$. That is,

$$\underset{\substack{\text{comp.}\\ \text{network}}}{\min} E\{2V\} = \underset{\substack{\text{comp.}\\ \text{network}}}{\min} E\left\{ \int_0^T (y'Qy + u'Ru)dt \right\} \quad *$$

If the equations of motion which describe the system are manipulated into the vector form

$$\dot{x} = Fx + Gu$$

where F and G have entries which are random variables with the joint density p(F, G), this problem can be stated as

$$\underset{\substack{\text{comp.}\\ \text{network}}}{\min} \iint p(F,G) \left[\int_0^T (y'Qy + u'Ru)dt \right] dF dG$$

In this section, it is not our intent to formulate the problem in the most general manner possible (that is, consider the multi-controller and multi-output situation) since it is desirable that we gradually work our way into the problem and develop a "feel" for the mathematics involved. It will suffice to

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*In this report expressions such as $\underset{u}{\min} E\{2V\}$ will read as "The control **u** which minimizes the expected value of 2V.

say that all the tools required to extend the single-input, single-output case to the multi-controller case are already available within the present framework of the calculus of variations. Thus we will be content with outlining the background philosophy, solving three simple illustrative problems, and applying what has been developed to the task of investigating the characteristics of the optimum compensating network required for a flexible booster (only the first bending mode will be considered).

The equations which define the conditions of optimality will be derived using frequency domain methods. Appendix B considers some of the difficult points which arise with nonminimum phase systems since the transfer functions which describe the booster have both poles and zeros in the right-half plane.

Preliminary Comments

To begin, assume the feedback control system shown in Figure 5.1.

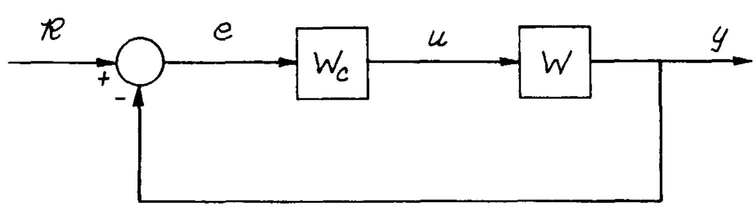

Figure 5.1 Forward Loop Compensation

The compensating network, W_c, is to be designed in such a fashion that the expression

$$2V = \int_0^\infty \left[e(t)^2 + r u(t)^2 \right] dt \tag{5-1}$$

is minimized. In Figure 5.1, u represents the control (or input) to the fixed elements of the system W(s). The above statement may be expressed in the form

$$\min_{W_c} \int_0^\infty \left[e(t)^2 + r u(t)^2 \right] dt \tag{5-2}$$

Although it may be possible, it is believed that the analytical difficulties involved in minimizing Equation 5-1 by solving directly for the optimum compensating network are insurmountable. In the past, some writers (Reference 8) have attempted to sidestep the difficulties involved by reformulating the problem in an open-loop manner, and then solving for an open-loop compensating network. That is, they formulated the problem as depicted in Figure 5.2.

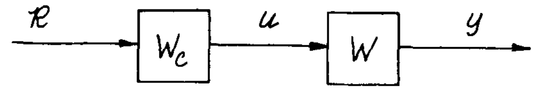

Figure 5.2 Open-Loop Configuration

This, of course, is completely unsatisfactory when W(s) contains nonminimum phase components.

In the event that W(s) is completely known, the analytical difficulties involved in finding the optimum compensating network may be avoided by solving for the optimum value of some other system variable such as the optimum control or the optimal error. These problems turn out to be quite tractable and present no insurmountable analytical road-blocks. Once either the optimal error or the optimal control has been found, it is a simple matter of algebra to solve for the optimal compensating network. As an alternate procedure, one might elect to solve for both the optimal error and optimal control and then define

$$W_c = \frac{\min\limits_{u} \int_0^\infty (e^2 + ru^2)\,dt}{\min\limits_{e} \int_0^\infty (e^2 + ru^2)\,dt}$$

All of these procedures obviously yield the same expression for W_c when W is completely deterministic.

When W(s) contains uncertain coefficients, described only by probability density functions, it seems reasonable to find the compensating network which minimizes the expected value of the performance index. However, one cannot solve for just the optimal control which minimizes the $E\{2V\}$ and then algebraically manipulate the blocks to find W_c since W is unknown (for any given system). Nevertheless, it can still make sense to define

$$W_c = \frac{\min\limits_{u} E\left\{\int_0^\infty (e^2 + ru^2)\,dt\right\}}{\min\limits_{e} E\left\{\int_0^\infty (e^2 + ru^2)\,dt\right\}} \tag{5-3}$$

That is, define the compensating network to be the ratio of the control which minimizes the expected value of the performance index to the error which minimizes the expected value of the performance index. Defining the compensation in this manner is a subjective decision which is certainly open to discussion. However, one feels that the final decision as to the "goodness" of the choice should rest with the results achieved -- namely, does it design "acceptable" systems.

It is, of course, difficult enough to decide when a system is acceptable even when all the parameters are known precisely -- thus we expect that our difficulty in deciding this question must increase as our knowledge of the system becomes less precise. However, the simple illustrative examples which follow do seem to indicate that there is merit in the idea of finding the compensating network which minimizes the expected value of the performance index. Notice that this does not necessarily imply that the performance will be less sensitive to parameter variations nor does it guarantee that any particular system which uses the average compensation prescribed by the analysis will be stable. (Of course, one would hope that this might be a by-product of the design procedure.) It would thus seem worthwhile to expend additional effort to determine the relationships, if any, which exist between parameter sensitivity, stability and the minimum of the expected value of the performance index.

We will take Equation 5-3 as the definition of the compensating network to be employed for the system depicted in Figure 5.1. If a different feedback configuration is desired, say that of Figure 5.3, W_c might be defined as

$$W_c = \frac{\mathcal{R} - \min\limits_{u} E\left\{\int_0^\infty (e^2 + ru^2)dt\right\}}{\mathcal{R} - \min\limits_{e} E\left\{\int_0^\infty (e^2 + ru^2)dt\right\}} \tag{5-4}$$

where

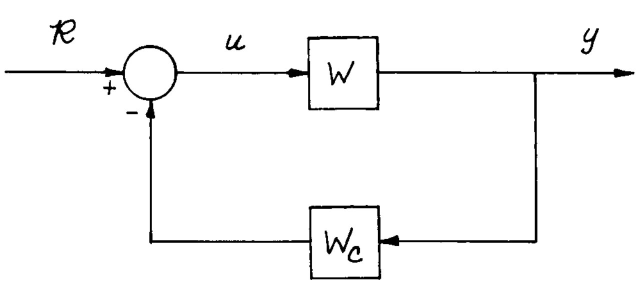

Figure 5.3 Feedback Compensation

It is still necessary, in any alternate compensating scheme, to solve for at least two optimal variables of the system.

In the deterministic case, one may show that the following Wiener-Hopf equations define the optimal conditions (References 2 and 9):

$$\left.\begin{aligned}
\left[r + W\bar{W}\right] u_0 - \mathcal{R}\bar{W} &= \mathfrak{z}_1(s) \\
\left[1 + \frac{r}{W\bar{W}}\right] e_0 - \frac{r\mathcal{R}}{W\bar{W}} &= \mathfrak{z}_2(s) \\
\left[1 + \frac{r}{W\bar{W}}\right] y_0 - \mathcal{R} &= \mathfrak{z}_3(s)
\end{aligned}\right\} \tag{5-5}$$

where $W = W(s)$, $\bar{W} = W(-s)$ and $\mathfrak{z}_1$, $\mathfrak{z}_2$, and $\mathfrak{z}_3$ have inverse Laplace transforms which are zero for $t \geq 0$. In the event that $W(s)$ is a minimum phase system, it suffices to say that $\mathfrak{z}_1(s)$, $\mathfrak{z}_2(s)$, and $\mathfrak{z}_3(s)$ are analytic in the left-half plane. Statements such as $\mathfrak{z}_1(t) = 0$ for $t \geq 0$ are sufficient conditions which, when satisfied, insure a minimum value for the performance index. It may occur, for example, that $\mathfrak{z}_2(s)$ = constant. In this case the sufficient condition breaks down and one must resort to a more general condition which is both necessary and sufficient (refer to Appendix B).

In the nondeterministic case, precisely the same sort of analysis leads to the following expressions (again refer to Appendix B)

$$\left.\begin{aligned}
\left[r + E\{W\bar{W}\}\right] u_0 - \mathcal{R}E\{\bar{W}\} &= \mathfrak{z}_1(s) \\
\left[1 + rE\left\{\frac{1}{W\bar{W}}\right\}\right] e_0 - r\mathcal{R}E\left\{\frac{1}{W\bar{W}}\right\} &= \mathfrak{z}_2(s) \\
\left[1 + rE\left\{\frac{1}{W\bar{W}}\right\}\right] y_0 - \mathcal{R} &= \mathfrak{z}_3(s)
\end{aligned}\right\} \tag{5-6}$$

where $E\{\ \}$ represents the operation of taking the expected value of the quantity in brackets. Again $\mathcal{L}^{-1}[\mathfrak{z}_1(s)]$, etc. are required to be zero for $t \geq 0$.

In the multi-control case, if the equations which represent the system are given in the first-order form

$$\dot{x} = Fx + Gu \qquad y = Hx$$

with

$$2V = \int_0^\infty (y'Qy + u'\mathcal{R}u)\,dt$$

and where F and G have entries which are random variables with the joint density p(F, G), the analogous procedure is simply to find the optimal control which minimizes the expected value of the performance index as well as the expressions for the optimal state variables which minimize the expected value of the performance measure. Thus, the sum of the convolutions of unknown network impulse responses with the optimal state variables, which are set equal to the optimal control, will serve to define the required feedback. The tools for this type of an analysis are already available within the framework of the classical calculus of variations.

Illustrative Example No. 1

Experience indicates that the simple first-order system shown in Figure 5.4 is adequate to demonstrate the important features of the theory.

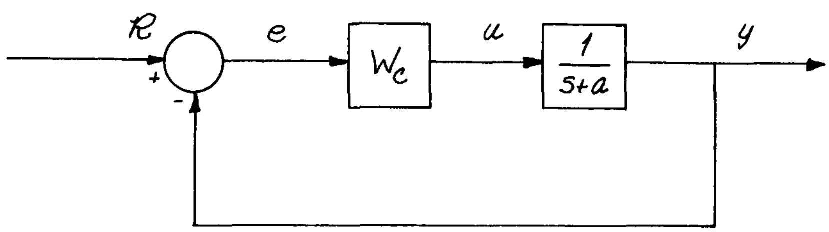

Figure 5.4 First-Order System

It is assumed that the exact value of a is unknown, being described only by the uniform probability density function shown in Figure 5.5 and that $\mathcal{R}(s)$ is a step input.

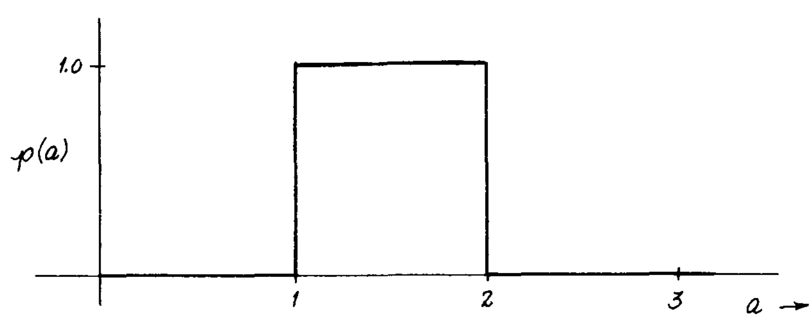

Figure 5.5 Uniform Probability Density

In order to specify W_c, we solve for both the optimal error and the optimal control. The Wiener-Hopf equation which must be solved in order to determine the error which minimizes the expected value of the performance index is

$$\left[\frac{1}{r} + E\left\{\frac{1}{W\bar{W}}\right\}\right]e_0 - RE\left\{\frac{1}{W\bar{W}}\right\} = \mathfrak{z}$$

Since
$$\frac{1}{W\bar{W}} = -s^2 + a^2,$$

$$E\left\{\frac{1}{W\bar{W}}\right\} = \int_{-\infty}^{\infty} (-s^2 + a^2)\,p(a)\,da$$

$$= -s^2 + E\{a^2\}.$$

$$E\{a^2\} = \int_1^2 a^2 da = \frac{a^3}{3}\Big]_1^2 = \frac{7}{3}$$

$$\therefore E\left\{\frac{1}{W\bar{W}}\right\} = -s^2 + \frac{7}{3}$$

$$\doteq (-s + 1.527)(s + 1.527)$$

The Wiener-Hopf equation becomes, after letting $R(s) = 1/s$,

$$\left[-s^2 + \frac{7}{3} + \frac{1}{r}\right] e_0 - \frac{1}{s}\left(-s^2 + \frac{7}{3}\right) = z$$

or

$$\left[(s+\alpha)(-s+\alpha)\right] e_0 - \frac{1}{s}\left(-s^2 + \frac{7}{3}\right) = z \tag{5-7}$$

Let
$$e_0 = \frac{C_0 s + C_1}{s(s+\alpha)} \; ^*$$

where C_o and C_1 are unknown coefficients and substitute into Equation 5-7:

$$\frac{(-s+\alpha)(C_0 s + C_1) - (-s^2 + 7/3)}{s} = z(s) \tag{5-8}$$

In order for $\int_0^\infty z(t) e_1(t)\, dt = 0$, it is sufficient that the numerator on the left-hand side of Equation 5-8 be zero for $s = 0$ and $s = \infty$. (Refer to Appendix B where the case for which $z(s)$ = constant is discussed.)

At $s = 0$: $\alpha C_1 = 7/3$

$C_1 = 7/3\alpha$

As $s \rightarrow \infty$, $-C_0^2$ must equal $-s^2$.
Therefore,

$C_1 \equiv 1.$

---

* In finding the optimal error, a direct method described in Reference 9 is being used.

The final result is

$$e_0 = \frac{s + \frac{7}{3\alpha}}{s(s+\alpha)}$$

Notice that this result gives $z(s) = (\alpha - 7/3\alpha)$, which is analytic in the left-half plane. Hence the e_0 given is the optimal one.

It is important to note that one may employ a root square locus analysis (Reference 2) on Equation 5-7 to find the optimal closed-loop roots. This would be done for a more complicated example, but it is not at all necessary for this simple case since

$$\alpha = \sqrt{\frac{7}{3} + \frac{1}{r}} \tag{5-9}$$

We proceed now to find the optimal control by solving the Wiener-Hopf equation

$$\left[r + E\{W\bar{W}\}\right]u_0 - \mathcal{R}E\{\bar{W}\} = z_1 \tag{5-10}$$

This will turn out to be a more difficult task in which one is confronted with the task of analyzing Laplace transforms which are transcendental functions of s.

The first task is to evaluate the

$$E\{W\bar{W}\} = E\left\{\frac{1}{-s^2+a^2}\right\}$$

$$= \int_{-\infty}^{\infty} \frac{p(a)da}{-s^2+a^2} = \int_1^2 \frac{da}{-s^2+a^2}$$

Thus

$$E\{W\bar{W}\} = -\frac{1}{2s} \ln \left|\frac{s+a}{s-a}\right|_1^2 = -\frac{1}{2s} \ln \left|\frac{s^2+s-2}{s^2-s-2}\right| \tag{5-11}$$

Using the series

$$\ln x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots\right] \tag{5-12}$$

valid for $x > 0$, one obtains

$$E\{W\bar{W}\} = -\frac{1}{s}\left[\frac{s}{s^2-2} + \frac{1}{3}\left(\frac{s}{s^2-2}\right)^3 + \frac{1}{5}\left(\frac{s}{s^2-2}\right)^5 + \dots\right], \quad \left|\frac{s^2+s-2}{s^2-s-2}\right| > 0 \tag{5-13}$$

Checking the series, it is found that the first two terms afford an excellent approximation to the log function and one may use

$$E\{W\bar{W}\} \doteq \frac{[(s+1.385)^2+(.283)^2][(-s+1.385)^2+(.283)^2]}{(s+\sqrt{2})^3(-s+\sqrt{2})^3}$$

Notice that the $\sqrt{2}$ is the geometric mean of the density function.

The next task is to evaluate $E\left\{\frac{1}{-s+a}\right\}$:

$$E\left\{\frac{1}{-s+a}\right\} = \int_1^2 \frac{da}{-s+a} = \ln\left|-s+a\right|_1^2 = \ln\left|\frac{-s+2}{-s+1}\right| \qquad (5\text{-}14)$$

Using Equation 5-12 gives

$$E\left\{\frac{1}{-s+a}\right\} = 2\left[\frac{\frac{1}{2}}{-s+\frac{3}{2}} + \frac{\left(\frac{1}{2}\right)^3}{3\left(-s+\frac{3}{2}\right)^3} + \cdots\right]$$

$$= \frac{1}{-s+\frac{3}{2}}\left[1 + \frac{\left(\frac{1}{2}\right)^2}{3\left(-s+\frac{3}{2}\right)^2}\right]$$

$$\doteq \frac{1}{-s+\frac{3}{2}}\left[\frac{s^2-3s+\frac{9.333}{4}}{s^2-3s+\frac{9}{4}}\right]$$

or

$$E\left\{\frac{1}{-s+a}\right\} \doteq \frac{1}{s+\frac{3}{2}}\left[\frac{\left(-s+\frac{3}{2}\right)^2+(.289)^2}{\left(-s+\frac{3}{2}\right)^2}\right]$$

The Wiener-Hopf equation for the optimal control is now

$$\left\{r + \frac{[(s+1.385)^2+(.283)^2][(-s+1.385)^2+(.283)^2]}{(s+\sqrt{2})^3(-s+\sqrt{2})^3}\right\}u_o - \frac{R\left[\left(-s+\frac{3}{2}\right)^2+(.289)^2\right]}{\left(-s+\frac{3}{2}\right)^3} = \mathfrak{z} \qquad (5\text{-}15)$$

The root square locus for this system, plotted as $|s|$ vs relative damping in Figure 5.6 shows that for values of $1/r$ of approximately one, the equation

$$(s+1.385)^2 + (.283)^2 = 0$$

defines two of the optimal roots of the system.

Solving for the optimal control, using the equation (Reference 9)

$$u_o = \frac{1}{Y}\left[\frac{G}{\bar{Y}}\right]_+$$

where $Y\bar{Y}u_o - G = z$

one finds, when $1/r = 68/3$, for example,

$$u_o \doteq \frac{(s+\sqrt{2})^3}{(s+5.001)\left[(s+1.385)^2+(.283)^2\right]}\left[\frac{\left(\frac{68}{3}\right)(-s+\sqrt{2})^3\left[\left(-s+\frac{3}{2}\right)^2+(.289)^2\right]}{s(-s+5.001)\left(-s+\frac{3}{2}\right)^3\ (-s+1.385)^2+(.238)^2}\right]_+ \tag{5-16}$$

$$\doteq \frac{4.44\,(s+\sqrt{2})^3}{s(s+5.001)\left[(s+1.385)^2+(.283)^2\right]} \tag{5-17}$$

Thus

$$W_c = \frac{u_o}{e_o} = 4.44\left(\frac{s+5}{s+5.001}\right)\left\{\frac{(s+\sqrt{2})^2}{(s+1.385)^2+(.283)^2}\right\}\left(\frac{s+\sqrt{2}}{s+7/15}\right) \tag{5-18}$$

since $e_o = \dfrac{s+7/15}{s\,(s+5)}$ when $\dfrac{1}{r} = \dfrac{68}{3}$

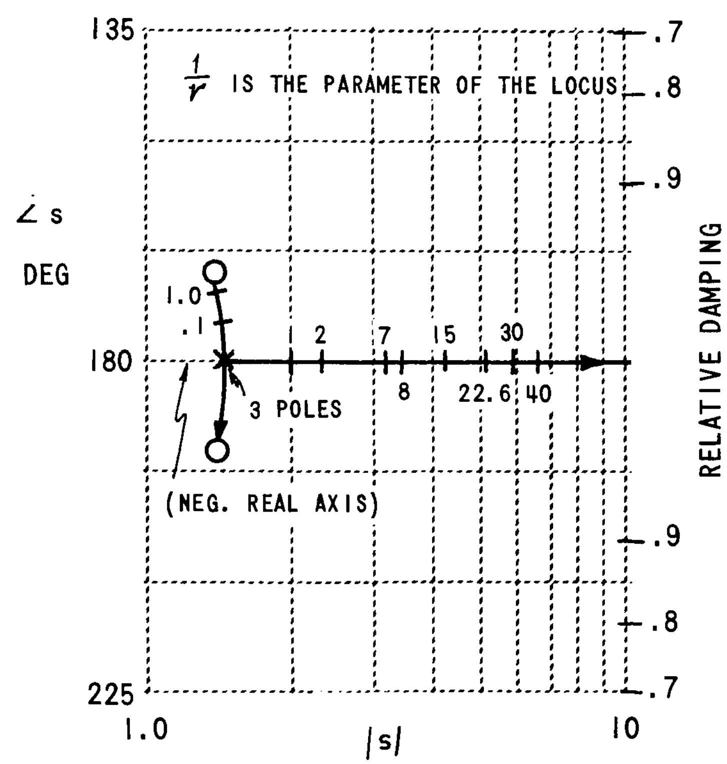

Figure 5.6 Root Square Locus for Example 1

If p(a) had the form given in Figure 5.7,

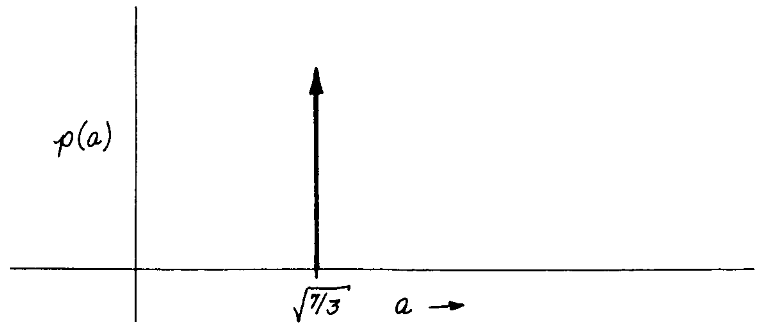

Figure 5.7 Delta Function Density

the optimal compensating network would have been

$$\frac{(4.53)(s+\sqrt{7/3})}{(s+7/15)} \tag{5-19}$$

It is apparent that there will be no essential difference between the closed-loop system which uses the compensation network given by Equation 5-18 and the system using the compensation of Equation 5-19. That is, the parameter uncertainty was not enough to cause any drastic change in the compensating network.

Illustrative Example No. 2

It is fairly obvious, from the first example, that the order of the compensating network will increase as the information concerning the system becomes more "distributed". This is not a surprising result and we expect it to hold true for all realistic density functions. To illustrate this, the optimal compensating network for the density function of Figure 5.8 and the open-loop system of the previous example will be found.

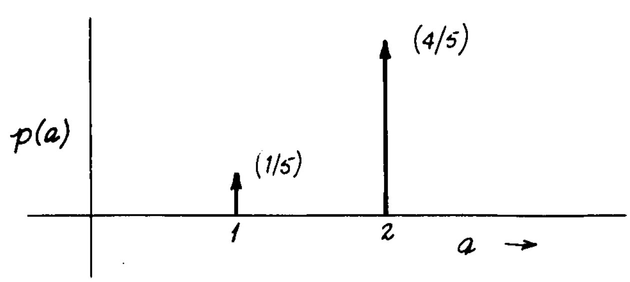

Figure 5.8 Density Function for Example 2

The various averages are:

$$E\left\{\frac{1}{W\bar{W}}\right\} = \int_{-\infty}^{\infty}(-s^2+a^2)\left[\frac{1}{5}\,\delta(a-1)+\frac{4}{5}\,\delta(a-2)\right]da = -s^2+\frac{17}{5}$$

$$E\left\{W\bar{W}\right\} = \int_{-\infty}^{\infty}\frac{\left[\frac{1}{5}\,\delta(a-1)+\frac{4}{5}\,\delta(a-2)\right]da}{-s^2+a^2} = \frac{-s^2+\frac{8}{5}}{(-s^2+1)(-s^2+4)} \tag{5-20}$$

$$E\left\{\bar{W}\right\} = \int_{-\infty}^{\infty}\frac{\left[\frac{1}{5}\,\delta(a-1)+\frac{4}{5}\,\delta(a-2)\right]da}{-s+a} = \frac{-s+\frac{6}{5}}{(-s+1)(-s+2)}$$

The expression for the optimal control, when $\mathcal{R} = 1/s$, becomes:

$$\left[1+\frac{\frac{1}{r}\left(-s^2+\frac{8}{4}\right)}{(-s^2+1)(-s^2+4)}\right]u_0 - \frac{\frac{1}{r}}{s}\;\frac{\left(-s+\frac{6}{5}\right)}{(-s+1)(-s+2)} = \mathfrak{z}_1 \tag{5-21}$$

If $r = 2/15$:

$$\left[\frac{(s^2+4.25s+4)(s^2-4.25s+4)}{(s+1)(s+2)(-s+1)(-s+2)}\right]u_0 - \frac{\frac{15}{2}(-s+1.2)}{s(-s+1)(-s+2)} = \mathfrak{z}_1$$

The optimal control is

$$u_0 = \frac{225(s+1)(s+2)}{s(s+3.333)(s+1.2)} \tag{5-22}$$

The Wiener-Hopf equation for the optimal error is:

$$\left[7.5+\left(-s^2+\frac{17}{5}\right)\right]e_0 - \mathcal{R}\left[-s^2+\frac{17}{5}\right] = \mathfrak{z}_2$$

$$(-s+3.3)(s+3.3)\,e_0 - \frac{1}{s}\left(-s^2+\frac{17}{5}\right) = \mathfrak{z}_2$$

Let

$$e_0 = \frac{as+b}{s(s+3.3)}$$

Then

$$\frac{(-s+3.3)(as+b)-\left(-s^2+\frac{17}{5}\right)}{s} = \mathfrak{z}_2 \tag{5-23}$$

Let $s = 0$ $\qquad 3.3b = \frac{17}{5}$

$$b = \frac{17}{16.5} \doteq 1.03$$

Everything is well behaved at $s = +\infty$ if $-as^2 + s^2 = 0$.

Therefore a = 1

and $e_o = \frac{s+1.03}{s(s+3.3)}$

$$W_c = \frac{u_o}{e_o} = \frac{2.25(s+1)(s+2)(\cancel{s})(s+3.3)}{\cancel{s}(s+3.333)(s+1.2)(s+1.03)}$$

$$= 2.25\left(\frac{s+1}{s+1.03}\right)\left(\frac{s+2}{s+1.2}\right)\left(\frac{s+3.3}{s+3.333}\right) \tag{5-24}$$

If p(a) had been as given in Figure 5.9,

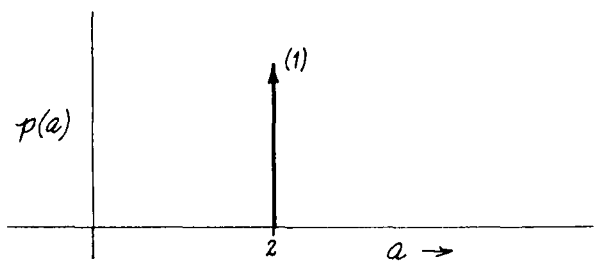

Figure 5.9 Alternate Density for Example 2

the optimal compensating network would be

$$W_c = \frac{u_o}{e_o} = 2.22\,\frac{(s+2)}{(s+1.18)} \tag{5-25}$$

which is very similar to that of Equation 5-24.

The importance of this simple example should not be overlooked. It indicates that if one can feel justified in approximating a given density function by a weighted set of delta functions, no additional analytical difficulties will be experienced in solving for the optimal compensating network than would occur in the completely deterministic case. Moreover, it is seen that the order of the optimal control expression increases by one for each additional delta function which is added to the approximation of the true density function.

In addition, all the optimal theory tools such as the root square locus and Bode plots (Reference 2), can be brought to bear on the problem, giving the engineer a clearer insight as to the effect of any particular uncertain coefficient on the system's performance.

Illustrative Example No. 3

One intuitively feels that the more realistic density functions for describing uncertain parameters in real physical systems should monotonically decrease from some peak value and go to zero for two finite values of the argument. That is, they should have a form such as that shown in Figure 5.10 where the peak value occurs at $a = a_1$ and the density function is identically zero for $a \geq a_2$ and $a \leq a_0$. A probable density function which satisfies these requirements and, in addition, does not introduce excessive analytical difficulties into the problem is the "Beta" density function. This function is described by Equation 5-26.

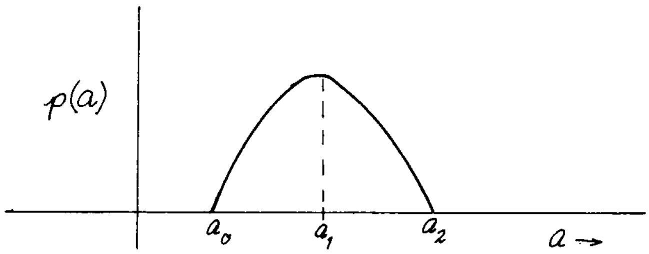

Figure 5.10 A Nonuniform Density

$$p(a) = \begin{cases} A(a-a_0)^b(a_2-a)^c\,, & a_0 \leq a \leq a_2 \\ 0 \text{ elsewhere} \end{cases} \tag{5-26}$$

The scale factor A is obtained by integrating the density and setting the result equal to one. That is,

$$\int_{a_0}^{a_2} p(a)\,da = 1.$$

Thus

$$A = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\,\Gamma(c+1)} \tag{5-27}$$

In addition, the peak of the density function occurs for

$$a_1 = \frac{ba_2 + ca_0}{c+b} \tag{5-28}$$

For illustrative purposes, let $b = c = 1$, $a_0 = 1$ and $a_2 = 2$. The density function becomes

$$p(a) = 6(-a^2 + 3a - 2) \tag{5-29}$$

Using the system of the previous problem the various expectation operations yield (again let $\mathcal{R}(s) = 1/s$):

$$\bar{W}_2 = E\{\bar{W}\} = E\left\{\frac{1}{-s+a}\right\} \doteq \frac{\frac{3}{2}}{-s+\frac{3}{2}}$$

$$E\left\{\frac{1}{W\bar{W}}\right\} = E\{-s^2+a^2\} = -s^2 + 2.3$$

$$E\{W\bar{W}\} = E\left\{\frac{1}{-s^2+a^2}\right\} = -6 + 9\ln\left|\frac{-s^2+4}{-s^2+1}\right| + \frac{(3s^2+6)}{s}\ln\left|\frac{s^2+s-2}{s^2-s-2}\right| \tag{5-30}$$

The series given in Equation 5-12 yields the following approximation for Equation 5-30:

$$E\left\{\frac{1}{-s^2+a^2}\right\} \doteq W_1\bar{W}_1$$

where

$$W_1 = \frac{(s+.3527)\left[s^2+2(.9994)(1.501)s+(1.501)^2\right]\left[s^2+2(.8992)(2.088)s+(2.088)^2\right]}{(s+\sqrt{2})^3(s+\sqrt{5/2})^2} \tag{5-31}$$

With this information one may follow the analysis procedures used in the previous examples to show that

$$e_o = \frac{s+\frac{2.3}{\alpha}}{s(s+\alpha)} \tag{5-32}$$

where $\alpha = \sqrt{2.3+\frac{1}{r}}$

results from the solution of the Wiener-Hopf equation

$$\left[\frac{1}{r} + E\left\{\frac{1}{W\bar{W}}\right\}\right] e_o - \mathcal{R}E\left\{\frac{1}{W\bar{W}}\right\} = \mathfrak{z}_2 .$$

The equation for the optimal control is now

$$\left[1 + \frac{1}{r} W_1\bar{W}_1\right] u_o - \frac{\mathcal{R}}{r}\bar{W}_2 = \mathfrak{z}_1 \tag{5-33}$$

The root square locus for

$$1 + \frac{1}{r} W_1\bar{W}_1 = 0$$

is given in Figure 5.11. It is seen that when $1/r \geq 10$ the equation

$$\left[s^2 + 2(.9994)(1.501)s + (1.501)^2\right]\left[s^2 + 2(.8992)(2.088)s + (2.088)^2\right]$$

$$= \begin{bmatrix} \omega_n = 1.501 \\ \zeta = .9994 \end{bmatrix} \begin{bmatrix} \omega_n = 2.088 \\ \zeta = .8992 \end{bmatrix} = 0$$

defines four of the roots of the optimal control. Rewriting Equation 5-33 as

$$\frac{\Delta \bar{\Delta}}{D_1 \bar{D}_1} u_0 - \frac{1/r\ \bar{N}_2}{s\ \bar{D}_2} = z,$$

for which the solution is

$$u_0 = \frac{D_1}{\Delta}\left[\frac{(1/r)\bar{N}_2 \bar{D}_1}{s \bar{D}_2 \bar{\Delta}}\right]_+ , \qquad (5\text{-}34)$$

one finds

$$u_0 = \frac{K\,(s+\sqrt{2})^3\,(s+\sqrt{5/2})^3}{s \begin{bmatrix} \omega_n = 2.088 \\ \zeta = .8992 \end{bmatrix} \begin{bmatrix} \omega_n = 1.501 \\ \zeta = .9994 \end{bmatrix} (s+.43)(s+5)} \qquad (5\text{-}35)$$

when, for example, $1/r = 20$

$$K = \left[\frac{1/r\ \bar{N}_2 \bar{D}_1}{\bar{D}_1 \bar{\Delta}}\right]_{s=0} = \left[\frac{20\left(\frac{3}{2}\right)(\sqrt{2})^3\left(\sqrt{5/2}\right)^3}{\left(\frac{3}{2}\right)(1.501)^2(2.088)^2(.43)(5)}\right] \doteq 10.6$$

and

$$W_c = \frac{10.6\,(s+\sqrt{2})^3\,(s+\sqrt{5/2})^3\,(s+4.72)}{(s+.43)(s+.488)(s+5) \begin{bmatrix} \omega_n = 2.88 \\ \zeta = .8992 \end{bmatrix} \begin{bmatrix} \omega_n = 1.501 \\ \zeta = .9994 \end{bmatrix}} \qquad (5\text{-}36)$$

since

$$e_0 = \frac{s + 0.488}{s(s+4.72)}$$

when $1/r = 20$.

Thus the Beta density function forces the use of a filter which is higher than that found for the uniform probability density function (compare Equation 5-36 with Equation 5-18).

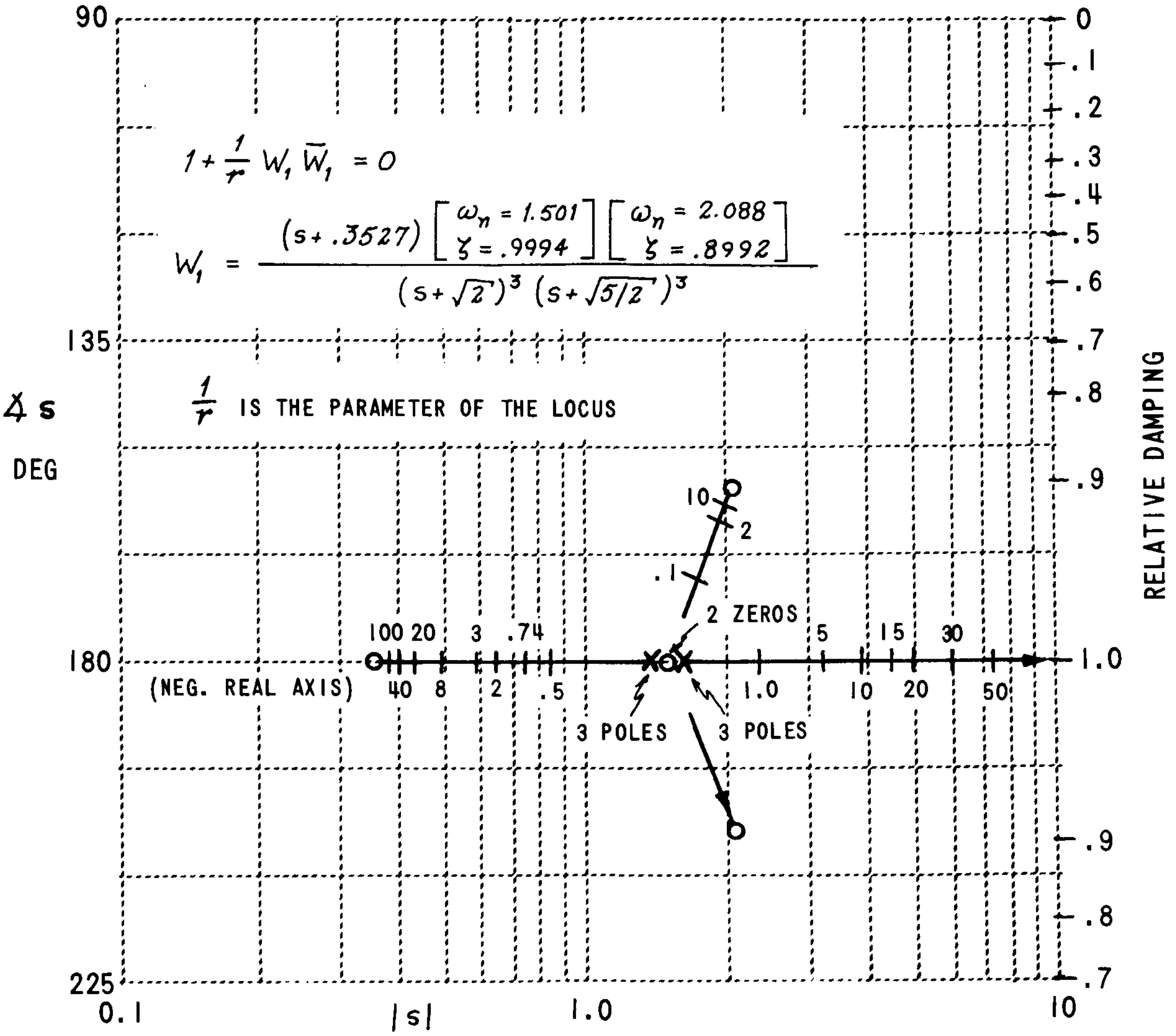

Figure 5.11 Root Square Locus for Example 3

Booster With One Bending Mode

The theoretical development in this section has been restricted to the single control variable case. The basic idea behind this approach was to keep the theory as simple as possible in the hope of coming to some clear-cut conclusion as to whether or not "acceptable" control systems could be designed using the approach. We have seen, in the previous examples, that the theory yields reasonable results which approach the usual ones as the density functions become sharper and sharper. Conversely, as the density functions become broader, the distributed nature of the information forces one to hedge on the design by putting in a higher order compensating network.

The single control variable theory is, however, of more than just academic interest. One specific problem to which it can be applied directly is that of controlling the bending modes in a flexible booster where the only control available comes from the gimbaled engines. In this example, transfer functions obtained from Appendix A will be used. These are listed at an appropriate point below.

The performance index will be

$$2V = \int_0^\infty \left\{ q\left[\mathcal{R}(t) - \phi_x(t)\right]^2 + r\beta_c^{\,2} \right\} dt \tag{5-37}$$

and will pertain to the block diagram of Figure 5.12.

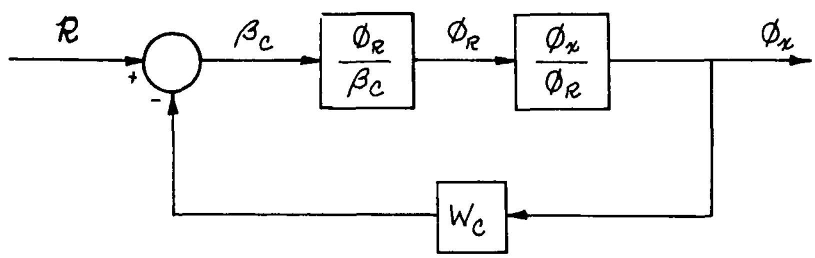

Figure 5.12 Booster Block Diagram

In Figure 5.12, ϕ_R / β_c is the transfer function which relates the rigid-body pitch angle (ϕ_R) to the gimbal deflection (β_c).

It is to be assumed that a position gyro is located at x meters and that the pitch angle sensed by this gyro is given by the equation

$$\phi_{P.G.} = \phi_x = \phi_R - \sum_{i=1}^{n} Y_i' \eta_i$$

where Y_i' is the slope of the ith bending modes and η_i is the ith normal bending mode. For the purpose of this study, it will be assumed that

$$\phi_x = \phi_R - Y_1' \eta_1 \tag{5-38}$$

It is not our intent, in using Equation 5-38, to lightly dismiss the important effects of the higher order modes. However, our primary purpose here is to examine the nature of the solutions and for this reason we restrict ourselves to only the first bending mode.

At any rate,

$$W = \frac{\phi_x}{\beta_c} = \frac{\phi_R}{\beta_c} - Y_1' \frac{\eta_1}{\beta_c}$$

The Wiener-Hopf equations which must be solved are:

$$\left[r + q\,E\{W\bar{W}\}\right]\beta_{c_0} - q\,\mathcal{R}E\{\bar{W}\} = \gamma_1 \tag{5-39}$$

and

$$\left[q + r\,E\left\{\frac{1}{W\bar{W}}\right\}\right]\phi_{x_0} - \mathcal{R} = \gamma_2 \tag{5-40}$$

where β_{c_0} = optimal control

ϕ_{x_0} = optimal output

The compensating network is then

$$W_c = \frac{\mathcal{R} - \beta_{c_0}}{\phi_{x_0}}$$

It will be assumed that everything is known except the value of Y_1', which is described only by a probability density function. This is actually very close to the truth for a numerical analysis of the equations of motion from which the ϕ/β_c and η_1/β_c transfer functions were derived indicates that Y_1' is the one factor which dominates the uncertainty in the W transfer function. At the most forward possible instrument location on the booster (x = 122.5 meters), the data listed in Table 4-1 indicates that

$$Y_1' = -.123 \pm 25\%$$

It is now assumed that all the values within this range are equally probable and described by the uniform probability density function of Figure 5.13. (Note: the density in Figure 5.13 is actually based on $Y_1' = -.123 \pm 35\%$).

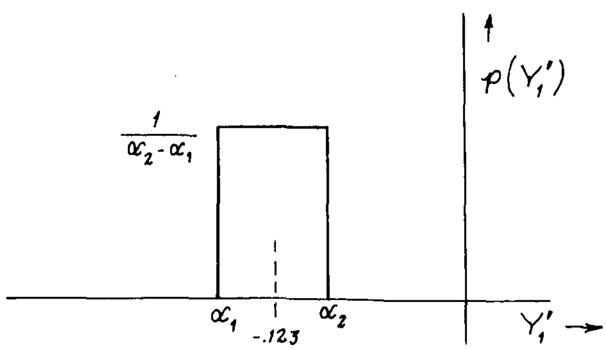

Figure 5.13 Probability Density Function for Y_1'

The numbers of interest are

$$\alpha_2 = -.08$$
$$\alpha_1 = -.166$$

and
$$\frac{1}{\alpha_2-\alpha_1} = \frac{1}{.086} = 11.63$$

Other numbers of interest are:

$$E\left\{Y_1'\right\} = -.123$$

$$\left[E\left\{Y_1'\right\}\right]^2 = .01513$$

$$E\left\{(Y_1')^2\right\} = .0157$$

The geometric mean of the density function is $\sqrt{(-.166)(-.08)} = .11524$

Thus, the $\left[E\left\{Y_1'\right\}\right]^2$ is, to 5% accuracy, equal to $E\left\{(Y_1')^2\right\}$.

As a first task, we elect to solve for the optimal control, β_{c_o} . The Wiener-Hopf equation is

$$\left[r + qE\left\{W\bar{W}\right\}\right]\beta_{c_o} - \mathcal{R}qE\left\{\bar{W}\right\} = z$$

$$W = \frac{\phi_x}{\beta_c} = \frac{\phi_R}{\beta_c} - Y_1' \frac{\eta_1}{\beta_c} \tag{5-41}$$

Let $\dfrac{\phi_R}{\beta_c} = \dfrac{N_1}{D}$ and $\dfrac{\eta_1}{\beta_c} = \dfrac{N_2}{D}$

For convenience, these transfer functions are listed here (refer to Appendix A)

$$\left.\begin{aligned}
\frac{\phi_R}{\beta_c}(s) &= \frac{-2.14\left(\frac{s}{.014}+1\right)\left[\left(\frac{s}{2.317}\right)^2 + \frac{2(.005)}{2.317}s+1\right]}{D} \\
\frac{\eta_1}{\beta_c}(s) &= \frac{9.18\left(\frac{s}{.041}+1\right)\left(\frac{-s}{.454}+1\right)\left(\frac{s}{.498}+1\right)}{D} \\
D &= \left(\frac{-s}{.042}+1\right)\left(\frac{-s}{.242}+1\right)\left(\frac{s}{.294}+1\right)\left(\frac{s}{17.9}+1\right)\left[\left(\frac{s}{2.32}\right)^2+\frac{2(.005)}{2.32}s+1\right]
\end{aligned}\right\} \tag{5-42}$$

Since

$$W = \frac{N_1 - Y_1' N_2}{D}, \quad \bar{W}\ \frac{\bar{N}_1 - Y_1'\bar{N}_2}{\bar{D}},$$

we find
$$W\bar{W} = \frac{N_1\bar{N}_1 - Y_1'(\bar{N}_1 N_2 + N_1\bar{N}_2) + (Y_1')^2 N_2\bar{N}_2}{D\bar{D}}$$

and
$$E\{W\bar{W}\} = \frac{N_1\bar{N}_1 - E\{Y_1'\}(\bar{N}_1 N_2 - N_1\bar{N}_2) + E\{(Y_1')^2\} N_2\bar{N}_2}{D\bar{D}} \tag{5-43}$$

The approximation is made that
$$[E\{Y_1'\}]^2 = E\{(Y_1')^2\}$$

$$\therefore\ E\{W\bar{W}\} = \frac{(N_1 - E\{Y_1'\}N_2)}{D} \cdot \frac{(\bar{N}_1 - E\{Y_1'\}\bar{N}_2)}{\bar{D}} \tag{5-44}$$

or
$$E\{W\bar{W}\} = W_1\bar{W}_1 \tag{5-45}$$

where
$$W_1 = \frac{N_1 - E\{Y_1'\}N_2}{D} \tag{5-46}$$

Also,
$$E\{\bar{W}\} = E\left\{\frac{\bar{N}_1 - Y_1'\bar{N}_2}{\bar{D}}\right\}$$

$$\therefore\ E\{\bar{W}\} = \bar{W}_1 \tag{5-47}$$

The expression for the optimal control has now become, after division by r
$$\left[1 + \frac{q}{r} W_1\bar{W}_1\right]\beta_{c_0} - \mathcal{R}\frac{q}{r}\bar{W}_1 = \mathfrak{z} \tag{5-48}$$

Using the tabulated transfer functions, the expression for W_1 is:
$$W_1 = \frac{-2.027\left(\frac{s}{.0131}+1\right)\left[\left(\frac{s}{3.1}\right)^2 + \frac{2(.01)}{3.1}s+1\right]}{\left(\frac{-s}{.042}+1\right)\left(\frac{-s}{.242}+1\right)\left(\frac{s}{.294}+1\right)\left(\frac{s}{17.9}+1\right)\left[\left(\frac{s}{2.3}\right)^2 + \frac{2(.005)}{2.3}s+1\right]} \tag{5-49}$$

Prior experience with the bending mode problem indicates that a desirable situation has been achieved. Note that the complex zeros of the W_1 transfer function have been increased in their natural frequency from the 2.3 rad/sec of the ϕ/β_c transfer function to 3.1 rad/sec. Previous experience with this problem indicates that it will now be possible to introduce considerable damping (relatively speaking) into the first bending mode.

The root square locus for this situation is shown in Figure 5.14*.

*Again we have used a log $|s|$ vs damping plot similar to those obtained from the ESIAC.

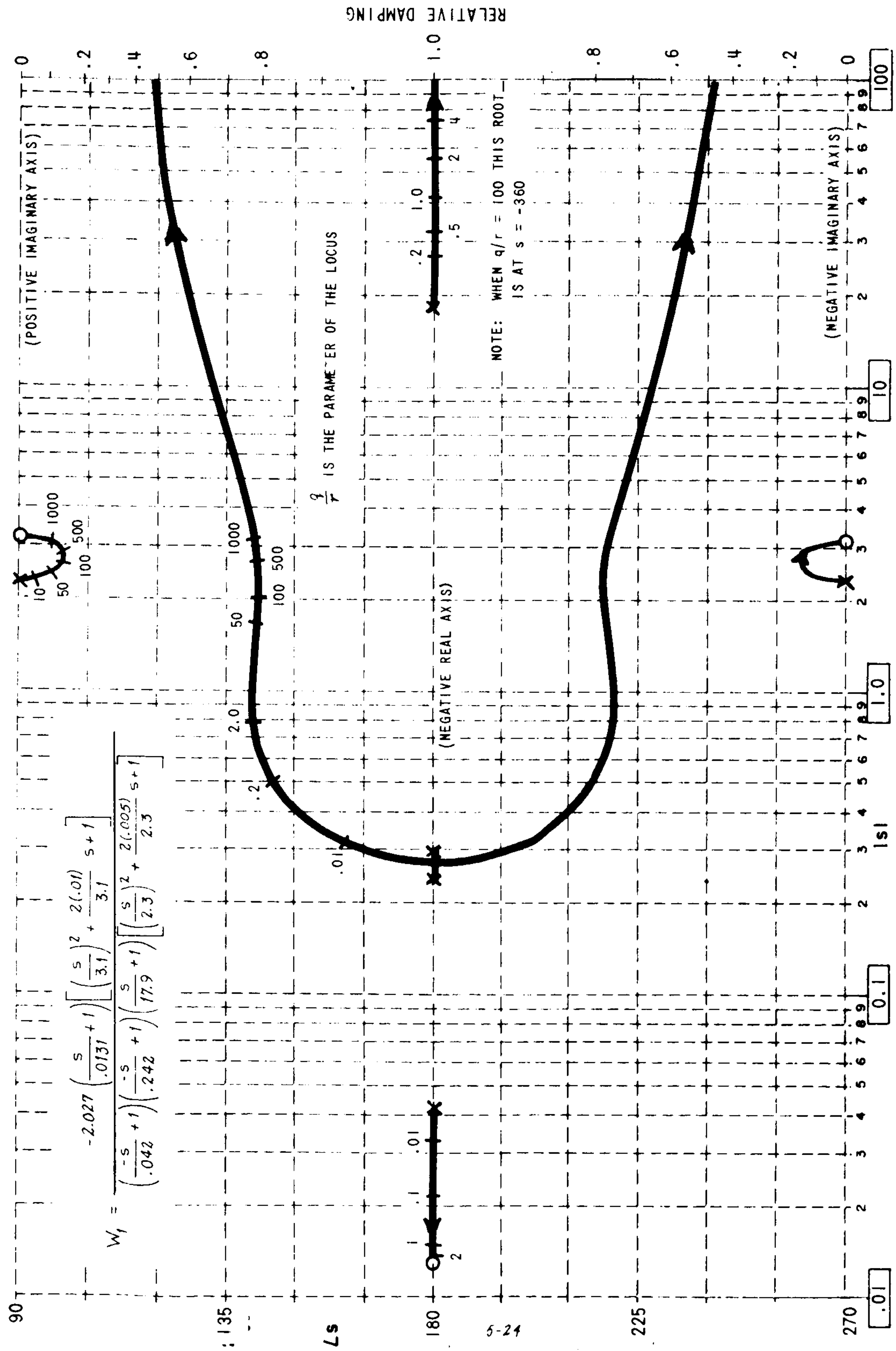

Figure 5.14 Root Square Locus for First Bending Mode

Equation 5-48 is now rewritten in the form

$$\frac{\Delta\bar{\Delta}}{D\bar{D}}\,\beta_{c_0} - \mathcal{R}\,\frac{q}{r}\,\frac{\bar{N}_1}{\bar{D}} = \mathfrak{z}$$

for which the solution is

$$\beta_{c_0} = \frac{D}{\Delta}\left[\frac{\mathcal{R}(q/r)\bar{N}_1}{\bar{\Delta}}\right]_+ \tag{5-50}$$

If q/r = 100, from the root locus plot (Figure 5.14),

$$\Delta \approx \left(\frac{s}{.013}+1\right)\left[\left(\frac{s}{2}\right)^2 + \frac{2(.8)}{2}\,s+1\right]\left[\left(\frac{s}{2.5}\right)^2 + \frac{2(.12)}{2.5}\,s+1\right]\left(\frac{s}{360}+1\right) \tag{5-51}$$

The damping factor of the first bending mode has been increased from .005 to approximately 0.12. Ignoring the secondary effects of the root at s = -360, the remaining three roots constitute an approximation to a "drift minimum" configuration. (The drift minimum model ideally has a pole at s = 0.)

Substituting Equation 5-51 into Equation 5-50, and of course, correctly taking into account the scale factor when $\mathcal{R}(s) = 1/s$, one finds

$$\beta_{c_0} = \frac{-(.49)\left(\frac{-s}{.042}+1\right)\left(\frac{-s}{.294}+1\right)\left(\frac{s}{17.1}+1\right)\left[\left(\frac{s}{2.3}\right)^2 + \frac{2(.005)}{2.3}\,s+1\right]}{s\left(\frac{s}{.013}+1\right)\left[\left(\frac{s}{2}\right)^2 + \frac{2(.8)}{2}\,s+1\right]\left[\left(\frac{s}{2.5}\right)^2 + \frac{2.12}{2.5}\,s+1\right]\left(\frac{s}{360}+1\right)} \tag{5-52}$$

when q/r = 100.

The above expression is the correct one for the optimal control, even though the W_1 transfer function has right half plane poles. Refer to Appendix B for a more thorough discussion of this point.

Before proceeding to the problem of finding the optimal output ($\phi_{122.5}$) it is pointed out that the result achieved (within the limitations of the approximations made) is precisely the same as would be obtained had the value of Y_1' been exactly known and equal to - .123. This is due to the fact that the density function was still sharp enough, and the effect of Y_1' was not critical enough, for the uncertainty to enter the analysis in any sensitive manner.

The next problem is to find the optimal value of $\phi_{122.5}$ which minimizes the expected value of the performance index. The Wiener-Hopf equation is now

$$\left[1 + \frac{r}{q}\,E\left\{\frac{1}{W\bar{W}}\right\}\right]\phi_{x_0} - \mathcal{R} = \mathfrak{z} \tag{5-53}$$

$$\frac{1}{W} = \frac{D}{(N_1 - Y_1' N_2)}\,, \qquad \frac{1}{W\bar{W}} \quad \frac{D\bar{D}}{(N_1 - Y_1' N_2)(\bar{N}_1 - Y_1'\bar{N}_2)}$$

$$E\left\{\frac{1}{W\bar{W}}\right\} = D\bar{D}\,E\left\{\frac{1}{N_1\bar{N}_1 - Y_1'(\bar{N}_1 N_2 + N_1\bar{N}_2) + (Y_1')^2 N_2\bar{N}_2}\right\} \tag{5-54}$$

$$= \frac{\left(\frac{1}{\alpha_2-\alpha_1}\right)D\bar{D}}{(\bar{N}_1 N_2 - N_1\bar{N}_2)}\ \ln\left|\frac{Y_1' - \frac{\bar{N}_1}{\bar{N}_2}}{Y_1' - \frac{N_1}{N_2}}\right|_{\alpha_1}^{\alpha_2} \tag{5-55}$$

$$= \frac{\left(\frac{1}{\alpha_2-\alpha_1}\right)D\bar{D}}{(\bar{N}_1 N_2 - N_1\bar{N}_2)}\ \ln\left|\frac{\xi\bar{\xi} - (\alpha_1\bar{\xi} + \alpha_2\xi) + \alpha_1\alpha_2}{\xi\bar{\xi} - (\alpha_1\xi + \alpha_2\bar{\xi}) + \alpha_1\alpha_2}\right|, \quad \xi = \frac{N_1}{N_2}$$

Again use the series

$$\ln x = 2\left(\frac{x-1}{x+1}\right)\left[1 + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^2 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^4 + \ldots\right] \tag{5-56}$$

to obtain $E\left\{\frac{1}{W\bar{W}}\right\} = \dfrac{D\bar{D}}{N_1\bar{N}_1 + \frac{(\alpha_1+\alpha_2)}{2}(\bar{N}_1 N_2 + N_1\bar{N}_2) + \alpha_1\alpha_2 N_2\bar{N}_2} \cdot \psi$

where

$$\psi = \left[1 + \frac{(\alpha_2-\alpha_1)^2}{12}\ \frac{(\bar{N}_1 N_2 - N_1\bar{N}_2)}{\left[\bar{N}_1 N_1 + \frac{(\alpha_1+\alpha_2)}{2}(\bar{N}_1 N_2 - N_1\bar{N}_2) + \alpha_1\alpha_2 N_2\bar{N}_2\right]} + \ldots\right] \tag{5-57}$$

Since

$$\frac{(\alpha_2-\alpha_1)^2}{12} = 6.2 \times 10^{-4}$$

we will ignore the higher order terms in the series expansion of Equation 5-57. There is a small element of danger in doing this since $\bar{N}_1 N_2 - N_1\bar{N}_2$ has a constant term which is identically zero and therefore the discarded terms in the series contain free powers of s. For this reason the approximation should be checked as a function of frequency. However, experience indicates that the denominator will, as a function of frequency, continue to dominate.

Thus

$$E\left\{\frac{1}{W\bar{W}}\right\} \doteq \frac{D\bar{D}}{N_1\bar{N}_1 - .123(\bar{N}_1 N_2 - N_1\bar{N}_2) + \underbrace{.0133}_{\alpha_1\alpha_2} N_2\bar{N}_2} \tag{5-58}$$

where $(\alpha_1 + \alpha_2)/2$ is the mean value of $Y_1' = -.123$.

If the denominator of Equation 5-58 were expanded, one would see that the effect on the coefficients of the term $.0133\ N_2 \bar{N}_2$ is generally small and we may safely set

$$\alpha_1 \alpha_2 = \left[E\left\{ Y_1' \right\} \right]^2 \tag{5-59}$$

For example, this approximation causes a 1.5% error in the constant term of the denominator of Equation 5-58 .

Using this approximation, one obtains

$$E\left\{ \frac{1}{W\bar{W}} \right\} = \frac{1}{W_1 \bar{W}_1}$$

or

$$E\left\{ \frac{1}{W\bar{W}} \right\} \doteq \frac{1}{E\{W\bar{W}\}}$$

At this point, the problem will be terminated, since the Wiener-Hopf equation has become

$$\left[1 + \frac{r}{q} \frac{1}{W_1 \bar{W}_1} \right] \phi_{x_0} - \mathcal{R} = \mathcal{Z}$$

and the compensating network will be the same as would occur were we dealing with a completely deterministic system in which $Y_1' = -.123$. That is, we may as well find the compensating network using the conventional optimal theory.

To summarize the results of this section, the expected value of the performance index

$$2V = \int_0^\infty \left[q(\mathcal{R} - \phi_x)^2 + r\beta_0^2 \right] dt \tag{5-60}$$

has been minimized by finding both the optimal control and the optimal output. The slope of the first bending mode was considered to be the only unknown coefficient in the equations of motion. The results indicate that one may introduce a damping of approximately 12% into the 1st bending mode and the required compensation will be essentially the same as that used when a completely deterministic system with Y_1' equal to the mean value of the assumed uniform probability density function. The density, which was assumed to have a spread of ±35%, was not broad enough to force the use of a higher order filter than would normally be used.

The reader should note that while the example considered a sixth-order system, only one uncertain coefficient was involved. However, it is not too difficult to visualize that the mathematics would remain essentially the same had additional uncertain coefficients been included -- namely it will continue to use the table of integrals of the rational and irrational algebraic functions to solve the problem.

Summary

In this section, simple illustrative examples have been used to obtain a "feel" for the types of problems which will be encountered when one applies linear optimal control to systems containing uncertain parameters. Towards this end, only a single-input and single-output feedback control system was considered.

The approach taken was to assume that the logical thing to do in the face of parameter uncertainty was to minimize the expected value of the performance index. Furthermore, the viewpoint was taken that it made no sense to find only the optimal control which accomplishes this minimization since one would be forcing an open-loop control law on a system which was originally postulated to be a closed-loop one. It was reasoned that the best approach to the problem was to find the compensating network which minimized the expected value of the performance index. However, in the face of the analytical difficulties encountered in attempting to solve directly for this network, the subjective decision was made to solve for the optimal variables at the input and output of the compensating filter and then define the compensation to be the ratio of the two. The indications are that this procedure leads to a theory which behaves in a reasonable manner as the information concerning the system becomes more distributed. Specifically, the design procedure forces one to hedge, by increasing the order of the compensating network, as the density functions become broader.

The basic new features which are added to the linear optimal theory in the face of parameter uncertainty are the necessity for the use of the tables of integrals for the rational and irrational algebraic functions and the problem of approximating irrational functions of s with rational ones in order that the root square locus may continue to be of use and that the compensating networks will be rational functions of s. However, the example which considered the weighted delta function density shows clearly that this type of density function introduces no additional analytical difficulties into the problem. That is, the Wiener-Hopf equations which are involved will, in general, be of a higher order than those in the deterministic case but no transcendental function of s will be involved. Thus, if the engineer feels justified in approximating the given density functions by a weighted set of delta functions, the vast body of knowledge concerning the solution of Wiener-Hopf equations can be easily modified to solve the uncertain parameter problem.

A specific application of the theory was given in the example which considered the design of the compensating network required for a flexible booster. It was shown that the uncertain parameter, which was the slope of the first bending mode, did not enter into the design in a critical fashion. Thus the resultant design would be the same as had one initially fixed the value of slope of the first bending mode at the mean value of the probability density function.

Only uniform, "Beta" and "weighted delta functions" densities were considered. It seems important to continue these studies for other types of probability density functions.

It would also be worthwhile to obtain equivalent results for the multi-controller problem. As noted in the main text, a good approach to this problem would be to find the networks which should be interposed between the optimal controls and the optimal state variables. Some preliminary work along these lines, not reported here, indicate that reasonable results will be obtained. For example, as the density functions become sharper and sharper, the "feedback" networks reduce to the gains that one normally associates with the feedback control law $u_0 = -Kx$.

SECTION 6
THE CONSTRAINED GAIN PROBLEM

Introduction

A problem of importance to optimal booster control system design is the so-called constrained gain problem. Briefly, for settings of control weighting and state-variable excursion weighting, acceptable closed-loop system response may be obtained. Once the weighting matrices are selected, the optimal feedback gains are uniquely specified. These gains may not be changed individually without loss of implementation of the optimal solution.

Often, some of the feedback gains specified by the optimal solution are too large for practical application, in which case, the solution must be discarded and a different set of weighting matrices chosen. The new weighting matrices will cause all feedback gains to change, even though only one or perhaps two were too large in the original design, and the new selection may still not satisfy the constraints on the gain magnitudes.

A more desirable approach to the constrained gain problem may be to directly constrain those gains that are considered too large (in the optimal solution) and then to determine the new optimal solution with these constraints imposed. Although large amplitude constraints on the feedback gains might solve the problem, it is unlikely, in that case, that any design aid such as the root square locus could be evolved. Moreover, the designer might have to resort to a dynamic programming solution with its inherent difficulties. Rather than work from the hard-constraint approach, a soft-constraint approach was undertaken on this project. In this case, the gains are constrained by a quadratic measure, or they are specified a priori with the help of the usual optimal regulator solution. The objective is to carry the analytical approach as far as possible before resorting to numerical or heuristic approaches.

In the study conducted on this project, a number of analytical approaches were worked out in detail. However, even the most promising of these is too difficult to use in a practical application, and therefore an experimental or heuristic approach was also tried. The major results of both the theoretical and the experimental approaches are described in this section.

Theoretical Development

Approach 1

The first approach will allow an analytical solution of the problem, which is difficult to use, and the feedback gains are apparently not constant. It is described briefly below.

Consider a system of the form

$$\dot{x} = Fx + Gu \;, \quad y = Hx \;, \quad u = -Kx \tag{6-1}$$

where the dimensions of the matrices are

$$x\ (n \times 1) \qquad G\ (n \times p) \qquad H\ (r \times n) \qquad y\ (r \times 1)$$
$$F\ (n \times n) \qquad u\ (p \times 1) \qquad K\ (p \times n)$$

Let

$$G \equiv \left[G_1 \,\vdots\, G_2 \,\vdots\, \cdots \,\vdots\, G_p \right]$$

$$u \equiv \begin{bmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_p \end{bmatrix} \qquad\qquad K \equiv \begin{bmatrix} K_1 \\ \hline K_2 \\ \hline \vdots \\ \hline K_p \end{bmatrix}$$

where each partition quantity has the dimensions

$G_q\ (n \times 1)$, $u_q\ (1 \times 1)$ and $K_q\ (1 \times n)$

Then

$$Gu = G_1 u_1 + G_2 u_2 + \ldots + G_p u_p$$

and

$$u_q = -K_q x$$

The performance index to be minimized is

$$2V = \int_0^\infty \left[y'Qy + K_1 S_1 K_1' + K_2 S_2 K_2' + \ldots + K_p S_p K_p' \right] dt \tag{6-2}$$

where $S_1, S_2, \ldots S_p$ are positive definite diagonal ($n \times n$) matrices. As long as these matrices are all positive definite, the control u is implicitly constrained. In addition a constraint on any individual feedback gain in may be increased by increasing the corresponding values of the diagonal element of the S_1 matrix.

The performance index of Equation 6-2 may be extremized subject to the constraints of Equation 6-1 by means of the calculus of variations. The Euler-Lagrange equations for the extremals are

$$\left.\begin{aligned} &H'QHx + (F - G_1 K_1 - G_2 K_2 - \ldots - G_p K_p)' \lambda + \dot{\lambda} = 0 \\ &-S_q K_q' + x G_q' \lambda = 0 \quad ; \quad q = 1, 2, \ldots, p \end{aligned}\right\} \tag{6-3}$$

where λ is an $(n \times 1)$ vector Lagrange multiplier. Rearrangement yields

$$\left.\begin{aligned} &H'QHx + F'\lambda - \sum_{q=1}^{p} S_q^{-1} x \left(G_q' \lambda\right)^2 + \dot{\lambda} = 0 \\ &K_q' = S_q^{-1} x \, G_q' \lambda \;\; ; \;\; q = 1, 2, \ldots, p \end{aligned}\right\} \qquad (6\text{-}4)$$

These equations, together with the equations of 6-1 can be solved for λ and x and then for each K_q. However, the feedback gains are functions of the state variable x so that apparently these gains will be significantly time-variable. Moreover, the equations of 6-4 are nonlinear, making a general solution difficult or impossible.

This approach to the constrained gain problem has been described because it exhibits the difficulty of the problem. A major conclusion that can be drawn is that the class of admissible feedback gains must be limited to those that are constant (in solution times). If the class is not so limited, the feedback gains will vary significantly -- a situation that may be undesirable for booster design. It may also be concluded that the feedback gain constraint problem leads to the solution of nonlinear equations, whether they be algebraic or differential. Therefore, every effort must be made to cast the problem in its simplest form so that complexity is minimized.

Approach 2

The second approach to the gain constraint problem incorporates in the problem statement the specification that the feedback gains are to be constant. In this approach, it is assumed that the usual optimal regulator problem has been solved, and that some (but not all) of the feedback gains are too large in magnitude. The feedback gains that are too large are then reduced to values that are at the limit of the magnitude constraints, and are thus specified at the outset of the constrained gain optimal regulator solution. The problem is to determine the remaining gains (those that did not violate the magnitude constraints in the usual optimal solution) so as to minimize the original performance measure.

Again consider a system of the form given in Equation 6-1, but assume that the dimensions of the matrices are

$$x\ (n \times 1) \qquad G\ (n \times 1) \qquad y\ (r \times 1) \qquad K\ (1 \times n) \qquad R\ (1 \times 1)$$

$$F\ (n \times n) \qquad u\ (1 \times 1) \qquad H\ (r \times n) \qquad Q\ (r \times r)$$

The performance measure to be minimized is

$$\begin{aligned} 2V &= \int_0^\infty (y'Qy + u'Ru)\,dt \\ &= \int_0^\infty x'\left[H'QH + K'RK\right] x\,dt \end{aligned} \qquad (6\text{-}5)$$

where K is considered to be a vector of constants. Some of the elements of K will be specified a priori by virtue of the magnitude constraints. The remainder of the elements must then be determined in such a way as to minimize the performance measure of Equation 6-5.

The minimization process is carried out by expressing the performance index in terms of $\mathcal{K}$ and then setting equal to zero the expression for the partial derivative of the measure with respect to each element of $\mathcal{K}$ that is to be determined. The Laplace transform of $x(t)$ is

$$\mathcal{L}\left[x(t)\right] \equiv X(s,\mathcal{K}) = \left[Is - F + G\mathcal{K}\right]^{-1} x_0 \tag{6-6}$$

where x_0 is the initial condition vector. If use is made of an extended form of Parseval's theorem, the performance measure may be written as

$$2V = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} X'(-s,\mathcal{K})\,B(\mathcal{K})\,X(s,\mathcal{K})\,ds \tag{6-7}$$

where $B(\mathcal{K}) \equiv H'QH + \mathcal{K}'R\mathcal{K}$

To obtain the extremal values of V for any given gain, the partial derivative with respect to that gain is taken and set equal to zero:

$$\frac{\partial V}{\partial k_i} = 0 \tag{6-8}$$

where k_i is one element of the vector $\mathcal{K}$

Equation 6-8 may be evaluated by making use of the equation

$$\begin{aligned} \frac{\partial}{\partial k_i}\left[X'(-s,\mathcal{K})B(\mathcal{K})X(s,\mathcal{K})\right] = {} & \frac{\partial}{\partial k_i}\left[X'(-s,\mathcal{K})B(\mathcal{K}_0)X(s,\mathcal{K}_0)\right] \\ & + \frac{\partial}{\partial k_i}\left[X'(-s,\mathcal{K}_0)B(\mathcal{K})X(s,\mathcal{K}_0)\right] \\ & + \frac{\partial}{\partial k_i}\left[X'(-s,\mathcal{K}_0)B(\mathcal{K}_0)X(s,\mathcal{K})\right] \end{aligned} \tag{6-9}$$

where $\mathcal{K}_0$ is equal to $\mathcal{K}$, but is treated as a constant while the partial derivative is being taken. In the derivative operation, it is necessary to differentiate inverse matrices with respect to k_i. The operation may be performed by using the formula

$$\frac{\partial}{\partial k_i}\left[M^{-1}\right] = -M^{-1}\left(\frac{\partial}{\partial k_i}M\right)M^{-1} \tag{6-10}$$

where $M \equiv M(s,\mathcal{K}) \equiv \left[Is - F + G\mathcal{K}\right]$

After a great deal of mathematical manipulation, it is found that the condition on k_i for minimization of the performance measure is

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \theta_i'\, X(s,\mathcal{K})X'(-s,\mathcal{K})\left[B(\mathcal{K})M^{-1}(s,\mathcal{K})G - \mathcal{K}'R\right] ds = 0 \tag{6-11}$$

where θ_i is an n vector with zeros for all elements except for the i[th] element, which is 1.

The optimal condition yields a scalar equation in the frequency domain. Equation 6-11 can be rewritten in the form

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{1}{|M(S,K)|^2\,|M(S,K)|}\left\{\theta_i' N(S,K) x_0 x_0' N'(S,K)\left[B(K)N(S,K)G - K'R|M(S,K)|\right]\right\} ds = 0 \tag{6-12}$$

where

$|M(S,K)|$ is the determinant of $M(S,K)$

and $N(S,K)$ is such that $M^{-1}(S,K) = \dfrac{N(S,K)}{|M(S,K)|}$

If the closed-loop control system is to be stable, then all of the roots of $|M(S,K)| = 0$ in S must be in the same half plane. Similarly, all of the roots of $|M(-S,K)| = 0$ must be in the opposite half plane. Moreover, if the left-hand side of Equation 6-12 is to be equal to zero, the total integrand must be analytic in one or the other of the two half planes. This condition can be met if, and only if, the bracketed factor in the integrand of Equation 6-12 contains a factor $|M(-S,K)|$ or two factors $|M(S,K)|$. Therefore, the condition for the optimal value of the gain k_i is

$$\left\{\theta_i' N(S,K) x_0 x_0' N'(-S,K)\left[B(K)N(S,K)G - KR|M(S,K)|\right]\right\} = 0 \tag{6-13}$$

under the condition that

a. $|M(-S,K)| = 0$

or

b. $|M(S,K)|^2 = 0$

One equation such as Equation 6-13 can be obtained for each feedback gain that is to be optimally adjusted. Since the remaining gains are to be specified a priori, the number of conditions obtained is equal to the number of optimal gains to be determined.

Unfortunately, the equations obtained are difficult to solve, except for very elementary problems. It is possible that further study of the analytical solution as given herein may yield additional simplification. Spectral factorization, for example, if properly applied might yield a simpler expression for the optimal gains.

Approach 3

The third theoretical approach makes use of scalar differential equations and previously developed tables for optimal feedback gain solution. It is, in some ways, a scalar equivalent of Approach 2, but is less general.

Consider a single controller system described by the differential equation

$$u = a_0 x + a_1 \dot{x} + \dots + a_n \overset{(n)}{x} \quad ; \; x(0), \dots, \overset{(n-2)}{x}(0) = 0 \; ; \; \overset{(n-1)}{x}(0) \quad \text{given} \tag{6-14}$$

The control variable u is to be a weighted sum of the state variables; that is,

$$u = -k_0 x - k_1 \dot{x} - \dots - k_{n-1} \overset{(n-1)}{x} \tag{6-15}$$

Some of the k_i 's will be specified a priori because they have exceeded the magnitude constraints when the usual optimal regulator solution was obtained. The remaining gains are to be chosen so as to be optimal.

Choose the performance measure:

$$\begin{aligned} V &= \int_0^\infty \left(q_0 x^2 + q_1 \dot{x}^2 + \dots + q_{n-1} \overset{(n-1)}{x}{}^2 + r_0 k_0^2 x^2 + \dots + r_{n-1} k_{n-1}^2 \overset{(n-1)}{x}{}^2 \right) dt \\ &= \int_0^\infty \left[(q_0 + r_0 k_0^2) x^2 + (q_1 + r_1 k_1^2) \dot{x}^2 + \dots + (q_{n-1} + r_{n-1} k_{n-1}^2) \overset{(n-1)}{x}{}^2 \right] dt \end{aligned} \tag{6-16}$$

where the q_i 's and r_i 's are non-negative constants.
If use is made of Parseval's theorem, the performance measure may be written as

$$V = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\sum_{i=1}^{n-1} (-1)^i (q_i + r_i k_i^2) s^{2i} x(s) x(-s) \right] ds \tag{6-17}$$

where $x(s)$ is the Laplace transform of $x(t)$ or x .

The performance measure may be written in the form

$$V_1 = \frac{V}{a_n^2 \left[\overset{(n-1)}{x}(0) \right]^2} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{g_n(s)}{h_n(s)\, h_n(-s)} ds \tag{6-18}$$

where

$$g_n(s) \equiv \sum_{i=0}^{n} (-1)^i (q_i + r_i k_i^2) s^{2i}$$

and

$$h_n(s) \equiv (a_0 + k_0) + (a_1 + k_1) s + \dots + (a_{n-1} + k_{n-1}) s^{n-1} + a_n s^n$$

The performance measure is now in a form that may be evaluated using the tables of Appendix D in Reference 8. These tables do not require the factorization of the numerator and therefore are well suited to this problem.

Once the performance measure has been evaluated, it may be minimized by partial differentiation with respect to each free gain k_i and each result set equal to zero; that is,

$$\frac{\partial V_i}{\partial k_i} = 0 \; ; \quad 0 \le i \le n-1 \tag{6-19}$$

Once again, one equation can be obtained for each feedback gain that is to be optimally adjusted. Since the remaining gains are to be specified a priori, the number of conditions obtained is equal to the number of optimal gains to be determined.

All three of the previous theoretical approaches are very difficult to use in practice when the order of the system is greater than three or four. The booster design problem, when bending modes are considered, is of considerably higher order than third or fourth. Theoretical approaches, such as those developed herein cannot be considered as adequate for solving the booster constrained gain problem because they are too complex.

Experimental Solutions and Applications

To obtain a practical solution of the booster constrained gain problem, it is apparent that one must resort to heuristic programming methods. In this case, one performs a systematic search for the minimum value of the performance measure. Either a digital or an analog computer may be used.

An experimental study was performed on this project to determine the usefulness of a given heuristic method. The dynamics of a booster considering actuator and one bending mode were programmed on an analog computer. The optimal regulator feedback control system and the computation of the performance measure for return from an initial condition were also programmed. The equations of motion and performance index used are given below:

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\alpha} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0733 & 0 & 0 & -.45 \\ -.0405 & 1 & -.0107 & 0 & 0 & -.0211 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5.453 & -5.37 & -.0232 & 15.83 \\ 0 & 0 & 0 & 0 & 0 & -17.9 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \alpha \\ \eta_1 \\ \eta_2 \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 17.9 \end{bmatrix} \beta_c \tag{6-20}$$

$$2V = \int_0^\infty \left[.01 (30 \phi_R + \eta_1)^2 + \beta_c^2 \right] dt \tag{6-21}$$

In order that minimization of the performance measure might be accomplished, the computer was switched to the repetitive operation mode. Then the feedback gains were adjusted until the final value of the performance measure was minimized. Each parameter was adjusted in sequence until there was no visible further reduction in the performance measure.

Initially the computer was programmed for the usual optimal regulator configuration. A check of this condition on the computer showed the analog computer values for the feedback gains to be very close to those computed by the Ricatti equation solution. After the optimal regulator problem was programmed, one or more of the feedback gains were arbitrarily reduced in value and the remaining gains were readjusted so as to minimize the performance measure. In this way, optimal solutions with constrained gains were obtained. The details of the various experiments are described below.

In the first experiment, the feedback gain from the state variable $\dot{\phi}_R$ was constrained at various values. In the unconstrained condition, the value of feedback gain from $\dot{\phi}_R$ is -3.74. Accordingly, solutions were obtained for k_2 = -3.74, -3.00, -2.50, -2.00, -1.00, and -0.50. Table 6.1 lists the optimal feedback gains for the various values of constraint on k_2. It is seen that there is little effect on the other feedback gains when k_2 is constrained. However, the locus of closed-loop poles is heavily affected by the constraint on k_2. This locus is plotted in Figure 6.1 for each value of k_2. The locus shows that the major effect of the constraint on the optimal closed-loop system is to reduce the damping of the resonant pole pair associated with the rigid-body dynamics. There is little change in natural frequency of this pole pair. Additionally, the damping of the bending mode poles is decreased somewhat as k_2 becomes heavily constrained. The results of this experiment show that analog computer solutions for the optimal gains under constraints produce accurate results. There is little deviation of the locus of closed-loop poles from the smooth curves of Figure 6.1.

The second experiment involved constraints on two feedback gains. First the feedback from the actuator state variable β was set to zero, thus k_6 = 0. Then the feedback from $\dot{\phi}_R$, that is, k_2, was again constrained at various values. Table 6.2 contains the optimal values of the remaining feedback gains: k_1, k_3, k_4, and k_5. It can be seen that the feedback gains are affected to a slightly greater extent when k_6 = 0 than when it is optimally chosen. Once again, the locus of closed-loop poles is greatly changed as a function of the constraint on k_2, as is seen in Figure 6.2. Both the damping and the natural frequency of the resonant pole pair of the rigid-body dynamics are reduced. It is clear that the lower values of k_2 produce an optimal solution that has low damping of this pole pair. In addition, the damping of the bending mode poles is decreased for low values of k_2. The locus, again, fits smooth curves accurately.

TABLE 6.1

OPTIMAL FEEDBACK GAINS WITH CONSTRAINT IMPOSED ON k_2

RUN NO.	k_1	k_2	k_3	k_4	k_5	k_6
1*	-1.98	-3.74	-.370	.0653	.0216	.131
2	-1.96	-3.00	-.370	.0684	.0242	.094
3	-1.84	-2.50	-.368	.0712	.0239	.083
4	-1.93	-2.00	-.364	.0656	.0270	.104
5	-2.02	-1.00	-.369	.0663	.0191	.124
6	-1.98	-0.50	-.364	.0684	.0147	.129

*No constraint on k_2 in Run 1.

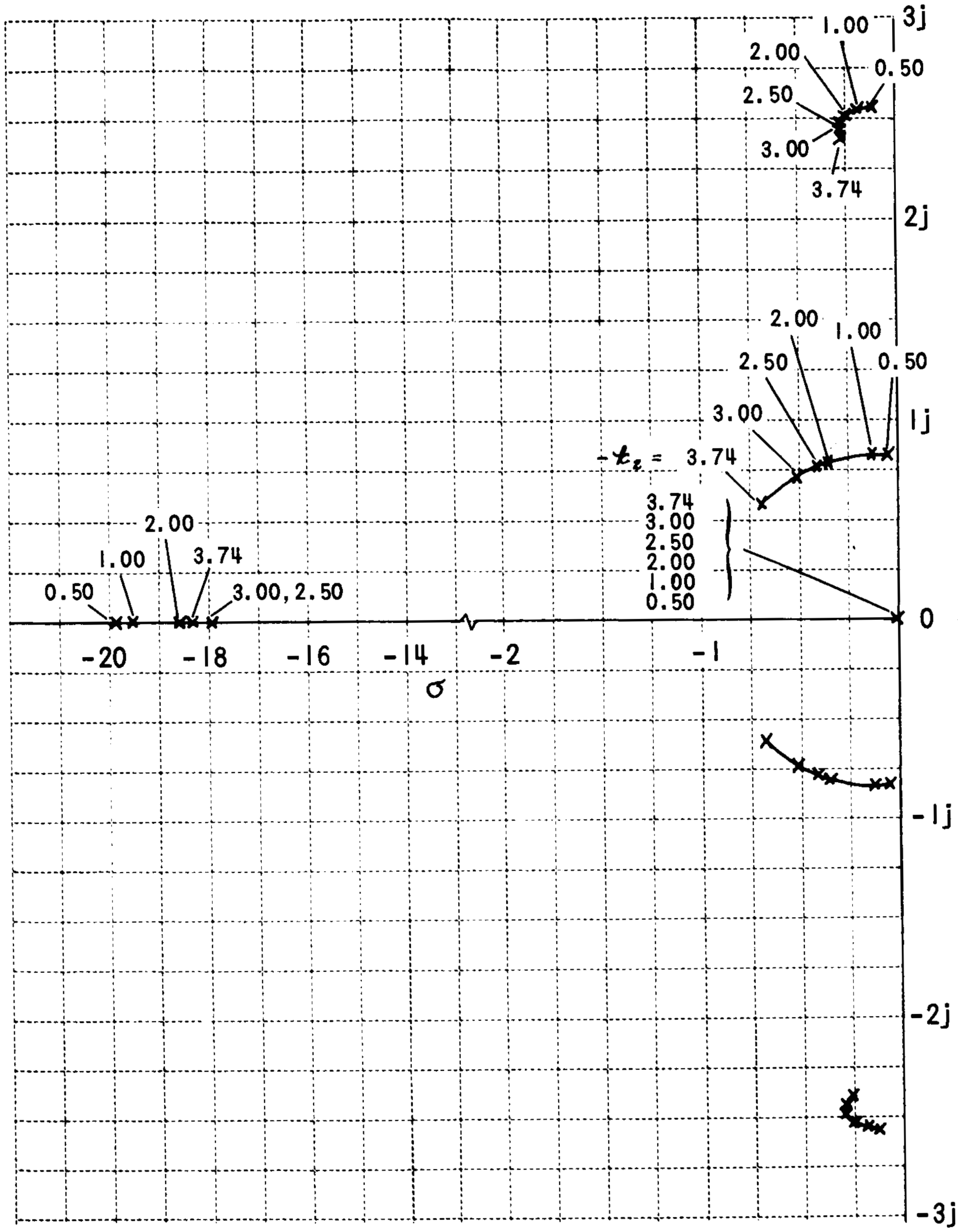

Figure 6.1 Optimal Closed-Loop Pole Locus as a Function of the Constraint on k_2

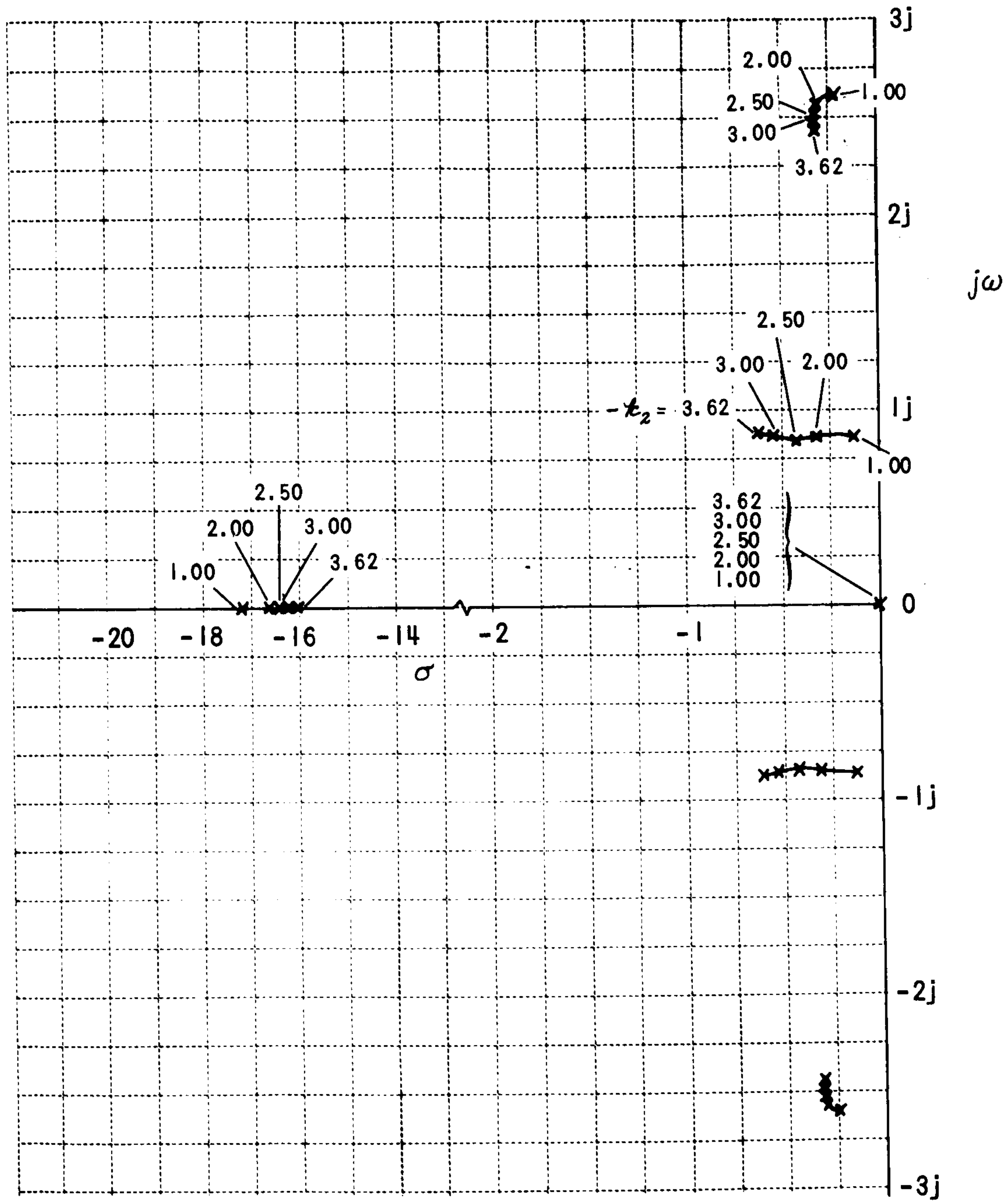

Figure 6.2 Optimal Closed-Loop Pole Locus as a Function of the Constraint on k_2 (k_6 = 0)

TABLE 6.2

OPTIMAL FEEDBACK GAINS WITH CONSTRAINT IMPOSED ON k_2, AND k_6 SET TO ZERO

RUN NO.	k_1	k_2	k_3	k_4	k_5	k_6
7*	-2.54	-3.62	-.376	.0658	.0203	0
8	-2.33	-3.00	-.369	.0696	.0208	0
9	-2.20	-2.50	-.369	.0708	.0230	0
10	-2.10	-2.00	-.367	.0754	.0249	0
11	-1.99	-1.00	-.361	.0755	.0224	0

*No constraint on k_2 in Run 7.

The third experiment was similar to the first, except that k_3 was constrained instead of k_2. It was noted in obtaining the previous solutions that the value of the performance measure was very sensitive to the setting of k_3, which is the feedback from the state variable, α. Therefore, one would expect large changes in the gains and in the closed-loop locus when k_3 is constrained. These large changes do, indeed, occur as Table 6.3 and Figure 6.3 show. All five remaining gains undergo large changes as k_3 is constrained. The value of k_6 changes sign. The resonant pole pair associated with the rigid body has a greatly reduced natural frequency, although the damping is not changed significantly. Moreover for the bending mode poles, the damping is reduced slightly and the natural frequency is increased slightly. It would have been desirable to obtain more solutions for larger constraints on k_3, but this would have required complete rescaling of the problem on the analog computer.

TABLE 6.3

OPTIMAL FEEDBACK GAINS WITH CONSTRAINT IMPOSED ON k_3

RUN NO.	k_1	k_2	k_3	k_4	k_5	k_6
1*	-1.98	-3.74	-.370	.0653	.0216	.131
12	-0.49	-1.66	-.300	.0643	.0158	.112
13	-0.37	-1.19	-.200	.0160	.0115	-.017

*No constraint on k_3 in Run 1.

Conclusions

The studies of the constrained gain problem that have been performed on this project have made certain aspects of the problem clear. First it can be seen from the complexities of the mathematical approaches that analytical solutions are of little value at this time. If the problem is expressed in its simplest form, a set of nonlinear, simultaneous algebraic equations must be solved. Since there are no general methods for solution of these nonlinear algebraic equations, the analytical solutions are too difficult to allow straightforward solution. It is possible that certain specific solutions could be worked

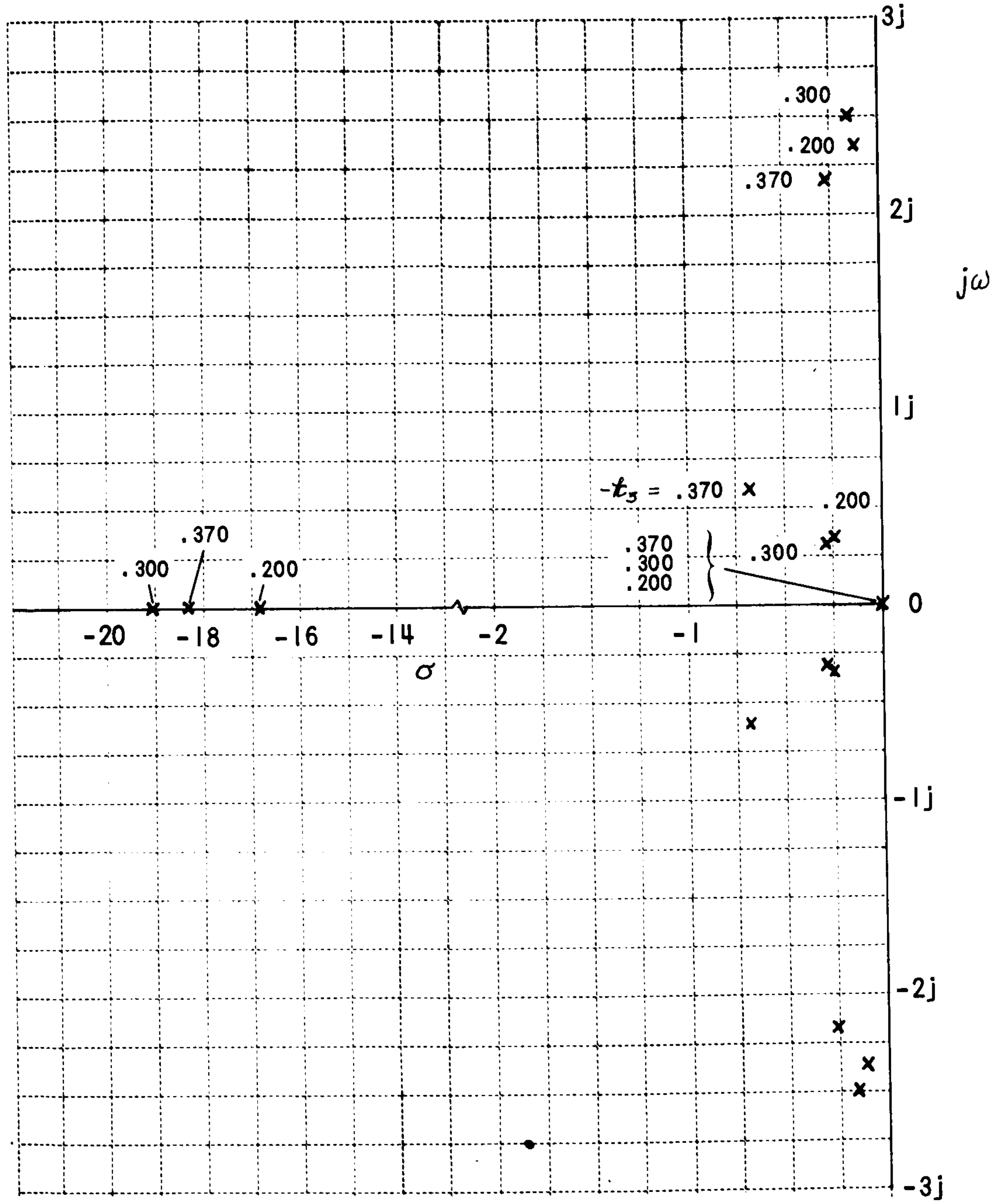

Figure 6.3 Optimal Closed-Loop Pole Locus as a Function of the Constraint on k_3

out by substitution and tedious computation, but these methods are not recommended for high order systems.

In order that heuristic programming methods might be evaluated in regard to application to the constrained gain problem, a sample booster problem was programmed on the analog computer. The performance index was minimized without difficulty when various gains were constrained. When the usual optimal regulator solution was used as a starting point, and one or more of the gains were gradually reduced so as to approach a desired constraint, a minimum could be found. No problems were encountered that could not be traced to the computer itself or its scalings.

It is clear that in some cases no constrained gain solution exists. For example, if the open-loop system is unstable, it may be possible to constrain one or more of the feedback gains (in magnitude) in such a way that there is no stable closed-loop solution. Physical reasoning will generally make the designer aware of such situations, and of course, they must be avoided in any heuristic programming method.

The experimental study has demonstrated the feasibility of developing a digital computer program which automatically performs a systematic search for optimal constrained gain solutions. A strategy could be developed for adjusting each parameter until the performance measure is minimized. If the program uses initial gain settings taken from the usual optimal regulator problem and then gradually shifts the constrained gains, an optimal solution can be obtained for most practical cases. The program could be developed in such a way as to accept matrix equations of motion and arbitrary quadratic performance measures similar to those used in the usual regulator problem. It is recommended that such a program be developed, since it may solve most constrained gain problems effectively and easily.

SECTION 7

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

1. Linear optimal control and associated matrix techniques can be used in a systematic and effective way to conceptually design satisfactory control systems for large, flexible launch vehicles.

2. The use of models in the design procedure simplifies the selection of suitable quadratic performance indices.

3. The optimal feedback control law for the flexible launch vehicle can be obtained in a direct, straightforward manner, not requiring the solution of a matrix Riccati or Wiener-Hopf equation.

4. The control law can be expressed in terms of quantities sensed on the body of the vehicle. The selection of the sensors and their locations is an important design consideration, for the feedback control law may not be realizable in all cases.

5. The optimally controlled system appears to be insensitive to variations of the bending mode shapes and slopes.

6. Optimal compensation networks can be specified for systems with parameters whose values are known only as random variables described on a sample space.

7. The "constrained gain" problem is difficult to solve in an analytical manner. However, it appears that solutions can be experimentally obtained quickly and easily using an analog or digital computer.

Recommendations

1. It has been demonstrated that linear optimal control techniques can be effectively used as a design tool to conceptually specify satisfactory feedback control systems for flexible launch vehicles. The techniques are relatively new, however, and all the unique aspects of the methods have not yet been investigated. In essence, the wealth of experience and background information collected for conventional design methods over the past three decades is not directly available to the optimal control system designer. A more measured and deliberate pace is therefore required when using optimal design methods. However, because of the promise and progress that these techniques have shown, optimal control studies for application to the flexible launch vehicle control problem should be continued.

2. Optimal control law synthesis problems should be investigated in more detail. Other control law approximation techniques should be studied with the objective of defining the extent to which approximations can be made without causing an unstable configuration.

3. The results concerning parameter variations are promising. A more thorough investigation should be conducted to extend the theory of optimal design of systems subject to parameter variations.

4. One of the important results of this study is the experimental determination that the optimal system is relatively insensitive to variations of the bending mode shapes and slopes. These results should be placed on a more firm theoretical basis for the elastic launch vehicle.

5. An investigation of the constrained gain problem should be continued. Relationships among the parameters in the performance index and the magnitudes of the feedback gains should be established. Relationships that exist among the closed-loop dynamics of an optimal system with one or more feedback gains fixed and with a preselected performance index should be established. A digital computer program may be the solution to this particular problem.

SECTION 8

REFERENCES

1. Marshall Space Flight Center Memorandum entitled "Model Vehicle No. 2 For Advanced Control Studies."

2. Rynaski, E. G. and Whitbeck, R. F.: "Theory and Application of Linear Optimal Control". CAL Report No. IH-1943-F-1; October 1965. (Also AFFDL-TR-65-28).

3. Kalman, R. E. and Englar, T.: "Fundamental Study of Adaptive Control Systems". Vol. I and II. ASD-TR-61-27, March 1961 and March 1962.

4. Chang, S. S. L.: "Synthesis of Optimum Control Systems", McGraw-Hill Book Company, Inc., 1961.

5. Rynaski, E. G.: "The Multivariable Root Square Locus -- An Optimal System Design Aid", Proceedings of the AIAA Guidance and Control Conference, August 1965.

6. Tyler, J. S., Jr.: "The Characteristics of Model-Following as Synthesized by Optimal Control", IEEE Transactions on Automatic Control, October 1965.

7. Penrose, R.: "A Generalized Inverse for Matrices", Proc. Camb. Phil. Soc. 51 (1955) 406-413.

8. Newton, G., Gould, L., and Kaiser, J.: "Analytical Design of Linear Feedback Controls", Wiley, New York, 1957.

9. Whitbeck, R. F.: "A Frequency Domain Approach to Linear Optimal Control". Proceedings of the AIAA Guidance and Control Conference, Minneapolis, Minn., August 1965.

10. Rynaski, E. G., Reynolds, P. A., and Shed, W. H.: "Design of Linear Flight Control Systems Using Optimal Control Theory", CAL Report No. IH-1648-F-1, October 1963 (Also ASD-TDR-63-376).

11. Fox, C.: "Introduction to the Calculus of Variations", Oxford Press, London, England, 1950.

APPENDIX A

EQUATIONS OF MOTION AND TRANSFER FUNCTIONS

The equations of motion used throughout this report were extracted from a Marshall Space Flight Center memo labeled "Model Vehicle No. 2 For Advanced Control Studies". The portions of the memo that are used in the body of this report are included in this appendix. The simplifications that were made are described and the pertinent transfer functions are included in this appendix.

The rigid-body coordinate system and the first elastic model geometry are shown in Figures A.1 and A.2, which are reproduced from Reference 1. Using the nomenclature of these two figures, the equations of motion are

Pitch Acceleration Equation

$$\ddot{\phi} + \frac{N' \ell_{cg}}{I_y}\alpha + \sum_i \left[\frac{F}{I_y} Y_{i(x_\beta)} - \frac{F \ell_{cg}}{I_y} Y'_{i(x_\beta)} - \frac{4(F-X)}{m}\frac{S_E}{I_y} Y'_{i(x_\beta)} \right] \eta_i$$

$$+ 4\left[\frac{\ell_{cg} S_E + I_E}{I_y} \right] \ddot{\beta} + \left[4 \frac{F-X}{m} \frac{S_E}{I_y} + \frac{R' \ell_{cg}}{I_y} \right] \beta = 0$$

Flight Path Curvature Equation

$$-\dot{\phi} + \frac{F-X}{m}\phi + \dot{\alpha} + \frac{N'}{mV}\alpha - \frac{F}{mV}\sum_b Y'_{i(x_\beta)}\eta_i + \frac{R'}{mV}\beta = 0$$

Bending Equation

$$-\frac{Q_{i_N}}{M_i} + \ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i - \frac{Q_{i_B}}{M_i} = 0$$

where

$$Q_{i_N} = \sum_{\eta} (q A \alpha) \frac{\partial C_{z_\alpha}}{\partial X_n} (\Delta X_n) Y_i(X_n)$$

$$M_i = \sum_{r=1}^{n} m_{x_r} \left(Y_{i(x_r)} \right)^2$$

$$Q_{i_\beta} = R'\beta Y_{i(x_\beta)} + \left(S_E Y_{i(x_\beta)} + I_E Y'_{i(x_\beta)} \right)\ddot{\beta}$$

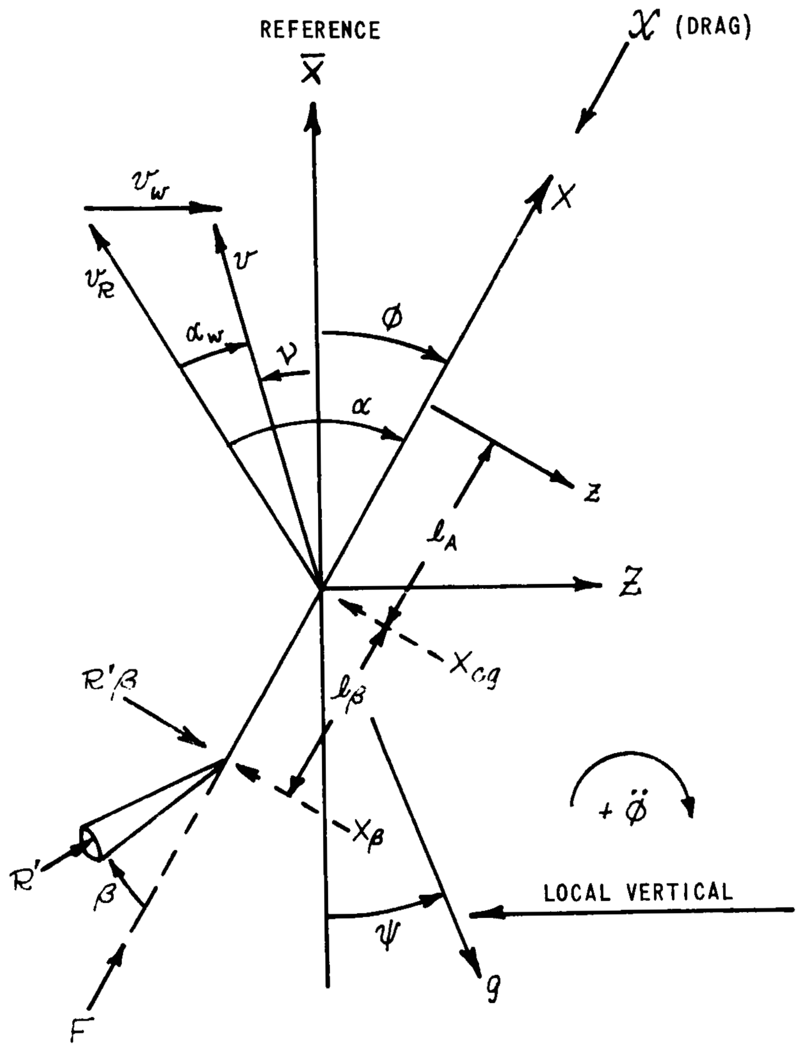

Figure A. 1 Rigid-Body Coordinate System

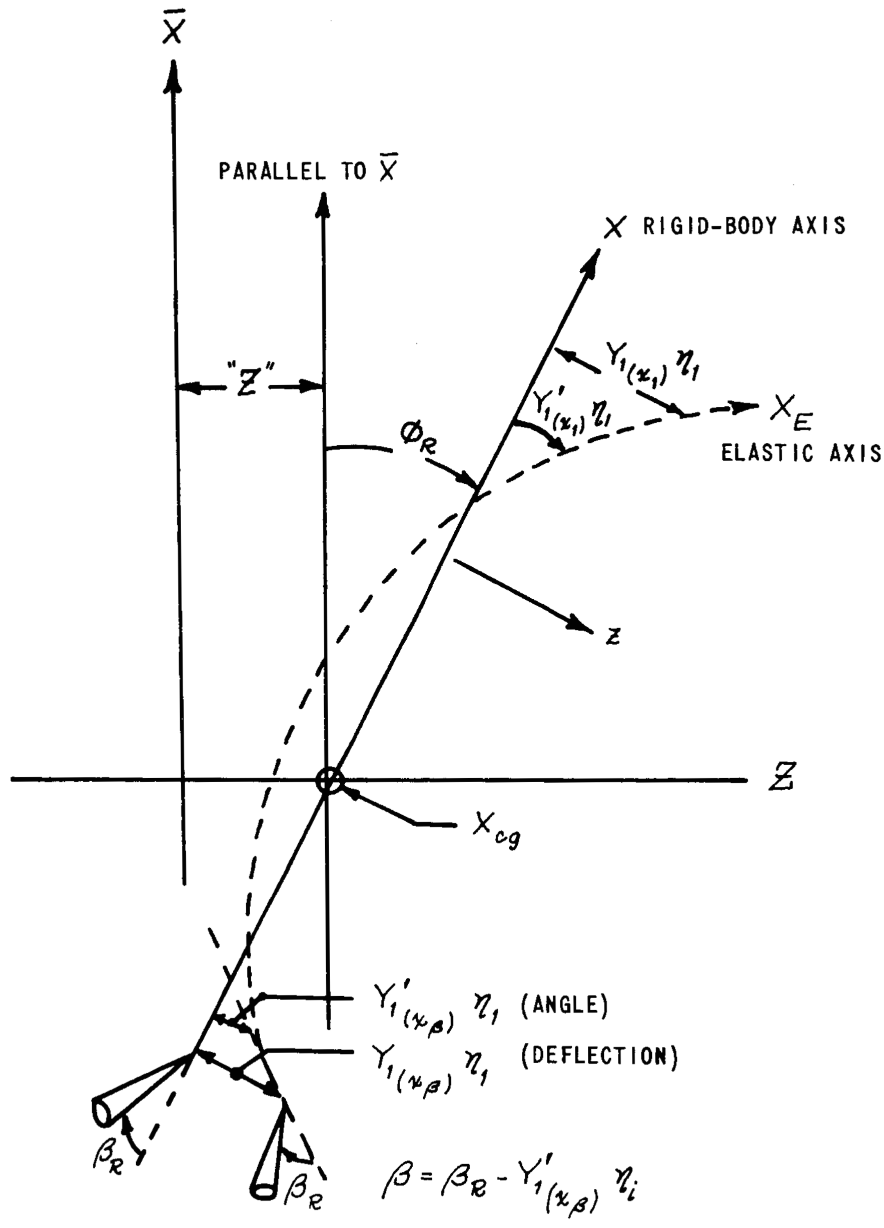

Figure A.2 First Bending Mode Geometry

The engine-actuator dynamics satisfy the differential equation

$$\dddot{\beta} + 23.7\ddot{\beta} + 2259\dot{\beta} + 31,130\,\beta = 31,130\,\beta_c$$

In addition to the equations of motion given above, the following subsidiary equations, which define the quantities measured by an attitude gyro, a rate gyro and accelerometer are used.

Output of an accelerometer

$$a_{\gamma_i} = -\ell_A \ddot{\phi}_R + \frac{N'}{m}\alpha + \sum_j Y_{j(x_i)}\ddot{\eta}_j - \sum_j \left[\frac{F}{m} Y'_{j(x_\beta)} - \frac{F-X}{m} Y'_{j(x_i)}\right]\eta_j + \frac{R'}{m}\beta$$

Rate Gyro

$$\dot{\phi}_{x_i} = \dot{\phi}_R - \sum_j Y'_{j(x_i)}\dot{\eta}_j$$

Attitude Gyro

$$\phi_{x_i} = \phi_R - \sum_j Y'_j \eta_j$$

The entire analysis in the report is performed at a fixed operating point during the boosters' launch trajectory. The point chosen was t = 80 seconds along the nominal trajectory. This operating point is within two seconds of the maximum dynamic pressure condition.

A number of simplifications and approximations were made to eliminate the minor terms in the equations of motion, when these minor terms add little in the frequency range of interest. The simplifications are:

1. The actuator and engine dynamics were approximated by the first-order equation

$$\dot{\beta} + 17.9\beta = 17.9\beta_c$$

The constant $1/\tau$ = 17.9 represents a first-order approximation to the frequency response and phase characteristics of the third-order engine dynamical equation.

2. The inertia of the gimbaled engines is small compared to the total inertia of the vehicle, therefore $4\left[\frac{\ell_{cg} S_E + I_E}{I_y}\right] = 0.$

3. The pitching acceleration due to engine reaction force is small compared to pitch acceleration due to control forces, i.e.,

$$\frac{R'\ell_{cg}}{I_y} >> 4\,\frac{F-X}{m}\,\frac{S_E}{I_y}, \quad \text{therefore} \quad \frac{4F-X}{m}\,\frac{S_E}{I_y} = 0.$$

4. The incremental pitching acceleration and normal acceleration

caused by vehicle flexibility are negligible. Therefore,

$$\frac{F\,Y_{i(x_\beta)}}{I_y} = \frac{F\,\ell_{cg}\,Y'_{i(x_\beta)}}{I_y} = \frac{F\,Y'_{i(x_\beta)}}{mV} = 0$$

Using the data given in the Model Vehicle No. 2 memo and using the assumptions given above, the equations of motion of the launch vehicle at t = 80 seconds were found to be

$\ddot{\phi}_R - .0733\alpha + .45\beta = 0$	pitching acceleration equation
$-\dot{\phi}_R + .0405\,\phi_R + \dot{\alpha} + .01067\alpha + .02106\beta = 0$	flight path curvature
$\dot{\beta} + 17.9\,\beta = 17.9\,\beta_c$	actuator dynamics
$-5.453\alpha + \ddot{\eta}_1 + .02317\,\dot{\eta}_1 + 5.37\,\eta_1 - 15.83\,\beta = 0$	1st bending mode
$-2.36\alpha + \ddot{\eta}_2 + .0564\,\dot{\eta}_2 + 31.8\,\eta_2 - 22.77\beta = 0$	2nd bending mode
$-11.8\alpha + \ddot{\eta}_3 + .0918\,\dot{\eta}_3 + 84.25\,\eta_3 - 26.25\,\beta = 0$	3rd bending mode

Most of the conceptual design work reported upon in the body of this report included two bending modes in the equations of motion. During the course of the design work, certain transfer functions were required. These transfer functions, which include only two bending modes, are given below:

$$\frac{\phi_R}{\beta_c}(s) = \frac{-2.14\left(1+\frac{s}{.0141}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]\left[1+\frac{2(.005)}{5.639}s+\frac{s^2}{(5.639)^2}\right]}{D_{B_2}}$$

$$\frac{\alpha}{\beta_c}(s) = \frac{6.14\left(1-\frac{s}{.04042}\right)\left(1+\frac{s}{21.41}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]\left[1+\frac{2(.005)}{5.639}s+\frac{s^2}{(5.639)^2}\right]}{D_{B_2}}$$

$$\frac{\eta_1}{\beta_c}(s) = \frac{9.18\left(1-\frac{s}{.04082}\right)\left(1-\frac{s}{.4543}\right)\left(1+\frac{s}{.4986}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]}{D_{B_2}}$$

$$\frac{\eta_2}{\beta_c}(s) = \frac{1.17\left(1-\frac{s}{.0412}\right)\left(1-\frac{s}{.3194}\right)\left(1-\frac{s}{.3691}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]}{D_{B_2}}$$

$$\frac{\beta}{\beta_c}(s) = \frac{1.0\left(1-\frac{s}{.04175}\right)\left(1-\frac{s}{.2417}\right)\left(1+\frac{s}{.2942}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]\left[1+\frac{2(.005)}{5.639}s+\frac{s^2}{(5.639)^2}\right]}{D_{B_2}}$$

$$D_{B_2}(s) = \left(1+\frac{s}{17.9}\right)\left(1+\frac{s}{.2942}\right)\left(1-\frac{s}{.2417}\right)\left(1+\frac{s}{.04175}\right)\left[1+\frac{2(.005)}{2.317}s+\frac{s^2}{(2.317)^2}\right]\left[1+\frac{2(.005)}{5.639}s+\frac{s^2}{(5.639)^2}\right]$$

APPENDIX B
THE WIENER-HOPF EQUATIONS

This appendix discusses the derivations which are pertinent to the theory contained in Section 5 of the report. First, a brief review of the derivation of the Wiener-Hopf equation, which specifies the optimal control will be given for the case where the plant is precisely known. In the derivation, the effect of having a plant with non-minimum phase components will be emphasized and then an example will be given to demonstrate the various aspects of the problem. After this, the Wiener-Hopf equation which defines the optimal control that minimizes the expected value of the performance index will be derived.

It seems necessary to carry out the proofs for only the optimal control. Those readers who are interested in deriving, for example, the expression which defines the optimal error may mimic the procedure used for finding the control. The one difficult problem which seems to affect the optimal error theory to a greater extent than it does, for example, the optimal control theory, is the point of whether or not the sufficient condition for a minimum is satisfied when $\gamma(s)$ is a constant. This point is considered at an appropriate point in the development.

The Optimal Control in the Deterministic Case

Consider the block diagram of Figure B.1

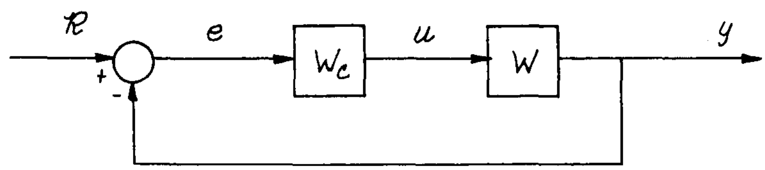

Figure B.1 Closed-Loop System

In Figure B.1, W represents the fixed elements of the system.

The performance index

$$2V = \int_0^{\infty} (e^2 + ru^2)\,dt , \tag{B-1}$$

in which r is the positive weighting on the control, is to be minimized by solving for the optimal control.

Equation B-1 can be rewritten in the equivalent frequency domain form

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{(\mathcal{R} - Wu)(\bar{\mathcal{R}} - \bar{W}\bar{u}) + ru\bar{u}\right\} ds \tag{B-2}$$

since

$$y(s) = W(s)\,u(s) \tag{B-3}$$

In Equation B-2, W = W(s), and $\overline{W}$ = W(-s), etc.

One now takes a variation on Equation B-2 by letting

$$u = u_o + \lambda u_1 \qquad \text{(B-4)}$$

where u_o is the optimal control and u_1 is physically realizable, but otherwise arbitrary. (If this derivation were carried out in the time domain, it would probably be more evident that the minimuzation procedure also requires that $u_{1\,(t=o)} = u_{1\,(t=\infty)} = 0$.) The result is

$$2V = J_a + \lambda(J_b + J_c) + \lambda^2 J_d \qquad \text{(B-5)}$$

where

$$J_a = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{(R - Wu_o)(\overline{R} - \overline{W}\,\overline{u}_o) + r u_o \overline{u}_o\right\} ds$$

$$J_b = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{(r + W\overline{W})\,\overline{u}_o - W\overline{R}\right\} u_1\, ds$$

$$J_c = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{(r + W\overline{W})\,u_o - \overline{W}R\right\} \overline{u}_1\, ds$$

$$J_d = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{(r + W\overline{W})\,u_1 \overline{u}_1\right\} ds$$

J_a is the optimum value of the performance index, $J_d > 0$ and $J_b = J_c$ since $J_b(-s) = J_c(s)$. From this it follows that a necessary and sufficient condition for u_o to give the lowest value of J is, for arbitrary u_1, $J_c = 0$ (see, for example, Reference 4).

Thus one must consider the equation

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{(r + W\overline{W})u_o - \overline{W}R\right\} \overline{u}_1\, ds \qquad \text{(B-6)}$$

in which all the poles of $\overline{u}_1$ are in the right-half plane. It is at this point that care must be exercised because one is tempted to say that a sufficient condition for Equation B-6 being identically zero is that all of the poles of the expression

$$(r + W\overline{W})u_o - \overline{W}R$$

must also be inside the right-half plane when the path of integration is completed to the left. This sufficient condition is usually expressed as

$$(r + W\bar{W})u_0 - \bar{W}\mathcal{R} = \mathfrak{z}(s) \tag{B-7}$$

where $\mathfrak{z}(s)$ is analytic in the left-half plane and, moreover, it assumes that the integrand of Equation B-6 has a sufficient excess of poles over zeros such that the integrand is forced to zero along the left-half plane contour. Equation B-7 is a scalar Wiener-Hopf equation for which one will obtain the correct answer by requiring that $\mathfrak{z}(s)$ have right-half plane poles only when W(s) is minimum phase. In the event that W(s) is nonminimum phase the correct sufficient condition is

$$\mathcal{L}^{-1}[\mathfrak{z}(s)] = \mathfrak{z}(t) = 0 \quad \text{for } t \geq 0 \tag{B-8}$$

for then

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathfrak{z}(s)\,\bar{u}_1\,ds = \int_0^{\infty} \mathfrak{z}(t)\,u_1(t)\,dt \tag{B-9}$$

is identically zero. The basic difficulty is due to the fact that, when there are poles in the right-half plane, W(s) has an impulse response which goes to ∞ as $t \to \infty$ while the impulse response associated with W(-s) goes to ∞ as $t \to -\infty$. The danger exists that one will attempt to solve the problem in a manner that identifies all right-half plane poles with time functions which exist only for $t < 0$. Clearly a right-half plane pole can also represent an unstable system for $t \geq 0$ and one must be careful to associate poles with the correct type of time function. Taking these precautions, one may now rewrite Equation B-7 in the form

$$\frac{[rD\bar{D} + N\bar{N}]}{D\bar{D}}\,u_0 - \frac{\bar{N}}{\bar{D}}\mathcal{R} = \mathfrak{z} \tag{B-10}$$

where
$$W = \frac{N}{D}.$$

D represents the open-loop poles of the system and will contain right-half plane poles when the open-loop system is unstable. Nevertheless, the time function associated with D is identically zero for $t < 0$ while the time function associated with $\bar{D}$ is zero for $t \geq 0$. $[rD\bar{D} + N\bar{N}]$ is now written as the product of a right-half plane component and a left-half plane component, that is,

$$[rD\bar{D} + N\bar{N}] = (rD\bar{D} + N\bar{N})^{+}(rD\bar{D} + N\bar{N})^{-} = \Delta\bar{\Delta}$$

where, for example, Δ gives rise to a stable time function which exists for $t \geq 0$. That is, we insist that the poles be in the left-hand plane.

Now define
$$\left[\frac{\bar{N}\mathcal{R}}{\bar{\Delta}}\right] = \left[\frac{\bar{N}\mathcal{R}}{\bar{\Delta}}\right]_{+} + \left[\frac{\bar{N}\mathcal{R}}{\bar{\Delta}}\right]_{-}$$

$[\bar{N}\mathcal{R}/\bar{\Delta}]$ has been decomposed into a sum in which the first term on the right is associated with a "positive time" function and the second term is associated with a "negative time" function. Moreover, because of the definition of Δ,

$[\bar{N}\mathcal{R}/\Delta]_+$ will be a stable positive time function. In making this statement, one assumes that $\mathcal{R}$ has left-half plane poles. This decomposition can be achieved using a partial fraction expansion.

One can now set

$$u_0 = \frac{D}{(rD\bar{D}+N\bar{N})^+}\left[\frac{\bar{N}\mathcal{R}}{(rD\bar{D}+N\bar{N})^-}\right]_+ \tag{B-11}$$

and verify that this is truly the optimal control by substituting back into Equation B-10 to obtain

$$z(s) = -\frac{(rD\bar{D}+N\bar{N})^-}{\bar{D}}\left[\frac{\bar{N}\mathcal{R}}{(rD\bar{D}+N\bar{N})^-}\right]_- \tag{B-12}$$

This is a function for which

$$z(t) = \mathcal{L}^{-1}\left[z(s)\right] = 0 \quad for\ t \geq 0$$

Equation B-11 is correct even when W(s) has poles and zeros in the right-half plane and is also correct when W(s) has a time delay if one interprets it properly. That is,

$$W(s) = \frac{N}{D}e^{-sT} = \frac{N}{De^{sT}} = \frac{N}{D_1} \tag{B-13}$$

(We will avoid the trap of talking about the "prediction" problem which is sometimes incorrectly stated as

$$W(s) = \frac{N}{D}e^{sT}$$

since

$$\mathcal{L}^{-1}\left[W(s)\right] \neq 0 \quad for\ t < 0).$$

To illustrate the preceding discussion, consider a simple example for which

$$W(s) = \frac{(-s+1)e^{-sT}}{s(-s+2)} \quad and\ \mathcal{R} = \frac{1}{s} \tag{B-14}$$

Let $N = (-s+1)$ and $D = s(-s+2)e^{sT}$ and use Equation B-11:

$$u_0 = \frac{s(-s+2)e^{sT}}{\left[r(-s^2)(-s^2+4)+(-s^2+1)\right]^+}\left[\frac{s+1}{s\left[r(-s^2)(-s^2+4)+(-s^2+1)\right]^-}\right]_+$$

$$= \frac{s(-s+2)e^{sT}}{\sqrt{r}\,(s^2+as+b)}\left[\frac{s+1}{s\sqrt{r}\,(s^2-as+b)}\right]_+$$

where

$$b = \frac{1}{\sqrt{r}}$$

and

$$a = \sqrt{\frac{4r+1}{r} + \frac{2}{\sqrt{r}}}$$

The only "left-half plane pole" of concern in the partial fraction expansion is due to the step input. Therefore

$$u_0 = \frac{s(-s+2)e^{sT}}{\sqrt{r}\,(s^2+as+b)} \cdot \frac{K}{s}$$

where

$$K = \left.\frac{s+1}{\sqrt{r}\,(s^2-as+b)}\right|_{s=0} = 1$$

$$\therefore \qquad u_0 = \frac{(-s+2)e^{sT}}{\sqrt{r}\,(s^2+as+1/\sqrt{r})} \qquad \text{(B-15)}$$

If one were to use the "analytic in the left-half plane" requirement blindly, the result

$$u_0 = \frac{s+2}{\sqrt{r}\,(s^2+as+b)}$$

would be obtained. This <u>does not</u> satisfy the Wiener-Hopf condition.

Substituting Equation B-15 into the Wiener-Hopf equation

$$\left[r + W\overline{W}\right]u_0 - R\overline{W} = \mathfrak{z}$$

gives

$$\mathfrak{z}(s) = \frac{-\sqrt{r}\,(s^2-as+b)}{(s+1)e^{-sT}}\left[\frac{s+1}{s\sqrt{r}\,(s^2-as+b)}\right]_- \qquad \text{(B-16)}$$

Thus $\mathcal{L}^{-1}[\mathfrak{z}(s)] = 0$ for $t \geq 0$ because $(s+1)e^{-sT}$ is, by definition, a factor of $\overline{D}$ and gives rise to a time function which exists only for negative time.

In any situation where there is some doubt as to what to do (for example, does one associate e^{-sT} with N or e^{+sT} with D?) the only recourse is to check the possible alternatives and see whether or not Equation B-9 has been satisfied.

It may occur that $\mathfrak{z}(s)$ turns out to be a constant (K) which is, of course, analytic in the entire s plane. In this event, the sufficient condition, at least in its more general interpretation, breaks down, because

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathfrak{z}(s)\,\bar{u}_1(s)\,ds = \int_0^{\infty} \mathfrak{z}(t)\,u_1(t)\,dt$$

$$= K\int \delta(t)\,u_1(t)\,dt = u_{1(t=0)}$$

It is thus necessary to evoke the additional requirement that $u_{1(t=0)} = u_{1(t=\infty)} = 0.$

The results for the single variable case are easily extended to the multi-control case (see, for example, Reference 2) when nonminimum phase components are involved.

The Optimal Control in the Random Variable Case

In the random variable case we again start with the performance index

$$2V = \int_0^{\infty} (e^2 + ru^2)\,dt \qquad \text{(B-17)}$$

which is also equal to, by Parseval's theorem,

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (e\bar{e} + ru\bar{u})\,ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left\{ (R - Wu)(\bar{R} - \bar{W}\bar{u}) + ru\bar{u} \right\} ds$$

Therefore

$$E\left\{2V\right\} = \frac{1}{2\pi j}\cdot E\left\{ \int_{-j\infty}^{j\infty} \left\{ (R - Wu)(\bar{R} - \bar{W}\bar{u}) + ru\bar{u} \right\} ds \right\}$$

Since $E\left\{2V\right\}$ really means that one must multiply 2V by the multi-dimensional probability density function associated with all of the unknown system parameters, and integrate over the proper domain, the conditions under which one can interchange the order of integration and the expectation operation reduce down to the sufficient conditions one considers for inverting orders of integration in iterated integrals. For example, the formula

$$\int_a^b \left[\int_c^d f(t,u)\,dh(u) \right] dg(t) = \int_c^d \left[\int_a^b f(t,u)\,dg(t) \right] dh(u)$$

is valid if both sides exist and at least one converges absolutely. That is, at least one of the integrals

$$\int_a^b \left[\int_c^d |f(t,u)| \, |dh(u)| \right] |dg(t)|$$

$$\int_c^d \left[\int_a^b |f(t,u)| \, |dg(t)| \right] |dh(u)|$$

is finite.

Thus if we wish to write

$$E\left\{ \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (e\bar{e} + ru\bar{u}) \, ds \right\} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E\left\{ e\bar{e} + ru\bar{u} \right\} ds$$

both sides must exist and either

$$\int_{\mathcal{R}} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left| e\bar{e} + ru\bar{u} \right| ds \, dP(A)$$

or

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[\int_{\mathcal{R}} \left| e\bar{e} + ru\bar{u} \right| dP(A) \right] ds,$$

where P(A) represents the multi-dimensional cumulative probability function for the unknown parameters (described by the vector A), must exist (i.e., be less than infinity).

The writers know of only one practical case for which this condition is not satisfied. If the input to a system is such that a finite following error must result, then Equation B-17 is infinite (e.g., a ramp input into a type one system). However, even in this case it can be shown that the integrand of the performance index has a minimum steady state value. This point is discussed at some length in Section 6 of Reference 2.

We now assume that the sufficient conditions have been satisfied and write

$$E\{2V\} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} E\left\{ (\mathcal{R} - Wu)(\bar{\mathcal{R}} - \bar{W}\bar{u}) + ru\bar{u} \right\} ds \qquad \text{(B-18)}$$

Assuming that $\mathcal{R}$ is deterministic and that the control which minimizes the expected value of the performance index is to be found, one obtains

$$E\{2V\} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left\{\mathcal{R}\bar{\mathcal{R}} - \mathcal{R}E\{\bar{W}\}\bar{u} - \mathcal{R}E\{W\}u + E\{W\bar{W}\}u\bar{u} + ru\bar{u}\right\}ds$$

Again, let $u = u_0 + \lambda u_1$, substitute and clear through to obtain the optimal condition

$$\left[r + E\{W\bar{W}\}\right]u_0 - \mathcal{R}E\{\bar{W}\} = \mathfrak{z}_1(s) \qquad \text{(B-19)}$$

where $\mathcal{L}^{-1}[\mathfrak{z}(s)] = \mathfrak{z}(t) = 0$ for $t \geq 0$

It is easy to see that interchanging the expectation and integration operation will give the same answers as were obtained in the deterministic case with $W\bar{W}$ replaced by $E\{W\bar{W}\}$, etc. Thus the optimal error condition becomes

$$\left[1 + rE\left\{\frac{1}{W\bar{W}}\right\}\right]e_0 - rE\left\{\frac{1}{W\bar{W}}\right\} = \mathfrak{z}_2(s) \qquad \text{(B-20)}$$

and the optimal output condition becomes

$$\left[1 + rE\left\{\frac{1}{W\bar{W}}\right\}\right]y_0 - \mathcal{R} = \mathfrak{z}_3(s) \qquad \text{(B-21)}$$

Notice that the basic philosophy emphasized in this very straightforward and simple derivation is that W is fixed for any given system but not precisely known. We then subjectively decided to take the usual performance index and find its expected value. It is only fair to point out that this idea would lead to trivial results for the case where the system was deterministic but the input was a random variable for then

$$E\left\{\int_0^\infty (e^2 + ru^2)dt\right\} \qquad \text{(B-22)}$$

would give the result that $u_0 \equiv 0$ when $\mathcal{R}$ has a zero mean. In this situation, a different philosophy is adopted which uses the performance index

$$J_1 = e^2 + ru^2$$

and asks that one find the minimum of a time average. That is,

$$E\{J_1\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(e^2 + ru^2)dt.$$

When this problem is explored, one finds that the reasonable physical entity to minimize is the system impulse response $[F_0(t)]$. The various manipulations are then carried through completely in the time domain and yield the time Wiener-Hopf equation (see, for example, Reference 1).

$$\int_0^{\infty} F_0(\tau_2)\left[\phi_{RR}(\tau_1-\tau_2)+r\,\phi_{R_1R_1}(\tau_1-\tau_2)\right]d\tau_2-\phi_{RR}(\tau_1)=0 \qquad \text{(B-23)}$$

$$\text{for } \tau_1 \geq 0$$

where

ϕ_{RR} = autocorrelation function of the input

$\phi_{R_1R_1}$ = autocorrelation function of the input to the fixed elements of the system

$F_0(\tau_2)$ = the system impulse response which minimizes the $E\{J\}$.

Transforming into the frequency domain one obtains the frequency domain Wiener-Hopf equation

$$\left[1+\frac{r}{W\bar{W}}\right]\phi_{RR}(s)\,F_0(s)-\phi_{RR}(s)=\mathfrak{z} \qquad \text{(B-24)}$$

where

$\mathcal{L}^{-1}[\mathfrak{z}(s)]$ = 0 for $t \geq 0$

$\phi_{RR}(s)$ = the power spectral density of the input signal

$F_0(s)$ = the optimal transfer function.