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ON THE STABILITY OF NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

A method for determining a region of relative stability for a class of interpolation and extrapolation formulas is obtained. Some considerations are also given to the absolute stability of this class of formulas.

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SUMMARY

A method for determining a region of relative stability for a class of interpolation and extrapolation formulas is obtained. Some considerations are also given to the absolute stability of this class of formulas.

INTRODUCTION

This paper contains a theoretical development of a procedure for determining a region of relative stability for a given difference equation. A definition of relative stability is given in a paper by Ralston (ref. 1) which is concerned with the problem of integrating a single ordinary differential equation. The following results are concerned primarily with the problem of relative stability for integrating systems of ordinary differential equations, although some considerations are given to absolute stability as defined by Dahlquist (ref. 2) and others. Dahlquist's work, however, dealt with the existence of stable numerical solutions, whereas this procedure is concerned with restricting certain parameters in order to insure that the solution is stable.

SYMBOLS

A	matrix of partials $A = \left(a_{ij}\right)$
$a_i, b_i, D_{p+1}, \delta, \epsilon$	real constants
a_{ij}	$\partial f_i / \partial y_j$
C_i, α, λ	complex constants
$C^{(n)}$	class of functions with the property that their first n derivatives exist and are continuous

E	displacement operator
F	characteristic polynomial
$F_r, F_t, \rho_r, \rho_t, \sigma_r, \sigma_t$	partial derivatives with respect to the subscript
f, y	vector functions
f_i, y_j	components of f and y, respectively
$\bar{f}_n$	$f\left(x_n, \bar{y}_n\right)$
h	integration step-size
i, j, k, m, n, p	integers
P, Q, ρ, σ	polynomials
R	real numbers
$R \times R$	ordered pairs of real numbers
S, V	open sets in $R \times R$
t	real variable
x_n	$a + nh$
$\bar{y}_n$	an approximation to $y\left(x_n\right)$
ϵ_n	$\bar{y}_n - y\left(x_n\right)$
λ_j	characteristic root of A

DEFINITIONS

The following is concerned with the stability of the numerical solution of the system

$$y' = f(x, y), \quad y(a) = y_0 \tag{1}$$

obtained through the use of the difference equation

$$a_k\bar{y}_{n+k} + a_{k-1}\bar{y}_{n+k-1} + \dots + a_0\bar{y}_n = h\left(b_k\bar{f}_{n+k} + b_{k-1}\bar{f}_{n+k-1} + \dots + b_0\bar{f}_n\right) \tag{2}$$

where h is the integration step-size, $x_n = a + nh$, $\bar{y}_n$ is an approximation to $y\left(x_n\right)$, and $\bar{f}_n = f\left(x_n, \bar{y}_n\right)$. Associated with every equation of the form of equation (2) is a pair of polynomials

$$\left.\begin{aligned} \rho(E) &= a_0 + a_1E + \dots + a_kE^k \\ \sigma(E) &= b_0 + b_1E + \dots + b_kE^k \end{aligned}\right\} \tag{3}$$

If E is considered to be the displacement operator $E^k\bar{y}_n = \bar{y}_{n+k}$, then equation (2) can be represented by

$$\rho(E)\bar{y}_n = h\sigma(E)\bar{f}_n \tag{4}$$

In the following, it is assumed that:

(a) The coefficients a_i, b_i are real, and $a_k \neq 0$ $(i = 1, \dots, k)$

(b) No common factor exists for ρ and σ.

Definition 1. The difference equation (4) is said to have order p if, and only if, for $y \epsilon C^{(p+2)}$,

$$\rho(E)y(x) - h\sigma(E)y'(x) = D_{p+1}y^{(p+1)}(x)h^{p+1} + 0\left(h^{p+2}\right)$$

Definition 2. The difference equation (4) is consistent if its order p satisfies $p - 1 \geq 0$.

Definition 3. The difference equation (4) is said to be stable (in the sense of Dahlquist (ref. 2)) if

(a) The roots of ρ are located within or on the unit circle

(b) The roots on the unit circle are distinct.

Throughout the following, it is assumed that equation (4) satisfies assumptions (a) and (b) stated previously, and (a) and (b) of definition 3. The following theorems proved by Dahlquist (ref. 2) limit the order of stable methods of the type as equation (4).

Theorem 1. A necessary and sufficient condition that y_n converge to $y(x)$ is that the difference equation (4) be stable and consistent.

Theorem 2. The order of a stable operator cannot exceed $k+2$, where k is the degree of the polynomial ρ. Furthermore, if k is odd, the order cannot exceed $k+1$.

That stability and consistency are necessary conditions for convergence can be verified by applying a difference equation that does not satisfy definition 3 to the initial value problem $y' = 0$, $y(0) = 0$. (See ref. 3.) Stability and consistency are also sufficient in the sense that, if no errors other than truncation error were introduced, the approximate solution would converge to the true solution as h approaches zero. However, since round-off and other errors are unavoidable, and since a nonzero step-size must be chosen, it is necessary to define the meaning of stability in a neighborhood of zero.

Definition 4. Let λ_j, $j = 1, \ldots, m$ denote the characteristic roots of $A = \left(a_{ij}\right)$, and $a_{ij} = \partial f_i/\partial y_j$, where f_i and y_j are the ith and jth components of the vectors f and y, respectively. The difference equation (4) is stable if, for all λ_j with negative real parts, the roots of

$$\rho(r) - t\sigma(r) = 0, \; t = h\lambda_j \tag{5}$$

lie within or on the unit circle.

The equation

$$\left[\rho(E) - t\sigma(E)\right]\epsilon_n = 0 \tag{6}$$

gives the error in the numerical solution where $\epsilon_n = \overline{y}_n - y(x_n)$. Equation (5) is called the characteristic error equation. If the roots of equation (5) are distinct, the solution of equation (6) has the form

$$\epsilon_n = C_1 r_1{}^n + \ldots + C_k r_k{}^n \tag{7}$$

where the r_i are roots of equation (5). Note that the solution to equation (6) would be different if equation (5) had multiple roots. However, the form of the solution will not be of any concern in the following discussion.

In the solution of a differential equation obtained by solving an approximating difference equation, instability arises due to the presence of extraneous solutions, that is, solutions of the difference equation that are not related to the solution of the differential equation. In reference 3, it is shown that, if h is chosen so that the roots of equation (5) satisfy the condition of definition 4, errors introduced at some stage of the integration procedure are damped out in subsequent stages.

In certain types of problems, it is important that the errors introduced by extraneous solutions damp out faster than the true errors in the approximating solution. In these instances, the relative stability of equation (4) is important. Before relative stability is defined, the principal solution must be defined.

Definition 5. The continuous function $r_1(h)$, satisfying $r_1(0) = 1$ and equation (5), will be called the principal solution. All other solutions are extraneous.

The following definition is given by Ralston in reference 1.

Definition 6. A consistent numerical integration method of the type as equation (4) is said to be relatively stable on the interval $[a, b]$, which must include zero, if for all $h \epsilon [a, b]$,

$$\left| \frac{r_i(h)}{r_1(h)} \right| \leq 1; \; i = 2, \ldots, k$$

where r_1 denotes the principal solution of equation (5), and r_i, $i = 2, \ldots, k$, denote the remaining $k-1$ solutions. Furthermore, when $|r_1| = |r_i|$ then r_i is simple.

LOCATION OF THE PRINCIPAL SOLUTION

In this section the implicit function theorem for functions of two variables will be needed. For a given function $F(t, r)$, F_t and F_r will denote, respectively, $\partial F/\partial t$ and $\partial F/\partial r$.

Theorem 3 (implicit function theorem). Let $F(t, r)$ be a function defined on an open set S of $R \times R$, and let $(t_0, r_0) \epsilon S$.
Suppose that

(1) F has continuous first partial derivatives on S,

(2) $F(t_0, r_0) = 0$, and

(3) $F_r(t_0, r_0) \neq 0$

Then there exists an open neighborhood V of t_0 and a unique function $r \epsilon C^{(1)}$ satisfying the following conditions:

If $t \epsilon V$, then

(1) $F(t, r(t)) = 0$

(2) $(t, r(t)) \epsilon S$

(3) $dr/dt = -F_t/F_r$

Consider the characteristic polynomial (eq.(5)), which shall be denoted by

$$F(h, \alpha, r) = 0 \tag{8}$$

where α is an arbitrary complex parameter.

The requirement that equation (4) be consistent implies $F(0, \alpha, 1) = 0$. Also, as a consequence of condition (b) of definition 3 imposed on the polynomial ρ, $F_r(0, \alpha, 1) \neq 0$. Hence, by theorem 3, there exists some open region V containing zero in which $r_1(h)$ is unique and differentiable. Hence, $r_1(h)$ is the solution of the differential equation

$$dr/dh = \alpha\sigma(r)/(\rho_r - h\alpha\sigma_r) \tag{9}$$

with initial conditions $r(0) = 1$, where ρ_r and σ_r denote $\partial\rho/\partial r$ and $\partial\sigma/\partial r$, respectively.

Theorem 4. The solution of equation (9) is uniquely defined through any point (h, r) except possibly at those points (h, r) where r is a solution of

$$\left.\begin{aligned} \rho\sigma_r - \sigma\rho_r &= 0, \\ \text{and} \qquad & \\ h = \frac{\rho(r)}{\alpha\sigma(r)}, &\quad \alpha \neq 0 \end{aligned}\right\} \tag{10}$$

Proof. Let $\left(h_0, r_0\right)$ be a solution of equation (9). Then

$$\rho\left(r_0\right) - h_0\alpha\sigma\left(r_0\right) = 0 \tag{11}$$

and, if the solution is not unique, then

$$\rho_r\left(r_0\right) - h_0\alpha\sigma_r\left(r_0\right) = 0 \tag{12}$$

Case 1. Assume $\sigma(r_0)$ and $\sigma_r(r_0)$ are not zero. Then $\frac{\rho\left(r_0\right)}{\sigma\left(r_0\right)} = \frac{\rho_r\left(r_0\right)}{\sigma_r\left(r_0\right)}$ and, hence, r_0 is a root of equation (10). Now as a result of condition (b) placed on ρ and σ, $\sigma\left(r_0\right) \neq 0$.

Case 2. Assume $\sigma_r\left(r_0\right) = 0$.

If $\sigma_r\left(r_0\right) = 0$, then $\rho_r\left(r_0\right) = 0$ and r_0 is still a solution of equation (10).

Since $\sigma\left(r_0\right)$ cannot be zero, then for any solution $\left(h_0, r_0\right)$ of equation (8), h_0 must be defined by

$$h_0 = \frac{\rho\left(r_0\right)}{\alpha\sigma\left(r_0\right)} \tag{13}$$

This completes the proof.

Hence, the principal solution of the difference equation is given uniquely by equation (9) in any region that does not include a solution of equation (10).

The following well-known theorem is stated for completeness.

Theorem 5 (Rouche). If $f(z)$ and $g(z)$ are analytic interior to a simple closed Jordan curve C, and if they are continuous on C, and $|f(z)| < |g(z)|$ on C, then the function $F(z) = f(z) + g(z)$ has the same number of zeros interior to C as does $g(z)$.

The following theorem is an immediate consequence of theorem 5.

Theorem 6. Let C be a simple closed Jordan curve so that an nth degree polynomial $P(z)$ has exactly $n-1$ roots in the interior of C and $P(z) \neq 0$ on C. Furthermore, let $Q(z)$ be a polynomial and λ a real number so that

$$|P(z)| > |\lambda Q(z)|$$

on C; then if α is any complex number such that $|\alpha| \leq |\lambda|$, then $P(z) \pm \alpha Q(z)$ has exactly $n-1$ roots interior to C.

Now consider the polynomials ρ and σ where the degree of ρ is n and the degree of σ is less than or equal to n.

Theorem 7. Let C be any simple closed Jordan curve having $n-1$ roots of $\rho(r)$ in its interior, and the root $r_1 = 1$ on the exterior. Let λ be chosen such that

$$|\rho(r)| > |\lambda\sigma(r)|$$

for all r on C. Then $r_1(h)$ (the principal solution) is always on or outside C for all α such that $|\alpha| \leq |\lambda|$.

Proof. Certainly there is a neighborhood of zero such that $r_1(h)$ is outside C since $r_1(h)$ is a continuous function. Indeed, all the roots of $\rho(r) - \lambda\sigma(r)$ are continuous functions of the coefficients (see for example ref. 4). Now if $r_1(h)$ is ever interior to C, then there must exist real numbers h' and $\epsilon > 0$ such that $r_1(h')$ is on C, and either $r_1(h' + \delta)$ $\big($or $r_1(h' - \delta)\big)$ is interior to C for any δ such that $0 < \delta < \epsilon$. Since the remaining $n-1$ roots are interior to C, there exists a neighborhood of each root contained completely within C. Now consider an arbitrary δ such that $r_1(h' + \delta)$ is in C. Then some other root $r_2(h' + \delta)$ must be outside or on C which contradicts the fact that r_2 is continuous.

The main result of this paper follows as a direct consequence of the previous theorem and is stated as corollary 7.1.

Corollary 7.1. If λ is chosen as in theorem 7, and C' of theorem 7 is taken as a circle with center at the origin and radius less than 1, then the approximate solution to the system (eq. (1)) obtained from equation (4) is relatively stable if h is chosen such that $h\alpha \leq |\lambda|$ where α is a bound on the eigenvalues of the matrix of partial derivatives $A = \left(a_{ij}\right)$.

Another corollary concerning the absolute stability of the difference equation solution follows.

Corollary 7.2. Let C be any simple closed Jordan curve contained in the unit circle and excluding the point $z=1$ and let λ be chosen as in theorem 7. Then if $|h\alpha| \leq |\lambda|$ for all eigenvalues α of the matrix of partial derivatives, the absolute stability of the numerical solution is dependent only on the principal solutions of the system corresponding to each eigenvalue.

AN EXAMPLE

Before considering a simple illustration of the results of the preceding section, observe the following theorem.

Theorem 8. If $\rho(E) = h\sigma(E)f_n$ is a difference equation such that:

(1) ρ has all zero roots with the exception of the principal root

(2) r_i, $i = 1, \ldots, n$ are the roots of σ

(3) δ is a real number such that $0 < \delta < 1$

(4) α is a bound on the partial derivatives of A (the matrix of partial derivatives),

then the difference equation is relatively stable if

$$h < \frac{|\rho(\delta)|}{\alpha\left[\prod_{i=1}^{n}\left(\left|r_i\right| + \delta\right)\right]} \tag{14}$$

where h is the integration step-size.

In view of theorem 7, the proof of the above proposition is obvious since $|\rho(z)| \geq |\rho(\delta)|$ and $|\sigma(z)| \leq \prod_{i=1}^{n} \left(|r_i| + \delta\right)$ for $|z| = \delta$.

Now consider the system of equations

$$\left.\begin{aligned} \dot{x}_1 &= f_1\left(x_1, x_2\right) = x_2 \\ \dot{x}_2 &= f_2\left(x_1, x_2\right) = -x_1 \end{aligned}\right\} \tag{15}$$

and let the difference equation to be used be the trapezoidal formula

$$y_{n+1} - y_n = \frac{h}{2}\left(f_{n+1} + f_n\right) \tag{16}$$

For the system (eq. (15)) the matrix A is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

whose eigenvalues are obviously bounded by $\alpha = 1$. For the difference equation (16) the associated polynomials are

$$\rho(z) = z - 1$$

$$\sigma(z) = \frac{1}{2}(z + 1)$$

Consequently, by theorem 8, the numerical solution will be relatively stable if $h < 1$.

CONCLUSIONS

The primary purpose of this paper has been to determine methods for finding stability regions for numerical solutions of systems of differential equations using a difference equation. Very good bounds on stability regions are known or can be computed with relative ease for single differential equations. Thus this special case has not been explicitly mentioned. Note, however, that by using the results of the paper an optimum region of relative stability can be obtained. A method of determining a region of relative stability for the general case is given . While in general this region will not be optimal, it is at least easy to compute. For any step-size h such that the principal solution of the difference equation is a good approximation to the true error equation, stability of the difference equation solution is insured if the proper conditions on the eigenvalues are satisfied.

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REFERENCES

1. Ralston, A.: Relative Stability in the Numerical Solution of Ordinary Differential Equations. SIAM Review, Vol. 7, Jan. 1965, pp. 114-125.

2. Dahlquist, G.: Convergence and Stability in the Numerical Integration of Ordinary Differential Equations. Mathematica Scandinavia, Vol. 4, 1956, pp. 33-53.

3. Lea, R.: Stability of Multistep Methods in Numerical Integration. NASA TN D-2772, 1965.

4. Marden, M.: The Geometry of the Zeros. American Mathematical Society, 1956.

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