A METHOD FOR DETERMINING

OPTIMUM RE-ENTRY TRAJECTORIES

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AUBURN UNIVERSITY

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By

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SUMMARY

The Pontryagin Maximum Principle is used to formulate the problem of finding optimum atmospheric vehicular re-entry trajectories. The optimization problem is that of minimizing an integral which is a function of the state and control variables. The vehicle's motion is assumed to be influenced only by a gravitational force and an aerodynamic force. The problem is formulated and the necessary equations are developed simultaneously for three sets of Euler angles. Computational procedures are suggested so that numerical trajectories may be generated.

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LIST OF SYMBOLS

A	Projected cross-sectional area of vehicle
C.G.	Center of gravity
C.P.	Center of pressure
$C\alpha$	Cosine α
$C\alpha_y$	Cosine α_y
CP	Cosine ϕ_p
CR	Cosine ϕ_r
CY	Cosine ϕ_y
C_x, C_z	Vehicle configuration factors
$\overline{F}_a$	Aerodynamic force in the aerodynamic coordinate system
$\overline{F}_{am}$	Aerodynamic force in the missile system
$\overline{F}_g$	Gravitational force in the plumbline system
$\overline{F}_{r1} = -\overline{F}_{r2}$	Roll forces in the missile system
G	Gravitational constant
m	Mass of the vehicle

M	Mass of the earth
$\overline{M}_{am}$	Aerodynamic moment (missile system)
$\overline{M}_{rm}$	Roll couple (missile system)
q	Dynamic pressure
R_o	Earth's radius
$\lvert\overline{R}\rvert$	Absolute value of the plumbline position vector
$S\alpha$	Sine α
$S\alpha_y$	Sine α_y
SP	Sine ϕ_p
SR	Sine ϕ_r
SY	Sine ϕ_y
t	Time
T	Kinetic energy of the vehicle
$\overline{V}_r$	Relative velocity vector (Aerodynamic System)
$\overline{V}_{rm}$	Relative velocity vector (Missile System)
$\overline{V}_R$	Relative velocity vector (Plumbline System)
$\overline{W}$	Velocity vector for abnormal air movement in plumbline system

$\overline{X}$	Plumbline position vector
$\overline{X}_a$	Aerodynamic system position vector
$\overline{x}_{cp}$	Position of the center of pressure in the missile system
$\overline{X}_m$	Missile system position vector
$\overline{z}_r$	Roll jet positions in the missile system
α^*	Angle of attack
α_y	Yaw angle of attack
ϕ_p	Pitch angle
ϕ_r	Roll angle
ϕ_y	Yaw angle
ρ	Density of the Atmosphere
$\overline{\omega}$	Angular velocity vector of the vehicle in the missile system
$\overline{\omega}_e$	Angular velocity vector of the attracting body in the plumbline system

LIST OF FIGURES

I. INTRODUCTION

This paper is an extension of previous work done by Grady Harmon and W. A. Shaw, presented in NASA TM X-53024, March 14, 1964.

The objectives of this paper are (1) to present a method for treating optimum re-entry problems in a simplified manner and (2) to generalize the computational scheme outlined in the aforementioned paper. The computational scheme given allows for the optimization of any functional subject to the specified constraints. Atmospheric data, vehicle configuration and aerodynamic coefficients are incorporated in the computational scheme in tabular form. Thus, different vehicles and/or atmospheres may be considered by changing the appropriate tables. The governing equations are developed for three different gimbal sets. A computational scheme is outlined for each case.

No numerical results are available at present, but development of the computer deck is underway at the Electronics Research Center.

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II. STATEMENT OF THE PROBLEM

The problem is that of finding the optimum control process, $\alpha_y(t)$, that will transfer a vehicle from an initial state, at time t_o, in an atmosphere to a terminal state, at time t_1, in the same atmosphere so that the value of the functional

$$J = \int_{t_o}^{t_1} f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\phi}}, \phi_r, \phi_y, \phi_p, \alpha, \alpha_y, F_r)\, dt$$

is a minimum. The trajectory associated with this optimum control process is the optimum trajectory.

The rotational motion of the vehicle is treated in a simplified manner. The equations governing the vehicle's rotational motion are considered as a steady-state problem with only one component of the angular velocity vector present for any given gimbal set. A gimbal set is used to measure the Euler angles, ϕ_r, ϕ_y, and ϕ_p. The equations of motion are developed simultaneously for three different gimbal sets.

The problem is formulated as a Pontryagin initial value problem. The relative velocity equations appear as algebraic constraints. The yaw angle of attack, $\alpha_y(t)$, is the control variable.

Additional assumptions are made as follows:

1. The motion of the vehicle is influenced by an aerodynamic force that acts through the vehicle's center of pressure.
2. The attracting body is a rotating sphere with homogeneous mass.

3. The vehicle's centroid of mass and centroid of volume are not coincident.
4. The vehicle's center of mass is invariant with respect to the vehicle.
5. The center of pressure of the vehicle is invariant with respect to the vehicle.
6. A system of roll control jets is available on the vehicle that produce a pure roll couple as required by the optimum control process.

III. COORDINATE SYSTEMS

Three rectangular coordinate systems will be used in this paper. They are:

1. The plumbline space fixed coordinate system,
2. The vehicle fixed missile coordinate system,
3. The aerodynamic coordinate system.

A. Plumbline System

The plumbline system, Figure 1, has its origin at the earth's center with the Y-axis parallel to the gravity gradient at the launch point. The X-axis is parallel to the earth fixed launch azimuth and the Z-axis is chosen to form a right-handed system.

B. Missile System

The missile system, Figure 1, is located with its origin at the center of mass of the vehicle and its y_m axis parallel to the longitudinal axis of the vehicle. The x_m and z_m axes are chosen to form a right-handed system which is parallel to the plumbline system at the launch point.

As the vehicle moves along its trajectory, the missile system undergoes a displacement with respect to the plumbline system. This displacement is given by three Euler angles as measured by a gimbal set. The Euler angles uniquely specify the orientation of the vehicle at any time. Any particular orientation of the vehicle may be described by

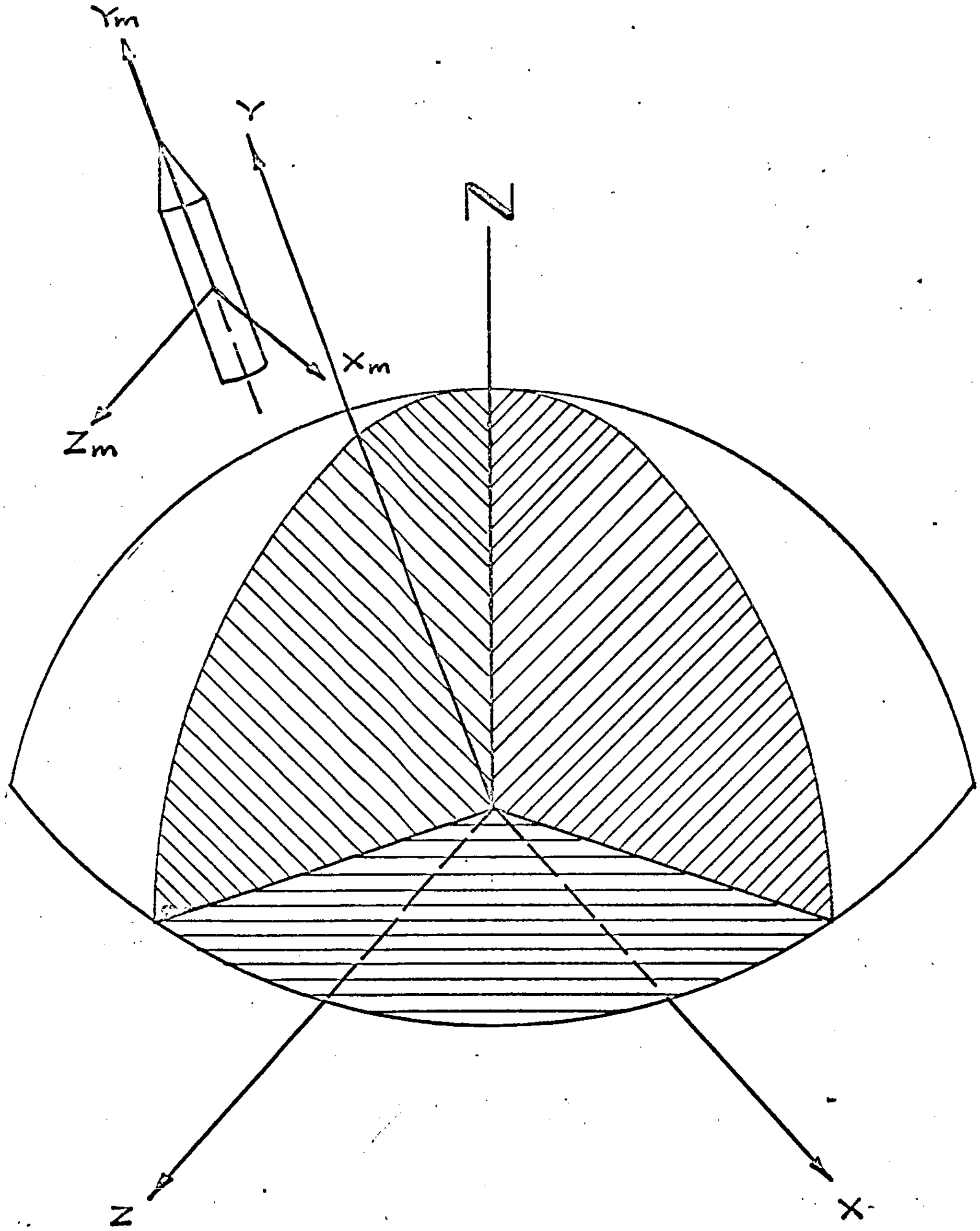

Fig. 1. Plumbline and missile coordinate systems

different sets of Euler angles depending solely on the sequence in which the angles are measured. Therefore, it is mandatory that a specific sequence be followed in measuring the Euler angles. The three Euler angles are referred to as the yaw angle, ϕ_y, the roll angle, ϕ_r, and the pitch angle, ϕ_p. The yaw angle is measured with respect to an X axis. The roll angle is measured with respect to a Y axis, and the pitch angle is measured with respect to a Z axis. An angle is considered positive counterclockwise when viewed from the positive end of the axis about which the rotation is taken. The angles are measured by a set of gimbals on the vehicle. A gimbal set measures the Euler angles in a specific sequence such as pitch, yaw, and roll. In this paper, equations that involve the angles yaw, roll, or pitch are developed simultaneously for three different sets of Euler angles. The angles are obtained from three gimbal sets. They will be referred to as follows:

1. A gimbal set which measures in the order of pitch, yaw, roll.
2. A gimbal set which measures in the order of pitch, roll, yaw.
3. A gimbal set which measures in the order of roll, yaw, pitch.

The Euler angles are shown in Figures 2, 3, and 4.

A position vector in the missile coordinate system may be written in terms of a position vector in the plumbline coordinate system.

The equations of transformation are given by the orthogonal rotation matrices

$$[\phi_y] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & CY & SY \\ 0 & -SY & CY \end{bmatrix} ,$$

$$[-\phi_r] = \begin{bmatrix} CR & 0 & SR \\ 0 & 1 & 0 \\ -SR & 0 & CR \end{bmatrix} ,$$

$$[\phi_p] = \begin{bmatrix} CP & SP & 0 \\ -SP & CP & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The particular combination of the above rotation matrices that relate a vector in the two coordinate systems is dependent on the gimbal set used. The relationship for gimbal set 1 is

$$\overline{X}_m = [-\phi_r] \; [\phi_y] \; [\phi_p] \; \overline{X} \qquad (1a)$$

or

$$\overline{X}_m = [A_d]_1 \; \overline{X} \qquad (1b)$$

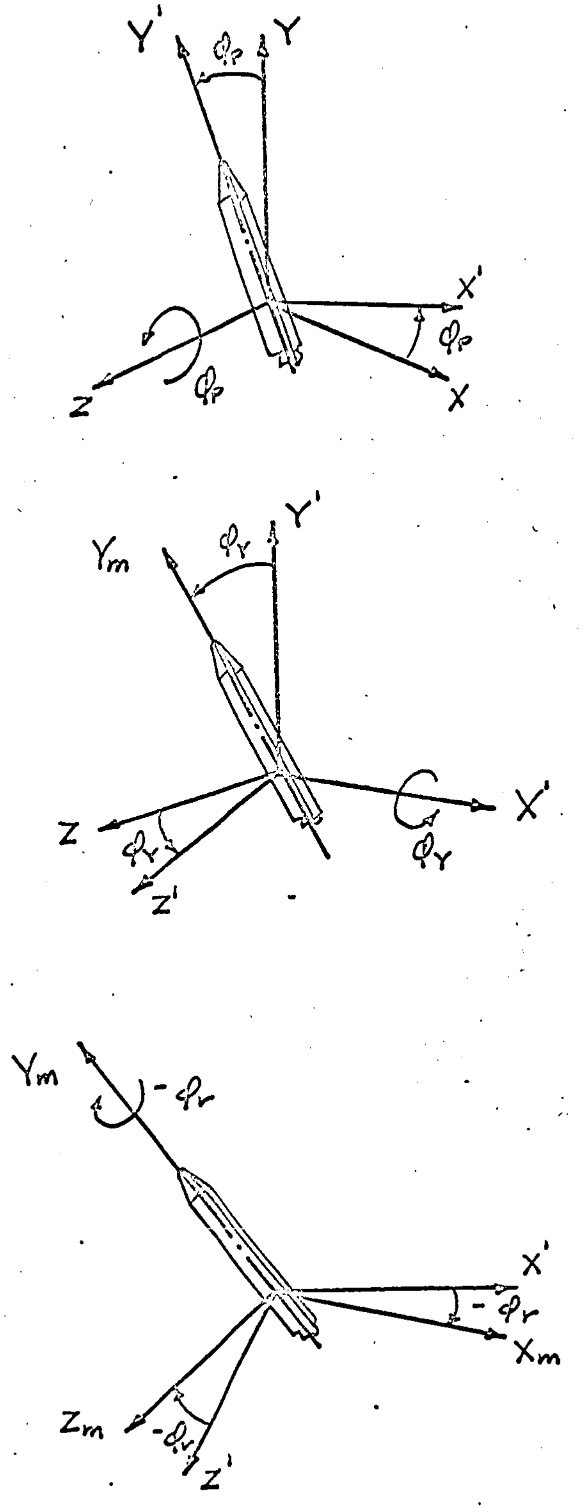

Fig. 2. Eulerian angles for gimbal set 1

where

$$[A_d]_1 = \begin{bmatrix} CRCP+SPSRSY & CRSP-SRSYCP & SRCY \\ -CYSP & CYCP & SY \\ -SRCP+CRSYSP & -SPSR-CRCPSY & CRCY \end{bmatrix} \qquad (1c)$$

is the combined product of the rotation matrices in equation (1a). When $\emptyset_y = 90°$ gimbal set 1 is oriented so that $\emptyset_r$ and $\emptyset_p$ are measured in the same direction, refer to Figure 2. This condition is referred to as gimbal lock.

The relationship for gimbal set (2) is

$$\overline{X}_m = [\emptyset_y] \; [-\emptyset_r] \; [\emptyset_p] \; \overline{X} \qquad (2a)$$

or

$$\overline{X}_m = [A_d]_2 \; \overline{X} \qquad (2b)$$

where

$$[A_d]_2 = \begin{bmatrix} CRCP & CRSP & SR \\ -CYSP-SYSRCP & CYCP-SYSRSP & SYCR \\ SYSP-CYSRCP & -SYCP-CYSRSP & CYCR \end{bmatrix} \qquad (2c)$$

is the combined product of the rotation matrices in equation (2a). Gimbal set 2 is locked when $\emptyset_r = 90°$. At this orientation, refer to Figure 3, $\emptyset_y$ and $\emptyset_p$ are measured in the same direction.

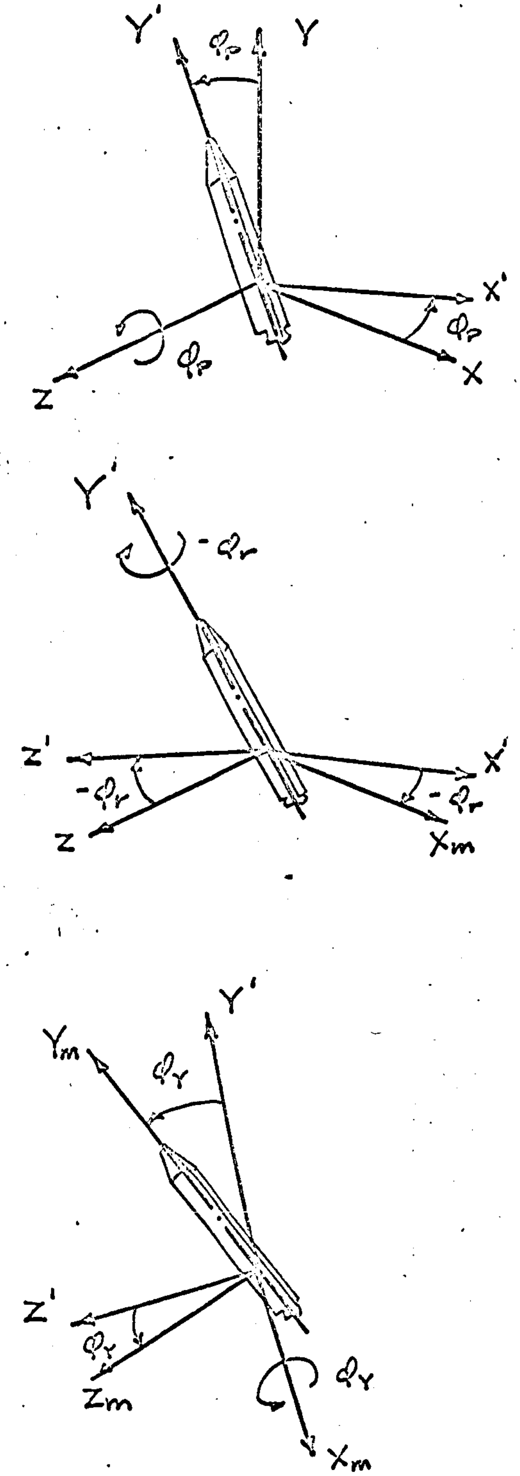

Fig. 3. Eulerian angles for gimbal set 2

The relationship for gimbal set (3) is

$$\overline{X}_m = [\emptyset_p]\ [\emptyset_y]\ [-\emptyset_r]\ \overline{X} \tag{3a}$$

or

$$\overline{X}_m = [A_d]_3\ \overline{X} \tag{3b}$$

where

$$[A_d]_3 = \begin{bmatrix} \text{CPCR-SPSRSY} & \text{SPCY} & \text{CPSR+SPSYCR} \\ \text{-SPCR-CPSRSY} & \text{CPCY} & \text{-SPSR+CPSYCR} \\ \text{-SRCY} & \text{-SY} & \text{CYCR} \end{bmatrix} \tag{3c}$$

is the combined product of the rotation matrices in equation (3a). Gimbal set 3 is locked when $\emptyset_y = 90°$. At this orientation, refer to Figure 4, $\emptyset_p$ and $\emptyset_r$ are measured in the same direction.

The transformation matrices (1c), (2c), and (3c) will be referred to as

$[A_d]_3$ where i = 1, 2, 3.

Equations (1b), (2b), and (3b) are restated as

$$\overline{X}_m = [A_d]_i\ \overline{X}. \tag{4}$$

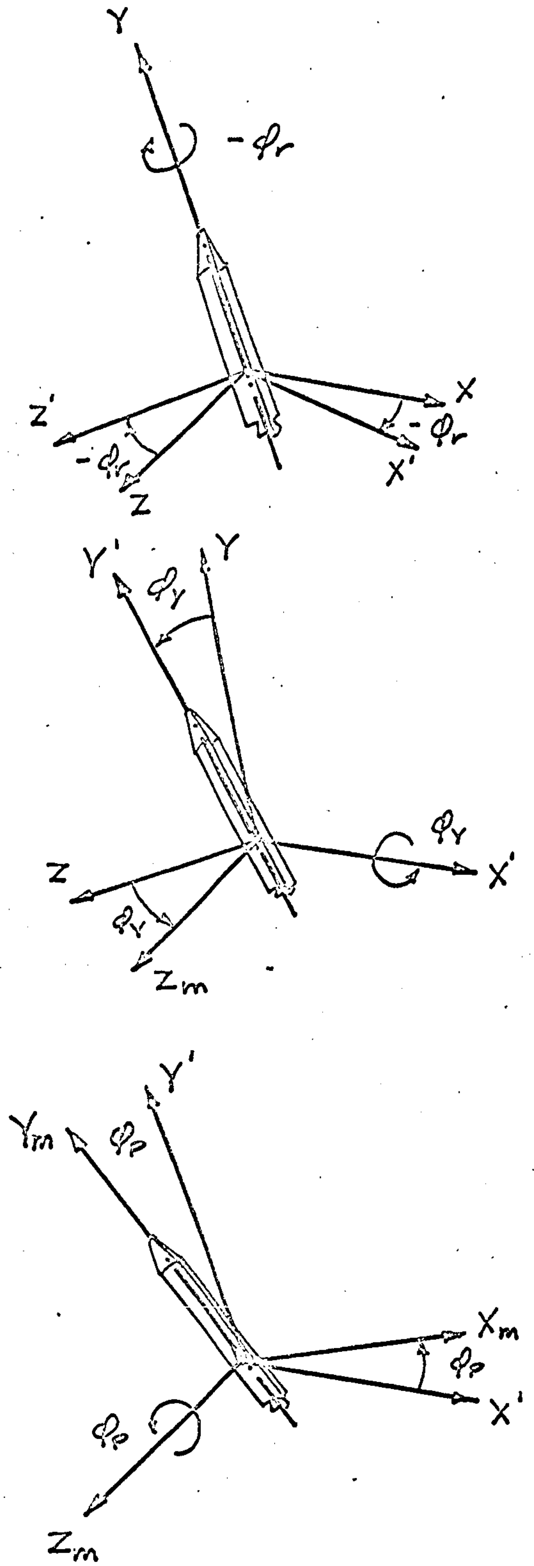

Fig. 4. Eulerian angles for gimbal set 3

C. Aerodynamic Coordinate System

The aerodynamic coordinate system is located as shown in Figure 5 with its origin at the center of pressure of the vehicle. The Y_a axis lies in the plane containing the vehicle longitudinal axis of symmetry and the relative velocity vector. The relative velocity vector, $\overline{V}_R$, is defined as the velocity of the air with respect to the vehicle as measured from the inertial reference. The X_a and Z_a axes are chosen to form a right-handed system. As the vehicle moves along its trajectory, there will be a relative displacement between the missile fixed coordinate system and the aerodynamic coordinate system. The direction of the Y_a axis is defined by the following rotations as shown in Figure 5:

1. Rotate the vehicle fixed reference frame about the Y_m axis so that the X_m axis lies in a plane parallel to the plane formed by the vehicle's longitudinal axis of symmetry and the relative velocity vector. The angle traversed is referred to as the yaw angle of attack, α_y.
2. Rotate about the new Z axis by the true angle of attack, α^*. This specifies the orientation of the aerodynamic coordinate system.

The true angle of attack, α^*, will be expressed in terms of the aerodynamic force in the next section.

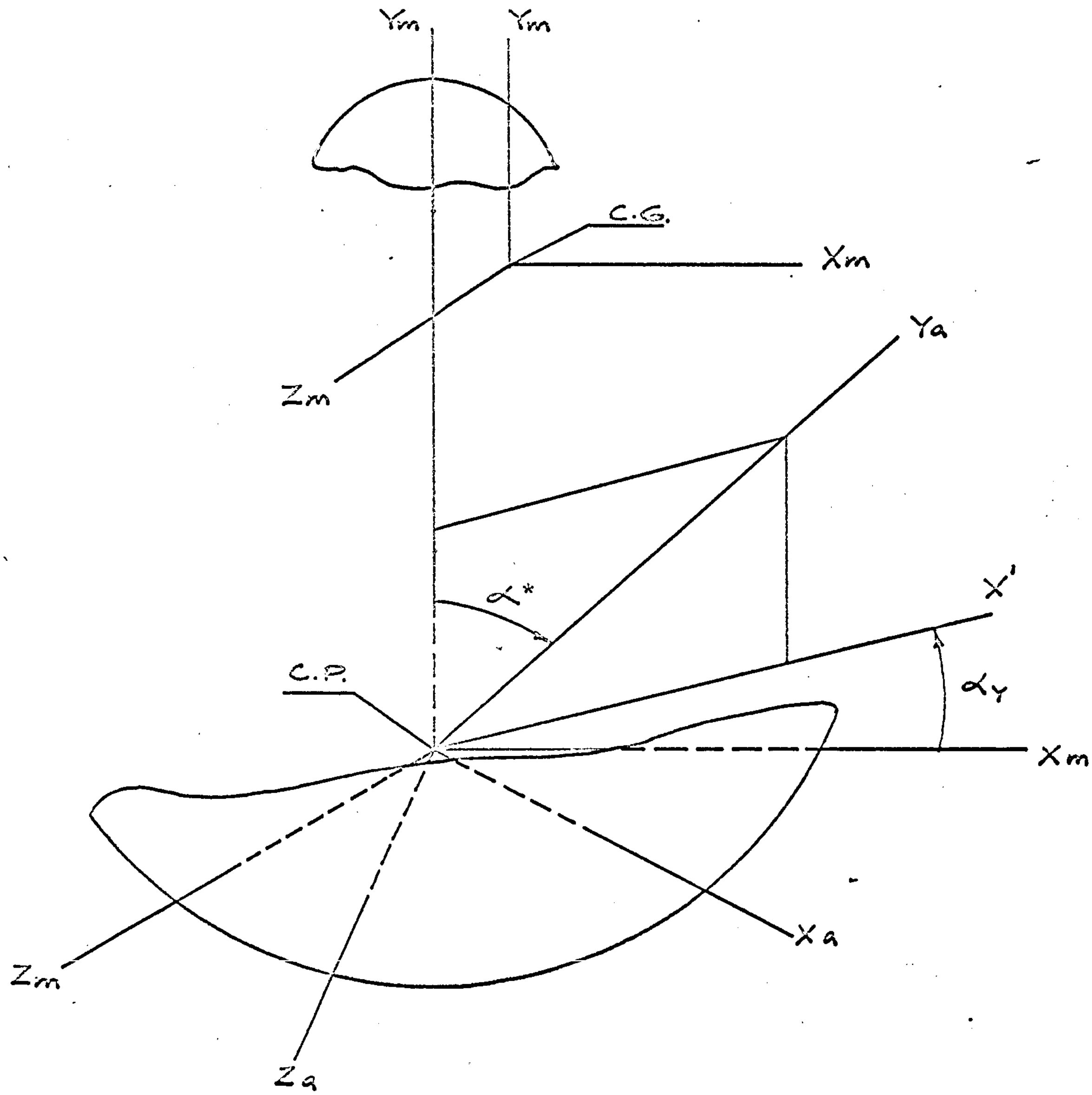

Fig. 5. Aerodynamic and missile coordinate systems

A position vector in the aerodynamic coordinate system may be written in terms of a position vector in the missile fixed coordinate system. The orthogonal transformation matrices are

$$[-\alpha^*] = \begin{bmatrix} C\alpha^* & -S\alpha^* & 0 \\ S\alpha^* & C\alpha^* & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\alpha_y] = \begin{bmatrix} C\alpha_y & 0 & -S\alpha_y \\ 0 & 1 & 0 \\ S\alpha_y & 0 & C\alpha_y \end{bmatrix} .$$

A positive vector in the aerodynamic coordinate system is expressed in terms of a position vector in the missile fixed reference as

$$\overline{X}_a = [-\alpha^*]\ [\alpha_y]\ \overline{X}_m , \qquad (5a)$$

or

$$\overline{X}_a = [A_a]\ \overline{X}_m , \qquad (5b)$$

where

$$[A_a] = \begin{bmatrix} C\alpha^*C\alpha_y & -S\alpha^* & -C\alpha^*S\alpha_y \\ S\alpha^*S\alpha_y & C\alpha^* & -S\alpha^*S\alpha_y \\ S\alpha_y & 0 & C\alpha_y \end{bmatrix} \qquad (5c)$$

is the combined product of the rotation matrices in equation (5a). The aerodynamic coordinate system transformation matrix (5c) is independent of the sequence used in measuring the angles yaw, roll, and pitch.

IV. MECHANICS

A. Forces

Two forces are assumed to act on the vehicle as it moves along its trajectory. It was assumed that the attracting body is a homogeneous sphere. Thus, an inverse square gravitational force is written in terms of the plumbline coordinates as

$$\overline{F}_g = \frac{-G\ M\ m\ \overline{X}}{|\overline{R}|^3} \quad . \qquad (6)$$

The vehicle's motion is also influenced by an aerodynamic force. The force lies in the plane formed by the vehicle longitudinal axis of symmetry and the relative velocity vector and passes through the center of pressure of the vehicle, as shown in Figure 6.

The components of the aerodynamic force are defined by the equations

$$F_x = A\ q\ C_x\ (\alpha^*) \quad , \qquad (7a)$$

and

$$F_z = A\ q\ C_z\ (\alpha^*) \quad . \qquad (7b)$$

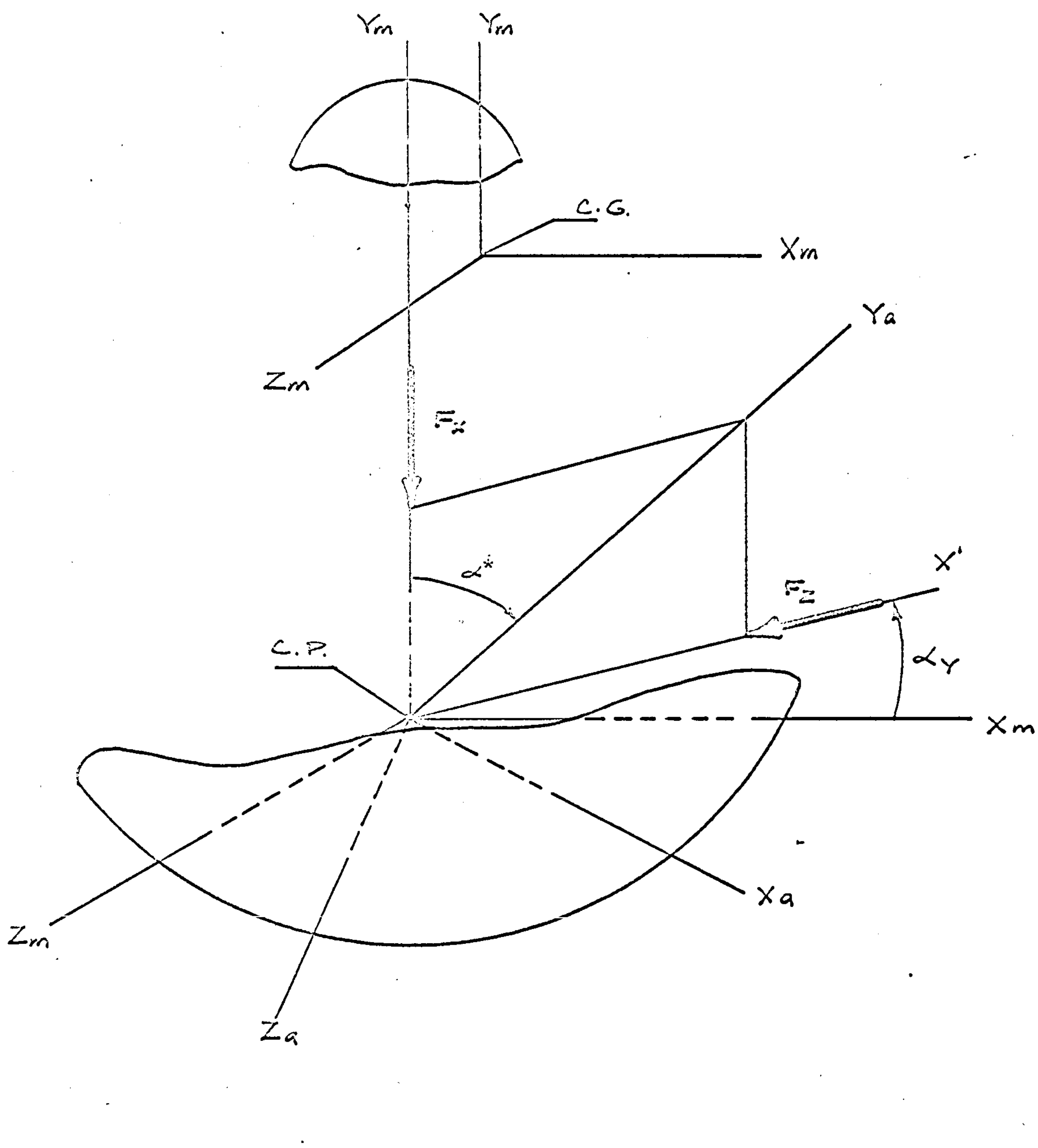

Fig. 6. Aerodynamic force components F_x and F_z

A is the projected cross-section area of the vehicle and q is the dynamic pressure. C_x and C_z are experimentally determined factors that are dependent on the vehicle's shape and the angle of attack. It is assumed that C_z and C_x are known. The aerodynamic force is expressed in the aerodynamic system as

$$\overline{F}_a = \begin{bmatrix} -F_z C\alpha^* + F_x S\alpha^* \\ -F_x C\alpha^* - F_z S\alpha^* \\ 0 \end{bmatrix} \quad .(8)$$

The aerodynamic force is expressed in terms of the missile fixed reference as

$$\overline{F}_{am} = [A_a]^T \, \overline{F}_a \quad .(9a)$$

(Note: The symbol $[A]^T$ is used to denote the transpose of matrix A.)

Equation (9a) can be written in component form as

$$\begin{bmatrix} F_{amx} \\ F_{amy} \\ F_{amz} \end{bmatrix} = \begin{bmatrix} C\alpha^* C\alpha_y & S\alpha^* C\alpha_y & S\alpha_y \\ -S\alpha^* & C\alpha^* & 0 \\ -C\alpha^* S\alpha_y & -S\alpha^* S\alpha_y & C\alpha_y \end{bmatrix} \begin{bmatrix} -F_a S\alpha C\alpha^* + F_a C\alpha S\alpha^* \\ -F_a C\alpha C\alpha^* - F_a S\alpha S\alpha^* \\ 0 \end{bmatrix} \quad .(9b)$$

When simplified, equation (9b) becomes

$$\begin{bmatrix} F_{amy} \\ F_{amy} \\ F_{amz} \end{bmatrix} = F_a \begin{bmatrix} -S\alpha C\alpha_y \\ -C\alpha \\ S\alpha S\alpha_y \end{bmatrix} \quad .(9c)$$

where the magnitude of the aerodynamic force is

$$F_a = \sqrt{F_x^2 + F_z^2} \quad , \qquad (10)$$

and α is expressed in terms of the components of the aerodynamic force through the equations

$$S\alpha = \frac{F_z}{|\overline{F}_a|} \quad , \qquad (11a)$$

$$C\alpha = \frac{F_x}{|\overline{F}_a|} \quad , \qquad (11b)$$

$$\tan \alpha = \frac{S\alpha}{C\alpha} = \frac{F_z}{F_x} = \frac{C_z(\alpha^*)}{C_x(\alpha^*)} \quad . \qquad (11c)$$

The magnitude of the aerodynamic force is related to the relative velocity through the dynamic pressure by the equation

$$q = 1/2\ \rho\ V_R^2 \quad . \qquad (12)$$

It is assumed that the atmosphere normally moves with the attracting body (6). Hence, at all times there is an air mass movement with respect to the plumbline coordinate system. $\overline{W}$ is a vector that represents any abnormal air movement. An equation expressing the velocity of the wind may be written as

$$\overline{V}_{wind} = \overline{\omega}_e \times \overline{X} + \overline{W} \quad . \qquad (13)$$

The relative velocity equation is

$$\dot{\overline{X}} = \overline{V}_{wind} + \overline{V}_R \quad . \qquad (14)$$

When equation (13) is substituted into equation (14), the result is

$$\overline{V}_R = \dot{\overline{X}} + \overline{X} \times \overline{\omega}_e - \overline{W} \quad , \qquad (15a)$$

or, in component form,

$$\begin{bmatrix} V_{RX} \\ V_{RY} \\ V_{RZ} \end{bmatrix} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \times \begin{bmatrix} \omega_{ex} \\ \omega_{ey} \\ \omega_{eZ} \end{bmatrix} - \begin{bmatrix} W_x \\ W_y \\ W_Z \end{bmatrix} \quad . \qquad (15b)$$

The relative velocity may be expressed in terms of the aerodynamic, missile, or plumbline coordinate system variables. The relative velocity vector is written in the missile coordinate system as

$$\overline{V}_{rm} = [A_d]_i \; \overline{V}_R = [A_a]^T \; \overline{V}_r \tag{16}$$

where

$$\overline{V}_{rm} = \begin{bmatrix} V_{rmx} \\ V_{rmy} \\ V_{rmz} \end{bmatrix}$$

is the relative velocity vector in the missile system and

$$\overline{V}_r = \begin{bmatrix} 0 \\ V_r \\ 0 \end{bmatrix}$$

is the relative velocity vector in the aerodynamic system. (Note that equation (16) represents three possible equations depending on i.)

The resultant force acting on the vehicle written in the plumbline coordinates is

$$\overline{F}_R = \overline{F}_g + [A_d]_i^T \; \overline{F}_{am} \quad . \tag{17}$$

B. Couples and Moments

The motion of the vehicle is influenced by a moment and a couple. It is assumed that the center of pressure and the center of mass are invariant with respect to the vehicle. Thus, the center of pressure is located by a constant position vector, $\overline{x}_{cp}$, in the missile fixed reference. The aerodynamic moment is given by the vector product of the position vector, $\overline{x}_{cp}$, and the aerodynamic force, $\overline{F}_{am}$. The aerodynamic moment is written in the missile fixed reference as

$$\overline{M}_{am} = \overline{x}_{cp} \times \overline{F}_{am} \quad , \qquad (18a)$$

or

$$\begin{bmatrix} M_{amx} \\ M_{amy} \\ M_{amz} \end{bmatrix} = \begin{bmatrix} F_a \quad y_{cp} S\alpha S\alpha_y & + & F_a z_{cp} C\alpha \\ -F_a \quad z_{cp} S\alpha C\alpha_y & - & F_a x_{cp} S\alpha S\alpha_y \\ -F_a \quad x_{cp} C\alpha & + & F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} \quad . \qquad (18b)$$

A system of roll jets is used to produce a pure roll control couple about the Y_m axis. The jets are located with respect to the missile fixed coordinate system so that

$$\overline{F}_{r1} = \begin{bmatrix} F_r \\ 0 \\ 0 \end{bmatrix} \text{ located at } \overline{Z}_r = \begin{bmatrix} 0 \\ 0 \\ Z_r \end{bmatrix}$$

and

$$\overline{F}_{r2} = \begin{bmatrix} -F_r \\ 0 \\ 0 \end{bmatrix} \quad \text{located at } -\overline{Z}_r = \begin{bmatrix} 0 \\ 0 \\ -Z_r \end{bmatrix}$$

yield a roll couple

$$\overline{M}_{rm} = 2\,(\overline{Z}_r \times \overline{F}_r) \quad , \tag{19a}$$

which may be expressed as

$$\overline{M}_{rm} = \begin{bmatrix} 0 \\ 2z_r F_r \\ 0 \end{bmatrix} \quad . \tag{19b}$$

The resultant moment about the center of mass of the vehicle in the missile fixed coordinate system is the sum of the roll couple and aerodynamic moment

$$\overline{M}_{Tm} = \overline{M}_{am} + \overline{M}_{rm} \quad . \tag{20}$$

When equations (18b) and (19b) are substituted into equation (20), the result is

$$\overline{M}_{Tm} = \begin{bmatrix} F_a y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\ -F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y \\ -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} + \begin{bmatrix} 0 \\ 2z_r F_r \\ 0 \end{bmatrix} \quad , \tag{21a}$$

which can be reduced to

$$\overline{M}_{Tm} = \begin{bmatrix} F_a y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\ 2Z_r F_r - F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y \\ -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} . \quad (21b)$$

C. Equations of Motion

It is possible to interpret the motion of a rigid body as the sum of two independent effects--the motion of the center of mass of the vehicle with respect to an inertial coordinate system and the rotational motion of the vehicle about its center of mass. The motion of a rigid body in general requires six independent coordinates to specify its orientation at any time. The six independent coordinates used in this problem are the three plumbline coordinates and three Eulerian angles.

The translational equations of motion are written for the center of mass in the inertial reference as

$$\overline{F}_R = m \ddot{\overline{X}} \quad (22a)$$

or

$$\overline{F}_g + [A_d]_i^T \, \overline{F}_{am} = m \ddot{\overline{X}} , \quad (22b)$$

where

$$\ddot{\overline{X}} = \begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix} .$$

When expression (6) is substituted into equation (22b), the second order translational equations of motion become

$$\ddot{\overline{X}} = - \frac{G\ M\ \overline{X}}{|\overline{R}|^3} + [A_d]_i^T \frac{\overline{F}_{am}}{m} \quad . \qquad (22c)$$

The three second order differential equations, (22c), may be reduced to six first order differential equations by a change of variables. Let

$$\overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \dot{\overline{X}} \quad . \qquad (23)$$

When the above transformation is used, the second order differential equations of motion, (22c), reduce to

$$\dot{\overline{u}} = - \frac{G\ M\ \overline{X}}{|\overline{R}|^3} + [A_d]_i^T \frac{\overline{F}_{am}}{m} \quad . \qquad (24)$$

For convenience, the following definitions are made:

$$g = - \frac{G\ M}{|\overline{R}|^3} \quad , \qquad (25)$$

$$[A_d]_i^T \frac{\overline{F}_{am}}{m} = \frac{F_a}{m} \overline{N}_i = F_a^* \overline{N}_i \quad , \qquad (26)$$

where

$$F_a^* = \frac{F_a}{m}$$

and

$$\overline{N}_1 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_1$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_1 = \begin{bmatrix} -(S\alpha C\alpha_y)(CRCP+SRSYSP) + C\alpha CYSP + (S\alpha S\alpha_y)(-SRCP+CRSYSP) \\ -(S\alpha C\alpha_y)(CRSP-SRSYCP) - C\alpha CYCP - (S\alpha S\alpha_y)(SPSR+CRCPSY) \\ -(S\alpha C\alpha_y)(SRCY) - C\alpha SY + (CRCYS\alpha S\alpha_y) \end{bmatrix} \quad (27a)$$

and

$$\overline{N}_2 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_2$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_2 = \begin{bmatrix} -(S\alpha C\alpha_y)(CRCP) + C\alpha\ (CYSP+SYSRCP) + S\alpha S\alpha_y\ (SYSP-CYSRCP) \\ -(S\alpha C\alpha_y)(CRSP) - C\alpha\ (CYCP-SYSRSP) - (S\alpha S\alpha_y)(SYCP+CYSRSP) \\ -(S\alpha C\alpha_y)(SR) - C\alpha\ (SYCR) + (S\alpha S\alpha_y)(CRCY) \end{bmatrix} (27b)$$

and

$$\overline{N}_3 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_3$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_3 = \begin{bmatrix} -(S\alpha C\alpha_y)(CPCR-SPSRSY) + C\alpha\ (SPCR+CPSRSY) - S\alpha S\alpha_y\ (CYSR) \\ -(S\alpha C\alpha_y)(SPCY) - C\alpha\ (CPCY) - S\alpha S\alpha_y\ (SY) \\ -(S\alpha C\alpha_y)(CPSR+SPSYCR) - C\alpha\ (-SPSR+CPSYCR) + (S\alpha S\alpha_y)(CYCR) \end{bmatrix} \tag{27c}$$

When these definitions are used in equation (24), it may be written as

$$\dot{\overline{u}} = g\ \overline{X} + F_a^*\ \overline{N}_i \tag{28}$$

It is convenient to write the rotational equations of motion in the Lagrangian form. When the Eulerian angles (pitch, roll, yaw) are generalized coordinates, the rotational equations of motion take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_j}\right) - \frac{\partial T}{\partial \phi_j} = M_{\phi_j}, \qquad j = p,\ y,\ r \tag{29}$$

T is the rotational kinetic energy of the vehicle and M_{ϕ_j} is the moment associated with the ϕ_j rotation. Based on the assumption of an offset center of mass, all components of the inertia matrix are assumed to be non-zero. The inertia matrix is

$$[\mu] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \tag{30}$$

The rotational kinetic energy may be expressed with respect to the missile fixed coordinate system as

$$T = \tfrac{1}{2}\,\bar{\omega}^T\,[\mu]\,\bar{\omega} \quad , \quad (31)$$

where $\bar{\omega}$ is the angular velocity of the vehicle in the missile fixed coordinate system.

When expressions (30) and (31) are substituted into equation (29), the result is

$$\frac{d}{dt}\left(\frac{\partial\bar{\omega}}{\partial\dot{\phi}_j}^T\right)[\mu]\bar{\omega} + \frac{\partial\bar{\omega}}{\partial\dot{\phi}_j}^T[\mu]\frac{d}{dt}(\bar{\omega}) - \frac{\partial\bar{\omega}}{\partial\phi_j}^T[\mu]\bar{\omega} = M_{\phi_j} \quad . \quad (32)$$

The angular velocity vector, $\bar{\omega}$, is obtained from a coordinate transformation of the angular velocity components $\dot{\phi}_y$, $\dot{\phi}_r$, and $\dot{\phi}_p$ into the missile fixed reference. The transformation is dependent on the gimbal set used. The transformation matrix is developed for gimbal set 1. (Gimbal set 1 measures the Euler angles in the order pitch, yaw, roll.) A coordinate transformation is not required for $\dot{\phi}_r$ since it is measured with respect to the missile coordinate system. The angular velocity component $\dot{\bar{\phi}}_y$ is expressed in the missile fixed reference by use of the rotation matrix $[-\phi_r]^T$. The transformation is

$$\dot{\bar{\phi}}_y \Big|_{missile} = [-\phi_r]^T \begin{bmatrix} \dot{\phi}_y \\ 0 \\ 0 \end{bmatrix} .$$

The angular velocity component $\dot{\bar{\phi}}_p$ is expressed in the missile fixed reference by use of two rotation matrices as follows

$$\dot{\bar{\phi}}_p \Big|_{missile} = [-\phi_r]^T \, [\phi_p]^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix} .$$

Thus, the angular velocity vector

$$\bar{\omega} = \dot{\bar{\phi}}_r + \dot{\bar{\phi}}_y \Big|_{missile} + \dot{\bar{\phi}}_p \Big|_{missile} \tag{33a}$$

or, in component form,

$$\begin{bmatrix} \omega_{xm} \\ \omega_{ym} \\ \omega_{zm} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\phi}_r \\ 0 \end{bmatrix} + [-\phi_r]^T \begin{bmatrix} \dot{\phi}_y \\ 0 \\ 0 \end{bmatrix} + [-\phi_r]^T \, [\phi_y]^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix} \tag{33b}$$

which may be expressed as

$$\bar{\omega} = [A_\omega]_1 \, \dot{\bar{\phi}} \quad , \qquad (33c)$$

where

$$\dot{\bar{\phi}} = \begin{bmatrix} \dot{\phi}_y \\ \dot{\phi}_r \\ \dot{\phi}_p \end{bmatrix} \quad ,$$

and the transformation matrix

$$[A_\omega]_1 = \begin{bmatrix} CR & 0 & SRCY \\ 0 & -1 & SY \\ -SR & 0 & CRCY \end{bmatrix} \quad . \qquad (34a)$$

A similar argument is used to develop transformation matrices for gimbal sets 2 and 3:

$$[A_\omega]_2 = \begin{bmatrix} 1 & 0 & SR \\ 0 & -CY & CRSY \\ 0 & SY & CRCY \end{bmatrix} \qquad (34b)$$

$$[A_\omega]_3 = \begin{bmatrix} CP & -SPCY & 0 \\ -SP & -CPCY & 0 \\ 0 & SY & 1 \end{bmatrix} \quad . \qquad (34c)$$

The angular velocity vector, $\overline{\omega}$, is restated for the three gimbal sets as

$$\overline{\omega} = [A_\omega]_i \, \dot{\overline{\phi}} \qquad i = 1, 2, 3 \qquad . \tag{35}$$

It should be noted that the transformation matrices $[A_\omega]_i$ are not orthogonal.

By use of the expressions obtained above, the rotational equations of motion become

$$\ddot{\overline{\phi}}_i = [C]_i \left\{ M_{\phi_i} - \left(\left\{ \frac{d}{dt}[A_\omega]_i^T \right\} [\mu][A_\omega]_i + [A_\omega]_i^T [\mu] \frac{d}{dt}[A_\omega]_i \right) \dot{\overline{\phi}}_i + \overline{B}_i \right\} \tag{36}$$

where

$$[C]_i = \left[[A_\omega]_i^T [\mu] [A_\omega]_i \right]^{-1} \quad , \tag{37a}$$

$$B_j = \dot{\overline{\phi}}_i^{\,T} \frac{\partial [A_\omega]_i^T}{\partial \phi_j} [\mu] [A_\omega] \dot{\overline{\phi}}_i \quad , \tag{37b}$$

$$\overline{M}_{\emptyset_i} = [A_\omega]_i^T \, \overline{M}_{Tm} \quad , \qquad (37c)$$

$$\overline{B}_i = \begin{bmatrix} B_y \\ B_r \\ B_p \end{bmatrix}_i \text{, and } \ddot{\overline{\emptyset}}_i = \begin{bmatrix} \ddot{\emptyset}_y \\ \ddot{\emptyset}_r \\ \ddot{\emptyset}_r \end{bmatrix} \quad . \qquad (37d)$$

$$i = 1,2,3 \qquad\qquad j = p,y,r$$

Definitions for $\dot{\overline{\emptyset}}$ and $\ddot{\overline{\emptyset}}$ are introduced that conform to the simplifications referred to in the problem statement. These definitions will be used throughout the remainder of the paper. For gimbal set 1:

$$\ddot{\overline{\emptyset}}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\overline{\emptyset}}_1 = \begin{bmatrix} 0 \\ \dot{\emptyset}_r \\ 0 \end{bmatrix} \quad ,$$

for gimbal set 2:

$$\ddot{\overline{\emptyset}}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\overline{\emptyset}}_2 = \begin{bmatrix} \dot{\emptyset}_y \\ 0 \\ 0 \end{bmatrix} \quad ,$$

and for gimbal set 3:

$$\ddot{\bar{\phi}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\bar{\phi}}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix} .$$

It is noted that each of the matrices in the matrix product of equation (37a) is non-singular. Thus, the product is non-singular, and the rotational equations of motion, (36), can be reduced to the following form for each gimbal set.

$$M_{Tm} = [A_\omega]_i^{-1} \left\{ \left(\left\{ \frac{d}{dt} [A_\omega]_i^T \right\} [\mu] \; [A_\omega]_i + [A_\omega]_i^T \; [\mu] \; \frac{d}{dt} \; [A_\omega]_i \right) \dot{\bar{\phi}}_i - \bar{B}_i \right\} \quad . \tag{38}$$

Three rotational equations of motion are obtained for each gimbal set from equation (38). The three equations may be solved for three unknowns. Because a particular computational procedure is anticipated, the equations for each gimbal set are solved for the roll force, F_r, the angle, α, and the angular velocity component that appears.

Gimbal Set 1

The three rotational equations of motion are:

$$
\begin{aligned}
-\dot{\phi}_r^{\,2} I_{zy} &= F_a y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\
0 &= -F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y + 2F_r z_r \quad . \qquad (39) \\
\dot{\phi}_r^{\,2} I_{xy} &= -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y
\end{aligned}
$$

The first and third of equations (39) are solved for

$$\alpha = \arctan\left[\frac{I_{zy} x_{cp} - I_{xy} z_{cp}}{y_{cp}(I_{zy} C\alpha_y + I_{xy} S\alpha_y)}\right] \quad . \qquad (42a)$$

The second of equations (39) is solved for

$$F_r = \frac{F_a S\alpha\,(x_{cp} S\alpha_y + z_{cp} C\alpha_y)}{2z_r} \quad , \qquad (43a)$$

and the third is solved for

$$\dot{\phi}_r = \pm\sqrt{\frac{F_a\,(y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha)}{I_{xy}}} \quad . \qquad (44a)$$

Gimbal Set 2

The rotational equations of motion are:

$$
\begin{aligned}
0 &= F_a Y_{cp} S\alpha S\alpha_y + F_a Z_{cp} C\alpha \\
\dot{\emptyset}_y^{\,2} I_{xz} &= -F_a Z_{cp} S\alpha C\alpha_y - F_a X_{cp} S\alpha S\alpha_y + 2F_r Z_r \\
-\dot{\emptyset}_y^{\,2} I_{xy} &= -F_a X_{cp} C\alpha + F_a Y_{cp} S\alpha C\alpha_y
\end{aligned}
\tag{40}
$$

The first of equations (40) is solved for

$$\alpha = \arctan \frac{-Z_{cp}}{Y_{cp} S\alpha_y} \quad . \tag{42b}$$

The third of equations (40) is solved for

$$\dot{\emptyset}_y = \pm\sqrt{\frac{F_a(X_{cp}C\alpha - Y_{cp}S\alpha C\alpha_y)}{I_{xy}}} \quad , \tag{43b}$$

and the second is solved for

$$F_r = \frac{\dot{\emptyset}_y^{\,2} I_{xz} + F_a S\alpha(Z_{cp}C\alpha_y + X_{cp}S\alpha_y)}{2Z_r} \quad . \tag{44b}$$

Gimbal Set 3

The three rotational equations of motion are:

$$\begin{aligned}
\dot{\phi}_p^{\,2}\, I_{yz} &= F_a Y_{cp} S\alpha S\alpha_y + F_a Z_{cp} C\alpha \\
-\dot{\phi}_p^{\,2}\, I_{xz} &= -F_a Z_{cp} S\alpha C\alpha_y - F_a X_{cp} S\alpha S\alpha_y + 2F_r Z_r \\
0 &= -F_a X_{cp} C\alpha + F_a Y_{cp} S\alpha C\alpha_y
\end{aligned} \tag{41}$$

The third of equations (41) is solved for

$$\alpha = \arctan \frac{X_{cp}}{Y_{cp} C\alpha_y} \quad . \tag{42c}$$

The first of equations (41) is solved for

$$\dot{\phi}_p = \pm\sqrt{\frac{F_a (Y_{cp} S\alpha S\alpha_y + Z_{cp} C\alpha)}{I_{yz}}} \quad , \tag{43c}$$

and the second is solved for

$$F_r = \frac{F_a S (Z_{cp} C\alpha_y + X_{cp} S\alpha_y) - \dot{\phi}_p^{\,2} I_{xz}}{2Z_r} \quad . \tag{44c}$$

V. THE RELATIVE VELOCITY CONSTRAINTS

The fact that the relative velocity vector may be written in terms of the three coordinate systems constitutes an algebraic constraint given by

$$\bar{V}_{rm} = [A_d]_i \bar{V}_R = [A_a]^T \bar{V}_r \tag{16}$$

where $\bar{V}_{rm}$ is the relative velocity vector expressed in the missile coordinate system. Vector equation (16) yields three equations for each gimbal set. The three equations of each set are not independent. Hence, they may not be solved for three unknowns. For each gimbal set, the three equations are solved for two angular displacements. The uniqueness of these angular displacements is discussed in Appendix B.

<u>Gimbal Set 1</u>

The constraint equations are:

$$\begin{aligned}
(CRCP+SRSYSP)\, V_{Rx} + (CRSP-SRSYCP)\, V_{Ry} + SRSYV_{RZ} &= V_{rmx} \\
(-CYSP)\, V_{Rx} + CYCP\, V_{Ry} + SYV_{RZ} &= V_{rmy} \qquad (45) \\
(-SRCP+CRSYSP)\, V_{Rx} - (SRSP+CRCPSY)\, V_{Ry} + CRCYV_{RZ} &= V_{rmz}
\end{aligned}$$

The first and third of equations (45) are solved for

$$SP = \frac{J\,V_{RY} - V_{RX}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} , \qquad (46a)$$

and

$$CP = \frac{J\,V_{RX} + V_{RY}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(R_{RX}^2 + V_{RY}^2)} \qquad (46b)$$

where

$$J = CR\ V_{rmx} - SR\ V_{rmz} ,$$

$$\phi_p = \arctan \frac{SP}{CP} . \qquad (46c)$$

As shown in Appendix B, equations (46) may be solved for a unique value of ϕ_p only if

$$-\Pi \leqq \phi_p \leqq \Pi .$$

The second set of equations (45) is solved for

$$SY = \frac{V_{rmy} V_{RZ} - K\sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} , \qquad (47a)$$

and

$$CY = \frac{V_{rmy} K + V_{RZ} \sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} , \quad (47b)$$

where

$$K = CP\ V_{RY} - SP\ V_{RX} ,$$

$$\phi_y = \arctan \frac{SY}{CY} . \quad (47c)$$

As shown in Appendix B, equations (47) may be solved for a unique value of ϕ_y only if

$$-\Pi \leq \phi_y \leq \Pi .$$

Gimbal Set 2

The constraint equations are:

$$V_{RX}CRCP + V_{RY}CRSP + V_{RZ}SR = V_{rmx}$$

$$-V_{RX}CPSRSY - V_{RX}CYSP + V_{RY}(CYCP-SYSRSP) + V_{RZ}SYCR = V_{rmy} \quad (48)$$

$$V_{RX}(SYSP-CYSRCP) - V_{RY}(SYCP+CYSRSP) + V_{RZ}CRCY = V_{rmz}$$

The second and third of equations (48) are combined to give

$$V_{RX}\,SP - V_{RX}\,CP = -V_{rmy}\,CY + V_{rmz}\,SY$$

which is solved for

$$SP = \frac{F\,V_{RX} + V_{RY}\sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad , \qquad (49a)$$

and

$$CP = \frac{F\,V_{RY} + V_{RX}\sqrt{V_{RX}^2 - F^2 - V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \qquad (49b)$$

where

$$F = -V_{rmy}\,CY + V_{rmz}\,SY \quad ,$$

$$\emptyset_p = \arctan\frac{SP}{CP} \quad . \qquad (49c)$$

As shown in Appendix B, equations (49) may be solved for a unique value of $\emptyset_p$ only if

$$-\Pi \leqq \emptyset_p \leqq \Pi \quad .$$

The first of equations (48) is solved for

$$SR = \frac{V_{RZ}\, V_{rmx} - G\sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 - G^2)} \quad , \qquad (50a)$$

and

$$SR = \frac{G\, V_{rmx} + V_{RZ}\sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \qquad (50b)$$

where

$$G = V_{RX}\, CP + V_{RY}\, SP \quad ,$$

$$\phi_r = \arctan \frac{SR}{CR} \quad . \qquad (50c)$$

As shown in Appendix B, equations (50) may be solved for a unique value of ϕ_r only if

$$-\Pi \leq \phi_r \leq \Pi \quad .$$

<u>Gimbal Set 3</u>

The constraint equations are:

$$V_{RX}(CPCR-SPSRSY) + V_{RY}SPCY + V_{RZ}(CPSR+SPSYCR) = V_{rmx}$$

$$-V_{RX}(SPCR+CPSRSY) + V_{RY}CPCY + V_{RZ}(-SPSR+CPSYCR) = V_{rmy} \qquad (51)$$

$$-V_{RX}CYSR - V_{RY}SY + V_{RZ}CYCR = V_{rmz}$$

The first and second of equations (51) are solved for

$$SR = \frac{V_{RZ} A - V_{RX} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} , \qquad (52a)$$

and

$$CR = \frac{V_{RX} A + V_{RZ} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} \qquad (52b)$$

where

$$A = CP\ V_{rmx} - SP\ V_{rmy} ,$$

$$\phi_r = \arctan \frac{SR}{CR} . \qquad (52c)$$

As shown in Appendix B, equations (52) may be solved for a unique value of ϕ_r only if

$$-\Pi \leqq \phi_r \leqq \Pi .$$

The third of equations (51) is solved for

$$SY = \frac{-V_{RY} V_{rmz} + B \sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} , \qquad (53a)$$

and

$$CY = \frac{B\ V_{rmz} + V_{RY}\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \qquad (53b)$$

where

$$B = V_{RZ}CR - V_{RX}SR \quad ,$$

$$\emptyset_y = \arctan \frac{SY}{CY} \quad . \qquad (53c)$$

As shown in Appendix B, equations (53) may be solved for a unique value of $\emptyset_y$ only if

$$-\Pi \leq \emptyset_y \leq \Pi \quad .$$

VI. FORMULATION OF THE OPTIMIZATION PROBLEM

The optimization problem is that of finding the optimum control process, $\alpha_y(t)$, that will transfer a vehicle from an initial state to a terminal state in an atmosphere in a manner so that the functional

$$J = \int_{t_o}^{t_1} f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\emptyset}}, \emptyset_r, \emptyset_y, \emptyset_p, \alpha, \alpha_y, F_r)\, dt$$

is a minimum. Since the Pontryagin formulation is to be used, it is necessary to write the Pontryagin H function for each gimbal set (5).

<u>Gimbal Set 1</u>

The Pontryagin H function is

$$H_1 = \overline{\lambda}_1 \cdot \dot{\overline{X}} + \overline{\lambda}_{II} \cdot \dot{\overline{u}} \pm \lambda_7 \dot{\emptyset}_r + \lambda_8 \dot{J} \quad , \quad (55a)$$

which may be expressed as

$$H_1 = \overline{\lambda}_I \cdot \dot{\overline{X}} + \overline{\lambda}_{II} \cdot (g\, \overline{X} + F_a^* \overline{N}_1)$$

$$\pm \lambda_7 \sqrt{\frac{F_a (y_{cp}\, S\alpha\, C\alpha_y - x_{cp}\, C\alpha)}{I_{xy}}} + \lambda_8 f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\emptyset}}_1, \overline{\emptyset}\, \alpha, \alpha_y, F_r) \quad (55b)$$

where

$$\overline{\lambda}_I = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} , \text{ and } \quad \overline{\lambda}_{II} = \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} .$$

The $\lambda(t)$ are auxiliary variables used in a manner analogous to Lagrangian multipliers in the classical calculus of variations.

Gimbal Set 2

The Pontryagin H function is

$$H_2 = \overline{\lambda}_{III} \cdot \dot{\overline{X}} + \overline{\lambda}_{IV} \cdot \dot{\overline{u}} \pm \lambda_{15} \dot{\emptyset}_y + \lambda_{16} \dot{J} , \tag{56a}$$

which may be expressed as

$$H_2 = \overline{\lambda}_{III} \cdot \dot{\overline{X}} + \overline{\lambda}_{IV} \cdot (g\overline{X} + F_a^* \overline{N}_2)$$

$$\pm \lambda_{15} \sqrt{\frac{F_a(x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}} + \lambda_{16} f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\emptyset}}_2, \overline{\emptyset}, \alpha, \alpha_y, F_r) . \tag{56b}$$

where

$$\overline{\lambda}_{III} = \begin{bmatrix} \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \end{bmatrix} , \text{ and } \quad \overline{\lambda}_{IV} = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix} .$$

The expressions for the auxiliary variables are obtained from the H functions as follows:

Gimbal Set 1

$$-\dot{\bar{\lambda}}_{I} = \frac{\partial H_1}{\partial \bar{X}}$$

$$= F_a^{*} \frac{\partial(\bar{\lambda}_{II} \cdot \bar{N}_1)}{\partial \bar{X}} + (\bar{\lambda}_{II} \cdot \bar{N}_1) \frac{\partial F_a^{*}}{\partial \bar{X}} - \bar{\lambda}_{II} g$$

$$+ (\bar{\lambda}_{II} \cdot \bar{X}) \frac{\partial g}{\partial \bar{X}} \pm \lambda_7 \sqrt{\frac{y_{cp} S\alpha C\alpha_y - X_{cp} C\alpha}{I_{xy}}} \left\{ \frac{\partial (F_a^{*})^{1/2}}{\partial \bar{X}} \right\}$$

$$+ \lambda_8 \frac{\partial F_a^{*}}{\partial \bar{X}}. \qquad (58a)$$

$$-\dot{\bar{\lambda}}_{II} = \frac{\partial H_1}{\partial \bar{u}}$$

$$= \bar{\lambda}_{I} + F_a^{*} \frac{\partial(\bar{\lambda}_{II} \cdot \bar{N}_1)}{\partial \bar{u}} + (\bar{\lambda}_{II} \cdot \bar{N}_1) \frac{\partial F_a^{*}}{\partial \bar{u}}$$

$$+ \lambda_7 \sqrt{\frac{y_{cp} S\alpha C\alpha_y - X_{cp} C\alpha}{I_{xy}}} \left\{ \frac{\partial (F_a^{*})^{1/2}}{\partial \bar{u}} \right\}$$

$$+ \lambda_8 \frac{\partial F_a^{*}}{\partial \bar{u}}. \qquad (59a)$$

Gimbal Set 3

The Pontryagin H function is

$$H_3 = \overline{\lambda}_V \cdot \dot{\overline{X}} + \overline{\lambda}_{VI} \cdot \dot{\overline{u}} \pm \lambda_{23}\dot{\emptyset}_p + \lambda_{24}\dot{J} \quad , \quad (57a)$$

which may be expressed as

$$H_3 = \overline{\lambda}_V \cdot \dot{\overline{X}} + \overline{\lambda}_{VI} \cdot (g\,\overline{X} + F_a^*\overline{N}_3)$$

$$\pm\lambda_{23}\sqrt{\frac{F_a(y_{cp}S\alpha S\alpha_y + z_{cp}C\alpha)}{I_{yz}}} + \lambda_{24}\, f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\emptyset}}_3, \overline{\emptyset}, \alpha, \alpha_y, F_r) \quad (57b)$$

where

$$\overline{\lambda}_V = \begin{bmatrix} \lambda_{17} \\ \lambda_{18} \\ \lambda_{19} \end{bmatrix}$$

and

$$\overline{\lambda}_{VI} = \begin{bmatrix} \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix}$$

$$-\dot{\lambda}_7 = \frac{\partial H_1}{\partial \phi_r} = F_a^* \left\{ \bar{\lambda}_{II} \cdot \frac{\partial \bar{N}_1}{\partial \phi_r} \right\}. \tag{60a}$$

$$-\dot{\lambda}_8 = \frac{\partial H_1}{\partial J} = 0. \tag{61a}$$

Gimbal Set 2

The expressions for the auxiliary variables are:

$$\begin{aligned} -\dot{\bar{\lambda}}_{III} &= \frac{\partial H_2}{\partial \bar{x}} \\ &= F_a^* \frac{\partial(\bar{\lambda}_{IV} \cdot \bar{N}_2)}{\partial \bar{x}} + (\bar{\lambda}_{IV} \cdot \bar{N}_2)\frac{\partial F_a^*}{\partial \bar{x}} \\ &\quad + \bar{\lambda}_{IV}\, g + (\bar{\lambda}_{IV} \cdot \bar{x})\frac{\partial g}{\partial \bar{x}} \\ &\quad \pm \lambda_{15} \sqrt{\frac{x_{cp} C\alpha - y_{cp} S\alpha\, C\alpha_y}{I_{xy}}} \left\{ \frac{\partial (F_a^*)^{1/2}}{\partial \bar{x}} \right\} \\ &\quad + \lambda_{16} \frac{\partial F_a^*}{\partial \bar{x}}. \end{aligned} \tag{58b}$$

$$-\dot{\bar{\lambda}}_{IV} = \frac{\partial H_2}{\partial \bar{u}}$$

$$= \bar{\lambda}_{III} + F_a^* \frac{\partial(\bar{\lambda}_{IV} \cdot \bar{N}_2)}{\partial \bar{u}} + (\bar{\lambda}_{IV} \cdot \bar{N}_2) \frac{\partial F_a^*}{\partial \bar{u}}$$

$$\pm \lambda_{15} \sqrt{\frac{x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y}{I_{xy}}} \left\{ \frac{\partial (F_a^*)^{1/2}}{\partial \bar{u}} \right\}$$

$$+ \lambda_{16} \frac{\partial F_a^*}{\partial \bar{u}}. \tag{59b}$$

$$-\dot{\lambda}_{15} = \frac{\partial H_2}{\partial \phi_y} = F_a^* \left(\bar{\lambda}_{IV} \cdot \frac{\partial \bar{N}_2}{\partial \phi_y}\right). \tag{60b}$$

$$-\dot{\lambda}_{16} = \frac{\partial H_3}{\partial J} = 0. \tag{61b}$$

<u>Gimbal Set 3</u>

The expressions for the auxiliary variables are:

$$-\dot{\bar{\lambda}}_{V} = \frac{\partial H_3}{\partial \bar{x}}$$

$$= F_a^* \frac{\partial(\bar{\lambda}_{VI} \cdot \bar{N}_3)}{\partial \bar{x}} + (\bar{\lambda}_{VI} \cdot \bar{N}_3) \frac{\partial F_a^*}{\partial \bar{x}} + \bar{\lambda}_{VI}\, g$$

$$+ (\bar{\lambda}_{VI} \cdot \bar{x}) \frac{\partial g}{\partial \bar{x}} \pm \lambda_{23} \sqrt{\frac{y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha}{I_{yz}}} \left\{ \frac{\partial (F_a^*)^{1/2}}{\partial \bar{x}} \right\}$$

$$+ \lambda_{24} \frac{\partial F_a^*}{\partial \bar{x}}. \tag{58c}$$

$$-\mathring{\bar{\lambda}}_{VI} = \frac{\partial H_3}{\partial \bar{u}}$$

$$= \bar{\lambda}_{V} + F_a^{*}\frac{\partial(\bar{\lambda}_{VI}\cdot\bar{N}_3)}{\partial \bar{u}} + (\bar{\lambda}_{VI}\cdot\bar{N}_3)\frac{\partial F_a^{*}}{\partial \bar{u}}$$

$$\pm \lambda_{23}\sqrt{\frac{y_{cp}S\alpha S\alpha_y + z_{cp}C\alpha}{I_{yz}}}\left\{\frac{\partial (F_a^{*})^{1/2}}{\partial \bar{u}}\right\}$$

$$+ \lambda_{24}\frac{\partial F_a^{*}}{\partial \bar{u}}. \qquad (59c)$$

$$-\mathring{\lambda}_{23} = \frac{\partial H_3}{\partial \phi_p} = F_a^{*}\left\{\bar{\lambda}_{VI}\cdot\frac{\partial \bar{N}_3}{\partial \phi_p}\right\}. \qquad (60c)$$

$$-\mathring{\lambda}_{24} = \frac{\partial H_3}{\partial J} = 0. \qquad (61c)$$

Equations (61a), (61b) and (61c) imply that λ_8, λ_{16}, and λ_{24} are constant. The constant in each case is taken equal to plus one. This insures that a minimization of the H function is also a minimization of the payoff function.

The necessary condition for a critical value of J is

$$\frac{\partial H_i}{\partial \alpha_y} = 0 \tag{62}$$

where

$$i = 1, 2, 3 \quad .$$

The inequality

$$\frac{\partial^2 H_i}{\partial \alpha_y^2} \geq 0 \tag{63}$$

must also be satisfied to insure a minimum of the payoff function. (Note: The criteria expressed in (62) and (63) are valid only if the H function is differentiable at each point on the trajectory.) Partial differentiation of the H functions as indicated in (62) and (63) produces the equations given on the following page.

$$\frac{\partial H_1}{\partial \alpha_y} = \bar{\lambda}_{\dot{II}} F_a^* \frac{\partial \bar{N}_1}{\partial \alpha_y} \pm \lambda_7 \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha)}{I_{xy}}}$$

$$+ \lambda_8 \frac{\partial}{\partial \alpha_y} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_1, \alpha_y, \alpha, F_r) = 0. \quad \text{(64a)}$$

$$\frac{\partial H_2}{\partial \alpha_y} = \bar{\lambda}_{\dot{IV}} F_a^* \frac{\partial \bar{N}_2}{\partial \alpha_y} \pm \lambda_{15} \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}}$$

$$+ \lambda_{16} \frac{\partial}{\partial \alpha_y} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_2, \alpha_y, \alpha, F_r) = 0. \quad \text{(64b)}$$

$$\frac{\partial H_3}{\partial \alpha_y} = \bar{\lambda}_{\dot{IV}} F_a^* \frac{\partial \bar{N}_3}{\partial \alpha_y} \pm \lambda_{23} \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha)}{I_{yz}}}$$

$$+ \lambda_{24} \frac{\partial}{\partial \alpha_y} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_3, \alpha_y, \alpha, F_r) = 0. \quad \text{(64c)}$$

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{II}} F_0^* \frac{\partial^2 \bar{N}_1}{\partial \alpha_y^2} \pm \lambda_7 \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{F_0 (y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha)}{I_{xy}}}$$

$$+ \lambda_8 \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_1, \alpha, \alpha_y, F_r) > 0. \qquad (65a)$$

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{IV}} F_0^* \frac{\partial^2 \bar{N}_2}{\partial \alpha_y^2} \pm \lambda_{15} \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{F_0 (x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}}$$

$$+ \lambda_{16} \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_2, \alpha, \alpha_y, F_r) > 0. \qquad (65b)$$

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{VI}} F_0^* \frac{\partial^2 \bar{N}_3}{\partial \alpha_y^2} \pm \lambda_{23} \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{F_0 (y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha)}{I_{yz}}}$$

$$+ \lambda_{24} \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \mathring{\bar{x}}, \bar{\phi}, \mathring{\bar{\phi}}_3, \alpha, \alpha_y, F_r) > 0. \qquad (65c)$$

The algebraic and differential constraint equations (28), (42), (43), (44), (46c), (47c), (49c), (50c), (52c), and (53c), and the characteristic equations (58), (59), (60), (61), and (64) form a complete set of equations for the problem. To insure that the payoff function has been minimized, the inequality (65) must also be satisfied.

VII. COMPUTATIONAL PROCEDURE

The problem formulated is of a general nature and the equations involved are quite complex. It is highly improbable that a closed form solution can be found. Therefore, no time has been spent in search of this type solution. A computational scheme is suggested in order that trajectories may be generated on a digital computer. For convenience in the discussion of the computational scheme, the principle equations are written in functional notation.

Gimbal Set 1

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66a}$$

$$\dot{\phi}_r = \pm \dot{\phi}_r \, (\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67a}$$

$$\phi_p = \phi_p(\phi_r, \overline{X}, \dot{\overline{X}}, \alpha, \alpha_y) \tag{68a}$$

$$\phi_y = \phi_y(\phi_p, \overline{X}, \dot{\overline{X}}, \alpha, \alpha_y) \tag{69a}$$

$$\ddot{\overline{X}} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y) \tag{70a}$$

$$F_r = F_r(\alpha, \alpha_y) \tag{71a}$$

$$H_1 = H_1(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y, \lambda_i) \tag{72a}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{73a}$$

$$\frac{\partial H_1}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_1(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74a}$$

<u>Gimbal Set 2</u>

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66b}$$

$$\dot{\emptyset}_y = \pm \dot{\emptyset}_y(\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67b}$$

$$\emptyset_r = \emptyset_r(\emptyset_y, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{68b}$$

$$\emptyset_p = \emptyset_p(\emptyset_r, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{69b}$$

$$\ddot{\overline{X}} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y) \tag{70b}$$

$$F_r = F_r(\emptyset_y, \alpha, \alpha_y) \tag{71b}$$

$$H_2 = H_2(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{72b}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{73b}$$

$$\frac{\partial H_2}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_2(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74b}$$

Gimbal Set 3

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66c}$$

$$\dot{\phi}_p = \pm \dot{\phi}_p(\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67c}$$

$$\phi_r = \phi_r(\phi_p, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{68c}$$

$$\phi_y = \phi_y(\phi_r, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{69c}$$

$$\ddot{\overline{X}} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y) \tag{70c}$$

$$F_r = F_r(\phi_y, \alpha, \alpha_y) \tag{71c}$$

$$H_3 = H_3(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y, \lambda_i) \tag{72c}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y, \lambda_i) \tag{73c}$$

$$\frac{\partial H_3}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_3(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74c}$$

A complete set of equations has been developed for each gimbal set. Therefore, three independent, but similar, computational procedures are written. All three computational procedures require the following initial data:

Atmospheric tables for ρ as a function of position

Atmospheric tables for $\overline{W}$ as a function of position

Aerodynamic tables for $C_x(\alpha^*)$ and $C_z(\alpha^*)$ as a function of α^*

Values for:

A	R_o
G	$\overline{x}_{cp}$
m	$\overline{z}_r$
M	$[\mu]$

Plumbline position, $\overline{X}_o$, and velocity, $\dot{\overline{X}}_o$, vectors at the initial point on the optimum trajectory

Computational procedure for Gimbal Set 1

Initial values for the auxiliary variables, λ_i, and the roll angle, ϕ_r, are required. It is assumed that these values are known. These initial data are referred to as:

$$\bar{\lambda}_{I_0} = \begin{bmatrix} \lambda_{1_0} \\ \lambda_{2_0} \\ \lambda_{3_0} \end{bmatrix} , \qquad \bar{\lambda}_{II_0} = \begin{bmatrix} \lambda_{4_0} \\ \lambda_{5_0} \\ \lambda_{6_0} \end{bmatrix} ,$$

$$\begin{aligned} &\lambda_{7_0} , \\ &\lambda_{8_0} = 1 , \\ &\phi_{r_o} . \end{aligned}$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180°$
2. Choose the positive sign in equation (67a) and compute:

   a. α from (66a); iterate (11c) for α^*

   b. $\dot{\phi}_r$ from (67a)

   c. ϕ_p from (68a)

d. ϕ_y from (69a)

e. $\ddot{X}$ from (70a)

f. H_1 from (72a)

g. $\dfrac{\partial H_1}{\partial \alpha_y}$ from (74a)

3. Choose $\alpha_y = \alpha_y + 5°$ and repeat step 2. Continue until $\alpha_y = +180°$.
4. Repeat steps 1 through 3 using the negative sign in equation (67a).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67a)	α_y	H_1	$\frac{\partial H_1}{\partial \alpha_y}$		Eqn. - (67a)	α_y	H_1	$\frac{\partial H_1}{\partial \alpha_y}$

A plot of H_1 vs α_y should yield insight as to the number of solutions that exist. In addition, this plot should yield a starting value of α_y for the iteration of equation (74a).

Preload Computation II

5. Use the positive sign in equation (67a) and the results of Preload Computation I to iterate equation (74a) for α_y.

6. Use the α_y computed in step 5 to compute

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} .$$

7. If the inequality

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} > 0$$

is satisfied, a minimum exists. Proceed to step 12. Use the positive sign in equation (67a) in all remaining calculations. If the inequality is not satisfied, proceed to step 8.

8. Use the negative sign in equation (67a) and the results from Preload Computation I to iterate equation (74a) for α_y.

9. Use the α_y found in step 8 to compute

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} > 0 .$$

11. Proceed to step 12. Use the negative sign in equation (67a) in all remaining calculations.

"N" line computation

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67a) to iterate (74a) for α_y.

13. Use the α_y computed in step 12 and the initial data to compute:

a. α from equation (66a); iterate (11c) for α^*

b. $\dot{\phi}_r$ from equation (67a)

c. ϕ_p from equation (68a)

d. ϕ_y from equation (69a)

e. $\ddot{\bar{X}}$ from equation (70a)

f. F_r from equation (71a)

g. H_1 from equation (72a)

h. $\dot{\bar{\lambda}}_I$ from equation (73a)

i. $\dot{\bar{\lambda}}_{II}$ from equation (73a)

j. $\dot{\lambda}_7$ from equation (73a)

14. Use a numerical integration technique to integrate

$$\ddot{\bar{X}} \text{ for } \dot{\bar{X}} \text{ for } \bar{X} \quad ,$$

$$\dot{\phi}_r \text{ for } \phi_r \quad ,$$

$$\dot{\bar{\lambda}}_I \text{ for } \bar{\lambda}_I \quad ,$$

$$\dot{\bar{\lambda}}_{II} \text{ for } \bar{\lambda}_{II} \quad ,$$

$$\dot{\lambda}_7 \text{ for } \lambda_7 \quad .$$

15. Use the integrated values from step 14 for the new initial values in the "N + 1" line computation.

Computational procedure for Gimbal Set 2

Initial values for the auxiliary variables and the yaw angle are required. It is assumed that these values are known. These initial data re referred to as:

$$\bar{\lambda}_{III_0} = \begin{bmatrix} \lambda_{9_0} \\ \lambda_{10_0} \\ \lambda_{11_0} \end{bmatrix} \quad , \qquad \bar{\lambda}_{IV_0} = \begin{bmatrix} \lambda_{12_0} \\ \lambda_{13_0} \\ \lambda_{14_0} \end{bmatrix} \quad ,$$

$$\lambda_{15_0} ,$$

$$\lambda_{16_0} = 1 ,$$

$$\dot{\phi}_{y_0} .$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180°$
2. Choose the positive sign in equation (67b) and compute:

   a. α from (66b); iterate (11c) for α^*

   b. $\dot{\phi}_y$ from (67b)

   c. ϕ_r from (68b)

   d. ϕ_p from (69b)

   e. $\ddot{X}$ from (70b)

   f. H_2 from (72b)

   g. $\dfrac{\partial H_2}{\partial \alpha_y}$ from (74b)

3. Choose $\alpha_y = \alpha_y + 5°$ and repeat step 2. Continue until $\alpha_y = +180°$.

4. Repeat steps 1 through 3 but use the negative sign in equation (67b).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67b)	α_y	H_2	$\frac{\partial H_2}{\partial \alpha_y}$		Eqn. - (67b)	α_y	H_2	$\frac{\partial H_2}{\partial \alpha_y}$

A plot of H_2 vs α_y should yield insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for α_y to be used in the iteration of equation (74b).

Preload Computation II

5. Use the positive sign in equation (67b) and the results of Preload Computation I to iterate equation (74b) for α_y.
6. Use the value of α_y found in step 5 to compute

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} .$$

7. If the inequality

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} > 0$$

is satisfied a minimum exists. Proceed to step 12. Use the positive sign in equation (67b) in all remaining calculations. If the inequality is not satisfied, proceed to step 8.

8. Use the negative sign in equation (67b) and the results from Preload Computation I to iterate equation (74b) for α_y.

9. Use the value of α_y found in step 8 to compute

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} > 0 .$$

11. Proceed to step 12. Use the negative sign in equation (67b) in all remaining calculations.

"N" line computation

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67b) to iterate equation (74b) for α_y.

13. Use the value of α_y computed in step 12 and the initial data to compute:

a. α from equation (66b) ; iterate (11c) for α^*

b. $\dot{\emptyset}_y$ from equation (67b)

c. $\emptyset_r$ from equation (68b)

d. $\emptyset_p$ from equation (69b)

e. $\ddot{\overline{X}}$ from equation (70b)

f. F_r from equation (71b)

g. H_2 from equation (72b)

h. $\dot{\overline{\lambda}}_{III}$ from equation (73b)

i. $\dot{\overline{\lambda}}_{IV}$ from equation (73b)

j. $\dot{\lambda}_{15}$ from equation (73b)

14. Use a numerical integration technique to integrate

$$\ddot{\overline{X}} \text{ for } \dot{\overline{X}} \text{ for } \overline{X} \quad ,$$

$$\dot{\emptyset}_y \text{ for } \emptyset_y \quad ,$$

$$\dot{\overline{\lambda}}_{III} \text{ for } \overline{\lambda}_{III} \quad ,$$

$$\dot{\overline{\lambda}}_{IV} \text{ for } \overline{\lambda}_{IV} \quad ,$$

$$\dot{\lambda}_{15} \text{ for } \lambda_{15} \quad .$$

15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.

Computational procedure for Gimbal Set 3

Initial values for the auxiliary variables and the pitch angle are required. It is assumed that these values are known. These initial data are referred to as:

$$\bar{\lambda}_{V_0} = \begin{bmatrix} \lambda_{17_0} \\ \lambda_{18_0} \\ \lambda_{19_0} \end{bmatrix} , \qquad \bar{\lambda}_{VI_0} = \begin{bmatrix} \lambda_{20_0} \\ \lambda_{21_0} \\ \lambda_{22_0} \end{bmatrix} ,$$

$$\lambda_{23_0} ,$$

$$\lambda_{24_0} = 1 ,$$

$$\phi_{r_0} .$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180°$
2. Choose the positive sign in equation (67c) and compute:

   a. α from (66c); iterate (11c) for α^*

   b. $\ddot{\phi}_p$ from (67c)

   c. ϕ_r from (68c)

   d. ϕ_y from (69c)

   e. $\ddot{\bar{X}}$ from (70c)

   f. H_3 from (72c)

   g. $\dfrac{\partial H_3}{\partial \alpha_y}$ from (74c)

3. Choose $\alpha_y = \alpha_y + 5°$ and repeat step 2. Continue until $\alpha_y = +180°$.
4. Repeat steps 1 through 3, but use the negative sign in equation (67c).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67c)	α_y	H_3	$\frac{\partial H_3}{\partial \alpha_y}$		Eqn. - (67c)	α_y	H_3	$\frac{\partial H_3}{\partial \alpha_y}$

A plot of H_3 vs α_y should give insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for α_y to be used in the iteration of equation (74c).

Preload Computation II

5. Use the positive sign in equation (67c) and the results of Preload Computation I to iterate equation (74c) for α_y.
6. Use the value of α_y found in step 5 to compute

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} .$$

7. If the inequality

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} > 0$$

is satisfied, a minimum exists. Proceed to step 12. Use the positive sign in equation (67c) in all remaining calculations. If the inequality is not satisfied proceed to step 8.

8. Use the negative sign in equation (67c) and the results from Preload Computation I to iterate equation (74c) for α_y.

9. Use the value of α_y found in step 8 to compute

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} > 0 .$$

11. Proceed to step 12. Use the negative sign in equation (74c) in all remaining calculations.

<u>"N" line computation</u>

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67c) to iterate equation (74c) for α_y.

13. Use the value of α_y computed in step 12 and the initial data to compute:

    a. α from equation (66c); iterate (11c) for α^*

    b. $\dot{\phi}_p$ from equation (67c)

c. ϕ_r from equation (68c)

d. ϕ_y from equation (69c)

e. $\ddot{\overline{X}}$ from equation (70c)

f. F_r from equation (71c)

g. H_3 from equation (72c)

h. $\dot{\overline{\lambda}}_V$ from equation (73c)

i. $\dot{\overline{\lambda}}_{VI}$ from equation (73c)

j. $\dot{\lambda}_{23}$ from equation (73c)

14. Use a numerical integration technique to integrate

$$\ddot{\overline{X}} \text{ for } \dot{\overline{X}} \text{ for } \overline{X} \quad ,$$

$$\dot{\phi}_p \text{ for } \phi_p \quad ,$$

$$\dot{\overline{\lambda}}_V \text{ for } \overline{\lambda}_V \quad ,$$

$$\dot{\overline{\lambda}}_{VI} \text{ for } \overline{\lambda}_{VI} \quad ,$$

$$\dot{\lambda}_{23} \text{ for } \lambda_{23}$$

15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.

VIII. DISCUSSION

The problem studied has application to cases involving the flight of any "unpowered" vehicle through any atmosphere--subject to the assumptions given in the problem statement. For example, any space vehicle returning to the earth's surface must pass through the earth's atmosphere. This paper provides a method for determining an optimum trajectory for the transfer of the vehicle through the atmosphere. The pay-off function to be minimized over the atmospheric trajectory is a function of the state and control variables. For example, it may be desirable to minimize quantities such as the accumulative aerodynamic drag or the aerodynamic heating.

In order to solve the rotational equations of motion for three unknowns, it was necessary to introduce particular definitions for the angular acceleration, $\ddot{\bar{\phi}}$, and the angular velocity, $\dot{\bar{\phi}}$, of the vehicle. The definitions essentially eliminate all angular acceleration and two of the three components of the angular velocity for any given gimbal set. Thus, response of equipment and/or crew on the vehicle to a particular angular velocity may dictate choice of gimbal sets.

In the numerical generation of a trajectory, it is possible that an Euler angle will be computed that produces gimbal lock. A trajectory that produces gimbal lock is not admissible since gimbals will not function when in the gimbal lock orientation. Should the situation of gimbal lock arise, a new set of initial values for the auxiliary variables may be selected and a new trajectory generated. A particular set of auxiliary variables will yield an optimum trajectory for each gimbal set. The trajectory generated will not be the same for each gimbal set even though the same initial values of the auxiliary variables are chosen. No attempt has been made in this paper to determine the initial values of the auxiliary variables for any of the gimbal sets.

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APPENDIX A

Experimentally Determined Values of C_x and C_z for a Typical Space Vehicle

α^* Degrees	C_z	C_x
± 0	± 0	+ 1.828
± 5	± .014	+ 1.812
± 10	± .028	+ 1.772
± 15	± .040	+ 1.710
± 20	± .052	+ 1.626
± 25	± .063	+ 1.520
± 30	± .074	+ 1.388
± 35	± .084	+ 1.246
± 40	± .096	+ 1.092
± 45	± .118	+ .932
± 50	± .146	+ .768
± 55	± .182	+ .588
± 60	± .224	+ .416
± 65	± .268	+ .256
± 70	± .318	+ .112
± 75	± .372	- .020
± 80	± .426	- .134
± 85	± .486	- .236
± 90	± .546	- .322
± 95	± .596	- .394
± 100	± .628	- .444
± 105	± .690	- .486
± 110	± .728	- .516
± 115	± .756	- .542
± 120	± .772	- .566
± 125	± .776	- .582
± 130	± .772	- .586
± 135	± .756	- .584
± 140	± .730	- .576
± 145	± .686	- .560
± 150	± .628	- .544
± 155	± .554	- .524
± 160	± .468	- .510
± 165	± .366	- .498
± 170	± .248	- .490
± 175	± .130	- .484
± 180	± 0	- .480

APPENDIX B

Uniqueness of Solution for the Euler Angles

The relative velocity constraint equations are solved for two Euler angles in each gimbal set. The identity

$$\sin^2 \phi + \cos^2 \phi = 1$$

is used. Thus, the question arises as to which sign should be used with the radical that appears. This question is answered for each gimbal set by considering the way in which the coordinate systems are defined.

Gimbal Set 1

A first algebraic solution of equations (45) for ϕ_p and ϕ_y yields

$$SP = \frac{J\, V_{RY} \pm V_{RX} \sqrt{V^2_{RX} - J^2 + V^2_{RY}}}{(V^2_{RX} + V^2_{RY})} , \qquad (B1)$$

and

$$CP = \frac{J\, V_{RX} \pm V_{RY} \sqrt{V^2_{RX} - J^2 + V^2_{RY}}}{(V^2_{RY} + V^2_{RY})} \qquad (B2)$$

where

$$J = CR\ V_{rmx} - SR\ V_{rmz} \quad .$$

$$SY = \frac{V_{rmy} V_{RZ} \pm K \sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} \quad , \qquad \text{(B3)}$$

and

$$CY = \frac{V_{rmy} K \pm V_{RZ} \sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} \qquad \text{(B4)}$$

where

$$K = CP\ V_{RY} - SP\ V_{RX} \quad .$$

The identity

$$SP^2 + CP^2 = 1$$

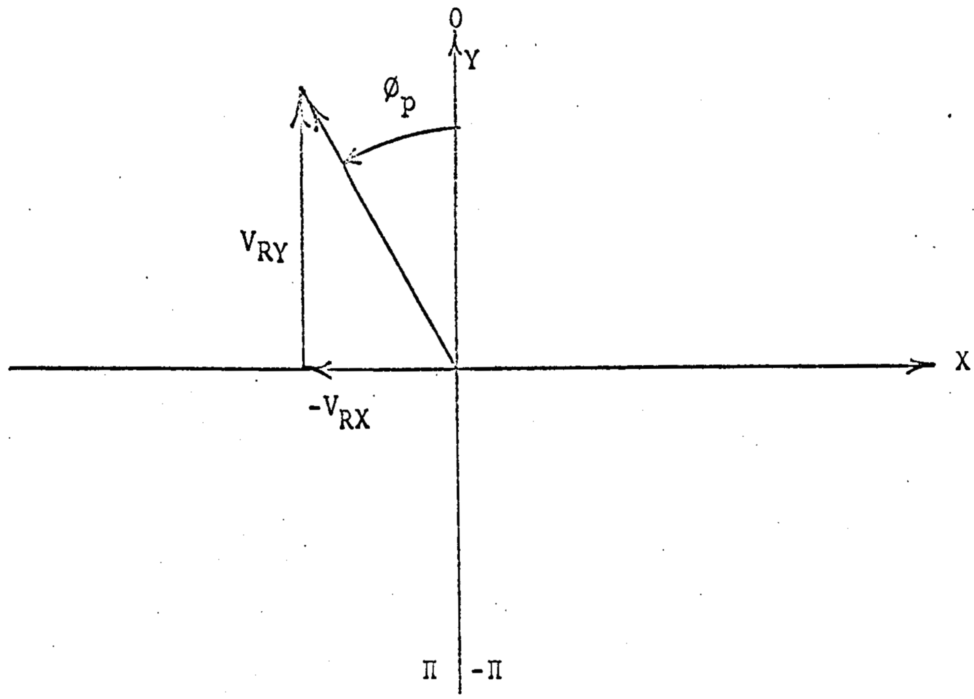
is satisfied only if opposite signs appear with the radical in (B1) and (B2). Let $\phi_r = \alpha = 0$. Then equations (B1) and (B2) reduce to

$$SP = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad , \qquad \text{(B5)}$$

and

$$CP = \frac{\pm V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad . \qquad \text{(B6)}$$

Consider the positive pitch angle, ϕ_p, shown in Appendix Figure 1. Now restrict ϕ_p, $-\Pi \leqq \phi_p \leqq \Pi$.



Coordinate System Showing A Positive Pitch Angle

Appendix Figure 1

Thus, the correct signs for the sine and cosine are

$$SP = \frac{-\ V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad , \qquad \text{(B7)}$$

and

$$CP = \frac{+\ V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad . \qquad \text{(B8)}$$

The identity

$$SY^2 + CY^2 + 1$$

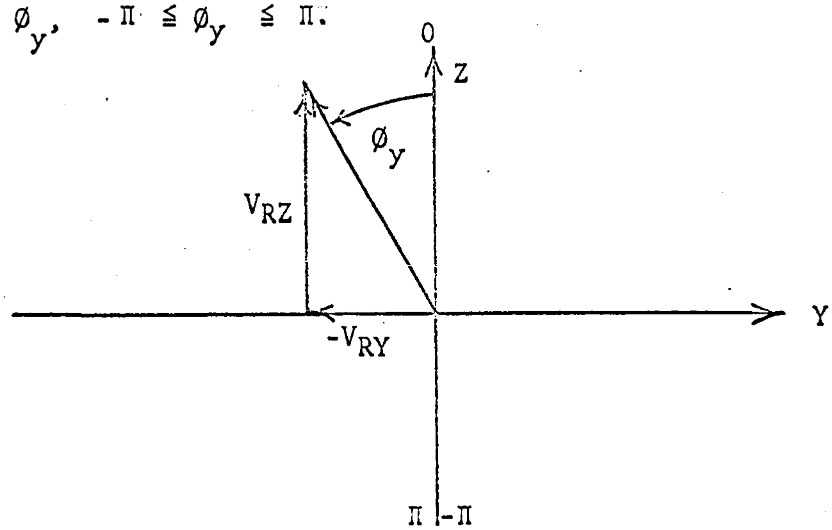
is satisfied only if opposte signs appear with the radical in (B3) and (B4). Let $\alpha_y = 90$ and $\emptyset_p = 0$. Then equations (B3) and (B4) reduce to

$$SY = \frac{\pm V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad (B9)$$

and

$$CY = \frac{\pm V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad (10)$$

Consider the positive yaw angle, $\emptyset_y$, shown in Appendix Figure 2. Now restrict $\emptyset_y$, $-\Pi \leqq \emptyset_y \leqq \Pi$.



Coordinate System Showing A Positive Yaw Angle $\emptyset_y$

Appendix Figure 2

Thus, the correct signs for the sine and cosine are

$$SY = \frac{-V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad (B11)$$

and

$$CY = \frac{+V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad (B12)$$

Gimbal Set 2

A first algebraic solution of equations (48) for ϕ_p and ϕ_r yields

$$SP = \frac{F\, V_{RX} \pm V_{RY} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} , \quad (B13)$$

and

$$CP = \frac{-F\, V_{RY} \pm V_{RX} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad (B14)$$

where

$$F = -V_{rmy}\, CY + V_{rmz}\, SY .$$

$$SR = \frac{V_{RZ} V_{rmx} \pm G \sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \quad , \qquad (B15)$$

and

$$CR = \frac{G V_{rmx} \pm V_{RZ} \sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \qquad (B16)$$

where

$$G = V_{RX} CP + V_{RY} SP \quad .$$

The identity

$$SP^2 + CP^2 = 1$$

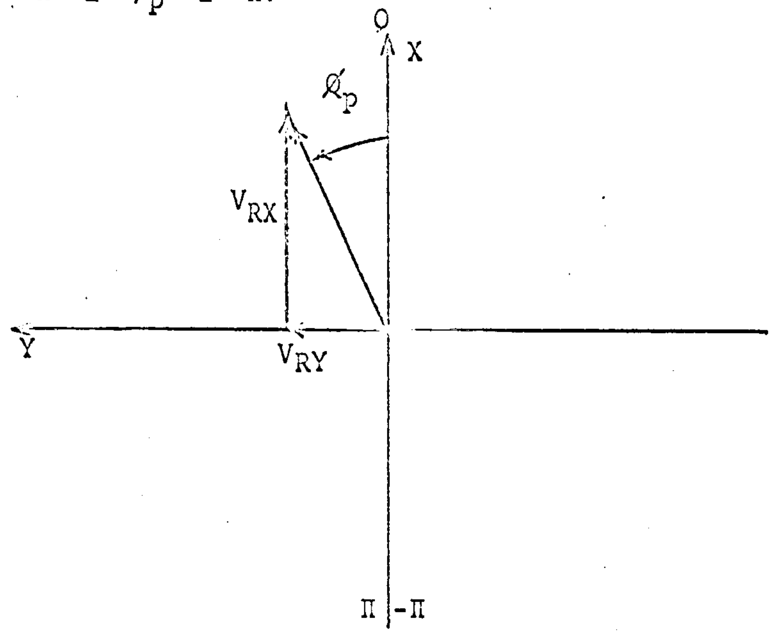
is satisfied only if the same sign appears with the radical in (B13) and (B14). Let $\alpha = 0$ and $\emptyset_y = 90°$. Then equations (B13) and (B14) reduce to

$$SP = \frac{\pm V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad , \qquad (B17)$$

and

$$CP = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad . \qquad (B18)$$

Consider the positive pitch angle, ϕ_p, shown in Appendix Figure 3. Now restrict ϕ_p, $-\Pi \leqq \phi_p \leqq \Pi$.



Coordinate System Showing A Positive Pitch Angle

Appendix Figure 3

Thus, the positive sign is chosen for both the sine and cosine.

$$SP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RX}^2}} , \tag{B19}$$

and

$$CP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RX}^2}} . \tag{B20}$$

The identity

$$SR^2 + CR^2 = 1$$

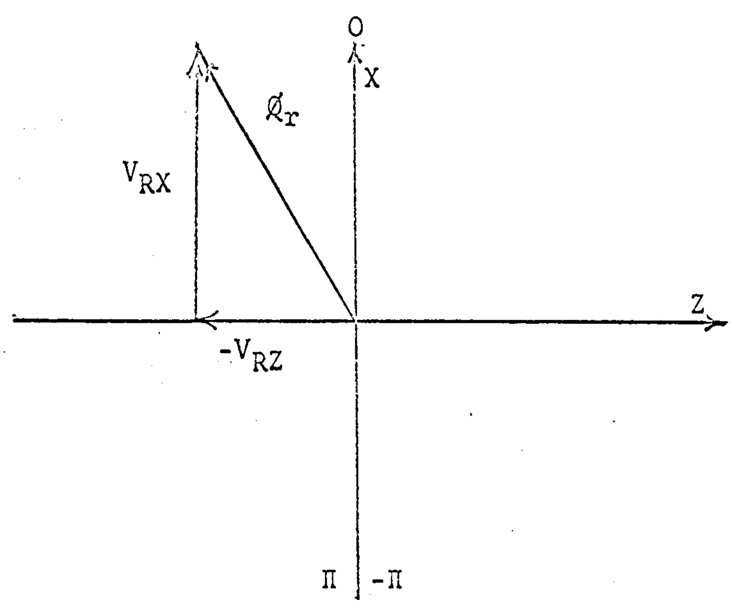
is satisfied only if opposite signs appear with the radical in (B15) and (B16). Let $\alpha = \phi_p = 0$. Then equations (B15) and (B16) reduce to

$$SR = \frac{\pm V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} \quad , \qquad \text{(B21)}$$

and

$$CR = \frac{\pm V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} \quad . \qquad \text{(B22)}$$

Consider the positive roll angle, ϕ_r, shown in Appendix Figure 4. Now restrict ϕ_r, $-\Pi \leqq \phi_r \leqq \Pi$.



Coordinate System Showing A Positive Roll Angle

Appendix Figure 4

Thus, the correct signs for the sine and cosine are

$$SR = \frac{-\ V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} , \qquad (B23)$$

and

$$CR = \frac{+\ V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} . \qquad (B24)$$

Gimbal Set 3

A first algebraic solution of equations (51) for ϕ_y and ϕ_r yields

$$SR = \frac{V_{RZ} A \pm V_{RX} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} , \qquad (B25)$$

where

$$CR = \frac{V_{RX} A \pm V_{RZ} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} \qquad (B26)$$

where

$$A = CP\ V_{rmx} - SP\ V_{rmy} .$$

$$SY = \frac{-V_{RY}V_{rmz} \pm B\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} , \quad (B27)$$

and

$$CY = \frac{B\,V_{rmz} \pm V_{RY}\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \quad (B28)$$

where

$$B = V_{RZ}\,CR - V_{RX}\,SR .$$

The identity

$$SR^2 + CR^2 = 1$$

is satisfied only if opposite signs appear with the radical in (B25) and (B26). Let $\alpha = \phi_p = 0$. Then equations (B25) and (B26) reduce to

$$SR = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RZ}^2}} , \quad (B29)$$

and

$$CR = \frac{\pm V_{RZ}}{\sqrt{V_{RX}^2 + V_{RZ}^2}} . \quad (B30)$$

Note, equations (B29) and (B30) are the same as (B21) and (B22). The identity

$$SR^2 + CR^2 = 1$$

is satisfied in the same way in each case. Hence, the signs for the sine and cosine are chosen the same as in equations (B23) and (B24). The identity

$$SY^2 + CY^2 = 1$$

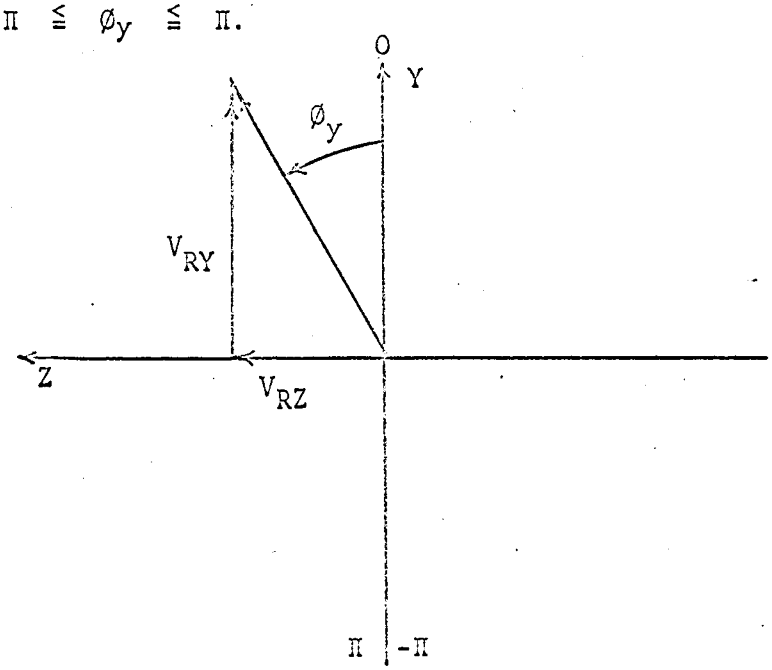
is satisfied only if the same sign appears with the radical in (B27) and (B28). Let $\alpha = \phi_r = 0$. Then equations (B27) and (B28) reduce to

$$SY = \frac{\pm V_{RZ}}{\sqrt{V_{RY}^2 + V_{RZ}^2}} \qquad , \quad \text{(B31)}$$

and

$$CY = \frac{\pm V_{RY}}{\sqrt{V_{RY}^2 + V_{RZ}^2}} \qquad . \quad \text{(B32)}$$

Consider the positive yaw angle, ϕ_y, shown in Appendix Figure 5. Now restrict ϕ_y, $-\Pi \leqq \phi_y \leqq \Pi$.



Coordinate System Showing A Positive Yaw Angle

Appendix Figure 5

Thus, the positive sign is chosen for both the sine and cosine.

$$SY = \frac{+ V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad (B33)$$

and

$$CY = \frac{+ V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad (B34)$$