TWO-DIMENSIONAL WINGS OF MAXIMUM LIFT-TO-DRAG RATIO IN HYPERSONIC FLOW(*)

by

DAVID G. HULL(**)

SUMMARY

The problem of maximizing the lift-to-drag ratio of a slender, two-dimensional, flat-top wing in hypersonic flow is considered under the assumptions that the pressure coefficient is modified Newtonian and the skin-friction coefficient is constant. Arbitrary conditions are imposed on the chord, the thickness, and the profile area; and the necessary conditions to be satisfied by an optimum wing are derived with the indirect methods of the calculus of variations. Then, several particular cases are analyzed and, for each case, analytical expressions are determined for the optimum shape and the maximum lift-to-drag ratio.

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(**) Research Associate in Astronautics, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

1. INTRODUCTION

For two-dimensional, hypersonic wings, two extremal problems are of interest, that is, to minimize the drag for a given lift and to maximize the lift-to-drag ratio for unconstrained lift. Since problems of the former type were considered in Ref. 1, the object of this paper is to consider problems of the latter type. With regard to the calculation of the lift-to-drag ratio, the hypotheses employed are as follows: (a) the wing is two-dimensional; (b) the upper surface is a plane parallel to the undisturbed flow direction; (c) the lower surface is slender in the chordwise sense; (d) the pressure coefficient is modified Newtonian; (e) the skin-friction coefficient is constant; (f) the base drag is neglected; and (g) the effect of tangential forces on the lift is negligible.

The maximum lift-to-drag ratio problem is formulated for arbitrary constraints imposed on the chord, the thickness, and the profile area. After the necessary conditions to be satisfied by the optimum shape are stated in general, several particular cases are analyzed in detail.

2. FORMULATION OF THE VARIATIONAL PROBLEM

In order to relate the lift-to-drag ratio and the profile area of a two-dimensional, flat-top wing to its geometry, the following Cartesian coordinate system is used: the origin O is the leading edge; the x-axis is in the direction of the undisturbed flow and is parallel to the upper surface; while the z-axis is normal to the x-axis and positive downward. If hypotheses (a) through (g) are considered and if the lower surface is represented by the relationship $z = z(x)$, the lift-to-drag ratio $E = L/D$ is given by (Refs. 1 and 2)

$$E = \frac{\int_0^c \dot{z}^2 \, dx}{\int_0^c (\dot{z}^3 + C_f/n) \, dx} \tag{1}$$

where c is the chord, $\dot{z}$ the derivative dz/dx, C_f the constant skin-friction coefficient, and n a factor modifying the Newtonian pressure distribution(*). This lift-to-drag ratio is to be maximized subject to the isoperimetric constraint of given profile area

$$A = \int_0^c z \, dx \tag{2}$$

(*) In other words, the pressure coefficient is assumed to be $C_p = 2n\dot{z}^2$.

and the inequality constraint

$$\dot{z} \geq 0 \tag{3}$$

which expresses the limits of validity of the Newtonian pressure law.

One way to account for Ineq. (3) is to transform it into an equality constraint by introducing an appropriate auxiliary variable. If this is done, one must expect the optimum contour to include subarcs $\dot{z} = 0$. From previous experience (see Chapter 14 of Ref. 3), it is known that the subarcs $\dot{z} = 0$ generally start at the initial point and terminate at the final point. Therefore, an alternate way to account for Ineq. (3) is to investigate the class of airfoils composed of a zero-slope shape followed by a regular shape followed by another zero-slope shape and, therefore, defined by (Fig. 1)

$$\begin{aligned}
&\dot{z} = 0 \quad , \quad z = z_i = 0 \quad , \quad 0 \leq x \leq x_i \\
&\dot{z} \geq 0 \quad , \quad z_i \leq z \leq z_f \quad , \quad x_i \leq x \leq x_f \\
&\dot{z} = 0 \quad , \quad z = z_f = t \quad , \quad x_f \leq x \leq c
\end{aligned} \tag{4}$$

where the initial abcissa x_i, the final abcissa x_f, the chord c, and the thickness t may be either prescribed or free. If this point of view is taken, the variational problem reduces to that of maximizing the expression

$$E = \frac{\int_{x_i}^{x_f} \dot{z}^2 dx}{\int_{x_i}^{x_f} \dot{z}^3 dx + C_f c/n} \tag{5}$$

subject to the isoperimetric constraint

$$A = \int_{x_i}^{x_f} z\, dx + t(c - x_f) \tag{6}$$

and certain prescribed boundary conditions.

3. NECESSARY CONDITIONS

According to Ref. 4, the proposed problem is equivalent to that of maximizing the functional

$$I = \int_{x_i}^{x_f} F(z, \dot{z}, E, \lambda)\, dx + G(x_f, c, t, \lambda) \tag{7}$$

with respect to the functions z(x) and the parameters c and t satisfying the isoperimetric constraint (6) and the prescribed boundary conditions. In Eq. (7), the functions F and G are defined as

$$F = \dot{z}^2 - E\dot{z}^3 + \lambda z$$
$$G = -\, EC_f c/n + \lambda t(c - x_f) \tag{8}$$

where the constant E is the maximum lift-to-drag ratio and λ is an undetermined, constant Lagrange multiplier.

Euler Equation. It is known that the function z(x) which extremizes the functional (7) must be a solution of the Euler equation (see, for instance, Chapter 1 of Ref. 3)

$$dF_{\dot{z}}/dx - F_z = 0 \tag{9}$$

Its explicit form

$$\frac{d}{dx}(3E\dot{z}^2 - 2\dot{z}) + \lambda = 0 \tag{10}$$

admits the first integral

$$3E\dot{z}^2 - 2\dot{z} + \lambda x = C \tag{11}$$

where C is a constant. A second integral can be obtained, but it is more convenient to derive it when analyzing particular cases.

Transversality Condition. The integration constants which appear in the general solution of the Euler equation can be determined by applying the prescribed boundary conditions and the natural boundary conditions. The latter are obtained from the transversality condition

$$\left[(F - \dot{z}F_{\dot{z}})\,\delta x + F_{\dot{z}}\,\delta z\right]_i^f + \delta G = 0 \tag{12}$$

which must be satisfied identically for every set of variations consistent with the prescribed boundary conditions. Since $\delta z_i = 0$ and $\delta z_f = \delta t$, the explicit form of Eq. (12) is given by

$$(2E\dot{z}^3 - \dot{z}^2)_i\,\delta x_i - (2E\dot{z}^3 - \dot{z}^2)_f\,\delta x_f$$

$$+ (EC_f/n - \lambda t)\,\delta c + (3E\dot{z}^2 - 2\dot{z} + \lambda x - \lambda c)_f\,\delta t = 0 \tag{13}$$

which, for the four classes of solutions

$$
\begin{array}{llll}
\underline{\text{Class I:}} & x_i = 0 & , & x_f = c \\
\underline{\text{Class II:}} & x_i > 0 & , & x_f = c \\
\underline{\text{Class III:}} & x_i = 0 & , & x_f < c \\
\underline{\text{Class IV:}} & x_i > 0 & , & x_f < c
\end{array}
\qquad (14)
$$

yields the following natural boundary conditions:

$$
\begin{array}{ll}
\underline{x_i \equiv \text{free:}} & 2E\dot{z}_i^3 - \dot{z}_i^2 = 0 \\
\underline{x_f \equiv \text{free:}} & 2E\dot{z}_f^3 - \dot{z}_f^2 = 0 \\
\underline{c \equiv \text{free:}} & EC_f/n - \lambda t = 0 \\
\underline{x_f = c \equiv \text{free:}} & 2E\dot{z}_f^3 - \dot{z}_f^2 - EC_f/n + \lambda t = 0 \\
\underline{t \equiv \text{free:}} & 3E\dot{z}_f^2 - 2\dot{z}_f + \lambda x_f - \lambda c = 0
\end{array}
\qquad (15)
$$

Weierstrass Condition. Once an extremal solution has been obtained, it is necessary to verify that it actually maximizes the functional (7). In this connection, the Weierstrass condition requires that

$$F(z, \dot{z}_*, E, \lambda) - F(z, \dot{z}, E, \lambda) - F_{\dot{z}}(z, \dot{z}, E, \lambda)(\dot{z}_* - \dot{z}) \leq 0 \qquad (16)$$

where z and $\dot{z}$ are the ordinate and the slope of the extremal arc and $\dot{z}_*$ is the slope of the comparison arc. The explicit form of this inequality

$$(\dot{z}_* - \dot{z})^2 (- E\dot{z}_* + 1 - 2E\dot{z}) \leq 0 \tag{17}$$

is satisfied for every choice of the comparison slope consistent with the constraint (3) providing

$$\dot{z} \geq 1/2E \tag{18}$$

at each point of the extremal solution.

4. NONDIMENSIONAL QUANTITIES

In the following sections, several particular cases are analyzed with the aid of the previous necessary conditions. Concerning the procedure to be followed, we first determine the solutions of Class I; then, if more solutions are needed, we investigate those of Class II, Class III, and Class IV. In order to present the results in the most compact way, it is convenient to introduce the nondimensional coordinates

$$\xi = x/c \quad , \quad \zeta = z/t \tag{19}$$

and the thickness ratio

$$\tau = t/c \tag{20}$$

Furthermore, the following nondimensional variables are defined:

$$\begin{aligned} E_* &= E(C_f/n)^{1/3} \\ \tau_* &= \tau(C_f/n)^{-1/3} \\ A_* &= Ac^{-2}(C_f/n)^{-1/3} \\ A_o &= At^{-2}(C_f/n)^{1/3} \end{aligned} \tag{21}$$

5. ABSOLUTE MAXIMUM LIFT-TO-DRAG RATIO

If the chord, the thickness, and the profile area are free, the optimum solution is of Class I and must satisfy the natural boundary conditions (15-4) and (15-5) with $\lambda = 0$. The first integral (11) combined with the boundary condition (15-5) implies that $C = 0$ and that

$$\dot{z} = 2/3E \tag{22}$$

Integrating this differential equation subject to the initial conditions, one obtains the relation

$$z = (2/3E)x \tag{23}$$

which, applied at the trailing edge, implies that the thickness is given by

$$t = (2/3E)c \tag{24}$$

Therefore, if the nondimensional coordinates (19) are introduced, the optimum airfoil shape is the wedge

$$\zeta = \xi \tag{25}$$

Finally, because of Eqs. (5), (21), (23), and (24), the optimum thickness ratio satisfies the relation

$$\tau_* = \sqrt[3]{2} \cong 1.26 \tag{26}$$

and the maximum lift-to-drag ratio can be obtained from

$$E_* = \sqrt[3]{4}/3 \cong 0.529 \tag{27}$$

It should be noted that the geometry of the optimum wing is completely determined in $\xi\zeta$-coordinate system but depends on a scaling factor in the xz-coordinate system. Therefore, there exist an infinite number of wings having the lift-to-drag ratio (27). However, if one geometric quantity is specified (the chord, the thickness, or the profile area), the optimum wing becomes unique. Should two or three geometric quantities be simultaneously specified, the geometry of the optimum wing would generally change, and a loss in the lift-to-drag ratio would occur with respect to that predicted by Eq. (27).

6. GIVEN THICKNESS AND CHORD

Since the profile area is free, the relationship $\lambda = 0$ holds, and the first integral (11) implies that the slope of the regular shape is constant; hence, the regular shape is a straight line.

Solutions of Class I (Figs. 2 through 4). For these solutions, the slope can be expressed as

$$\dot{z} = \tau \tag{28}$$

which, in the light of the initial conditions, can be integrated to give

$$z = \tau x \tag{29}$$

Hence, the optimum shape is a wedge whose maximum lift-to-drag ratio is given by

$$E_* = \tau_*^2/(\tau_*^3 + 1) \tag{30}$$

Finally, because of the Weierstrass condition (18), the solutions of Class I are valid providing

$$\tau_* \geq 1 \tag{31}$$

Solutions of Class III (Figs. 2 through 4). For these solutions, the natural boundary condition (15-2) allows one to write

$$\dot{z} = 1/2E \tag{32}$$

Integrating this equation subject to the initial conditions, one obtains the relation

$$z = (1/2E)x \tag{33}$$

which, at the final point, becomes

$$t = (1/2E)x_f \tag{34}$$

In a nondimensional form, the optimum airfoil shape is given by

$$\begin{aligned} &0 \le \xi \le \xi_f \quad , \quad && \zeta = \xi/\xi_f \\ &\xi_f \le \xi \le 1 \quad , \quad && \zeta = 1 \end{aligned} \tag{35}$$

and, because of Eqs. (5), (32), and (34), is characterized by the following relationships between the lift-to-drag ratio, the thickness ratio, and the transition abcissa from the regular shape to the constant thickness portion:

$$\tau_* = \xi_f^{\,2/3} \quad , \quad E_* = \sqrt[3]{\xi_f}\,/2 \tag{36}$$

Since the thickness ratio is given, it is convenient to rewrite Eqs. (35) and (36) explicitly as

$$
\begin{aligned}
0 \le \xi \le \xi_f \quad &, \quad \zeta = \xi/\tau_* \sqrt{\tau_*} \\
\xi_f \le \xi \le 1 \quad &, \quad \zeta = 1
\end{aligned}
\tag{37}
$$

and

$$
\xi_f = \tau_* \sqrt{\tau_*} \quad , \quad E_* = \sqrt{\tau_*}\,/2 \tag{38}
$$

with the understanding that

$$
\tau_* \le 1 \tag{39}
$$

In closing, it should be remarked that the solutions of this class are not unique; any combination of zero slope shapes and regular shapes having the slope (32) would yield a lift-to-drag ratio identical with that given by Eq. (38-2).

7. GIVEN PROFILE AREA AND CHORD

If the profile area and the chord are prescribed, the geometry of the regular shape is governed by the differential equation (11).

Solutions of Class I (Figs. 5 through 8). Since the thickness is free, the natural boundary condition (15-5) applies and, if combined with Eq. (11), implies that

$$C = \lambda c \tag{40}$$

Hence, in the light of the Weierstrass condition (18), Eq. (11) can be rewritten as

$$\dot{z} = (1/3E)\left\{1 + [1 - \alpha(1 - x/c)]^{1/2}\right\} \tag{41}$$

where

$$\alpha = -3E\lambda c \tag{42}$$

The integration of this differential equation subject to the end conditions leads to the relationships

$$z = (c/3E)\, G(\xi, \alpha) \quad , \quad t = (c/3E)\, G(1, \alpha) \tag{43}$$

where

$$G(\xi, \alpha) = \xi + (2/3\alpha)\left\{[1 - \alpha(1 - \xi)]^{3/2} - [1 - \alpha]^{3/2}\right\} \tag{44}$$

Forming the ratio of Eqs. (43), one finds the geometry of the optimum airfoil to be

$$\zeta = G(\xi, \alpha)/G(1, \alpha) \tag{45}$$

The next step is to relate the quantity α to the prescribed quantities A and c as well as to determine the maximum lift-to-drag ratio and the optimum thickness ratio. In this connection, if Eqs. (5), (6), (43-2), and (45) are combined, the following results can be obtained:

$$\tau_* = G(1, \alpha)/M(\alpha) \quad , \quad E_* = M(\alpha)/3 \quad , \quad A_* = N(\alpha)/M(\alpha) \tag{46}$$

where

$$M(\alpha) = \left\{2 - (2/5\alpha)\left[(4 + \alpha)(1 - \alpha)^{3/2} - 4\right]\right\}^{1/3}$$

$$N(\alpha) = 1/2 - (2/15\alpha^2)\left[(2 + 3\alpha)(1 - \alpha)^{3/2} - 2\right] \tag{47}$$

Consequently, if the quantity α is eliminated between Eqs. (45) and (46), the following functional relations are obtained:

$$\zeta = f_1(\xi, A_*)$$

$$\tau_* = f_2(A_*) \quad , \quad E_* = f_3(A_*) \tag{48}$$

Since the Weierstrass condition requires that $\alpha \leq 3/4$, these solutions are valid providing

$$0.543 \leq A_* \leq \infty \tag{49}$$

Solutions of Class II (Figs. 5 through 8). The remaining solutions are of Class II and must satisfy the natural boundary conditions (15-1) and (15-5). In particular, Eq. (15-5) and the first integral (11) imply that Eq. (40), and hence Eq. (41), is also valid here. On the other hand, Eq. (15-1) requires that

$$\dot{z}_i = 1/2E \tag{50}$$

which, with Eq. (41) evaluated at the initial point, allows one to write

$$\alpha = 3/4(1 - \xi_i) \tag{51}$$

As a consequence, the integration of Eq. (41) subject to the end conditions leads to the relations

$$z = (c/3E)H(\xi, \xi_i) \quad , \quad t = (c/3E)H(1, \xi_i) \tag{52}$$

where

$$H(\xi, \xi_i) = \xi - \xi_i + (1/9)(1 - \xi_i)\left\{[4 - 3(1 - \xi)/(1 - \xi_i)]^{3/2} - 1\right\} \tag{53}$$

The nondimensional shape of the optimum airfoil is then given by the ratio of

Eqs. (52), that is, by

$$
\begin{aligned}
0 \le \xi \le \xi_i \quad &, \quad \zeta = 0 \\
\xi_i \le \xi \le 1 \quad &, \quad \zeta = H(\xi, \xi_i)/H(1, \xi_i)
\end{aligned}
\tag{54}
$$

The next step is to combine Eqs. (5), (6), (52-2), and (54) to obtain the relationships

$$
\begin{aligned}
\tau_* &= (16/9)(60/229)^{1/3}(1 - \xi_i)^{2/3} \\
E_* &= (1/3)(229/60)^{1/3}(1 - \xi_i)^{1/3} \\
A_* &= (229/270)(60/229)^{1/3}(1 - \xi_i)^{5/3}
\end{aligned}
\tag{55}
$$

Consequently, from Eqs. (54) and (55), it is seen that the solutions of Class II

satisfy the functional relations

$$
\begin{gathered}
\zeta = g_1(\xi, A_*) \\
\xi_i = g_2(A_*) \quad , \quad \tau_* = g_3(A_*) \quad , \quad E_* = g_4(A_*)
\end{gathered}
\tag{56}
$$

which are valid in the range

$$
0 \le A_* \le 0.543 \tag{57}
$$

8. GIVEN PROFILE AREA AND THICKNESS

If the profile area and the thickness are prescribed, the geometry of the regular shape is governed by the differential equation (11), which is rewritten here as follows:

$$\dot{z} = (1/3E)\left\{1 + [1 - \beta + \alpha(x/c)]^{1/2}\right\} \tag{58}$$

where α is defined by Eq. (42) and where

$$\beta = - 3EC \tag{59}$$

Solutions of Class I (Figs. 9 through 13). The integration of the above differential equation subject to the end conditions leads to the following relations:

$$z = (c/3E)G(\xi, \alpha, \beta) \quad , \quad t = (c/3E)G(1, \alpha, \beta) \tag{60}$$

where

$$G(\xi, \alpha, \beta) = \xi + (2/3\alpha)[(1 - \beta + \alpha\xi)^{3/2} - (1 - \beta)^{3/2}] \tag{61}$$

As a consequence, the shape of the optimum airfoil can be expressed as

$$\zeta = G(\xi, \alpha, \beta)/G(1, \alpha, \beta) \tag{62}$$

The next step is to combine Eqs. (5), (6), (60-2), and (62) to obtain the relationships

$$\tau_* = G(1, \alpha, \beta)/M(\alpha, \beta)$$

$$E_* = M(\alpha, \beta)/3 \tag{63}$$

$$A_o = M(\alpha, \beta)\, N(\alpha, \beta)/G^2(1, \alpha, \beta)$$

where

$$M(\alpha, \beta) = \left\{2 + (2/5\alpha)[(4 + \beta - \alpha)(1 - \beta + \alpha)^{3/2} - (4 + \beta)(1 - \beta)^{3/2}]\right\}^{1/3}$$

$$N(\alpha, \beta) = 1/2 - (2/3\alpha)(1 - \beta)^{3/2} + (4/15\alpha^2)[(1 - \beta + \alpha)^{5/2} - (1 - \beta)^{5/2}] \tag{64}$$

Furthermore, since the chord is free, the natural boundary condition (15-4) must be satisfied and, if combined with Eqs. (58), (63-1), and (63-2), yields the result

$$15\alpha\beta - 2(4 + \beta + 5\alpha)(1 - \beta)^{3/2} + 2(4 + \beta - \alpha)(1 - \beta + \alpha)^{3/2} = 0 \tag{65}$$

In order to find the solutions of this equation, it is necessary to apply the Weierstrass condition (18) which, combined with Eq. (58), requires that

$$\beta - \alpha\xi \leq 3/4 \tag{66}$$

If this inequality is employed at the initial point ($\xi_i = 0$) and the final point ($\xi_f = 1$), the following results are obtained:

$$\beta \le 3/4 \quad , \quad \beta - \alpha \le 3/4 \tag{67}$$

Numerical analyses show that, for $0 \le \beta \le 3/4$, there exist two values of the parameter α which satisfy Eq. (65); thus, the solutions of Class I must be divided into two subclasses.

Solutions of Class I-A. These solutions are given by the upper curve in Fig. 9 and can be represented by the equation

$$\alpha = \alpha(\beta) \tag{68}$$

As a consequence, the parameters α and β can be eliminated between Eqs. (62), (63), and (68) to obtain the functional relationships

$$\begin{gathered} \zeta = f_1(\xi, A_o) \\ \tau_* = f_2(A_o) \quad , \quad E_* = f_3(A_o) \end{gathered} \tag{69}$$

which are valid in the range

$$0 \le A_o \le 0.396 \tag{70}$$

Solutions of Class I-B. These solutions are represented by the lower curve in Fig. 9, that is, by the relation

$$\alpha = 0 \tag{71}$$

Hence, the optimum airfoil shape is the wedge

$$\zeta = \xi \tag{72}$$

whose thickness ratio and lift-to-drag ratio are parametrically represented by the relationships

$$\begin{aligned} \tau_* &= [1 + (1 - \beta)^{1/2}][2 + (2 + \beta)(1 - \beta)^{1/2}]^{-1/3} \\ E_* &= (1/3)[2 + (2 + \beta)(1 - \beta)^{1/2}]^{1/3} \\ A_o &= (1/2)[2 + (2 + \beta)(1 - \beta)^{1/2}]^{1/3}[1 + (1 - \beta)^{1/2}]^{-1} \end{aligned} \tag{73}$$

Elimination of the parameter β between these equations leads to functional relations of the form (69-2) and (69-3) which are valid providing

$$0.396 \le A_o \le 0.5 \tag{74}$$

Solutions of Class III (Figs. 9 through 13). The remaining solutions are of Class III, and hence, must satisfy the natural boundary conditions (15-2) and (15-3). The first of these conditions requires that

$$\dot{z}_f = 1/2E \tag{75}$$

which, when combined with Eq. (58), implies that

$$\beta = 3/4 + \alpha\,\xi_f \tag{76}$$

Consequently, the differential equation governing the regular shape becomes

$$\dot{z} = (1/3E)\left\{1 + [1/4 - \alpha(\xi_f - \xi)]^{1/2}\right\} \tag{77}$$

and can be integrated subject to the end conditions to obtain the relations

$$z = (c/3E)\,H(\xi, \xi_f, \alpha) \quad , \quad t = (c/3E)\,H(\xi_f, \xi_f, \alpha) \tag{78}$$

where

$$H(\xi, \xi_f, \alpha) = \xi + (2/3\alpha)\left\{[1/4 - \alpha(\xi_f - \xi)]^{3/2} - [1/4 - \alpha\,\xi_f]^{3/2}\right\} \tag{79}$$

By forming the ratio of Eqs. (78), we can express the geometry of the optimum airfoil as

$$\zeta = H(\xi, \xi_f, \alpha)/H(\xi_f, \xi_f, \alpha) \tag{80}$$

The next step is to combine Eqs. (5), (6), (78-2), and (80) to obtain the relations

$$\begin{aligned} \tau_* &= H(\xi_f, \xi_f, \alpha)/P(\xi_f, \alpha) \\ E_* &= P(\xi_f, \alpha)/3 \\ A_o &= P(\xi_f, \alpha)\,Q(\xi_f, \alpha)/H^2(\xi_f, \xi_f, \alpha) \end{aligned} \tag{81}$$

where

$$P(\xi_f, \alpha) = \Big\{2\xi_f + (2/\alpha)[(1/4)^{3/2} - (1/4 - \alpha\xi_f)^{3/2}] - (2/5\alpha)[(1/4)^{5/2} - (1/4 - \alpha\xi_f)^{5/2}]\Big\}^{1/3}$$

$$Q(\xi_f, \alpha) = (1 - \xi_f)H(\xi_f, \xi_f, \alpha) + \xi_f^{2}/2 - (2\xi_f/3\alpha)(1/4 - \alpha\xi_f)^{3/2} + (4/15\alpha^2)[(1/4)^{5/2} - (1/4 - \alpha\xi_f)^{5/2}] \qquad (82)$$

Finally, the natural boundary condition (15-3) in combination with Eqs. (81-1) and (81-2) yield the following relation between α and ξ_f:

$$5\alpha(2 + 3\alpha)\xi_f + 10(1 + \alpha)[(1/4)^{3/2} - (1/4 - \alpha\xi_f)^{3/2}] - 2[(1/4)^{5/2} - (1/4 - \alpha\xi_f)^{5/2}] = 0 \qquad (83)$$

Since the Weierstrass condition (18) and Eq. (77) require that the inequality

$$\alpha \leq 0 \qquad (84)$$

be satisfied, the only solution of Eq. (83) is

$$\alpha = 0 \qquad (85)$$

Therefore, the shape of the optimum airfoil becomes

$$\begin{aligned} 0 \leq \xi \leq \xi_f \quad &, \quad \zeta = \xi/\xi_f \\ \xi_f \leq \xi \leq 1 \quad &, \quad \zeta = 1 \end{aligned} \qquad (86)$$

while the thickness ratio and the lift-to-drag ratio can be expressed parametrically as

$$\tau_* = \xi_f^{2/3} \quad , \quad E_* = \sqrt[3]{\xi_f}\,/2 \quad , \quad A_o = (2 - \xi_f)/2\xi_f^{2/3} \tag{87}$$

Finally, Eqs. (86) and (87) can be used to write the functional relations

$$\begin{gathered} \zeta = g_1(\xi, A_o) \\ \xi_f = g_2(A_o) \quad , \quad \tau_* = g_3(A_o) \quad , \quad E_* = g_4(A_o) \end{gathered} \tag{88}$$

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LIST OF CAPTIONS

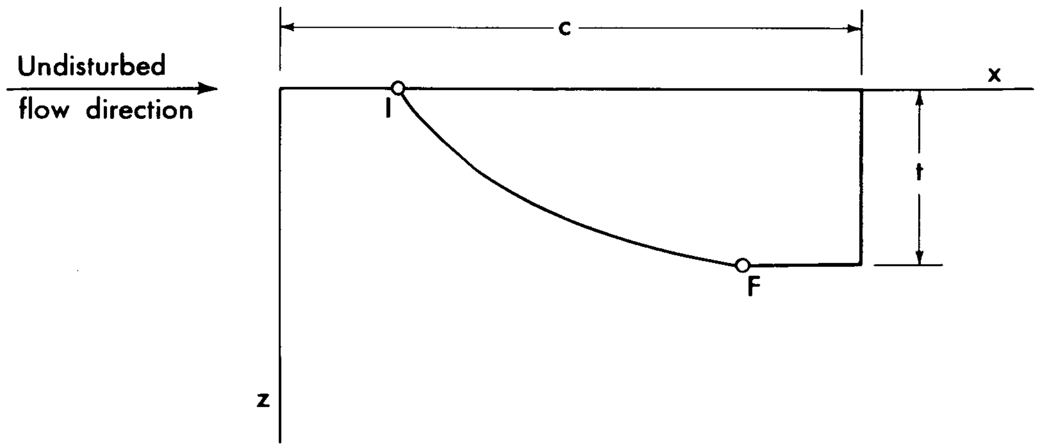

Fig. 1 Coordinate system.

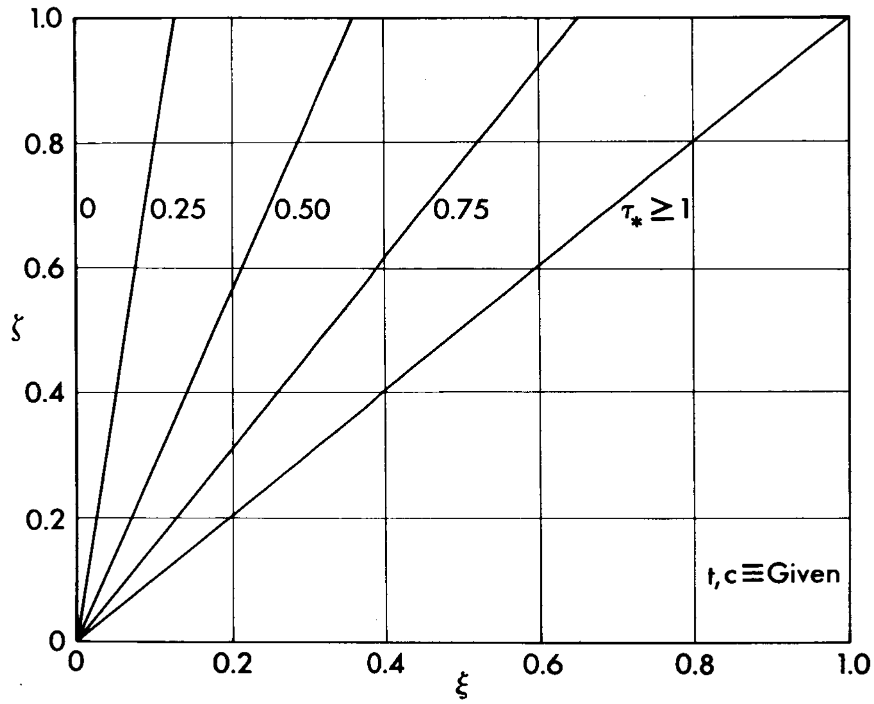

Fig. 2 Optimum shape.

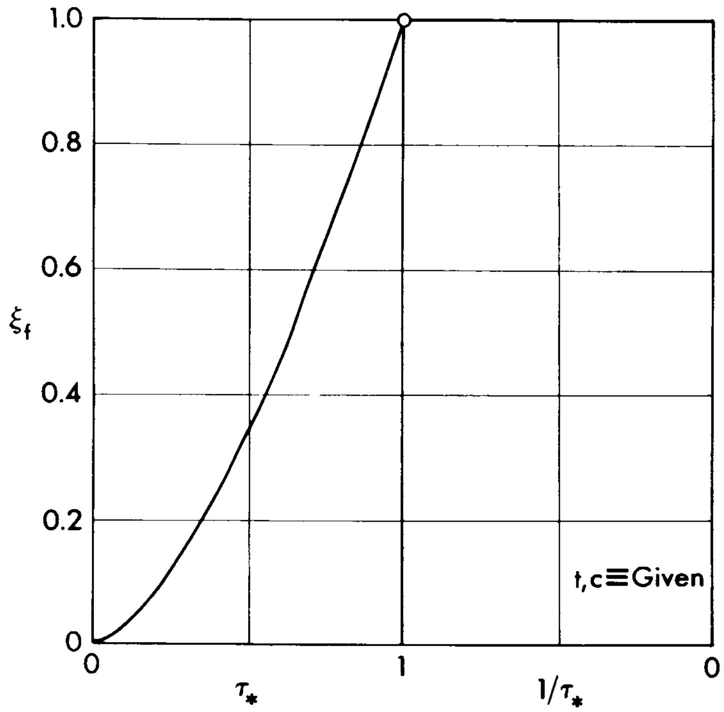

Fig. 3 Transition abscissa.

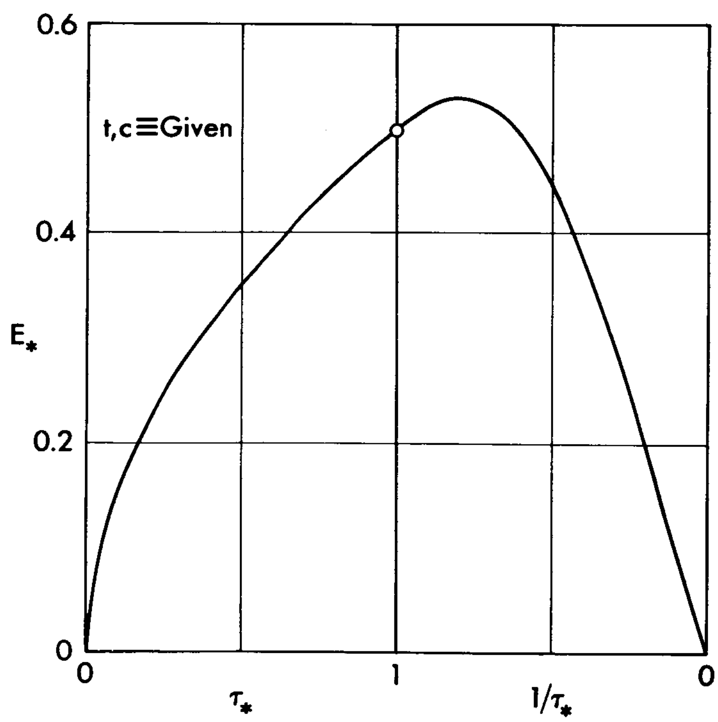

Fig. 4 Maximum lift-to-drag ratio.

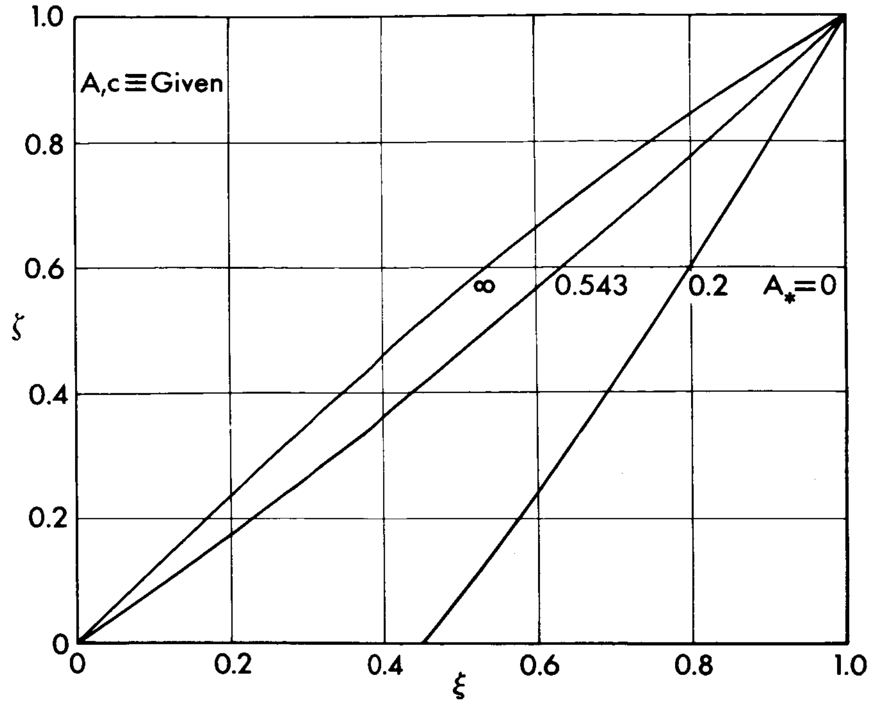

Fig. 5 Optimum shape.

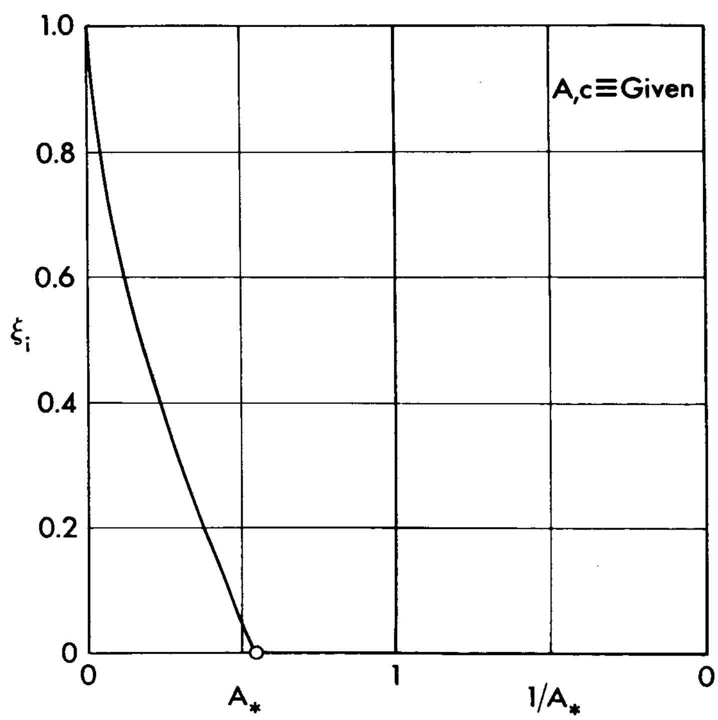

Fig. 6 Transition abscissa.

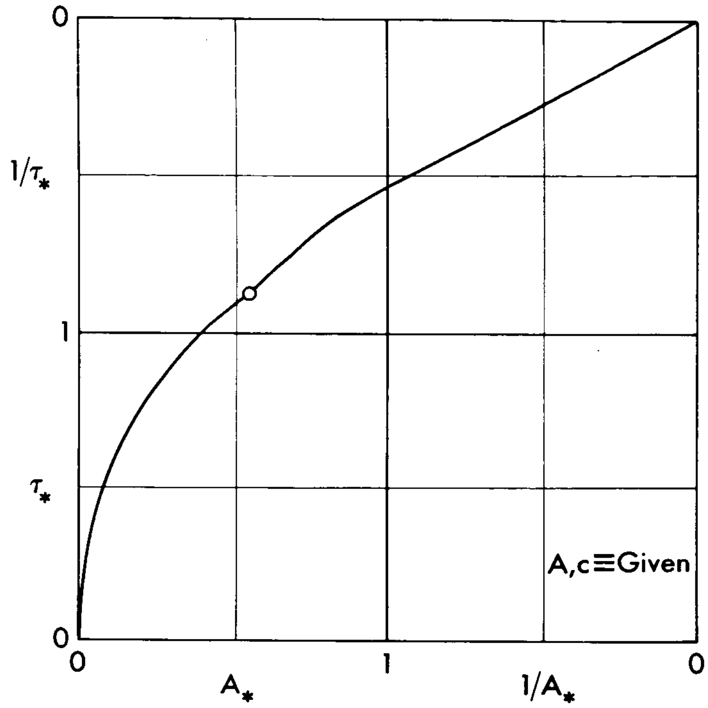

Fig. 7 Optimum thickness ratio.

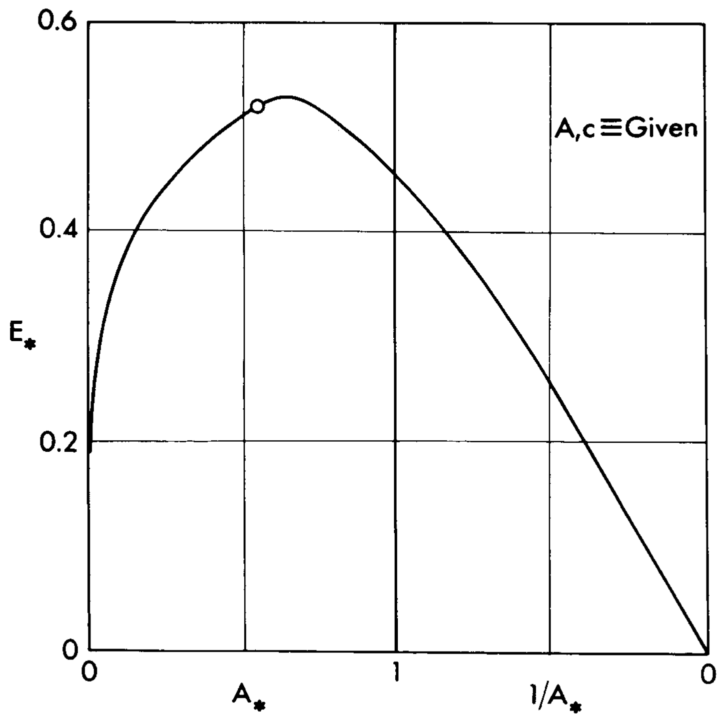

Fig. 8 Maximum lift-to-drag ratio.

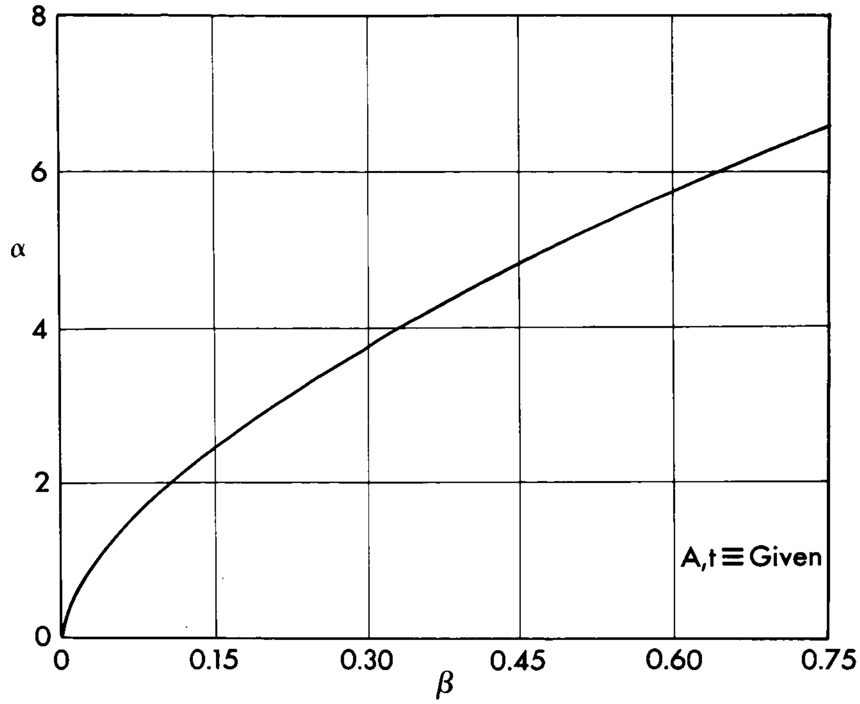

Fig. 9 Solutions of Eq. (65).

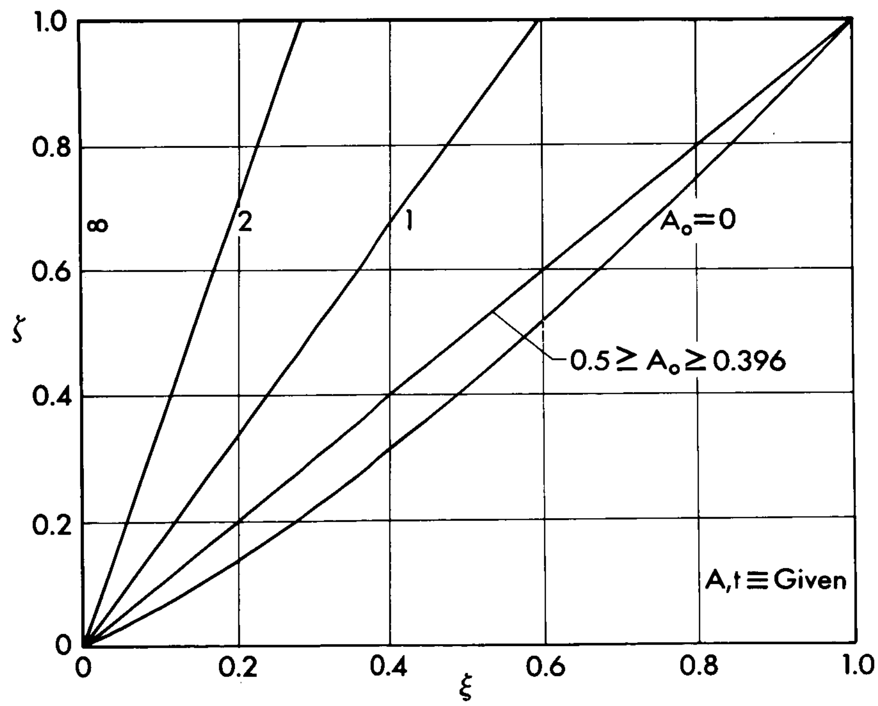

Fig. 10 Optimum shape.

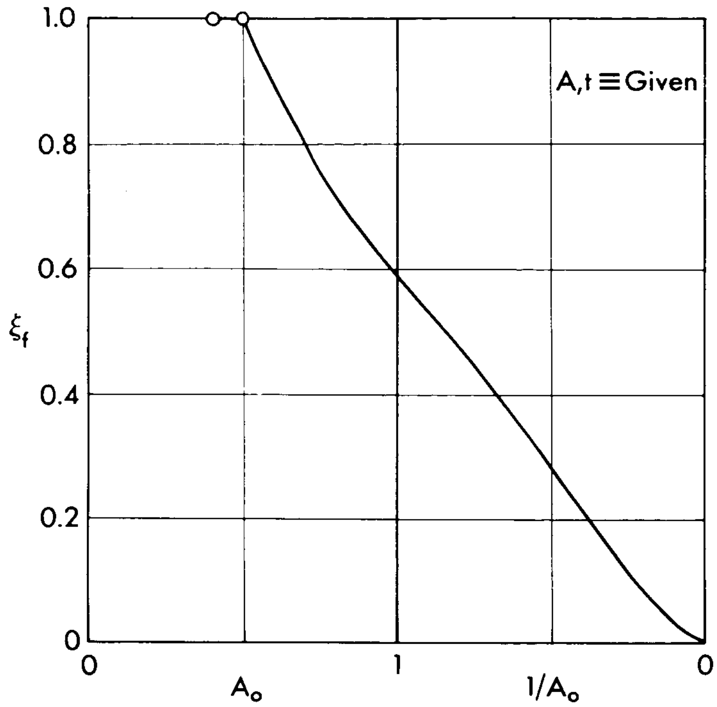

Fig. 11 Transition abscissa.

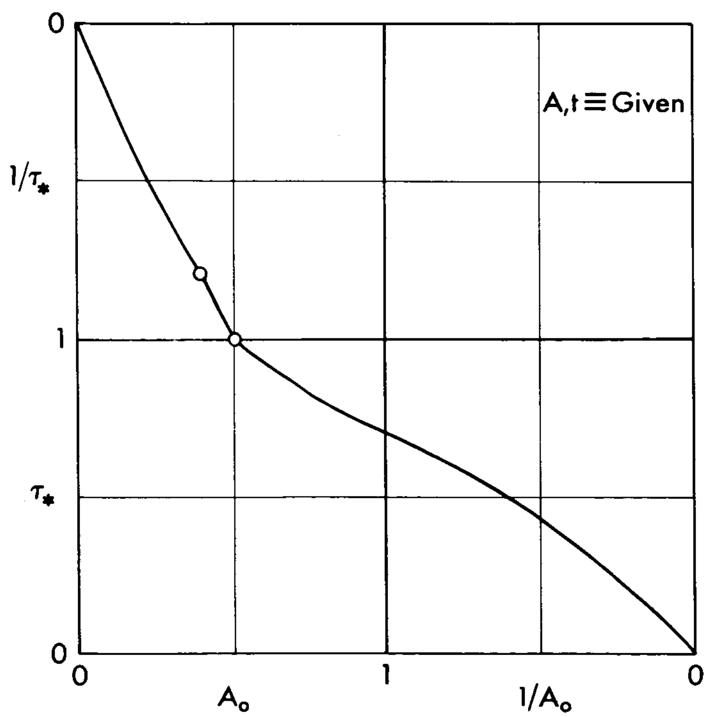

Fig. 12 Optimum thickness ratio.

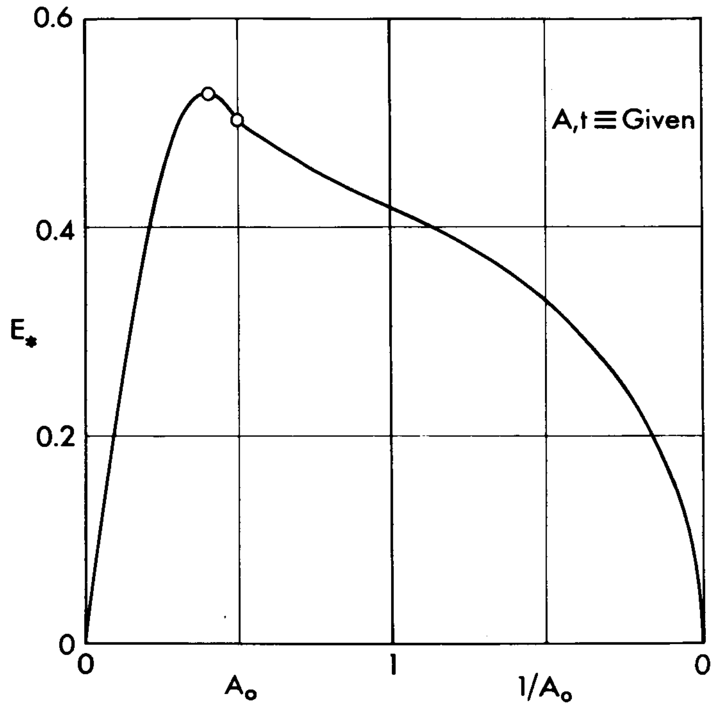

Fig. 13 Maximum lift-to-drag ratio.