X-640-66-481

NASA TM X- 55636

EXPOSITION OF SUNDMAN'S REGULARIZATION OF THE THREE BODY PROBLEM

BY

DONALD K. YEOMANS

DECEMBER 1966

NASA

GODDARD SPACE FLIGHT CENTER

GREENBELT, MARYLAND

X-640-66-481

EXPOSITION OF SUNDMAN'S REGULARIZATION OF THE THREE BODY PROBLEM

By
Donald K. Yeomans

December 1966

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

FOREWORD

The classical differential equations for the problem of three bodies remain valid only if there are no collisions or other discontinuities for real values of time. The equations of motion are not analytic when two or three of the bodies occupy coincident positions. In order to investigate collisions, the equations of motion must be made analytic by a suitable transformation of the independent variable. Once this transformation is carried out, the equations of motion are regularized. This paper is an exposition of Sundman's treatment of regularization of the three body problem. Although Sundman's work is the basis for this paper, related papers and discussions have been included. To my knowledge, this paper is the first complete exposition of Sundman's historic paper to be done in English and in vectorial form. The paper provides all developments in detail and leaves very little to be taken for granted.

EXPOSITION OF SUNDMAN'S

REGULARIZATION OF THE THREE BODY PROBLEM

by

Donald K. Yeomans

ABSTRACT

A complete exposition of Sundman's regularization of the three body problem is given. The equations of motion and the integrals of motion are derived. Double real collision is investigated and the vector joining the center of mass of the two colliding bodies and the non-participating third body is found to be bounded. The velocity and acceleration approach infinity as the distance between the two colliding bodies approaches zero. The unit vector approaches a limit near real collision. A new independent variable "u" is introduced which is seen to remove the singularity in the equation of motion for double collision. The mutual distances between the three bodies, along with the original independent variable are expanded into a power series in "u". A lower limit for the strip of convergence of these solution series is determined. Another independent variable "w" is introduced and it is seen to remove all singularities for any number of double real collisions between any of the three bodies. A lower limit is also determined for the strip of convergence of the power series solution with respect to this variable. The convergence of the power series solutions is investigated and found to be extremely slow.

ACKNOWLEDGMENTS

I would like to thank Dr. P. Musen of the Goddard Space Flight Center and University of Maryland, for unselfishly giving his time and direction throughout this project. In addition I am indebted to the Publications Section at Goddard Space Flight Center for the typing and printing of the final draft of this paper.

EXPOSITION OF SUNDMAN'S

REGULARIZATION OF THE THREE BODY PROBLEM

By
Donald K. Yeomans

Because of the interest in the material included in this paper by people working in the area of celestial mechanics in NASA, we are now making this material available as a Goddard report. We would like to thank the author, Mr. Yeomans, for permission to print this material prior to publication.

CONTENTS

EXPOSITION OF SUNDMAN'S REGULARIZATION OF THE THREE BODY PROBLEM

INTRODUCTION

The motion of the three body system is considered regular if the coordinates of the system are analytic functions of the independent variable. These equations of motion for the three body problem remain analytic as long as the mutual distances between the three bodies remain greater than zero. A singularity is encountered when two bodies collide. The series solutions for the coordinates in the three body problem, converge only in so far as there are no singularities. The main body of this paper will be concerned with the removal of the singularity brought about by collision of two bodies. This paper gives the exposition of Karl Sundman's "Memoire sur Le Probleme Des Trois Corps," although additional material and explanations are included. Sundman showed that the singularity of the differential equations which corresponds to a collision of two of the bodies is not of an essential character, and may be removed altogether by making a suitable change of the independent variable. The new independent variable is chosen in such a way that the differential equations of motion are regular and a real prolongation of the motion after collision is possible. The coordinates can then be specified for all values of time, whether collisions take place or not and a positive lower bound can be assigned to the two greater of the mutual distances. The coordinates of the three bodies, and the time are analytic functions of the new independent variable "τ" and they can be expanded as convergent series in powers of "τ" for all real values of the time.

THE EQUATIONS OF MOTION

If the masses of the three bodies are designated m_0, m_1 and m_2 and their position vectors from an arbitrary point "0" given by $\vec{r}_0$, $\vec{r}_1$ and $\vec{r}_2$, the equations of motion are written:

$$m_0 \frac{d^2 \vec{r}_0}{dt^2} = \frac{f\, m_0 m_1 (\vec{r}_1 - \vec{r}_0)}{\Delta_{01}^3} + \frac{f\, m_0 m_2 (\vec{r}_2 - \vec{r}_0)}{\Delta_{02}^3}$$

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \frac{f\, m_1 m_0 (\vec{r}_0 - \vec{r}_1)}{\Delta_{01}^3} + \frac{f\, m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{\Delta_{12}^3}$$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = \frac{f\, m_2 m_0 (\vec{r}_0 - \vec{r}_2)}{\Delta_{02}^3} + \frac{f\, m_2 m_1 (\vec{r}_1 - \vec{r}_2)}{\Delta_{12}^3} \quad (1)$$

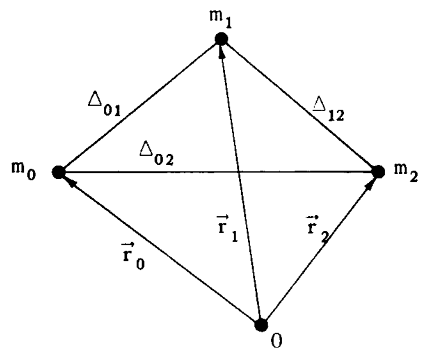

Figure 1

where Δ_{01}, Δ_{02}, Δ_{12} refer to distances between the masses. Adding Equations 1 we have;

$$m_0 \frac{d^2 \vec{r}_0}{dt^2} + m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} = 0 \quad (2)$$

Integrating we have;

$$m_0 \frac{d\vec{r}_0}{dt} + m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} = \vec{A} \tag{3}$$

$\vec{A}$ is a constant vector which provides three integrals of motion. Integrating once again we have;

$$m_0 \vec{r}_0 + m_1 \vec{r}_1 + m_2 \vec{r}_2 = \vec{A}t + \vec{B} \tag{4}$$

$\vec{B}$ is a constant vector. We now define $\vec{R}$ as the position vector of the center of mass of the system, where

$$\vec{R} = \frac{m_0 \vec{r}_0 + m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_0 + m_1 + m_2} \tag{5}$$

From Equation 3 we have

$$\frac{d\vec{R}}{dt} = \frac{\vec{A}}{M} \qquad M = m_0 + m_1 + m_2 \tag{6}$$

and from Equation 4 we have

$$\vec{R} = \frac{\vec{A}t + \vec{B}}{M} \tag{7}$$

From Equations 6 and 7 we see that the center of mass of the system moves in a straight line with constant velocity (i.e., inertial system). Switching to the center of mass system we have

$$\vec{R} = 0 \qquad \frac{d\vec{R}}{dt} = 0 \qquad \begin{array}{c} \vec{A} = \vec{B} = 0 \\ m_0 \vec{r}_0 + m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \end{array} \tag{8}$$

If we define

$$U = \frac{f\, m_0\, m_1}{\Delta_{01}} + \frac{f\, m_0\, m_2}{\Delta_{02}} + \frac{f\, m_1\, m_2}{\Delta_{12}} \tag{9}$$

We can rewrite Equations 1 as

$$m_0 \frac{d^2 \vec{r}_0}{dt^2} = \text{grad}_{\vec{r}_0} U$$

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \text{grad}_{\vec{r}_1} U$$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = \text{grad}_{\vec{r}_2} U \tag{10}$$

dot multiplying the first Equation in 10 by $d\vec{r}_0$, the second by $d\vec{r}_1$ and the third by $d\vec{r}_2$ we have,

$$m_0 \vec{v}_0 \cdot d\vec{v}_0 + m_1 \vec{v}_1 \cdot d\vec{v}_1 + m_2 \vec{v}_2 \cdot d\vec{v}_2 = d\vec{r}_0 \cdot \text{grad}_{r_0} U + d\vec{r}_1 \cdot \text{grad}_{r_1} U + d\vec{r}_2 \cdot \text{grad}_{r_2} U \tag{11}$$

or

$$\frac{1}{2} d\left[m_0 \vec{v}_0^2 + m_1 \vec{v}_1^2 + m_2 \vec{v}_2^2\right] = \frac{dU}{d\vec{r}_0} \cdot d\vec{r}_0 + \frac{dU}{d\vec{r}_1} \cdot d\vec{r}_1 + \frac{dU}{d\vec{r}_2} \cdot d\vec{r}_2 = dU \tag{12}$$

finally we have by integration the energy integral

$$\frac{1}{2} \left[m_0 \vec{v}_0^2 + m_1 \vec{v}_1^2 + m_2 \vec{v}_2^2\right] = U + \alpha \tag{13}$$

where α = constant. Cross multiplying the first Equation in 1 by $\vec{r}_0$, the second by $\vec{r}_1$ and the third by $\vec{r}_2$ we write,

$$\vec{r}_0 \times m_0 \frac{d\vec{v}_0}{dt} = \frac{f\, m_1 m_0}{\Delta_{01}^3} \vec{r}_0 \times \vec{r}_1 + \frac{f\, m_0 m_2 \left(\vec{r}_0 \times \vec{r}_2\right)}{\Delta_{02}^3}$$

$$\vec{r}_1 \times m_1 \frac{d\vec{v}_1}{dt} = \frac{f\, m_1 m_0}{\Delta_{01}^3} \vec{r}_1 \times \vec{r}_0 + \frac{f\, m_1 m_2\, \vec{r}_1 \times \vec{r}_2}{\Delta_{12}^3}$$

$$\vec{r}_2 \times m_2 \frac{d\vec{v}_2}{dt} = \frac{f\, m_2 m_0}{\Delta_{02}^3} \vec{r}_2 \times \vec{r}_0 + \frac{f\, m_2 m_1\, \vec{r}_2 \times \vec{r}_1}{\Delta_{12}^3} \tag{14}$$

adding

$$m_0 \vec{r}_0 \times \frac{d\vec{v}_0}{dt} + m_1 \vec{r}_1 \times \frac{d\vec{v}_1}{dt} + m_2 \vec{r}_2 + \frac{d\vec{v}_2}{dt} = 0 \tag{15}$$

Since $\vec{r} \times d\vec{v}/dt = d/dt\,(\vec{r} \times \vec{v})$ we can write 15 as

$$m_0 \frac{d}{dt}\left(\vec{r}_0 \times \vec{v}_0\right) + m_1 \frac{d}{dt}\left(\vec{r}_1 \times \vec{v}_1\right) + m_2 \frac{d}{dt}\left(\vec{r}_2 \times \vec{v}_2\right) = 0 \tag{16}$$

Integrating we get

$$m_0\left(\vec{r}_0 \times \vec{v}_0\right) + m_1\left(\vec{r}_1 \times \vec{v}_1\right) + m_2\left(\vec{r}_2 \times \vec{v}_2\right) = \vec{c} \tag{17}$$

Rewriting Equations 8, 13 and 17 the ten integrals of motion are

$$\frac{1}{2}\left(m_0 \vec{v}_0^{\,2} + m_1 \vec{v}_1^{\,2} + m_2 \vec{v}_2^{\,2}\right) = U + \alpha$$

$$m_0\left(\vec{r}_0 \times \vec{v}_0\right) + m_1\left(\vec{r}_1 \times \vec{v}_1\right) + m_2\left(\vec{r}_2 \times \vec{v}_2\right) = \vec{c}$$

$$m_0 \vec{r}_0 + m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

$$m_0 \vec{v}_0 + m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0 \tag{18}$$

The first equation is the integral of energy, the second equation (3 integrals) is the angular momentum integral and the last two equations (6 integrals) are referred to as the center of mass integrals.

It is evident that our problem consists of nine equations in 14 of second order and hence 18 degrees of freedom. With the corresponding 10 integrals of motion, there remains 8 integrals for a solution to the problem. These integrals are not known. However a reduction of the problem can be accomplished by using the last two equations in 18. This simplification reduces the degrees of freedom to 12. This reduction is accredited to Jacobi and is described below.

JACOBI'S REDUCTION

In the diagram (Figure 2) K is the center of mass of m_0 and m_1 while s represents the center of mass of the three body system. Our goal is to derive the equations of motion in terms of the two vectors $\vec{r}$ and $\vec{\rho}$, thus reducing the system to 12 degrees of freedom.

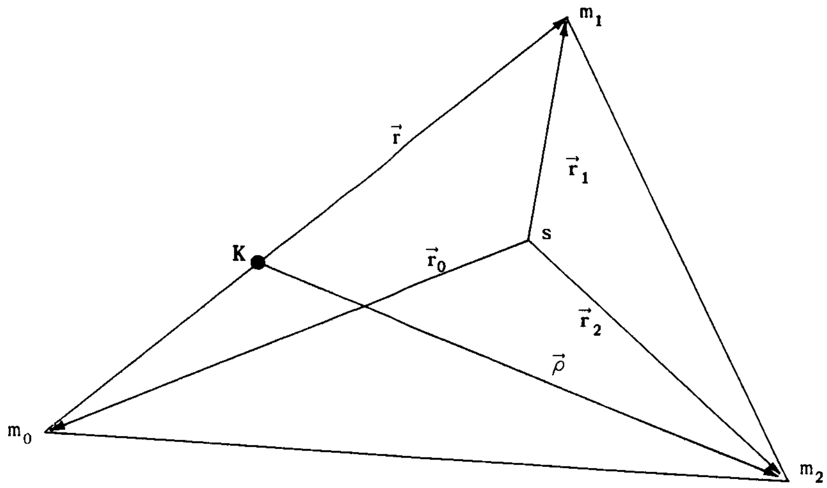

Figure 2

From the diagram:

$$\vec{r}_0 + \vec{r} = \vec{r}_1$$

$$\frac{m_1}{m_0 + m_1}\vec{r} + \vec{\rho} = \vec{r}_2 - \vec{r}_0 \qquad (19)$$

$$\vec{r}_1 - \vec{r}_0 = \vec{r}$$

$$\vec{r}_2 - \vec{r}_0 = \lambda\vec{r} + \vec{\rho}$$

where

$$\lambda = \frac{m_1}{m_0 + m_1}$$

$$m_0 \vec{r}_0 + m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \qquad (20)$$

$$\vec{r}_1 = \vec{r} + \vec{r}_0$$

$$\vec{r}_2 = \lambda\vec{r} + \vec{\rho} + \vec{r}_0$$

$$m_0 \vec{r}_0 + m_1 \left(\vec{r} + \vec{r}_0\right) + m_2 \left(\lambda\vec{r} + \vec{\rho} + \vec{r}_0\right) = 0$$

recombining terms

$$\vec{r}_0 \left(m_0 + m_1 + m_2\right) + m_1 \vec{r} + m_2 (\lambda \vec{r} + \vec{\rho}) = 0$$

or

$$\vec{r}_0 = - \frac{\vec{r}\left(m_1 + \lambda m_2\right) + \vec{\rho} m_2}{M}$$

where

$$M = m_1 + m_2 + m_3$$

$$r'_0 = - \frac{\vec{r}\left(m_1 + \frac{m_1 m_2}{m_0 + m_1}\right) + \vec{\rho} m_2}{M}$$

$$\vec{r}_0 = - \frac{\vec{r}\left[\frac{m_1 \left(m_0 + m_1 + m_2\right)}{m_0 + m_1}\right] + \vec{\rho} m_2}{M} \tag{21}$$

finally

$$\vec{r}_0 = - \vec{r}\lambda - \frac{m_2}{M} \vec{\rho} \tag{22}$$

Once again we write

$$\vec{r}_1 = \vec{r} + \vec{r}_0$$

Substituting Equation 22 for $\vec{r}_0$

$$\vec{r}_1 = \vec{r} - \lambda \vec{r} - \frac{m_2}{M} \vec{\rho}$$

$$\vec{r}_1 = \vec{r} - \frac{m_1}{m_0 + m_1} \vec{r} - \frac{m_2}{M} \vec{\rho} = \vec{r} \left[1 - \frac{m_1}{m_0 + m_1}\right] - \frac{m_2}{M} \vec{\rho} = \frac{m_0}{m_0 + m_1} \vec{r} - \frac{m_2 \vec{\rho}}{M}$$

letting

$$\frac{m_0}{m_0 + m_1} = \mu \tag{23}$$

$$\vec{r}_1 = \mu\vec{r} - \frac{m_2}{M}\vec{\rho} \tag{24}$$

Rewriting Equation 20

$$\vec{r}_2 = \lambda\vec{r} + \vec{\rho} + \vec{r}_0$$

Using Equation 22 we have

$$\vec{r}_2 = \lambda\vec{r} + \vec{\rho} - \lambda\vec{r} - \frac{m_2}{M}\vec{\rho} = \frac{m_0 + m_1}{M}\vec{\rho}$$

$$\vec{r}_2 = \frac{m_0 + m_1}{M}\vec{\rho} \tag{25}$$

rewriting Equation 25

$$\vec{\rho} = \frac{M}{m_0 + m_1}\vec{r}_2 \tag{26}$$

If we define

$$g = \frac{M}{m_2(m_0 + m_1)} \tag{27}$$

we have from 26

$$\vec{\rho} = g\, m_2\, \vec{r}_2 \tag{28}$$

Collecting formulas and definitions, remember that;

$$\vec{r}_0 = -\lambda\vec{r} - \frac{m_2}{M}\vec{\rho} \qquad \vec{r} = \vec{r}_1 - \vec{r}_0 \qquad \lambda = \frac{m_1}{m_1 + m_0}$$

$$\vec{r}_1 = \mu\vec{r} - \frac{m_2}{M}\vec{\rho} \qquad \vec{\rho} = g\, m_2 \vec{r}_2 \qquad \mu = \frac{m_0}{m_1 + m_0}$$

$$\vec{r}_2 = \frac{m_0 + m_1}{M}\vec{\rho} \qquad g = \frac{M}{m_2\left(m_0 + m_1\right)}$$

At this point Sundman changes notation. The distances between the bodies are denoted

$$\Delta_{01} = \left|\vec{r}_1 - \vec{r}_0\right| = r_2 = r$$

$$\Delta_{02} = \left|\vec{r}_2 - \vec{r}_0\right| = r_1 = \left|\vec{\rho} + \lambda\vec{r}\right|$$

$$\Delta_{12} = \left|\vec{r}_2 - \vec{r}_1\right| = r_0 = \left|\vec{\rho} - \mu\vec{r}\right| \tag{28'}$$

where the distance between m_1 and m_2 is denoted with the missing index (r_2). It should be emphasized that r_2 is not the absolute value of the vector $\vec{r}_2$. The notation r_2 and r are used interchangeably depending upon the circumstance.

Writing the equations of motion (1) using the new notation we have:

$$\frac{d^2\vec{r}_0}{dt^2} = \frac{f\, m_1\left(\vec{r}_1 - \vec{r}_0\right)}{r_2^3} + \frac{f\, m_2\left(\vec{r}_2 - \vec{r}_0\right)}{r_1^3}$$

$$\frac{d^2\vec{r}_1}{dt^2} = \frac{f\, m_0\left(\vec{r}_0 - \vec{r}_1\right)}{r_2^3} + \frac{f\, m_2\left(\vec{r}_2 - \vec{r}_1\right)}{r_0^3}$$

$$\frac{d^2\vec{r}_2}{dt^2} = \frac{f\, m_0\left(\vec{r}_0 - \vec{r}_2\right)}{r_1^3} + \frac{f\, m_1\left(\vec{r}_2 - \vec{r}_1\right)}{r_0^3} \tag{29}$$

We write for convenience,

$$\vec{r}_2 - \vec{r}_0 = \lambda\vec{r} + \frac{m_2}{M}\vec{\rho} + \frac{m_0 + m_1}{M}\vec{\rho} = \vec{\rho} + \lambda\vec{r} \tag{30}$$

$$\vec{r}_2 - \vec{r}_1 = \frac{m_0 + m_1}{M}\vec{\rho} - \mu\vec{r} + \frac{m_2}{M}\vec{\rho} = \vec{\rho} - \mu\vec{r} \tag{31}$$

$$\vec{r}_1 - \vec{r}_0 = \vec{r} \qquad r^2 = x^2 + y^2 + z^2 \tag{32}$$

Substracting the first Equation in 29 from the second and using 32 we write

$$\frac{d^2 \vec{r}}{dt^2} = -\frac{f m_0 \vec{r}}{r_2^3} - \frac{f m_1 \vec{r}}{r_2^3} + f m_2 \left[\frac{(\vec{r}_2 - \vec{r}_1)}{r_0^3} - \frac{(\vec{r}_2 - \vec{r}_0)}{r_1^3}\right]$$

$$= -f(m_0 + m_1)\frac{\vec{r}}{r^3} + f m_2 \left[\frac{\vec{\rho} - \mu\vec{r}}{r_0^3} - \frac{(\vec{\rho} + \lambda\vec{r})}{r_1^3}\right]$$

So that

$$\frac{d^2 r}{dt^2} = -f(m_0 + m_1)\frac{\vec{r}}{r^3} + f m_2 \left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\vec{\rho} - f m_2 \left(\frac{\lambda}{r_1^3} + \frac{\mu}{r_0^3}\right)\vec{r} \tag{33}$$

By Equation 26

$$\vec{\rho} = \frac{M}{m_0 + m_1}\vec{r}_2$$

differentiating we have

$$\frac{d^2 \vec{\rho}}{dt^2} = \frac{M}{m_0 + m_1}\frac{d^2 \vec{r}_2}{dt^2}$$

Substituting in the third equation in 29 and using 30, 31

$$\frac{d^2 \vec{\rho}}{dt^2} = \frac{M}{m_0 + m_1}\frac{d^2 \vec{r}_2}{dt^2} = \frac{M}{m_0 + m_1}\left[\frac{f m_0(-\vec{\rho} - \lambda\vec{r})}{r_1^3} + \frac{f m_1(-\vec{\rho} + \mu\vec{r})}{r_0^3}\right]$$

$$= f M\left[-\frac{\mu(\vec{\rho} + \lambda\vec{r})}{r_1^3} + \frac{\lambda(-\vec{\rho} + \mu\vec{r})}{r_0^3}\right]$$

$$= f M\left[-\vec{\rho}\left(\frac{\mu}{r_1^3} + \frac{\lambda}{r_0^3}\right) + \vec{r}\left(-\frac{\mu\lambda}{r_1^3} + \frac{\mu\lambda}{r_0^3}\right)\right]$$

$$\frac{d^2 \vec{\rho}}{dt^2} = f M\left[-\vec{\rho}\left(\frac{\mu}{r_1^3} + \frac{\lambda}{r_0^3}\right) + \mu\lambda\vec{r}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\right] \tag{34}$$

Rewriting Equations 33, 34 we have

$$\frac{d^2 \vec{r}}{dt^2} = -f(m_0 + m_1)\frac{\vec{r}}{r^3} + f m_2 \vec{\rho}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right) - f m_2 \vec{r}\left(\frac{\lambda}{r_1^3} + \frac{\mu}{r_0^3}\right)$$

$$\frac{d^2 \vec{\rho}}{dt^2} = fM\left[-\vec{\rho}\left(\frac{\mu}{r_1^3} + \frac{\lambda}{r_0^3}\right) + \mu\lambda\vec{r}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\right] \tag{35}$$

Looking at Equations 35 we have completed Jacobi's reduction. We now have 6 Equations of the second order for 12 degrees of freedom. Since we used the 6 integrals of the center of mass for the above reduction, we are left with 4 known integrals.

We note that in a collision of masses m_1 and m_0, r goes to zero while r_1 and r_0 remain relatively large and in fact equal to each other. This can be seen by referring to Figure 3.

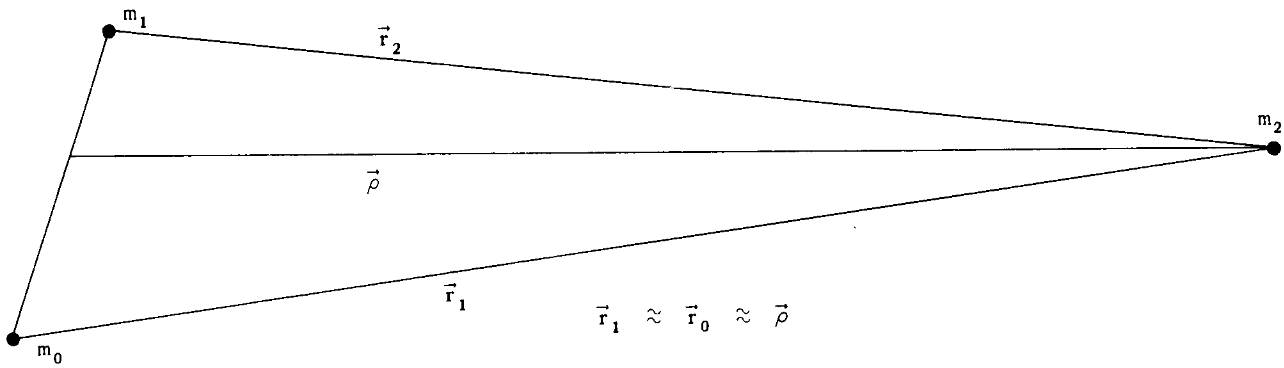

Figure 3

With these approximations the second Equation in 35 can be written

$$\frac{d^2 \vec{\rho}}{dt^2} = \frac{-f M\vec{\rho}}{\rho^3} \tag{36}$$

With this approximation the problem is reduced to the two body problem.

AREA INTEGRAL

The area integral will now be derived. From Equation 18 we rewrite

$$m_0 \vec{r}_0 \times \frac{d\vec{r}_0}{dt} + m_1 \vec{r}_1 \times \frac{d\vec{r}_1}{dt} + m_2 \vec{r}_2 \times \frac{d\vec{r}_2}{dt} = \vec{c}$$

Using Equations 22, 24, and 25

$$m_0\left(-\lambda\vec{r} - \frac{m_2}{M}\vec{\rho}\right)\times\frac{d}{dt}\left(-\lambda\vec{r} - \frac{m_2}{M}\vec{\rho}\right) + m_1\left(\mu\vec{r} - \frac{m_2}{M}\vec{\rho}\right)\times\frac{d}{dt}\left(\mu\vec{r} - \frac{m_2}{M}\vec{\rho}\right) + m_2\left(\frac{m_0+m_1}{M}\vec{\rho}\right)\times\frac{d}{dt}\left(\frac{m_0+m_1}{M}\vec{\rho}\right) = \vec{c}$$

$$m_0\lambda^2\vec{r}\times\frac{d\vec{r}}{dt} + \lambda\frac{m_0 m_2}{M}\vec{\rho}\times\frac{d\vec{r}}{dt} + \lambda\frac{m_0 m_2}{M}\vec{r}\times\frac{d\vec{\rho}}{dt} + \frac{m_2^2}{M^2}m_0\vec{\rho}\times\frac{d\vec{\rho}}{dt} + m_1\mu^2\vec{r}\times\frac{d\vec{r}}{dt} - \frac{m_1 m_2}{M}\mu\vec{\rho}\times\frac{d\vec{r}}{dt}$$

$$-\frac{m_1 m_2}{M}\mu\vec{r}\times\frac{d\vec{\rho}}{dt} + \frac{m_1 m_2^2}{M^2}\vec{\rho}\times\frac{d\vec{\rho}}{dt} + \frac{(m_0+m_1)^2}{M^2}m_2\vec{\rho}\times\frac{d\vec{\rho}}{dt} = \vec{c}$$

collecting terms

$$\vec{r}\times\frac{d\vec{r}}{dt}\left[m_0\lambda^2 + m_1\mu^2\right] + \vec{\rho}\times\frac{d\vec{\rho}}{dt}\left[\frac{m_2^2 m_0}{M^2} + \frac{m_1 m_2^2}{M^2} + \frac{(m_0+m_1)^2 m_2}{M^2}\right]$$

$$+\vec{\rho}\times\frac{d\vec{r}}{dt}\left[\frac{\lambda m_0 m_2}{M} - \frac{\mu m_1 m_2}{M}\right] + \vec{r}\times\frac{d\vec{\rho}}{dt}\left[\frac{\lambda m_0 m_2}{M} - \frac{\mu m_1 m_2}{M}\right] = \vec{c}$$

From Equations 20, 23 the last two terms are zero and

$$\vec{r}\times\frac{d\vec{r}}{dt}\left(m_0\lambda^2 + m_1\mu^2\right) + \vec{\rho}\times\frac{d\vec{\rho}}{dt}\left[\frac{m_2^2 m_0}{M^2} + \frac{m_1 m_2^2}{M^2} + \frac{(m_0+m_1)^2 m_2}{M^2}\right] = \vec{c} \tag{37}$$

now using 20 and 23 again

$$m_0\lambda^2 + m_1\mu^2 = \frac{m_0 m_1^2 + m_1 m_0^2}{(m_1+m_0)^2} = \frac{m_0 m_1(m_1+m_0)}{(m_1+m_0)^2} = \frac{m_0 m_1}{m_1+m_0} \tag{38}$$

and

$$\frac{m_2^2 m_0}{M^2} + \frac{m_1 m_2^2}{M^2} + \frac{(m_0+m_1)^2}{M^2}m_2 = \frac{m_2^2 m_0 + m_1 m_2^2 + m_2 m_0^2 + 2m_0 m_1 m_2 + m_2 m_1^2}{M^2}$$

$$= \frac{m_2(m_1+m_2+m_3)(m_0+m_1)}{M^2} = \frac{m_2(m_0+m_1)}{M} \tag{39}$$

If we define

$$h = \frac{m_0 + m_1}{m_0 m_1} \tag{40}$$

and from 27

$$g = \frac{M}{m_2 (m_0 + m_1)}$$

Using this notation, Equation 37 may be written

$$g\vec{r} \times \frac{d\vec{r}}{dt} + h\vec{\rho} \times \frac{d\vec{\rho}}{dt} = gh\vec{c} \qquad \text{Area integral} \tag{41}$$

ENERGY INTEGRAL

To derive the energy integral we start with Equation 13

$$\frac{1}{2}\left[m_0\left(\frac{d\vec{r}_0}{dt}\right)^2 + m_1\left(\frac{d\vec{r}_1}{dt}\right)^2 + m_2\left(\frac{d\vec{r}_2}{dt}\right)^2\right] = U + \alpha \tag{42}$$

where now from Equation 9

$$U = f\left(\frac{m_0 m_1}{r_2} + \frac{m_1 m_2}{r_0} + \frac{m_0 m_2}{r_1}\right) \tag{43}$$

Substituting Equations 22, 24, 25 into 42 we write

$$\frac{1}{2}\left[m_0\left(-\lambda\frac{d\vec{r}}{dt} - \frac{m_2}{M}\frac{d\vec{\rho}}{dt}\right)^2 + m_1\left(\mu\frac{d\vec{r}}{dt} - \frac{m_2}{M}\frac{d\vec{\rho}}{dt}\right)^2 + m_2\left(\frac{m_0 + m_1}{M}\frac{d\vec{\rho}}{dt}\right)^2\right] = U + \alpha$$

squaring the appropriate terms we have

$$\frac{1}{2}\left\{m_0\left[\lambda^2\left(\frac{d\vec{r}}{dt}\right)^2 + \frac{2\lambda m_2}{M}\frac{d\vec{r}}{dt}\frac{d\vec{\rho}}{dt} + \frac{m_2^2}{M}\left(\frac{d\vec{\rho}}{dt}\right)^2\right] + m_1\left[\mu^2\left(\frac{d\vec{r}}{dt}\right)^2 - \frac{2\mu m_2}{M}\frac{d\vec{r}}{dt}\frac{d\vec{\rho}}{dt} + \frac{m_2^2}{M^2}\left(\frac{d\vec{\rho}}{dt}\right)^2\right] + m_2\left[\frac{(m_0 + m_1)^2}{M^2}\left(\frac{d\vec{\rho}}{dt}\right)^2\right]\right\} = U + \alpha$$

Collecting terms

$$\frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2 \left(m_0 \lambda^2 + m_1 \mu^2\right) + \frac{1}{2}\left(\frac{d\vec{\rho}}{dt}\right)^2 \left(\frac{m_0 m_2^2}{M^2} + \frac{m_1 m_2^2}{M^2} + \frac{m_2 (m_0 + m_1)^2}{M^2}\right)$$

$$+ \frac{1}{2} \frac{d\vec{r}}{dt} \frac{d\vec{\rho}}{dt} \left(\frac{2m_0 \lambda m_2}{M} - \frac{2m_1 \mu m_2}{M}\right) = U ;$$

from Equations 20, 23, the last term on the left is zero and using Equations 38, 39, 40 and 27 we have

$$\frac{1}{2} g\left(\frac{d\vec{r}}{dt}\right)^2 + \frac{1}{2} h\left(\frac{d\vec{\rho}}{dt}\right)^2 = ghU + \alpha'$$

if we define

$$hgU = V \tag{44}$$

we have

$$g\left(\frac{d\vec{r}}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 = 2V - K \qquad \text{Energy integral} \tag{45}$$

The next few pages will be devoted to deriving some equations which will be useful in investigating the motion of the system. Differentiating Equation 45

$$2g \frac{d\vec{r}}{dt} \cdot \frac{d^2 \vec{r}}{dt^2} + 2h \frac{d\vec{\rho}}{dt} \cdot \frac{d^2 \vec{\rho}}{dt^2} = 2\frac{dV}{dr_2} \cdot \frac{dr_2}{dt} + 2 \frac{dV}{dr_1} \frac{dr_1}{dt} + 2 \frac{dV}{dr_0} \cdot \frac{dr_0}{dt}$$

this can be represented vectorially as

$$\left[v_r \hat{r} + v_\rho \hat{\rho}\right] \cdot \left[g \frac{d^2 \vec{r}}{dt^2} \hat{r} + h \frac{d^2 \vec{\rho}}{dt^2} \hat{\rho}\right] = \left(v_2 \hat{i} + v_1 \hat{j} + v_0 \hat{k}\right) \cdot \left(\hat{i} \frac{dV}{dr_2} + \hat{j} \frac{dV}{dr_1} + \hat{k} \frac{dV}{dr_0}\right)$$

or

$$\vec{v} \cdot \left[g \frac{d^2 \vec{r}}{dt^2} + h \frac{d^2 \vec{\rho}}{dt^2}\right] = \vec{v} \cdot \text{grad} V$$

we have

$$g\frac{d^2\vec{r}}{dt^2} = \text{grad}_{\vec{r}}V \qquad h\frac{d^2\vec{\rho}}{dt^2} = \text{grad}_{\vec{\rho}}V \tag{46}$$

The Euler Equation for a homogeneous function of the order n

$$\sum_i x_i \frac{\partial f}{\partial x_i} = mf$$

where

$$f = f(x_1, x_2, \cdots x_n)$$

Since V is a homogeneous function of order (-1)

$$\vec{r}\cdot\frac{\partial V}{d\vec{r}} + \vec{\rho}\cdot\frac{\partial V}{\partial\vec{\rho}} = (-1)V$$

from Equations 46 we have

$$g\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} + h\vec{\rho}\cdot\frac{d^2\vec{\rho}}{dt^2} = -V \tag{47}$$

adding Equations 45 and 47 we have

$$g\left[\left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\right] + h\left[\left(\frac{d\vec{\rho}}{dt}\right)^2 + \vec{\rho}\cdot\frac{d^2\vec{\rho}}{dt^2}\right] = V - K \tag{48}$$

which is equivalent to,

$$\frac{g}{2}\left[\frac{d^2(\vec{r}^2)}{dt^2}\right] + \frac{h}{2}\left[\frac{d^2(\vec{\rho}^2)}{dt^2}\right] = V - K$$

or

$$\frac{d^2}{dt^2}\left(g\vec{r}^2 + h\vec{\rho}^2\right) = 2(V - K) \tag{49}$$

If we define

$$R^2 = g\vec{r}^2 + h\vec{\rho}^2 \tag{50}$$

then from Equation 49

$$\frac{d^2 R^2}{dt^2} = 2(V - K) \tag{51}$$

Now we assert that

$$R^2 = \frac{r_0^2}{m_0} + \frac{r_1^2}{m_1} + \frac{r_2^2}{m_2} \tag{52}$$

where $r_0 = \Delta_{12}$, $r_1 = \Delta_{02}$, $r_2 = \Delta_{01}$ Equation 52 is justified if it can be reduced to Equation 50; using Equation 28' we have

$$R^2 = \frac{|\vec{\rho} - \mu\vec{r}|^2}{m_0} + \frac{|\vec{\rho} + \lambda\vec{r}|^2}{m_1} + \frac{|\vec{r}|^2}{m_2}$$

$$R^2 = \frac{\rho^2 - 2\mu\vec{\rho}\cdot\vec{r} + \mu^2 r^2}{m_0} + \frac{\rho^2 + 2\lambda\vec{\rho}\cdot\vec{r} + \lambda^2 r^2}{m_1} + \frac{r^2}{m_2}$$

$$R^2 = \rho^2\left[\frac{1}{m_0} + \frac{1}{m_1}\right] + r^2\left[\frac{\mu^2}{m_0} + \frac{\lambda^2}{m_1} + \frac{1}{m_2}\right] + \vec{\rho}\cdot\vec{r}\left[\frac{2\lambda}{m_1} - \frac{2\mu}{m_0}\right]$$

by Equations 20, 23 the last bracket is zero and

$$R^2 = \rho^2\left[\frac{m_1 + m_0}{m_1 m_0}\right] + r^2\left[\frac{m_0 + m_1}{(m_0 + m_1)^2} + \frac{1}{m_2}\right]$$

or

$$R^2 = \rho^2\left[\frac{m_1 + m_0}{m_1 m_0}\right] + r^2\left[\frac{m_2 + m_0 + m_1}{(m_0 + m_1)\, m_2}\right]$$

and using Equations 27, 40 we have $R^2 = gr^2 + h\rho^2$ which is Equation 50 and assertion 52 is justified. From Equation 51

$$\frac{d}{dt}\left[2R\,\frac{dR}{dt}\right] = 2R\,\frac{d^2R}{dt} + 2\left(\frac{dR}{dt}\right)^2 = 2(V-K)$$

and

$$R\,\frac{d^2R}{dt^2} + \left(\frac{dR}{dt}\right)^2 = V - K \tag{53}$$

We now differentiate R^2 (Equation 50) to get

$$R\,\frac{dR}{dt} = g\vec{r}\cdot\frac{d\vec{r}}{dt} + h\vec{\rho}\cdot\frac{d\vec{\rho}}{dt} \tag{54}$$

Squaring Equation 54

$$R^2\left(\frac{dR}{dt}\right)^2 = g^2r^2\left(\frac{dr}{dt}\right)^2 + h^2\rho^2\left(\frac{d\rho}{dt}\right)^2 + 2hg\vec{r}\cdot\vec{\rho}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d\vec{\rho}}{dt} \tag{55}$$

We now define

$$\underline{P} \equiv \frac{gh}{R^2}\left(r\,\frac{d\rho}{dt} - \rho\,\frac{dr}{dt}\right)^2 + \frac{g}{r^2}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)^2 + \frac{h}{\rho^2}\left(\vec{\rho}\times\frac{d\vec{\rho}}{dt}\right)^2 \tag{56}$$

Since $(\vec{a}\times\vec{b})^2 = \vec{a}^2\,\vec{b}^2 - (\vec{a}\cdot\vec{b})^2$ we can write

$$\underline{P} = \frac{gh}{R^2}\left(r\,\frac{d\rho}{dt} - \rho\,\frac{dr}{dt}\right)^2 + \frac{g}{r^2}\left[r^2\left(\frac{dr}{dt}\right)^2 - \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)^2\right] + \frac{h}{\rho^2}\left[\rho^2\left(\frac{d\rho}{dt}\right)^2 - \left(\vec{\rho}\cdot\frac{d\vec{\rho}}{dt}\right)^2\right] \tag{57}$$

Since $r\,dr/dt = \vec{r}\cdot d\vec{r}/dt$ and $\rho\,d\rho/dt = \vec{\rho}\cdot d\vec{\rho}/dt$ the last two brackets in 57 are identically zero and

$$\underline{P} = \frac{gh}{R^2}\left(r\,\frac{d\rho}{dt} - \rho\,\frac{dr}{dt}\right)^2 \tag{58}$$

using Equations 55 and 58 we write

$$\left(\frac{dR}{dt}\right)^2 + \underline{P} = \frac{1}{R^2}\left[g^2 r^2 \left(\frac{dr}{dt}\right)^2 + h^2 \rho^2 \left(\frac{d\rho}{dt}\right)^2 + 2gh\vec{r}\cdot\vec{\rho}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d\vec{\rho}}{dt} + gh\left(r\,\frac{d\rho}{dt} - \rho\,\frac{dr}{dt}\right)^2\right]$$

$$= \frac{1}{R^2}\left[g^2 r^2 \left(\frac{dr}{dt}\right)^2 + h^2 \rho^2 \left(\frac{d\rho}{dt}\right)^2 + 2gh\vec{r}\cdot\vec{\rho}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d\vec{\rho}}{dt} + ghr^2\left(\frac{d\rho}{dt}\right)^2 - 2gh\vec{r}\cdot\vec{\rho}\cdot\frac{d\vec{\rho}}{dt}\cdot\frac{d\vec{r}}{dt} + gh\rho^2\left(\frac{dr}{dt}\right)^2\right]$$

$$= \frac{1}{R^2}\left\{\left(gr^2 + h\rho^2\right)\left[g\left(\frac{dr}{dt}\right)^2 + h\left(\frac{d\rho}{dt}\right)^2\right]\right\}$$

from Equation 50 and Equation 45

$$\left(\frac{dR}{dt}\right)^2 + \underline{P} = 2V - K \tag{59}$$

It is evident from Equation 53 and Equation 59 that

$$\underline{P} = R\,\frac{d^2 R}{dt^2} + V$$

or

$$R\,\frac{d^2 R}{dt^2} = \underline{P} - V \tag{60}$$

adding Equations 60 to Equation 53 we have

$$2R\,\frac{d^2 R}{dt^2} + \left(\frac{dR}{dt}\right)^2 = \underline{P} - K \tag{61}$$

In an effort to express $\underline{P}$ (Equation 56) in a different form we introduce

$$\frac{hR^2}{gr^2\rho^2}\left(\vec{\rho}\times\frac{d\vec{\rho}}{dt} - \frac{gh}{R^2}\,\rho^2\,\vec{c}\right)^2 + \frac{g^2 h^2}{R^2}\,\vec{c}^{\,2} \tag{62}$$

from Equation 41 we denote

$$gh\vec{c} = \vec{c}_1 + \vec{c}_2$$

$$\vec{c}_1 = g\vec{r} \times \frac{d\vec{r}}{dt}$$

where

$$\vec{c}_2 = h\vec{\rho} \times \frac{d\vec{\rho}}{dt} \qquad (63)$$

Substituting Equations 63 into expression 62

$$\frac{hR^2}{gr^2\rho^2}\left[\frac{\vec{c}_2}{h} - \frac{\rho^2}{R^2}\left(\vec{c}_1 + \vec{c}_2\right)\right]^2 + \frac{\left(\vec{c}_1 + \vec{c}_2\right)^2}{R^2}$$

squaring

$$\frac{hR^2}{gr^2\rho^2}\left[\left(\frac{c_2}{h}\right)^2 - \frac{2\rho^2}{hR^2}\vec{c}_1 \cdot \vec{c}_2 - \frac{2\rho^2}{hR^2}c_2^2 + \frac{\rho^4}{R^4}\left(c_1^2 + c_2^2 + 2\vec{c}_1 \cdot \vec{c}_2\right)\right] + \frac{c_1^2 + c_2^2 + 2\vec{c}_1 \cdot \vec{c}_2}{R^2}$$

rearranging terms

$$c_1^2\left[\frac{hR^2}{gr^2\rho^2}\frac{\rho^4}{R^4} + \frac{1}{R^2}\right] + c_2^2\left[\frac{hR^2}{gr^2\rho^2h^2} - \frac{2hR^2\rho^2}{gr^2\rho^2hR^2} + \frac{hR^2}{gr^2\rho^2}\frac{\rho^4}{R^4} + \frac{1}{R^2}\right]$$

$$+ \vec{c}_1 \cdot \vec{c}_2\left[-\frac{hR^2}{gr^2\rho^2}\frac{2\rho^2}{hR^2} + \frac{hR^2\,2\rho^4}{gr^2\rho^2R^4} + \frac{2}{R^2}\right] \qquad (64)$$

the coefficient of $\vec{c}_1 \cdot \vec{c}_2$ by Equation 50 is written

$$-\frac{2}{gr^2} + \frac{2h\rho^2}{gr^2R^2} + \frac{2}{R^2} = -\frac{2}{gr^2} + \frac{2h\rho^2 + 2gr^2}{gr^2R^2} = -\frac{2}{gr^2} + \frac{2R^2}{gr^2R^2} = 0$$

therefore 64 is written

$$c_1^2\left[\frac{h\rho^2}{gr^2R^2}+\frac{1}{R^2}\right]+c_2^2\left[\frac{R^2}{gr^2\rho^2h}-\frac{2}{gr^2}+\frac{h\rho^2}{gr^2R^2}+\frac{1}{R^2}\right]$$

Since by Equation 50

$$\frac{h\rho^2}{gr^2R^2}+\frac{1}{R^2}=\frac{gr^2+h\rho^2}{gr^2R^2}=\frac{1}{gr^2}$$

we write

$$c_1^2\left[\frac{1}{gr^2}\right]+c_2^2\left[\frac{R^2}{gr^2\rho^2h}-\frac{2}{gr^2}+\frac{1}{gr^2}\right]$$

or

$$\frac{c_1^2}{gr^2}+c_2^2\left[\frac{R^2}{gr^2\rho^2h}-\frac{1}{gr^2}\right]=\frac{c_1^2}{gr^2}+c_2^2\left[\frac{R^2-\rho^2h}{gr^2\rho^2h}\right]$$

by Equation 50 again we have

$$\frac{c_1^2}{gr^2}+c_2^2\left[\frac{gr^2+h\rho^2-h\rho^2}{gr^2\rho^2h}\right]=\frac{c_1^2}{gr^2}+\frac{c_2^2}{h\rho^2} \qquad (62')$$

using Equations 63

$$\frac{c_1^2}{gr^2}+\frac{c_2^2}{h\rho^2}=\frac{g}{r^2}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)^2+\frac{h}{\rho^2}\left(\vec{\rho}\times\frac{d\vec{\rho}}{dt}\right)^2$$

now this equation is equal to the expression 62' and upon further inspection it is also the last two terms in the original definition of $\underline{P}$ (Equation 56). Therefore we can rewrite Equation 56 as

$$\underline{P}=\frac{gh}{R^2}\left(r\frac{d\rho}{dt}-\rho\frac{dr}{dt}\right)^2+\frac{hR^2}{gr^2\rho^2}\left(\vec{\rho}\times\frac{d\vec{\rho}}{dt}-\frac{gh\rho^2}{R^2}\vec{c}\right)^2+\frac{g^2h^2}{R^2}\vec{c}^2 \qquad (65)$$

Using the derived equations we shall now investigate double and triple collisions. Some important theorems will result.

DOUBLE COLLISION—ρ REMAINS BOUNDED

Remembering Equation 51

$$\frac{d^2 R^2}{dt^2} = 2(V - K)$$

we shall first investigate a collision of m_0 and m_1 (double collision). In this case $r \to 0$ as $t \to t_1$ where t_1 is the time of collision now since $V = hgU$ (Equation 44) and U is proportional to r^{-1}, V necessarily goes to infinity. We write if $r \to 0 \Rightarrow V \to \infty$, $t \to t_1$ and from Equation 51

$$\frac{d^2 R^2}{dt^2} > 0 \qquad t_1 - \delta_0 < t < t_1 \tag{66}$$

where $t_1 - \delta_0$ is an interval about t_1; from 66 it is evident that dR^2/dt is increasing and we have two possibilities

a) $\qquad \dfrac{dR^2}{dt} > 0$

b) $\qquad \dfrac{dR^2}{dt} < 0 \qquad (67)$

Case a) implies R^2 is smoothly increasing with time and case b) implies R^2 is smoothly decreasing with time.

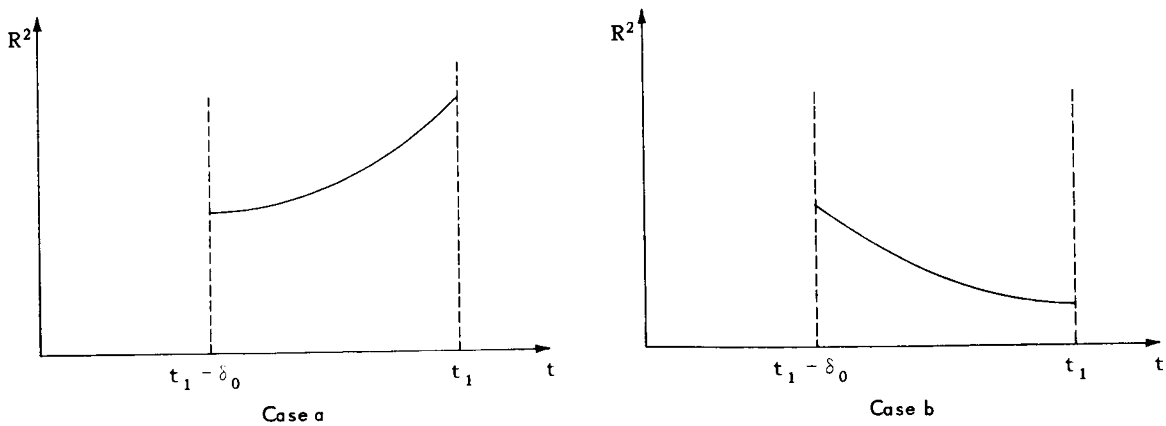

Figure 4

Either 67 a) or 67 b) could be true in a double collision.

We will now show that as $r \to 0$, $\rho \to \rho_1$ where ρ_1 is finite. This proof will be done using the method of reductio ad absurdum. For a double collision, $r \to 0$ and ρ is finite. *We shall assume that ρ does not approach a limit.* Since $R^2 = gr^2 + h\rho^2$ (Equation 50) it is evident that R^2 does not approach a limit. Since R^2 is continuous with continuous derivatives, and is without limit by assumption, R^2 must oscillate. However dR^2/dt, by 67 a and b, has a constant sign and R^2 cannot oscillate. We have a contradiction and the original assumption is incorrect. Therefore ρ has a limit which we denote ρ_1. This completes our proof and we write Theorem 1:

$$\lim_{r \to 0} \rho = \rho_1$$

From 28' we write

$$r_0 = |\vec{\rho} - \mu\vec{r}|$$

$$r_1 = |\vec{\rho} + \lambda\vec{r}|$$

$$r_2 = |\vec{r}|$$

as $t \to t_1$, $\vec{r} \to 0$ and

$$r_0 = \rho_1$$

$$r_1 = \rho_1$$

$$r_2 = 0$$

This trivial conclusion for a double collision was obtained from the signs of the first and second derivatives of R^2.

FOR A TRIPLE COLLISION, THE AREA INTEGRAL (C) EQUALS ZERO

In the case of a triple collision we have only to consider case b of 67

$$\frac{dR^2}{dt} < 0$$

and

$$R^2 \to 0$$

writing Equation 61

$$2R \frac{d^2 R}{dt^2} + \left(\frac{dR}{dt}\right)^2 = \underline{P} - K$$

and Equation 65

$$\underline{P} = \frac{gh}{R^2}\left(r \frac{d\rho}{dt} - \rho \frac{dr}{dt}\right)^2 + \frac{hR^2}{gr^2 \rho^2}\left(\vec{\rho} \times \frac{d\vec{\rho}}{dt} - \frac{gh\rho^2}{R^2} \vec{c}\right)^2 + \frac{g^2 h^2}{R^2} \vec{c}^{\,2}$$

If we let the first two terms of $\underline{P}$ = F, we have

$$\underline{P} = \frac{g^2 h^2 c^2}{R^2} + F \tag{68}$$

where $F \geq 0$. Substituting into Equation 61

$$2R \frac{d^2 R}{dt^2} + \left(\frac{dR}{dt}\right)^2 = \frac{g^2 h^2 c^2}{R^2} + F - K$$

or

$$2R \frac{d^2 R}{dt^2} + \left(\frac{dR}{dt}\right)^2 - \frac{g^2 h^2 c^2}{R^2} + K = F \tag{69}$$

Multiplying both sides of Equation 69 by dR/dt

$$2R \frac{dR}{dt} \frac{d^2 R}{dt^2} + \left(\frac{dR}{dt}\right)^3 - \frac{g^2 h^2 c^2}{R^2} \frac{dR}{dt} + K \frac{dR}{dt} = F \frac{dR}{dt} \tag{70}$$

For convenience we define

$$H = R\left(\frac{dR}{dt}\right)^2 + KR + \frac{g^2 h^2 c^2}{R} \tag{71}$$

differentiating, we have

$$\frac{dH}{dt} = \frac{dR}{dt}\left(\frac{dR}{dt}\right)^2 + 2R\frac{dR}{dt}\frac{d^2R}{dt^2} + K\frac{dR}{dt} - \frac{g^2h^2c^2}{R^2}\frac{dR}{dt}$$

Since dH/dt equals the left hand side of Equation 70 we write

$$\frac{dH}{dt} = F\frac{dR}{dt} \quad \text{or} \quad dH = F\frac{dR}{dt}\cdot dt \tag{72}$$

We now stop to introduce some convenient notation; when $t = t'$, $R = R'$ and $H = H'$ likewise when $t = t''$, $R = R''$ and $H = H''$

$$t' < t < t'' \tag{73}$$

Integrating 72 and using the above notation we have

$$H'' - H' = \int_{t'}^{t''} F\frac{dR}{dt}\,dt \tag{74}$$

we now consider two subcases a) and b)

a) $$\frac{dR}{dt} > 0 \quad \text{for} \quad t' < t < t''$$

therefore R is increasing and $R' < R''$; from 74

$$\int F\frac{dR}{dt}\,dt > 0$$

and $H' \leq H''$

b) $$\frac{dR}{dt} < 0 \quad \text{for} \quad t' < t < t'' \tag{75}$$

R is decreasing and $R' \geq R''$, $H' \geq H''$. Since we are considering a triple collision, R must be decreasing and we only consider case b) that is;

$$\frac{dR}{dt} < 0$$

and $R \to 0$ as $t \to t_1$, $H'' \le H'$, $R'' \le R'$. Rewriting Equation 71

$$H = R\left(\frac{dR}{dt}\right)^2 + KR + \frac{g^2 h^2 c^2}{R}$$

Since R is positive

$$H \ge KR + \frac{g^2 h^2 c^2}{R}$$

and

$$KR'' + \frac{g^2 h^2 c^2}{R''} \le H''$$

Since $H'' \le H'$

$$KR'' + \frac{g^2 h^2 c^2}{R''} \le H'$$

now as $t'' \to t_1$, $R'' \to 0$ if we are to have a triple collision. If R'' is then zero we have $\infty \le H'$. Since H' is finite, we must conclude that $c = 0$. This is one of Sundman's important theorems. That is; Theorem 2: For a triple collision, the area integral (c) equals zero. This is a necessary, but not sufficient condition.

BEHAVIOR OF VELOCITY VECTOR NEAR DOUBLE COLLISION

We now return to the case of the double collision in an effort to investigate the velocity near collision, i.e., to investigate $d\vec{r}/dt$ as $t \to t_1$. For double collision $r \to 0$ as $t \to t_1$. In an interval $t_1 - \delta_0 \le t < t_1$ there exists a number b such that

$$r_0 > b$$

$$r_1 > b$$

when

$$r_2 = r \to 0 \qquad (76)$$

Rewriting Equations 35,

$$\frac{d^2\vec{r}}{dt^2} = -f(m_0+m_1)\frac{\vec{r}}{r^3} + f\,m_2\,\vec{\rho}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right) - f\,m_2\,\vec{r}\left(\frac{\lambda}{r_1^3} + \frac{\mu}{r_0^3}\right)$$

$$\frac{d^2\vec{\rho}}{dt^2} = fM\left[-\vec{\rho}\left(\frac{\mu}{r_1^3} + \frac{\lambda}{r_0^3}\right) + \mu\lambda\vec{r}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\right]$$

The second of these equations can be rewritten

$$\frac{d^2\vec{\rho}}{dt^2} = -\frac{fM}{r_1^2}\mu\frac{(\vec{\rho}+\lambda\vec{r})}{r_1} - \frac{fM\lambda}{r_0^2}\frac{(\vec{\rho}-\mu\vec{r})}{r_0}$$

By the triangle inequality

$$\left|\frac{d^2\vec{\rho}}{dt^2}\right| \leq \frac{fM}{r_1^2}\mu\left|\frac{\vec{\rho}+\lambda\vec{r}}{r_1}\right| + \frac{fM\lambda}{r_0^2}\left|\frac{\vec{\rho}-\mu\vec{r}}{r_0}\right|$$

by Equation 30 and 31 this reduces to

$$\left|\frac{d^2\vec{\rho}}{dt^2}\right| \leq fM\left[\frac{\mu}{r_1^2} + \frac{\lambda}{r_0^2}\right]$$

and from 76

$$\left|\frac{d^2\vec{\rho}}{dt^2}\right| \leq \frac{fM}{b^2} \tag{77}$$

in the interval $t_1 - \delta_0 \leq t < t_1$.

We note that this acceleration is bounded. If we define $t' = t_1 - \delta_0 \leq t < t_1$ then we note that

$$\frac{d\vec{\rho}}{dt} = \int_{t'}^{t}\frac{d^2\vec{\rho}}{dt^2}\,dt + \left(\frac{d\vec{\rho}}{dt}\right)'$$

where the prime indicates $d\vec{\rho}/dt$ has been evaluated at time t'. By the triangle inequality

$$\left|\frac{d\vec{\rho}}{dt}\right| \leq \left|\left(\frac{d\vec{\rho}}{dt}\right)'\right| + \int_{t'}^{t} \left|\frac{d^2\vec{\rho}}{dt^2}\right| dt$$

by 77

$$\left|\frac{d\vec{\rho}}{dt}\right| \leq \left|\left(\frac{d\vec{\rho}}{dt}\right)'\right| + \frac{Mf}{b^2}(t - t')$$

as $t \to t_1$

$$\left|\frac{d\vec{\rho}}{dt}\right| \leq \left|\left(\frac{d\vec{\rho}}{dt}\right)'\right| + \frac{Mf}{b^2}\delta_0 \tag{78}$$

Since

$$t_1 - t' = t_1 - t_1 + \delta_0$$
$$= \delta_0$$

We see that $|(d\vec{\rho}/dt)'|$ is finite because it is evaluated before the time of collision and $|d\vec{\rho}/dt|$ is seen to be bounded. At a later time, $|d\vec{\rho}/dt|$ will be shown to have a definite limit at collision.

Rewriting the energy integral (Equation 45)

$$g\left(\frac{dr}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 = 2V - K$$

as $t \to t_1$, V has been shown to approach infinity and the above discussion leaves $d\vec{\rho}/dt$ finite. Therefore we realize that dr/dt, or the velocity of the colliding masses along the vector joining them, approaches infinity as $t \to t_1$. Thus far, we have seen that ρ has a definite limit, $d\vec{\rho}/dt$ is bounded and $d\vec{r}/dt$ approaches infinity as $t \to t_1$ in the case of a double collision. The following section will be devoted to further investigation of $d\vec{r}/dt$ as the masses m_0 and m_1 approach collision.

We start with $r^2 = \vec{r} \cdot \vec{r}$; differentiating we have

$$\frac{dr^2}{dt} = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$$

dividing through by 2 and differentiating again we obtain

$$\frac{1}{2}\frac{d^2 r^2}{dt^2} = \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r} \cdot \frac{d^2 \vec{r}}{dt^2} \tag{79}$$

The energy integral is once again written

$$g\left(\frac{d\vec{r}}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 = 2V - K$$

where V can be written as

$$V = fM\left[\frac{1}{m_2 r} + \frac{1}{m_1 r_1} + \frac{1}{m_0 r_0}\right] \tag{80}$$

i.e., $V = ghU$. Upon further inspection, in the above energy integral, $d\vec{\rho}/dt$ remains finite and the last two terms in the bracket of Equation 80 are finite if $r \to 0$ as $t \to t_1$. Under these conditions, the energy integral and Equation 80 yield

$$\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{2f(m_0 + m_1)}{r} - L_1 \tag{81}$$

L_1 = finite as $t \to t_1$. We note that $d\vec{r}/dt$ behaves roughly as the inverse root of r. Dot multiplying the first equation in 35 by $\vec{r}$, we have

$$\vec{r} \cdot \frac{d^2 r}{dt^2} = -\frac{f(m_0 + m_1)}{r} - L_2 \tag{82}$$

L_2 = finite as $t \to t_1$. Adding Equations 81 and 82 and making use of 79 we have

$$\frac{1}{2}\frac{d^2 r^2}{dt^2} = \frac{f(m_0 + m_1)}{r} - L' \tag{83}$$

where $L' = L_1 + L_2$. It is clear from 83 that as $r \to 0$ (double collision) $d^2 r^2/dt^2 \to \infty$ and

$$\frac{d^2 r^2}{dt^2} > 0 \tag{84}$$

This tells us that dr^2/dt is increasing and we have two subcases

$$\text{a)} \qquad \frac{dr^2}{dt} < 0$$

$$\text{b)} \qquad \frac{dr^2}{dt} > 0 \tag{85}$$

Case b) insures that r^2 is increasing so that r does not approach zero. Since we are considering collision we reject this case and consider only case a. In this case r^2 decreases monotonically.

From the vector indentity $\vec{a}^2\vec{b}^2 = (\vec{a}\cdot\vec{b})^2 + (\vec{a}\times\vec{b})^2$ we let $\vec{a} = \vec{r}$, $\vec{b} = d\vec{r}/dt$ and write

$$\vec{r}^2\left(\frac{d\vec{r}}{dt}\right)^2 = \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)^2 + \left(\vec{r}\times\frac{d\vec{r}}{dt}\right)^2$$

or

$$\vec{r}^2\left(\frac{d\vec{r}}{dt}\right)^2 = r^2\left(\frac{dr}{dt}\right)^2 + \left(\vec{r}\times\frac{d\vec{r}}{dt}\right)^2 \tag{86}$$

From 81 we see that as $t \to t_1$

$$r\left(\frac{d\vec{r}}{dt}\right)^2 \to 2f\left(m_0 + m_1\right)$$

and

$$r^2\left(\frac{d\vec{r}}{dt}\right)^2 \to 0 \tag{87}$$

Now from Equations 86 we see that we have the sum of two positive quantities equal to zero and hence they are each identically zero. Therefore

$$r\,\frac{dr}{dt} \to 0$$

$$\vec{r}\times\frac{d\vec{r}}{dt} \to 0 \tag{88}$$

as $t \to t_1$.

We will now show that

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) \to 0$$

as $t \to t_1$. From Equation 35

$$\frac{d^2 \vec{r}}{dt^2} = -f(m_0 + m_1)\frac{\vec{r}}{r^3} + f m_2 \vec{\rho}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right) - f m_2 \vec{r}\left(\frac{\lambda}{r_1^3} + \frac{\mu}{r_0^3}\right)$$

We cross each side of this equation by $\vec{r}$ so that

$$\vec{r} \times \frac{d^2 \vec{r}}{dt^2} = f m_2 \vec{r} \times \vec{\rho}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

and

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = f m_2 \vec{r} \times \vec{\rho}\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ we write

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = f m_2 \vec{r} \times \vec{\rho}\left(\frac{1}{r_0} - \frac{1}{r_1}\right)\left(\frac{1}{r_0^2} + \frac{1}{r_0 r_1} + \frac{1}{r_1^2}\right)$$

or

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = f m_2 \vec{r} \times \vec{\rho}\left(\frac{r_1 - r_0}{r_0}\right)\left(\frac{1}{r_1 r_0^2} + \frac{1}{r_0 r_1^2} + \frac{1}{r_1^3}\right)$$

taking absolute values

$$\left|\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)\right| = f m_2 \left|\vec{r} \times \vec{\rho}\right| \frac{\left|r_1 - r_0\right|}{r_0}\left(\frac{1}{r_1 r_0^2} + \frac{1}{r_0 r_1^2} + \frac{1}{r_1^3}\right) \qquad (89)$$

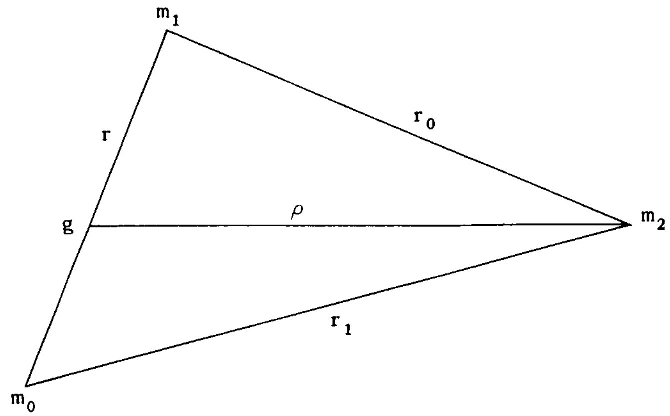

Figure 5

In the time interval $t_1 - \delta_0 \leq t < t_1$, $\rho < r_0 + gm_1 < r_0 + r$ where $r = \min(r_0, r_1, r_2)$ also since $r < r_0$

$$\rho < 2r_0 \tag{90}$$

in the neighborhood of collision and

$$|r_1 - r_0| < r \tag{91}$$

$$|\vec{r} \times \vec{\rho}| < r\rho \tag{92}$$

from 76 $r_0 > b$, $r_1 > b$ and

$$\frac{1}{r_0} < \frac{1}{b} \qquad \frac{1}{r_1} < \frac{1}{b} \tag{93}$$

Substituting the inequalities 90, 91, 92, 93 into Equation 89

$$\left|\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)\right| < \frac{6f\, m_2\, r^2}{b^3} \tag{94}$$

We note that from 88 and 94 both $\vec{r} \times d\vec{r}/dt$ and its derivative go to zero as r goes to zero. We now take two instants of time (t'', t') such that $t_1 - \delta_0 \leq t'' < t' < t_1$ and

$$\int_{t''}^{t'} \frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) dt = \left(\vec{r} \times \frac{d\vec{r}}{dt}\right)' - \left(\vec{r} \times \frac{d\vec{r}}{dt}\right)''$$

We see that

$$\left|\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)' - \left(\vec{r}\times\frac{d\vec{r}}{dt}\right)''\right| < \int_{t''}^{t'}\left|\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\right| dt$$

from 94

$$\int_{t''}^{t'}\left|\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\right| dt < \int_{t''}^{t'}\frac{6f\, m_2\, r^2}{b^3}\, dt = \frac{6f\, m_2\, r^2}{b^3}\,(t' - t'')$$

We see that

$$\left|\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)' - \left(\vec{r}\times\frac{d\vec{r}}{dt}\right)''\right| < \frac{6f\, m_2\, r^2\,(t' - t'')}{b^3}$$

now as $t' \to t_1$ (collision)

$$\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)' \to 0$$

and

$$\left|\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)''\right| < \frac{6f\, m_2\, r^2\left(t_1 - t''\right)}{b^3} \tag{95}$$

Here we see that $\vec{r}\times d\vec{r}/dt$ not only goes to zero as r goes to zero, but it does it very rapidly since as $r \to 0$, $t'' \to t_1$ the numerator in 95 goes to zero rapidly.

The unit vector $\hat{r}$ as $\vec{r} \to 0$ will reveal the type of motion near collision. In an attempt to investigate this unit vector we write.

$$\frac{-\vec{r}\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)}{r^3} = \frac{-\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) + \frac{d\vec{r}}{dt}\left(r^2\right)}{r^3} = \frac{r^2\frac{d\vec{r}}{dt} - \vec{r}\, r\,\frac{dr}{dt}}{r^3} = \frac{r\,\frac{d\vec{r}}{dt} - \vec{r}\,\frac{dr}{dt}}{r^2} = \frac{d\hat{r}}{dt}$$

Therefore we write

$$\frac{d\hat{r}}{dt} = \frac{1}{r^3}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)\times\vec{r} \tag{96}$$

now from 95

$$\left|\frac{d\hat{r}}{dt}\right| < \left|\frac{1}{r^3}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) r\right| < \frac{6f\, m_2\left(t_1 - t\right)}{b^3} \tag{97}$$

We introduce two moments of time t' and t'' such that $t_1 - \delta_0 < t'' < t' < t_1$ now

$$\hat{r}' - \hat{r}'' = \int_{t''}^{t'} \frac{d\hat{r}}{dt}\, dt$$

and by 97

$$|\hat{r}' - \hat{r}''| < \int_{t''}^{t'} \left|\frac{d\hat{r}}{dt}\right| dt < \int_{t''}^{t'} \frac{6f\, m_2 \left(t_1 - t\right)}{b^3}\, dt \tag{98}$$

We now evaluate the integral

$$\int_{t''}^{t'} (t_1 - t)\, dt = \int_{t''}^{t'} t_1\, dt - \int_{t''}^{t'} t\, dt = t_1(t' - t'') - \left(\frac{t'^2}{2} - \frac{t''^2}{2}\right) = t_1(t' - t'') - \frac{1}{2}(t' - t'')(t' + t'') = (t' - t'')\left[t_1 - \frac{1}{2}(t' + t'')\right]$$

from 98

$$|\hat{r}'' - \hat{r}'| < \frac{6f\, m_2}{b^3}(t' - t'')\left[t_1 - \frac{1}{2}(t' + t'')\right] \tag{99}$$

as $t', t'' \to t_1$, $|\hat{r}'' - \hat{r}'| \to 0$ and by the Cauchy criterion, $\hat{r}$ approaches a limit. If we denote this limit by $\vec{\psi}$ we have

$$\lim_{t \to t_1} \hat{r} = \vec{\psi}, \qquad |\vec{\psi}| = 1 \tag{100}$$

Because the unit vector is constant near collision we conclude that the motion is smooth and dismiss the possibility of m_0 and m_1 spiraling into collision with each other. In addition, $(-t)$ can be substituted into Equation 35 without any effect. This suggests that the motion is symmetrical about the t axis. The two above conditions suggest a smooth and symmetric motion before and after collision. see Figure 6

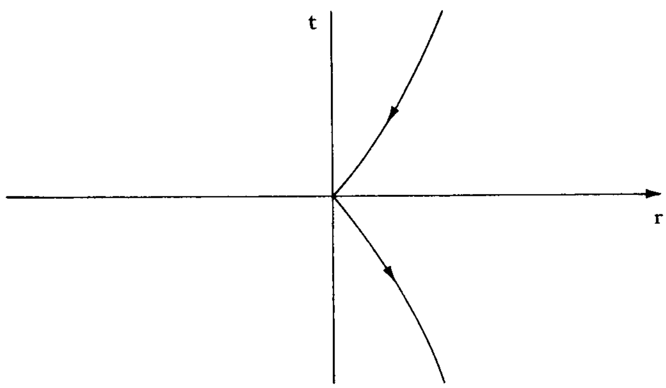

Figure 6

Since

$$\vec{r} = r \cdot \hat{r}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\hat{r} + r\,\frac{d\hat{r}}{dt}$$

and

$$\left(\frac{d\vec{r}}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\hat{r}}{dt}\right)^2 \tag{101}$$

From 87

$$r\left(\frac{d\vec{r}}{dt}\right)^2 \to 2\,f\left(m_0 + m_1\right) \tag{102}$$

so that

$$r\left(\frac{dr}{dt}\right)^2 + r^3\left(\frac{d\hat{r}}{dt}\right)^2 \to 2\,f\left(m_0 + m_1\right) \tag{103}$$

and as $t \to t_1$ we see from 97 that

$$r^3\left(\frac{d\hat{r}}{dt}\right)^2 \to 0$$

which leaves

$$r\left(\frac{dr}{dt}\right)^2 \to 2\,f\left(m_0 + m_1\right) \tag{104}$$

or

$$\sqrt{r}\,\frac{dr}{dt} \to -\sqrt{2\,f\left(m_0 + m_1\right)} \tag{105}$$

from 101

$$\sqrt{r}\,\frac{d\vec{r}}{dt} = \sqrt{r}\,\frac{dr}{dt} \cdot \hat{r} + r\sqrt{r}\,\frac{d\hat{r}}{dt} \tag{106}$$

and since the unit vector $\hat{r}$ is constant near collision the last term in 106 is zero. From 100, 105 and 106 we have

$$\sqrt{r}\,\frac{d\vec{r}}{dt} \to -\vec{\psi}\sqrt{2f(m_0+m_1)} \qquad (107)$$

and

$$\frac{d\vec{r}}{dt} \sim -\frac{\vec{\psi}}{\sqrt{r}}\sqrt{2f(m_0+m_1)} \qquad (108)$$

From 107, we can gain insight as to the behavior of the velocity vector near collision.

BEHAVIOR OF "r" NEAR DOUBLE COLLISION AND INTRODUCTION OF REGULARIZATION VARIABLE "u"

In order to investigate the asymptotic behavior of r in the neighborhood of collision, we write down 105

$$\sqrt{r}\,\frac{dr}{dt} \to -\sqrt{2f(m_0+m_1)}$$

which is equivalent to

$$\frac{2}{3}\frac{dr^{3/2}}{dt} + \sqrt{2f(m_0+m_1)} \to 0 \qquad (109)$$

we see that

$$\left|\frac{2}{3}\frac{dr^{3/2}}{dt} + \sqrt{2f(m_0+m_1)}\right| < \epsilon \qquad (110)$$

if $|t-t_1| < \eta$ where ϵ and η are arbitrary constants and $\epsilon \to 0$ as $t \to t_1$ now

$$\frac{2}{3}r^{3/2} - \sqrt{2f(m_0+m_1)}\,(t_1-t) = \int_{t_1}^{t}\left[\frac{2}{3}\frac{dr^{3/2}}{dt} + \sqrt{2f(m_0+m_1)}\right]dt$$

Since $r_{t=t_1} = 0$: also

$$\left|\frac{2}{3}r^{3/2} - \sqrt{2f(m_0+m_1)}\,(t_1-t)\right| < \int_{t}^{t_1}\left|\frac{2}{3}\frac{dr^{3/2}}{dt} + \sqrt{2f(m_0+m_1)}\right|dt$$

from Equation 110

$$\left|\frac{2}{3} r^{3/2} - \sqrt{2 f(m_0 + m_1)}\,(t_1 - t)\right| < \epsilon (t_1 - t)$$

if $|t_1 - t| < \eta$ and

$$\left|\frac{2r^{3/2}}{3\sqrt{2 f(m_0 + m_1)}(t_1 - t)} - 1\right| < \frac{\epsilon}{\sqrt{2 f(m_0 + m_1)}}$$

if $|t_1 - t| < \eta$ as $t \to t_1$, $\epsilon(t)$ goes to zero and

$$\frac{2r^{3/2}}{3\sqrt{2 f(m_0 + m_1)}\,(t_1 - t)} \to 1 \tag{111}$$

if we define

$$\frac{2}{3}\frac{1}{\sqrt{2 f(m_0 + m_1)}} = \frac{1}{A^{3/2}} \tag{112}$$

then

$$\frac{r^{3/2}}{A^{3/2}(t_1 - t)} \to 1$$

$t \to t_1$ and

$$r \to A(t_1 - t)^{2/3} \tag{113}$$

From 113 we note the asymptotic behavior of r, as a function of t, near collision. From 105 and 113 we write

$$\frac{dr}{dt} \sim -\frac{\sqrt{2 f(m_0 + m_1)}}{\sqrt{A(t_1 - t)^{2/3}}}$$

and if we define

$$B = -\frac{\sqrt{2f(m_0 + m_1)}}{\sqrt{A}} \qquad (114)$$

we have

$$\frac{dr}{dt} \sim B(t_1 - t)^{-1/3} \qquad (115)$$

From 115 the velocity, as a function of time, is evident in the neighborhood of collision. Once again we note that as $t \to t_1$ the velocity becomes infinite.

We shall show that r (see 113) can be expanded into the following series

$$r = a_2 (t_1 - t)^{2/3} + a_3 (t_1 - t)^{3/3} + a_4 (t_1 - t)^{4/3} + \cdots$$

if we let $(t_1 - t)^{1/3} = u$

$$r = a_2 u^2 + a_3 u^3 + a_4 u^4 + \cdots \qquad (116)$$

and

$$\frac{du}{dt} = \frac{1}{3} (t_1 - t)^{-2/3}$$

or

$$\frac{du}{dt} = \frac{1}{r}$$

where

$$u = \int_{t_0}^{t} \frac{dt}{r} \qquad (117)$$

now

$$du = \frac{dt}{r} \tag{118}$$

from 113, 115 and 117

$$\frac{dr}{du} = \frac{dr}{dt}\frac{dt}{du} \sim \frac{r}{\sqrt{r}}$$

and

$$\frac{dr}{du} \sim \sqrt{r} \tag{119}$$

The regularization variable u, as defined by 117, is referred to as the pseudo-time. This new pseudo-time is seen to remove the singularity in the velocity, for now, the new velocity is proportional to the square root of r and as $r \to 0$, for collision, the velocity does not go to infinity. The velocity is now an analytic function.

In an effort to investigate the convergence of the Taylor series solution, we state the Cauchy-Picard Theorem without proof.

CAUCHY-PICARD THEOREM

Theorem 3: Let $Q_i \left(q_1, q_2 \cdots q_n\right)$ $i = 1, 2, \cdots n$ be analytic functions which do not contain t explicitly and which are developable into Taylor series in the powers of differences of $q_i - \bar{q}_i$ and these series are convergent if;

a) $$\left|q_i - \bar{q}_i\right| < q_i'$$

Then there exists positive and finite quantities Q_j' such that when a) is satisfied

b) $$\left|Q_j\left(q_1 \cdots q_n\right)\right| < Q_j' \qquad j = 1, 2, \cdots n \tag{120}$$

Under conditions a) and b) the system of differential equations

$$\frac{dq_i}{dt} = Q_j\left(q_1 \cdots q_n\right) \tag{121}$$

admits one and only one analytic solution such that q_i goes to a finite limit $\bar{q}_i$ when $t \to \bar{t}$.

Condition a) assures the functions are analytic within the radius q_i'. In this solution, the unknowns (q_i) are developable into Taylor series in powers of $t - \bar{t}$ which are convergent for $|t - \bar{t}| \leq T'$ where T' is minimum

$$\frac{q_1'}{Q_1'}, \frac{q_2'}{Q_2'}, \ldots \frac{q_n'}{Q_n'} \tag{122}$$

At a time $t = \bar{t}$ let the components of q and $\bar{q}$ be (x_i, y_i, z_i) and $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ respectively and similarly let the components of $\dot{q}$ and $\dot{\bar{q}}$ be $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ and $\dot{\bar{x}}_i, \dot{\bar{y}}_i, \dot{\bar{z}}_i$. Condition a) assures that

$$|x_i - \bar{x}_i|, \quad |y_i - \bar{y}_i|, \quad |z_i - \bar{z}_i| < k_0$$

$$|\dot{x}_i - \dot{\bar{x}}_i|, \quad |\dot{y}_i - \dot{\bar{y}}_i|, \quad |\dot{z}_i - \dot{\bar{z}}_i| < k_0' \tag{123}$$

where k_0 and k_0' are minimum radii of convergence. From 28'

$$r_0^2 = |\vec{r}_2 - \vec{r}_1|^2 = \left|(\vec{\bar{r}}_2 - \vec{\bar{r}}_1) + \left[(\vec{r}_2 - \vec{\bar{r}}_2) - (\vec{r}_1 - \vec{\bar{r}}_1)\right]\right|^2$$

$$= |\vec{\bar{r}}_2 - \vec{\bar{r}}_1|^2 + 2(\vec{\bar{r}}_2 - \vec{\bar{r}}_1)\left[(\vec{r}_2 - \vec{\bar{r}}_2) - (\vec{r}_1 - \vec{\bar{r}}_1)\right] + \left[(\vec{r}_2 - \vec{\bar{r}}_2) - (\vec{r}_1 - \vec{\bar{r}}_1)\right]^2 \tag{124}$$

We let

$$\underline{P}_0 = (\vec{\bar{r}}_2 - \vec{\bar{r}}_1)\left[(\vec{r}_2 - \vec{\bar{r}}_2) - (\vec{r}_1 - \vec{\bar{r}}_1)\right] + \left[(\vec{r}_2 - \vec{\bar{r}}_2) - (\vec{r}_1 - \vec{\bar{r}}_1)\right]^2 \tag{125}$$

so that

$$r_0^2 \geq \bar{r}_0^2 - |P_0| \tag{126}$$

from 125

$$|P_0| \leq 2|\vec{\bar{r}}_2 - \vec{\bar{r}}_1| \left[|\vec{r}_2 - \vec{\bar{r}}_2| + |\vec{r}_1 - \vec{\bar{r}}_1|\right] + \left[|\vec{r}_2 - \vec{\bar{r}}_2| + |\vec{r}_1 \quad \vec{\bar{r}}_1|\right]^2 \tag{127}$$

We know from 123

$$|\vec{r}_2 - \vec{\bar{r}}_2| \leq |x_2 - \bar{x}_2| + |y_2 - \bar{y}_2| + |z_2 - \bar{z}_2| < 3k_0 \tag{128}$$

and similarly

$$\left|\vec{r}_1 - \vec{\bar{r}}_1\right| < 3k_0 \tag{128'}$$

We see that

$$\left|\vec{r}_2 - \vec{\bar{r}}_2\right| = \sqrt{(x_2 - \bar{x}_2)^2 + (y_2 - \bar{y}_2)^2 + (z_2 - \bar{z}_2)^2} < \sqrt{3k_0^2} = \sqrt{3}\, k_0 \tag{129}$$

likewise

$$\left|\vec{r}_1 - \vec{\bar{r}}_1\right| < \sqrt{3}\, k_0\,, \qquad \left|\vec{r}_0 - \vec{\bar{r}}_0\right| < \sqrt{3}\, k_0 \tag{129'}$$

Using 128, 128', 129, 129' and 127 we find

$$\left|P_0\right| \leq 12\,\bar{r}_0 k_0 + 12\,k_0^2$$

and from 126

$$r_0^2 > \bar{r}_0^2 - 12\left(\bar{r}_0 k_0 + k_0^2\right) \tag{130}$$

r_0^2 must remain positive so that

$$12\,k_0^2 + 12\,\bar{r}_0 k_0 - \bar{r}_0^2 < 0 \tag{131}$$

Since k_0 must be greater than zero we have

$$0 < k_0 < \frac{(4\sqrt{3} - 6)}{12}\,\bar{r}_0 = \frac{\bar{r}_0}{4\sqrt{3} + 6} = \frac{\bar{r}_0}{12.928} \cdots$$

The inequality is strengthened if the denominator is set equal to 14

$$\bar{r}_0 > 14\,k_0$$

and similarly

$$\bar{r}_1 \, , \, \bar{r}_2 > 14\, k_0 \tag{132}$$

from 124 and 125

$$r_0^2 = \bar{r}_0^2 + P_0$$

or

$$r_0 = \bar{r}_0 \sqrt{1 + \frac{P_0}{\bar{r}_0^2}}$$

and

$$\frac{1}{r_0} = \frac{1}{\bar{r}_0}\left(1 + \frac{P_0}{\bar{r}_0^2}\right)^{-1/2} \tag{133}$$

Here we see that $1/r_0$, which appears in the equations of motion, remains analytic for $\bar{r}_0 > 14\, k_0$. It must be remembered that $\bar{x}_i$, $\bar{y}_i$, $\bar{z}_i$ were chosen arbitrarily and that the radius of convergence k_0 depends upon these initial conditions. As $\bar{r}_0$ is taken smaller and smaller, the radius of convergence decreases. Rewriting 130

$$r_0^2 \geq \bar{r}_0^2 - \left(12\, \bar{r}_0\, k_0 + 12\, k_0^2\right)$$

using 132

$$r_0^2 \geq \bar{r}_0^2 - \left(12\, \bar{r}_0\, \frac{\bar{r}_0}{14} + 12\, \frac{\bar{r}_0^2}{196}\right)$$

or

$$r_0^2 \geq \bar{r}_0^2 - 12\, \bar{r}_0^2 \left[\frac{15}{196}\right] = \frac{4\, \bar{r}_0^2}{49}$$

so that

$$r_0 \geq \frac{2\,\bar{r}_0}{7}$$

and similarly

$$r_1 \geq \frac{2\,\bar{r}_0}{7}, \qquad r_2 \geq \frac{2\,\bar{r}_0}{7}$$

so that

$$\frac{1}{r_0} \leq \frac{7}{2\,\bar{r}_0}, \qquad \frac{1}{r_1} \leq \frac{7}{2\,\bar{r}_0}, \qquad \frac{1}{r_2} \leq \frac{7}{2\,\bar{r}_0} \tag{134}$$

These three inequalities hold for the given initial conditions 123. Using these same initial conditions the disturbing force will be investigated. We write;

$$|x_2 - x_1| = \left|(x_2 - \bar{x}_2) - (x_1 - \bar{x}_1) + (\bar{x}_2 - \bar{x}_1)\right|$$

$$|x_2 - x_1| \leq |x_2 - \bar{x}_2| + |x_1 - \bar{x}_1| + |\bar{x}_2 - \bar{x}_1|$$

Using 123 and 132

$$|x_2 - x_1| \leq k_0 + k_0 + \bar{r}_0 \leq \frac{\bar{r}_0}{14} + \frac{\bar{r}_0}{14} + \bar{r}_0 = \frac{8\,\bar{r}_0}{7} \tag{135}$$

from 134, 135 and 132

$$\left|\frac{x_2 - x_1}{r_0^3}\right| \leq \frac{8\,\bar{r}_0}{7}\left(\frac{7}{2\,\bar{r}_0}\right)^3 \leq \frac{1}{4\,k_0^2}$$

similarly

$$\left|\frac{x_1 - x_0}{r_2^3}\right|, \quad \left|\frac{x_2 - x_0}{r_1^3}\right| \leq \frac{1}{4\,k_0^2} \tag{136}$$

Now from 129, 129' we see that condition 120 a) of the Cauchy-Picard Theorem is satisfied.

The disturbing potential 43 is written

$$U = f\left(\frac{m_0 m_1}{r_2} + \frac{m_1 m_2}{r_0} + \frac{m_0 m_2}{r_1}\right)$$

now

$$\frac{1}{m_1}\frac{\partial U}{\partial x_1} = \frac{f}{m_1}\left[\frac{m_0 m_1}{r_2^3}\left(x_1 - x_0\right) + \frac{m_1 m_2}{r_0^3}\left(x_2 - x_1\right)\right]$$

and from 136

$$\frac{1}{m_1}\frac{\partial U}{\partial x_1} \leq f\left(m_0 + m_1\right)\left(\frac{1}{4k_0^2}\right) < \frac{f\left(m_0 + m_1 + m_2\right)}{4k_0^2}$$

In a similar way, it can be shown that

$$\left|\frac{1}{m_i}\frac{\partial U}{\partial x_i}\right|, \quad \left|\frac{1}{m_i}\frac{\partial U}{\partial y_i}\right|, \quad \left|\frac{1}{m_i}\frac{\partial U}{\partial z_i}\right| < \frac{fM}{4k_0^2} \tag{137}$$

From 27, 40 and 44

$$U = \frac{m_0 m_1 m_2}{M} V \tag{138}$$

and from 43 and the instant $t = \bar{t}$

$$\bar{U} = f\left[\frac{m_0 m_1}{\bar{r}_2} + \frac{m_0 m_2}{\bar{r}_1} + \frac{m_1 m_2}{\bar{r}_0}\right]$$

and using 132

$$\bar{U} \leq f\left[\frac{m_0 m_1 + m_0 m_2 + m_1 m_2}{14k_0}\right]$$

so that from 138

$$\frac{m_0 m_1 m_2}{M} \bar{V} \leq \frac{f(m_1 m_2 + m_2 m_0 + m_1 m_0)}{14k_0} \tag{139}$$

Two useful inequalities will now be derived. It is evident that

$$(m_1 - m_2)^2 + (m_2 - m_0)^2 + (m_1 - m_0)^2 > 0$$

or

$$2m_0^2 + 2m_1^2 + 2m_2^2 - 2(m_1 m_2 + m_2 m_0 + m_1 m_0) > 0$$

So that

$$m_0^2 + m_1^2 + m_2^2 > m_1 m_2 + m_2 m_0 + m_1 m_0$$

adding $2(m_1 m_2 + m_2 m_0 + m_1 m_0)$ to both sides of this inequality; $(m_0 + m_1 + m_2)^2 > 3(m_1 m_2 + m_2 m_0 + m_1 m_0)$ or

$$M^2 > 3(m_1 m_2 + m_2 m_0 + m_1 m_0) \tag{140}$$

Now if

$$4m_i m_j < M^2 \tag{141}$$

then

$$4m_0 m_1 < (m_0 + m_1 + m_2)^2$$

$$4m_0 m_1 < (m_0 + m_1)^2 + 2m_2(m_0 + m_1) + m_2^2$$

$$0 < (m_0 - m_1)^2 + 2m_2(m_0 + m_1) + m_2^2$$

Since the final step is correct, then the steps can be reversed and assumption 141 is correct. From 139 and 140

$$\frac{m_0\, m_1\, m_2}{M}\, \bar{V} \leq \frac{fM^2}{42k_0} \tag{142}$$

from 141

$$\left|\frac{m_i\, m_j}{M}\, K\right| < \frac{M}{4}\, |K| \tag{143}$$

where K is the energy integral constant. The energy integral can be written

$$\sum_{i=0}^{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) = -\frac{m_0\, m_1\, m_2}{M}\, K + \frac{2m_0\, m_1\, m_2}{M}\, V$$

so that

$$m_i\, \dot{x}_i^2 < \left|\frac{m_0\, m_1\, m_2\, K}{M}\right| + \left|\frac{2m_0\, m_1\, m_2\, V}{M}\right|$$

Since this is true at any moment of time

$$m_0\, \bar{\dot{x}}_0^2 < \left|\frac{m_0\, m_1\, m_2\, K}{M}\right| + \left|\frac{2m_0\, m_1\, m_2\, \bar{V}}{M}\right|$$

from 142 and 143

$$m_0\, \bar{\dot{x}}_0^2 < \frac{m_0\, M}{4}\, |K| + \frac{fM^2}{21k_0}$$

or

$$\bar{\dot{x}}_0^2 < \frac{M}{4}\, |K| + \frac{fM^2}{21m_0\, k_0} \tag{144}$$

If we let $m = \text{minimum}\ (m_0, m_1, m_2)$, 144 can be written

$$\bar{\dot{x}}_0^{\,2} < \frac{M}{4}\,|K| + \frac{fM^2}{21mk_0}$$

or

$$\left|\bar{\dot{x}}_0\right| < \sqrt{\frac{M}{4}\,|K| + \frac{fM^2}{21mk_0}}$$

It is obvious that a similar procedure will give the same results for $\left|\bar{\dot{x}}_i\right|$, $\left|\bar{\dot{y}}_i\right|$ and $\left|\bar{\dot{z}}_i\right|$ so that

$$\left|\bar{\dot{x}}_i\right|,\ \left|\bar{\dot{y}}_i\right|,\ \left|\bar{\dot{z}}_i\right| < \sqrt{\frac{M}{4}\,|K| + \frac{fM^2}{21mk_0}} \tag{145}$$

Since from 123

$$\left|\dot{x}_i - \bar{\dot{x}}_i\right| < k_0' \tag{146}$$

We let

$$k_0' = \sqrt{\frac{M|K|}{4} + \frac{fM^2}{21mk_0}} \tag{147}$$

and 145, 146, 147 yield

$$\left|\dot{x}_i\right| = \left|\dot{x}_i - \bar{\dot{x}}_i\right| + \left|\bar{\dot{x}}_i\right| < 2k_0' \tag{148}$$

We have k_0' for the Cauchy-Picard condition 123. Note that k_0' is the radius of convergence for a series, in powers of differences in velocities, which represents the right side of our differential equations. One should realize that k_0' can be arbitrarily chosen. The Taylor series representations of the right side of our differential equations, in powers of the differences in coordinates and powers of the differences in velocities, have been investigated. The radii of convergence were also determined. Each power series representation is valid up to the singularity at the point of collision. A new independent variable will be defined such that the singularity at the point of collision is removed.

RADIUS OF CONVERGENCE FOR TIME SERIES WHEN MUTUAL DISTANCES ARE ALL LARGER THAN $14k_0$

Our final Taylor series solution for the right hand side of our equations of motion, is expressed in powers of $t - \bar{t}$ which is convergent for $t - \bar{t} \leq T'$ where T' is minimum of q_1'/Q_1', q_2'/Q_2', . . . q_n'/Q_n'. (see 121) In order to investigate the radius of convergence for the solution series, we take two representative equations of motion:

$$\frac{dx_0}{dt} = \dot{x}_0$$

$$\frac{d\dot{x}_0}{dt} = \ddot{x}_0 \qquad (149)$$

from 123, $|x_0 - \bar{x}_0| < k_0$ and from 148 $|\dot{x}_0| < 2k_0'$ so that

$$\frac{q_0'}{Q_0'} = \frac{k_0}{2k_0'} \qquad (150)$$

from 123 $|\dot{x}_0 - \dot{\bar{x}}_0| < k_0'$ and from 137 $|\ddot{x}_0| < fM/4k_0^2$ so that

$$\frac{q_1'}{Q_1'} = \frac{4k_0' k_0^2}{fM} \qquad (151)$$

We know that T' is the minimum of 150 and 151. Now

$$\frac{4k_0^2 k_0'}{M} - \frac{k_0}{2k_0'} = \frac{k_0}{2Mk_0'} \left(8k_0 k_0'^2 - M\right)$$

and from 147

$$\frac{4k_0^2 k_0'}{M} - \frac{k_0}{2k_0'} = \frac{k_0}{2k_0'} \left[\frac{8M}{21m} + 2k_0 |K| - 1\right] \qquad (152)$$

Since the largest m can be is $M/3$, we substitute this into 152 and find 152 to be positive, so that $k_0/2k_0'$ is the minimum of 150 and 151. We see that the series solution, in powers of $t - \bar{t}$, is convergent for

$$|t - \bar{t}| \leq T' = \frac{k_0}{\sqrt{\dfrac{4M^2}{21mk_0} + M|K|}} \qquad \bar{r}_0, \bar{r}_1, \bar{r}_2 \geq 14k_0 \qquad (153)$$

Motion is regular in the given interval, if the coordinates of the bodies are analytic functions of the time in this interval.

This analysis has been done under the restriction that all the mutual distances are greater than $14k_0$.

THE EQUATIONS OF MOTION USING NEW INDEPENDENT VARIABLE "u"

If the scalar components of $\vec{r}$ (25) and $\vec{\rho}$ (26) are designated (x, y, z) and (ξ, η, ζ) respectively, equations 35 are written

$$\ddot{x} + \frac{f(m_0 + m_1)x}{r^3} = fX = -f m_2 x\left(\frac{\mu}{r_0^3} + \frac{\lambda}{r_1^3}\right) + f m_2 \xi\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

$$\ddot{y} + \frac{f(m_0 + m_1)y}{r^3} = fY = -f m_2 y\left(\frac{\mu}{r_0^3} + \frac{\lambda}{r_1^3}\right) + f m_2 \eta\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

$$\ddot{z} + \frac{f(m_0 + m_1)z}{r^3} = fZ = -f m_2 z\left(\frac{\mu}{r_0^3} + \frac{\lambda}{r_1^3}\right) + f m_2 \zeta\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

$$\ddot{\xi} = f\Xi = -fM\xi\left(\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right) + f\lambda\mu Mx\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

$$\ddot{\eta} = f\mathcal{H} = -fM\eta\left(\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right) + f\lambda\mu My\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)$$

$$\ddot{\zeta} = f\mathcal{Z} = -fM\zeta\left(\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right) + f\lambda\mu Mz\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right) \qquad (154)$$

The definitions of $(X, Y, Z, \Xi, \mathcal{H}, \mathcal{Z})$ should be evident from the above equations. From equations 41 we write in scalar form.

$$g(x\dot{y} - y\dot{x}) + h(\xi\dot{\eta} - \eta\dot{\xi}) = gh\, c_0$$

$$g(y\dot{z} - z\dot{y}) + h(\eta\dot{\zeta} - \zeta\dot{\eta}) = gh\, c_1$$

$$g(z\dot{x} - x\dot{z}) + h(\zeta\dot{\xi} - \xi\dot{\zeta}) = gh\, c_2 \qquad (155)$$

and from 45 we have

$$g\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + h\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) = 2V - K \qquad (156)$$

Now from 117, 118 we remember

$$dt = r\,du$$

$$u = \int_{t_0}^{t} \frac{dt}{r}$$

$$t - t_0 = \int_0^u r\,du \qquad (157)$$

With these definitions in mind we define different derivitives of a function w. Let

$$\frac{dw}{dt} = \dot{w} \quad \text{and} \quad \frac{dw}{du} = w' \qquad (158)$$

so that

$$\dot{w} = \frac{1}{r} w' , \qquad \ddot{w} = \frac{1}{r^2} w'' - \frac{r'}{r^3} w' \qquad (159)$$

and

$$w' = r\dot{w} , \qquad w'' = r^2 \ddot{w} + r \dot{r} \dot{w} \qquad (160)$$

Now from 160, $x'' = r^2 \ddot{x} + r \dot{r} \dot{x}$ and from 159 $x'' = r^2 \ddot{x} + r' x'/r$. Similar equations exist for y'' and z'' so that from 154

$$\frac{dx'}{du} = x'' = \frac{r'}{r} x' - \frac{f(m_0 + m_1) x}{r} + f r^2 X$$

$$\frac{dy'}{du} = y'' = \frac{r'}{r} y' - \frac{f(m_0 + m_1) y}{r} + f r^2 Y$$

$$\frac{dz'}{du} = z'' = \frac{r'}{r} z' - \frac{f(m_0 + m_1) z}{r} + f r^2 Z \qquad (161)$$

using Equations 156 and 159 we have

$$g(x'^2 + y'^2 + z'^2) + hr^2(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) = r^2(2V - K) \qquad (162)$$

From 160

$$r'' = r^2 \ddot{r} + r \dot{r}^2 \tag{163}$$

using the energy integral (45)

$$r \dot{r}^2 = -\frac{hr}{g} \left(\frac{d\vec{\rho}}{dt}\right)^2 + \frac{2Vr}{g} - \frac{K}{g} r$$

or using 27

$$r \dot{r}^2 = r\left[-\frac{h}{g}\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) - \frac{K}{g} + 2\left(m_0 + m_1\right) f\left(\frac{1}{r} + \frac{m_2}{m_0 r_0} + \frac{m_2}{m_1 r_1}\right)\right] \tag{164}$$

Now $r^2 \ddot{r}$ will be evaluated. We assert that

$$r^2 \ddot{r} = -f\left(m_0 + m_1\right) + r f (xX + yY + zZ) \tag{165}$$

since from 154

$$r^2 \ddot{r} = -f\left(m_0 + m_1\right) + r\left[x\ddot{x} + \frac{f\left(m_0 + m_1\right)}{r^3} x^2 + y\ddot{y} + \frac{f\left(m_0 + m_1\right)}{r^3} y^2 + z\ddot{z} + \frac{f\left(m_0 + m_1\right)}{r^3} z^2\right]$$

$$= -f\left(m_0 + m_1\right) + \frac{f\left(m_0 + m_1\right)}{r^2}\left(x^2 + y^2 + z^2\right) + r(x\ddot{x} + y\ddot{y} + z\ddot{z})$$

$$= r^2 \ddot{r}$$

using 164, 165 to evaluate 163 we may write

$$r'' = \frac{dr'}{du} = f\left(m_0 + m_1\right) + rL \tag{166}$$

where

$$L = f(xX + yY + zZ) + \frac{2f\, m_2\left(m_0 + m_1\right)}{m_0 r_0} + \frac{2f\, m_2\left(m_0 + m_1\right)}{m_1 r_1} - \frac{h}{g}\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) - \frac{K}{g}$$

(167)

We now let

$$\alpha = \frac{r'}{r} x' - \frac{f(m_0 + m_1) x}{r}$$

$$\beta = \frac{r'}{r} y' - \frac{f(m_0 + m_1) y}{r}$$

$$\gamma = \frac{r'}{r} z' - \frac{f(m_0 + m_1) z}{r} \qquad (168)$$

differentiating α with respect to u

$$\frac{d\alpha}{du} = \frac{r'}{r} x'' + \frac{x'}{r}\left(r'' - \frac{r'^2}{r}\right) - \frac{f(m_0 + m_1)}{r}\left[x' - \frac{xr'}{r}\right]$$

Substituting x'' (161) and r'' (166) we have

$$\frac{d\alpha}{du} = \alpha' = \frac{r'^2}{r^2} x' - \frac{f(m_0 + m_1)}{r^2} xr' + f\,rr'\,X + \frac{x'}{r}\left[f(m_0 + m_1) + rL - \frac{r'^2}{r}\right] - \frac{f(m_0 + m_1)}{r} x' + \frac{f(m_0 + m_1)}{r^2} xr'$$

after canceling appropriate terms we find that

$$\alpha' = fX\,rr' + Lx'$$

and similarly

$$\beta' = fY\,rr' + Ly'$$

$$\gamma' = fZ\,rr' + Lz' \qquad (169)$$

From 160 and 159

$$x'' = r^2 \ddot{x} + r\dot{r}\dot{x} = r^2 \ddot{x} + \frac{r'}{r} x' \qquad (170)$$

also

$$\alpha + fr^2 X = \frac{r'}{r} x' - \frac{f(m_0 + m_1)x}{r} + r^2 \ddot{x} + \frac{f(m_0 + m_1)x}{r} = \frac{r'}{r} x' + r^2 \ddot{x}$$

so that

$$\frac{dx'}{du} = x'' = \alpha + f r^2 X$$

and similarly

$$\frac{dy'}{du} = y'' = \beta + f r^2 Y$$

$$\frac{dz'}{du} = z'' = \gamma + f r^2 Z \qquad (171)$$

Collecting equations, we have

$$\frac{dr}{du} = r' \qquad \frac{dr'}{du} = r'' = f(m_0 + m_1) + rL \qquad \frac{dt}{du} = r$$

$$\frac{dx}{du} = x' \qquad \frac{dx'}{du} = \alpha + f r^2 X \qquad \frac{d\alpha}{du} = fX r r' + Lx'$$

$$\frac{dy}{du} = y' \qquad \frac{dy'}{du} \;\; \beta + f r^2 Y \qquad \frac{d\beta}{du} = fY r r' + Ly'$$

$$\frac{dz}{du} = z' \qquad \frac{dz'}{du} = \gamma + f r^2 Z \qquad \frac{d\gamma}{du} = fZ r r' + Lz'$$

$$\frac{d\xi}{du} = r\dot{\xi} \qquad \frac{d\eta}{du} = r\dot{\eta} \qquad \frac{d\zeta}{du} = r\dot{\zeta}$$

$$\frac{d\dot{\xi}}{du} = f r \Xi \qquad \frac{d\dot{\eta}}{du} = f r \mathcal{H} \qquad \frac{d\dot{\zeta}}{du} = f r Z \qquad (172)$$

For a given set of initial conditions $(x, y, z, x', y', z', \xi, \eta, \zeta, \xi', \eta', \zeta')$ α, β, γ can be determined from 168. In addition since

$$r^2 \quad x^2 + y^2 + z^2$$

$$r r' \quad x x' + y y' + z z' \qquad (173)$$

r and r' are determined from the initial conditions.

In order to determine L (167), we must find K from 162. It is necessary to divide by r^2, which will go to zero near collision. If the initial conditions are chosen near collision, then numerical difficulties arise. Given the initial conditions, the eighteen equations of motion (172) are now soluable.

Our new pseudo-time variable "u" (157) must go to collision simultaneously with "t". That is, from its definition, "u" appears to go to infinity. In the following exposition, which differs from

Sundman's, "u" will be shown to be limited. From 105

$$\sqrt{r}\,\frac{dr}{dt} = -\sqrt{2f(m_0 + m_1)}\,(1+\epsilon)^{-1} \tag{174}$$

where $|\epsilon| \to 0$ as $r \to r_1$ so that

$$dt = -\frac{\sqrt{r}}{\sqrt{2f(m_0 + m_1)}}(1+\epsilon)\,dr \tag{175}$$

from 157

$$u = \int_{t_0}^{t} \frac{dt}{r} = \int_{t_1-\delta''}^{t} \frac{dt}{r} + \int_{t_0}^{t_1-\delta''} \frac{dt}{r} \tag{176}$$

where the first integral is the interval near collision and the second is arbitrary since t_0 is arbitrary. From 175, 176

$$\int_{t_1-\delta''}^{t} \frac{dt}{r} = \int_{r''}^{r} -\frac{1}{\sqrt{2f(m_0 + m_1)}}(1+\epsilon)\frac{dr}{\sqrt{r}} = \frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{dr}{\sqrt{r}} + \frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{\epsilon\,dr}{\sqrt{r}} \tag{177}$$

evaluating the first integral we have

$$\frac{1}{\sqrt{2f(m_0 + m_1)}}\,2\left(\sqrt{r''} - \sqrt{r}\right) \tag{178}$$

which is finite as $r \to r_1$ the second integral

$$\left|\frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{\epsilon\,dr}{\sqrt{r}}\right| < \frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{|\epsilon|\,dr}{\sqrt{r}} \tag{179}$$

there exists a number "η" such that $|\epsilon| < \eta$ since $|\epsilon| \to 0$. With this in mind

$$\frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{|\epsilon|\,dr}{\sqrt{r}} < \frac{1}{\sqrt{2f(m_0 + m_1)}}\int_{r}^{r''}\frac{\eta\,dr}{\sqrt{r}} \tag{180}$$

The first integral in 177 is finite and the second integral, from 179 and 180 is bounded, so that "u" is bounded and cannot go to infinity as $t \to t_1$.

The question arises whether the equations of motion 29, 35 or 172 best describe the motion near collision. The distance r doesn't enter into the denominator in the right hand members of equations 172 and this system will be generally preferable when r is small with respect to the other two distances.

Corresponding to our eighteen equations 172 we have

$$r,\ r',\ t,\ x,\ x',\ \alpha,\ y,\ y',\ \beta,\ z,\ z',\ \gamma,\ \xi,\ \eta,\ \zeta,\ \dot{\xi},\ \dot{\eta},\ \dot{\zeta} \tag{181}$$

as our eighteen unknowns. If the distances r_0, r_1, r_2 are greater than zero for $t = t_0$, the variables in 29 and consequently the unknowns 181 are developable in powers of $t - t_0$ if $|t - t_0|$ is sufficiently small. This follows from 117. Since r is not zero for $t = t_0$, $1/r$ is analytic and can be expressed as a power series in $t - t_0$. Integrating this power series from t to t_0, u is expressed analytically as a power series in $t - t_0$. By rewriting 117 as

$$t - t_0 = \int_0^u r\,du$$

it is evident that the variable t is developable in powers of u when $|u|$ is smaller than a certain value. Upon substituting for t, this series in the series expression for the unknowns 181, we obtain functions of the variable u which verify the equations 172. Furthermore, upon substituting the series expression for u into the solution of the system, we get the solution of the system 29 from which we started. The equations of motion 172, are equivalent to equation 35 from which they were derived, and equations 35 were in turn derived from 29. The initial conditions for the system 172, are

$$x,\ y,\ z,\ x',\ y',\ z',\ \xi,\ \eta,\ \zeta,\ \xi',\ \eta',\ \zeta' \tag{182}$$

and are regulated by 162, 168 and 173. By recombination of the equations 172 the system is seen to have the following integrals,

$$r\left[\alpha - \frac{r'}{r}x' + \frac{(m_0 + m_1)x}{r}\right] = \bar{\alpha} \qquad r\left[\beta - \frac{r'}{r}y' + \frac{(m_0 + m_1)y}{r}\right] = \bar{\beta} \qquad r\left[\gamma - \frac{r'}{r}z' + \frac{(m_0 + m_1)z}{r}\right] = \bar{\gamma}$$

which are seen to be zero by 168. It remains for us to see that "r", which is amoung the unknowns of the system 172, as well as the constant K, which enters into the same system, both have the same significance as in 154 and 156.

Rewriting Equation 173,

$$rr' \qquad xx' + yy' + zz'$$

we see that this equation exists for all values of t or u and upon integrating

$$r^2 - \left(x^2 + y^2 + z^2\right)$$

is equal to a constant of integration. This constant is seen to be zero so that the unknown "r" satisfies Equation 32. Upon introducing "t" into 162 we get 45, from which one concludes that the constant of kinetic energy K is the same for both systems.

We would now like to see the unknowns 181 developed into a series in powers of $u - u_1$ and to determine the lower limit of the radii of convergence of these developments. To do this, we must find the upper limits for the unknowns when $t = t_1$ as well as the absolute values of the second members of the Equations 172. From these values, we then can form the ratios q_1'/Q_1', q_2'/Q_2' . . . needed for the Cauchy-Picard Theorem.

Suppose now that the motion is regular in the interval $0 \leq t < t_1$ where t_1 designates the moment of collision. Recall the quantities K:

$$r = r_2,\ r_0,\ r_1,\ r',\ x,\ y,\ z,\ x',\ y',\ z'$$

$$\rho,\ \xi,\ \eta,\ \zeta,\ \xi',\ \eta',\ \zeta,\ \alpha,\ \beta,\ \gamma$$

whose values, as t approaches t_1 are denoted

$$(r)_0,\ (r_0)_0,\ (r_1)_0,\ (r')_0,\ x_0,\ y_0,\ z_0,\ x_0',\ y_0',\ z_0'$$

$$\rho_0,\ \xi_0,\ \eta_0,\ \zeta_0,\ \xi_0',\ \eta_0',\ \zeta_0',\ \alpha_0,\ \beta_0,\ \gamma_0$$

We assume furthermore that

$$(r)_0 < \frac{k_1}{2} \tag{183}$$

and

$$\rho_1 \geq 14k_1 \tag{184}$$

where k_1 designates a positive constant whose value will be determined later.

We would like to have the unknowns 181 expressed as power series in $u - u_1$ and to determine a lower limit of the radii of convergence of these developments. This will be the goal of the following exposition. As we approach collision

$$(r_0)_0, (r_1)_0 > \rho_1 - (r)_0 \tag{185}$$

from 183 and 184

$$(r_0)_0, (r_1)_0 > 14k_1 - \frac{k_1}{2}$$

or

$$(r_0)_0, (r_1)_0 > \frac{27k_1}{2} \tag{186}$$

also

$$\frac{fM}{m_0 (r_0)_0} + \frac{fM}{m_1 (r_1)_0} < \frac{2fM (m_0 + m_1)}{27k_1 m_0 m_1} \tag{187}$$

we let

$$\Lambda_1 = \frac{m_2 f (m_0 + m_1)^2}{27 m_0 m_1 k_1} + \frac{|K|}{4g} \tag{188}$$

and from 162

$$x'^2 + y'^2 + z'^2 = \frac{r^2}{g} (2V - K) - \frac{h}{g} r^2 (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$

using 80 we have

$$x'^2 + y'^2 + z'^2 = \frac{2r^2 fM}{g} \left[\frac{1}{m_0 r_0} + \frac{1}{m_1 r_1} + \frac{1}{m_2 r_2}\right] + \frac{r^2 K}{g} - \frac{h}{g} r^2 (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$

incorporating 187

$$x'^2 + y'^2 + z'^2 < \frac{2r^2}{g} \frac{2Mf (m_0 + m_1)}{27 m_0 m_1 k_1} + \frac{2r fM}{gm_2} + \frac{r^2 K}{g}$$

using 188 and the definition of g (27)

$$x'^2 + y'^2 + z'^2 < 4(r)_0^2 \Lambda_1 + 2(r)_0 f\left(m_0 + m_1\right)$$

or

$$x_0'^2 + y_0'^2 + z_0'^2 < 2(r)_0 \left[f\left(m_0 + m_1\right) + 2\Lambda_1 (r)_0\right]$$

it follows that

$$|x_0'|, \; |y_0'|, \; |z_0'| \quad \text{and} \quad |r_0'| < \sqrt{2(r)_0 \left[f\left(m_0 + m_1\right) + 2\Lambda_1 (r)_0\right]}$$

from 168 and the above inequality

$$|\alpha_0|, \; |\beta_0|, \quad \text{and} \quad |\gamma_0| < 3f\left(m_0 + m_1\right) + 4\Lambda_1 (r)_0$$

using 183

$$|x_0'|, \; |y_0'|, \; |z_0'| \quad \text{and} \quad |(r')_0| < \sqrt{k_1 \left[f\left(m_0 + m_1\right) + \Lambda_1 k_1\right]} \tag{189}$$

and

$$|\alpha_0|, \; |\beta_0| \quad \text{and} \quad |\gamma_0| < 3f\left(m_0 + m_1\right) + 2\Lambda_1 k_1 \tag{190}$$

also from 183 we have

$$|x_0|, \; |y_0| \quad \text{and} \quad |z_0| \leq \frac{k_1}{2} \tag{191}$$

If we now consider the case where $(r)_1 = 0$ for $t = t_1$

$$x_1 = y_1 = z_1 = (r)_1 = 0 \tag{192}$$

by using 118 with 105

$$x_1' \qquad y_1' \qquad z_1' \qquad r_1' \qquad 0 \tag{193}$$

from 107 (in scalar form) we have

$$\sqrt{r}\,\frac{d}{dt} \qquad \sqrt{2f(m_0 + m_1)} \qquad \sqrt{r}\,\frac{dy}{dt} \quad \chi \sqrt{2f(m_0 + m_1)} \qquad \sqrt{r}\,\frac{dz}{dt} \qquad \sqrt{2f(m_0 + m_1)} \tag{194}$$

where $\ ,\chi,\ $ are components of the unit vector $\ $ and are further defined by

$$\lim_{t \to t_1} \frac{x}{r} \quad \phi \qquad \lim_{t \to t_1} \frac{y}{r} \quad \chi \qquad \lim_{t \to t_1} \frac{z}{r} \quad \psi \tag{195}$$

using 160, 168 and 195

$$\alpha_1 \quad (m_0 + m_1)\,\phi \qquad \beta_1 \quad (m_0 + m_1)\,\chi \qquad \gamma_1 \quad (m_0 + m_1)\,\psi \tag{196}$$

The values of $\ _1,\ _1,\ _1$ are seen to satisfy 190.

We now turn to look at the upper limits of the absolute value of the second member of Equations 172, and state that the unknowns verify the conditions:

$$|r - (r)_1|,\ |x - x_1|,\ |y - y_1|,\ |z - z_1|,\ |\xi - \xi_1|,\ |\eta - \eta_1|$$

and

$$|\zeta - \zeta_1| < \frac{k_1}{2}$$

$$|r' - (r')_1|,\ |x' - x_1'|,\ |y' - y_1'|$$

and

$$|z' - z_1'| \quad k'$$

$$|\dot{\xi} - \dot{\xi}_1|,\ |\dot{\eta} - \dot{\eta}_1|$$

and

$$|\dot{\ } - \dot{\ }_1| < k'$$

$$|\alpha - \alpha_1|,\ |\beta - \beta_1|$$

and

$$|\gamma - \gamma_1| \quad k$$

$$|\ \ t_1| \quad \tau' \tag{197}$$

The constants k', k'', $\bar{k}$ and ρ' are finite, positive quantities and we will determine their values later.

Using the last relation in 28' in scalar form

$$r_0^2 = (\xi - \mu x)^2 + (\eta - \mu y)^2 + (\zeta - \mu z)^2$$

which is equivalent to

$$\begin{aligned} r_0^2 = \rho_1^2 &+ 2\xi_1\left[(\xi - \xi_1) - \mu(x - x_1) - \mu x_1\right] + \left[(\xi - \xi_1) - \mu(x - x_1) - \mu x_1\right]^2 \\ &+ 2\eta_1\left[(\eta - \eta_1) - \mu(y - y_1) - \mu y_1\right] + \left[(\eta - \eta_1) - \mu(y - y_1) - \mu y_1\right]^2 \\ &+ 2\zeta_1\left[(\zeta - \zeta_1) - \mu(z - z_1) - \mu z_1\right] + \left[(\zeta - \zeta_1) - \mu(z - z_1) - \mu z_1\right]^2 \end{aligned}$$

and we write

$$r_0^2 = \rho_1^2 + \underline{P}_1 \qquad (198)$$

where $\underline{P}_1$ is a polynomial in

$$x - x_1,\ y - y_1,\ z - z_1,\ \xi - \xi_1,\ \eta - \eta_1,\ \zeta - \zeta_1,\ x_1,\ y_1,\ z_1$$

Since

$$|\xi_1|,\ |\eta_1|$$

and

$$|\zeta_1| \leq \rho_1 \qquad (199)$$

We see that by 191 and 197

$$|\underline{P}_1| < 12\rho_1 k_1 + 12k_1^2 \qquad (200)$$

and from 184

$$k_1 \leq \frac{\rho_1}{14}$$

so that

$$|\underline{P}_1| < \frac{45}{49}\rho_1^2 \tag{201}$$

We now see from 198, 184 and 201 that $1/r_0$ is developable into a series, of powers in $x - x_1$, $y - y_1$, $z - z_1$, $\xi - \xi_1$, $\eta - \eta_1$ and $\zeta - \zeta_1$. From 198

$$r_0^2 > \rho_1^2 - \underline{P}_1$$

and using 201

$$|r_0| > \frac{2}{7}\rho_1 \tag{202}$$

Similarly, if we start with

$$r_1^2 = (\xi + \lambda x)^2 + (\eta + \lambda y)^2 + (\zeta + \lambda z)^2$$

we can show

$$|r_1| > \frac{2}{7}\rho_1 \tag{203}$$

Using 184, 202 and 203 become

$$|r_0|, |r_1| > 4k_1 \tag{204}$$

Here it is evident that both $1/r_0$ and $1/r_1$ are analytic in the interval before collision. We state that

$$\left|\frac{\xi}{r_0^3}\right|, \left|\frac{\eta}{r_0^3}\right|, \left|\frac{\zeta}{r_0^3}\right|, \left|\frac{\xi}{r_1^3}\right|, \left|\frac{\eta}{r_1^3}\right|, \left|\frac{\zeta}{r_1^3}\right| < \frac{15}{64k_1^2}$$

$$\left|\frac{x}{r_0^3}\right|, \left|\frac{y}{r_0^3}\right|, \left|\frac{z}{r_0^3}\right|, \left|\frac{x}{r_1^3}\right|, \left|\frac{y}{r_1^3}\right|, \left|\frac{z}{r_1^3}\right| < \frac{1}{64k_1^2} \tag{205}$$

The first inequality is shown as follows: from 197 and 184

$$|\xi - \xi_1| < k_1 \leq \frac{\rho_1}{14}$$

whereby from 199

$$|\xi| < \frac{\rho_1}{14} + \rho_1 = \frac{15\rho_1}{14}$$

and by using the above inequality and 202

$$\left|\frac{\xi}{r_0^3}\right| < \frac{735}{16\rho_1^2}$$

or finally by 184

$$\left|\frac{\xi}{r_0^3}\right| < \frac{15}{64k_1^2}$$

The other inequalities of the first line in 205 are handled analogously. The inequalities involved in the second line of 205, follow from 204, and 191, 197 which together show that

$$|x|, |y|, |z| < k_1 \tag{206}$$

Using 154 to define $|fX|$, $|fY|$, $|fZ|$, $|f\Xi|$, $|fH|$, $|f\Sigma|$, remembering that the sum of absolute values is greater than the absolute value of the sums, and employing 205 we find,

$$|fX|, |fY|, |fZ| < \frac{m_2}{2k_1^2}$$

$$|f\Xi|, |fH|, |f\Sigma| < \frac{M}{2k_1^2} \tag{207}$$

Now using 167 to define $|L|$, and employing 204, 206, 207 we note that

$$|L| < \lambda_1 \tag{208}$$

where

$$\lambda_1 = \frac{m_2}{2k_1}\left[3 + \frac{(m_0 + m_1)^2}{m_0\, m_1}\right] + \frac{h}{g}\left(v_1^2 + 6v_1\, k_1' + 3k_1'^2\right) + \frac{|K|}{g} \tag{209}$$

where the factor $6v_1\, k_1' + 3k_1'^2$ has been arbitrarily added and the velocity v_1 is given by

$$v_1 = \sqrt{\dot{\xi}_1^2 + \dot{\eta}_1^2 + \dot{\zeta}_1^2} \tag{210}$$

Now from 183, 189, 190, 197, 210

$$|r| < k_1$$

$$|x'|,\ |y'|,\ |z'|,\ |r'| < k' + \sqrt{k_1\left(m_0 + m_1 + \Lambda_1\, k_1\right)}$$

$$|\dot{\xi}|,\ |\dot{\eta}|,\ |\dot{\zeta}| < v_1 + k''$$

$$|\alpha|,\ |\beta|,\ |\gamma| < \bar{k} + 3f\left(m_0 + m_1\right) + 2\,\Lambda_1\, k_1 \tag{211}$$

from 208, 211

$$\left|m_0 + m_1 + rL\right| < m_0 + m_1 + \lambda_1\, k_1$$

from 207, 211

$$|\alpha + r^2\, fX|,\ |\beta + r^2\, fY|,\ |\gamma + r^2\, fZ| < \bar{k} + 3f\left(m_0 + m_1\right) + 2\,\Lambda_1\, k_1 + \frac{m_2}{2}$$

from 207, 208, 211

$$|fX\, r\, r' + Lx'|,\ |fY\, r\, r' + Ly'|,\ |fZ\, r\, r' + Lz'| < \left(\lambda_1 + \frac{m_2}{2k_1}\right)\left[k' + \sqrt{k_1\left(m_0 + m_1 + \Lambda_1\, k_1\right)}\right]$$

from 211

$$|r\dot{\xi}|,\ |r\dot{\eta}|,\ |r\dot{\zeta}| < k_1\left(v_1 + k''\right)$$

from 207 and 211

$$|rf\Xi|,\ |rfH|,\ |rf\dot{Z}| < \frac{M}{2k_1}$$

Returning to the system 172, we conclude that the quantities which correspond to the ratios q_1'/Q_1', q_2'/Q_2' + · · · , q_n'/Q_n' in the Cauchy-Picard Theorem, in the present case are;

$$\frac{k_1}{2\left[k' + \sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}\right]}$$

$$\frac{k'}{m_0 + m_1 + \lambda_1 k_1}$$

$$\frac{k'}{\bar{k} + \frac{m_2}{2} + 3\left(m_0 + m_1\right) + 2\Lambda_1 k_1}$$

$$\frac{\bar{k}}{\left(\lambda_1 + \frac{m_2}{2k_1}\right)\left[k' + \sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}\right]}$$

$$\frac{1}{2\left(v_1 + k''\right)}$$

$$\frac{2k_1 k'}{M}$$

$$\frac{\tau'}{k_1}$$

Since τ' can be arbitrarily fixed, we give it a value such that τ'/k_1 is larger than the other ratios and we choose for k', k'' and $\bar{k}$ the values

$$k' = \sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}$$

$$k'' = \frac{1}{2}\sqrt{\frac{M}{k_1}}$$

$$\bar{k} = m_0 + m_1 + \Lambda_1 k_1 \qquad (212)$$

These choices of k', k'', $\bar{k}$ bring the above ratios to the following form

$$\frac{\sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}}{4\left(m_0 + m_1 + \Lambda_1 k_1\right)}$$

$$\frac{\sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}}{m_0 + m_1 + \lambda_1 k_1}$$

$$\frac{\sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}}{\frac{m_2}{2} + 4\left(m_0 + m_1\right) + 3\Lambda_1 k_1}$$

$$\frac{\sqrt{k_1\left(m_0 + m_1 + \Lambda_1 k_1\right)}}{2\lambda_1 k_1 + m_2}$$

$$\frac{1}{2v_1 + \sqrt{\frac{M}{k_1}}}$$

$$\sqrt{\frac{k_1}{M}}$$

We shall designate by Q_2' the smallest of these quantities. Then by the Cauchy-Picard Theorem we know

1) that, in the solution of the equations 172, which were deduced from 29, the quantities (K), are developable in series with powers of $u - u_1$

2) that these series converge for

$$|u - u_1| < Q_2' \tag{213}$$

3) that the inequalities 197 will be such that "u" verifies the inequality 213.

We now consider the case where one of the distances (r) goes to zero as $t \to t_1$. From 50

$$\lim_{t \to t_1} R = R_1$$

such that

$$R_1 = \sqrt{h}\,\rho_1$$

and from 184 as $t \to t_1$

$$\rho_1 = 14k_1$$

so that

$$k_1 = \frac{R_1}{14\sqrt{h}} \tag{214}$$

As $t \to t_1$, $r \to 0$ and

$$(r_0)_1 = (r_1)_1 = \rho_1 \tag{215}$$

Using the Equations 172, and the method of undetermined coefficients the following series solutions are obtained. These results can also be obtained using the asymptotic behavior of $d\vec{r}/dt$ and dr/dt. Only the first terms are given.

$$\xi = \xi_1 + \frac{f(m_0 + m_1)}{6} \dot{\xi}_1 (u - u_1)^3 + \cdots \qquad \dot{\xi} = \dot{\xi}_1 - \frac{Mf(m_0 + m_1)}{6\rho_1^3} \xi_1 (u - u_1)^3 + \cdots$$

$$\eta = \eta_1 + \frac{f(m_0 + m_1)}{6} \dot{\eta}_1 (u - u_1)^3 + \cdots \qquad \dot{\eta} = \dot{\eta}_1 - \frac{Mf(m_0 + m_1)}{6\rho_1^3} \eta_1 (u - u_1)^3 + \cdots$$

$$\zeta = \zeta_1 + \frac{f(m_0 + m_1)}{6} \dot{\zeta}_1 (u - u_1)^3 + \cdots \qquad \dot{\zeta} = \dot{\zeta}_1 - \frac{Mf(m_0 + m_1)}{6\rho_1^3} \eta_1 (u - u_1)^3 + \cdots$$

$$\alpha = \alpha_1 + \cdots \qquad \beta = \beta_1 + \cdots \qquad \gamma = \gamma_1 + \cdots$$

$$x = \frac{f(m_0 + m_1)}{2} \phi(u - u_1)^2 + \cdots \qquad x' = f(m_0 + m_1)\, \phi(u - u_1) + \cdots$$

$$y = \frac{f(m_0 + m_1)}{2} \chi(u - u_1)^2 + \cdots \qquad y' = f(m_0 + m_1)\, \chi(u - u_1) + \cdots$$

$$z = \frac{f(m_0 + m_1)}{2} \psi(u - u_1)^2 + \cdots \qquad z' = f(m_0 + m_1)\, \psi(u - u_1) + \cdots$$

$$r = \frac{f(m_0 + m_1)}{2} (u - u_1)^2 + \cdots \qquad r' = f(m_0 + m_1)(u - u_1) + \cdots \tag{216}$$

$$t - t_1 = \frac{f(m_0 + m_1)}{6}(u - u_1)^3 + \cdots \tag{217}$$

Looking at the series 217, we can see that $u - u_1$ can be developed into a series of $(t - t_1)^{1/3}$ and substituting this series in place of $u - u_1$ in the formulas 216, we find that the quantities ξ, η, ζ, $\cdots$ are also developable in powers of $(t - t_1)^{1/3}$. The quantities u, ξ, η, $\cdots$ considered as functions of t, have singular points at $t = t_1$.

The same series can be used to describe the movement of the bodies after collision.

From 216 we note that the ratios x/r, y/r, z/r tend toward the same limits (i.e., ϕ, χ, ψ) when t goes to t_1 in an increasing or decreasing fashion. We must conclude from this, that the motion approaching and exiting from collision forms a cusp. Of course, investigating the motion of a colliding body, after collision, is only of mathematical interest and the fact that any motion at all can be investigated after collision points up a divorce of mathematics and physical motion.

DETERMINING A LOWER LIMIT FOR "R"–DOUBLE COLLISION

It will now be our purpose to reinvestigate R, (Equation 50) with the idea of finding its lower limit for the case when the constant of the area integral is not equal to zero (two body collision). From Equation 50, we see that, in the case of two body collision, R goes toward a finite and non zero limit. If we define an interval $t' < t < t''$, then from 75b.

$$H'' \leq H'$$

$$R'' \leq R'$$

and from Equation 71

$$H'' = R''\left(\frac{dR''}{dt}\right)^2 + KR'' + \frac{g^2 h^2 c^2}{R''}$$

so that

$$H' \geq R''\left(\frac{dR''}{dt}\right)^2 + KR'' + \frac{g^2 h^2 c^2}{R''} \tag{218}$$

and

$$\frac{g^2 h^2 c^2}{R''} < H' - KR'' < H' + |K| R'$$

and finally if R doesn't have a minimum for $t = t'$

$$R'' \geq \frac{g^2 h^2 c^2}{H' + |K| R'} \tag{219}$$

The inequality 219 remains valid until t'' goes through $\bar{t}$, where dR/dt changes sign, and for the minimum value of R', where $dR'/dt = 0$, the equality

$$H' = R' \left(\frac{dR'}{dt}\right)^2 + KR' + \frac{g^2 h^2 c^2}{R'}$$

gives us

$$R' > \frac{g^2 h^2 c^2}{H' + |K| R'}$$

for $K > 0$ or $K < 0$. We see that the relation 219 holds for the minimum value of R so that we can generalize and say, Theorem 4:

$$R \geq \frac{g^2 h^2 c^2}{H' + |K| R'} \tag{220}$$

is valid up to the point where $R = R_{max}$.

If $K \leq 0$, Equation 51 shows that $d^2 R^2/dt^2$ is never negative from which one concludes that R doesn't have a maximum. In this case, R goes to infinity when t goes to infinity, and 220 therefore gives a lower limit to R which remains valid for all values of time, If $K > 0$ and R doesn't have a maximum for finite values of t_1 Equation 220 will again give the lower limit for R. This particular limit can be expressed in the following manner. We multiply the numerator and denominator of 220 by R and employ Equation 71, to get

$$R \geq \frac{g^2 h^2 c^2 R}{R\left(\frac{dR}{dt}\right)^2 + 2KR^2 + g^2 h^2 c^2}$$

If we let R^0 and dR^0/dt be the values of R and dR/dt for $t = 0$ and let

$$g^2 h^2 c^2 = f_1^2 \tag{221}$$

we have Theorem 5:

$$R \geq \frac{f^2 R^0}{\left(R^0 \frac{dR^0}{dt}\right)^2 + f_1^2} \tag{222}$$

if $K \leq 0$ and

no maxima

$$R \geq \frac{f^2 R^0}{\left(R^0 \frac{dR^0}{dt}\right)^2 + 2KR^{02} + f_1^2} \tag{223}$$

if $K > 0$ and R has no maximum for finite values of t.

It remains for us to find a lower limit of R when $K > 0$ and R has a maximum for a finite value of time (i.e., "S" expression 289). We will find this lower limit on R, for all values of time, such that it only depends upon f_1 and K. We assume that R has a maximum R' for $t = t'$ so that we consider the case where

$$K > 0 \tag{224}$$

and

$$\frac{dR'}{dt} = 0 \tag{225}$$

Now from 71 and 225 at $t = t'$

$$H' = KR' + \frac{f_1^2}{R'} \tag{226}$$

Since R' is a maximum, there exists in the neighborhood of t', and instant t'' such that the derivative dR/dt does not change sign and that $R < R'$ in the interval from t' to t''. From 218 and 226

$$R''\left(\frac{dR''}{dt}\right)^2 + KR'' + \frac{f_1^2}{R''} \leq KR' + \frac{f_1^2}{R'} \tag{227}$$

so that

$$K(R' - R'') \geq f_1^2\left(\frac{1}{R''} - \frac{1}{R'}\right) = \frac{f_1^2 (R' - R'')}{R'' R'}$$

and

$$K \geq \frac{f_1^2}{R' \, R''}$$

$$KR'^2 > f_1^2 \qquad (228)$$

and finally by taking the square root of both sides and multiplying by f_1 we have

$$\frac{f_1^2}{R'} < f_1 \sqrt{K} \qquad (229)$$

now from 226, 229 we have

$$H' < f_1 \sqrt{K} + KR' \qquad (230)$$

Starting with 223, and using 227, 226 and 230 respectively we see that

$$R > \frac{f_1^2}{R'' \left(\frac{dR''}{dt}\right)^2 + 2KR'' + \frac{f_1^2}{R''}} > \frac{f_1^2}{2KR' + \frac{f_1^2}{R'}} = \frac{f_1^2}{H' + KR'} > \frac{f_1^2}{f_1 \sqrt{K} + 2KR'}$$

or

$$R > \frac{f^2}{f_1 \sqrt{K} + 2KR'} \qquad (231)$$

Now this lower limit on R will become smaller as t goes to $\pm\infty$ if R' gets larger as t goes to $\pm\infty$.

We wish to find a positive, fixed limit which remains valid however large the maximum R'. With this in mind we note 156 and write

$$2V - K \geq 0$$

and from 80

$$\frac{1}{m_0 \, r_0} + \frac{1}{m_1 \, r_1} + \frac{1}{m_2 \, r_2} \geq \frac{K}{2M} \qquad (232)$$

where f is set $= 1$. If we designate r_m as the smallest distance r_i, we have from 232

$$\frac{1}{r_m}\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right) \geq \frac{K}{2M} \tag{233}$$

Now we note that $r_m \leq q$ where

$$q = \frac{2M}{K}\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right) \tag{234}$$

The movement of the three bodies will be such that one of the distances r_i will be the smallest. Since the distances r_i are continuous functions of time, it is evident that each time that a certain distance ceases to be the smallest, it will become equal to another distance. At this instant, both distances are considered the minimum distance and 234 insures they are both $\leq q$. The third distance of the isosoles triangle would be $\leq 2q$. To simplify our following formulas we will say that all the distances are $<q\sqrt{5}$. From 52 and the fact that r_0, r_1, and $r_2 \leq \sqrt{5}\, q$ we write

$$R < R_0 \tag{235}$$

where R_0 is denoted by the positive radical of the equation

$$R_0^2 = 5q^2\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right) \tag{236}$$

We conclude that in an interval of time where $R \geq R_0$ a single distance remains constantly $<q$.

We now, consider the movement in an interval of time where the inequality

$$R \geq R_0 \tag{237}$$

is constantly verified. We also let $r_2 \equiv r$ be the distance which remains smaller than q, or

$$r < q \tag{238}$$

We will now deduce from 237 and 238 some other needed inequalities. We shall also determine a certain value $\bar{R}_0$ of the quantity R which will play an important role. With this in mind, it can be shown that

$$R_0^2 = \left(g + \sigma^2 h\right) q^2 \tag{239}$$

where g and h are defined in 27, 40 and

$$\sigma^2 = 4 + \frac{4m_0 m_1}{m_2 \left(m_0 + m_1\right)} + \frac{m_0^2 + m_0 m_1 + m_1^2}{\left(m_0 + m_1\right)^2} \tag{240}$$

These relations are proved by substituting 240 into 239 and winding up with 236. We can see from 240 that

$$\sigma > 2 \tag{241}$$

From 50, 237, 239 and 238 respectively

$$gr^2 + h\rho^2 \geq R_0^2 = \left(g + \sigma^2 h\right) q^2 > gr^2 + \sigma^2 hq^2$$

or

$$\rho > \sigma q \tag{242}$$

and from 238

$$r < \frac{\rho}{\sigma} \tag{243}$$

We see from 185, that $r_0 > \rho - r$, $r_1 > \rho - r$ and from 243 and 242

$$\left.\begin{aligned} r_0 &> \frac{(\sigma - 1)}{\sigma} \rho > (\sigma - 1) q \\ r_1 &> \frac{(\sigma - 1)}{\sigma} \rho > (\sigma - 1) q \end{aligned}\right\} \tag{244}$$

Rewriting Equation 50 as

$$h\rho^2 = R^2 - gr^2 \tag{245}$$

so that

$$\rho \leq \frac{R}{\sqrt{h}} \tag{246}$$

Using 239, 237, 238

$$gr^2 < gq^2 = \frac{g R_0^2}{g + \sigma^2 h} \leq \frac{g R^2}{g + \sigma^2 h} \tag{247}$$

we now can see from 245, 247 that

$$h\rho^2 > R^2 - \frac{g R^2}{g + \sigma^2 h} = R^2 \left[1 - \frac{g}{g + \sigma^2 h}\right] = R^2 \left[\frac{\sigma^2 h}{g + r^2 h}\right]$$

or

$$\rho > \frac{eR}{\sqrt{h}} \tag{248}$$

where the constant

$$e = \sqrt{\frac{\sigma^2 h}{g + \sigma^2 h}} \tag{249}$$

is smaller than one.

We now define

$$\bar{R}_0 = \frac{R_0}{e} \tag{250}$$

where we see that $\bar{R}_0 > R_0$. Designating by ρ_0, $\bar{\rho}_0$ the values of ρ which correspond to the values R_0 and $\bar{R}_0$ of R, we see that after 246 and 248,

$$\rho_0 \leq \frac{R_0}{\sqrt{h}}$$

$$\bar{\rho}_0 > e \frac{\bar{R}_0}{\sqrt{h}} \tag{251}$$

and consequently $\bar{\rho}_0 > \rho_0$. One must conclude from 251 that in the interval of time in which R decreases from $\bar{R}_0$ to R_0, there exists an instant $\bar{t}$ where the inequality

$$\frac{d\rho}{dt} < 0 \tag{252}$$

is verified. That is to say, that ρ is decreasing at an instant $\bar{t}$. Letting $\bar{R}$, $\bar{\rho}$, $\cdots$ designate the values of R, ρ, $\cdots$ for an instant $\bar{t}$, and using 252, 250 and 251 we have

$$\frac{d\bar{\rho}}{dt} < 0 \tag{253}$$

$$R_0 \leq \bar{R} \leq \bar{R}_0 \tag{254}$$

$$e^2 \frac{\bar{R}_0}{\sqrt{h}} < \bar{\rho} \leq \frac{\bar{R}_0}{\sqrt{h}} \tag{255}$$

The differential equations of the movement (35) remain invariable when "t" is replaced by "-t". From this we deduce that the lower limits are independent of t and are valid before and after the maximum of R. We shall study the values of R after the moment t' where R passes through a maximum R'. It will be necessary to divide the maxima into three classes, according to the sizes of the maxima R' and the minima R'' which follow it.

The first class will refer to the maxima which verify the condition

$$R' \leq \bar{R}_0 \tag{256}$$

and after 231 we have

$$R > \frac{f_1^2}{f_1 \sqrt{K} + 2K\bar{R}_0}$$

The second class will include the maxima for which

$$R' > \bar{R}_0$$

and

$$R'' \geq R_0 \tag{257}$$

and since R is in the interval of maxima R' and minima R'',

$$R \geq R_0$$

The first two classes are seen to have lower limits. The third class satisfies the conditions that

$$R' > \bar{R}_0$$

and

$$R'' < R_0 \tag{258}$$

The next section will be concerned with finding a lower limit for R in this third case.

Consider a maximum of the third case. R will diminish constantly from R′ to a value smaller than R_0 since the minimum is less than R_0. This third case corresponds to the relations 253, 254, 255. To find an inferior limit for R, we will seek a superior limit for H (Equation 71) when $t = \bar{t}$. In order to accomplish this, we must find a limit for the absolute value of the derivitive $d\bar{R}/dt$. Since from Equation 54,

$$\bar{R}\frac{d\bar{R}}{dt} = g\bar{r}\frac{d\bar{r}}{dt} + h\bar{\rho}\frac{d\bar{\rho}}{dt} \tag{259}$$

we see that the superior limits for the expressions

$$\left|\bar{r}\frac{d\bar{r}}{dt}\right|$$

and

$$\left|\bar{\rho}\frac{d\bar{\rho}}{dt}\right|$$

are necessary. In an effort to determine the upper limit of the former relation, we note that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \geq \left(\frac{dr}{dt}\right)^2 \tag{260}$$

and

$$\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 \geq \left(\frac{d\rho}{dt}\right)^2 \tag{261}$$

and from 156 it follows that

$$g\left(\frac{dr}{dt}\right)^2 + h\left(\frac{d\rho}{dt}\right)^2 \leq 2V - K \tag{262}$$

If $R \geq R_0$ (237) we see from 244, 241 and 234

$$\frac{2M}{m_0 r_0} + \frac{2M}{m_1 r_1} < \frac{2M}{(\sigma - 1)q}\left(\frac{1}{m_0} + \frac{1}{m_1}\right) = \frac{\frac{1}{m_0} + \frac{1}{m_1}}{\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}} \frac{K}{\sigma - 1} < K \tag{263}$$

Using the definition of V (Equation 80) and setting the constant $f = 1$ for simplicity, we see from 263 that

$$2V - K < \frac{2M}{m_2 r} \tag{264}$$

Now incorporating 264 into 262 we have

$$g\left(\frac{dr}{dt}\right)^2 + h\left(\frac{d\rho}{dt}\right)^2 < \frac{2M}{m_2 r}$$

or

$$\left(\frac{dr}{dt}\right)^2 < \frac{2M}{g m_2 r}$$

so that

$$\left|\bar{r} \frac{d\bar{r}}{dt}\right| < \sqrt{\frac{2Mr}{g m_2}}$$

and from 234

$$\left|\bar{r} \frac{d\bar{r}}{dt}\right| < \sqrt{\frac{2Mq}{g m_2}} \tag{265}$$

It remains for us to find an upper limit for $|\bar{\rho}\, d\bar{\rho}/dt|$. To this effect we write the definition $\rho^2 = \xi^2 + \eta^2 + \zeta^2$ and differentiate twice with respect to t and find that,

$$\rho \frac{d^2 \rho}{dt^2} + \left(\frac{d\rho}{dt}\right)^2 = \xi \frac{d^2 \xi}{dt^2} + \eta \frac{d^2 \eta}{dt^2} + \zeta \frac{d^2 \zeta}{dt^2} + \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2$$

and after 261

$$\rho \frac{d^2 \rho}{dt^2} \geq \xi \frac{d^2 \xi}{dt} + \eta \frac{d^2 \eta}{dt^2} + \zeta \frac{d^2 \zeta}{dt^2}$$

so that by means of 154

$$\rho \frac{d^2 \rho}{dt^2} \geq - M\left[\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right] \rho^2 + \lambda\mu M\left[\frac{1}{r_0^3} - \frac{1}{r_1^3}\right](\xi x + \eta y + \zeta z)$$

or

$$\rho \frac{d^2 \rho}{dt^2} \geq - M\left[\frac{\lambda}{r_0^2} \frac{\rho^2 - \mu(x\xi + y\eta + z\zeta)}{r_0} + \frac{\mu}{r_1^2} \frac{\rho^2 + \lambda(x\xi + y\eta + z\zeta)}{r_1}\right] \tag{266}$$

Now writing 28' in scalar form

$$r_0^2 = (\xi - \mu x)^2 + (\eta - \mu y)^2 + (\zeta - \mu z)^2$$

we have

$$r_0^2 = \rho^2 + \mu^2 r^2 - 2\mu(x\xi + y\eta + z\zeta)$$

and it follows that

$$\rho^2 r_0^2 = \rho^4 + \mu^2 r^2 \rho^2 - 2\mu \rho^2 (x\xi + y\eta + z\zeta)$$

or

$$\rho^2 r_0^2 = \left[\rho^2 - \mu(x\xi + y\eta + z\zeta)\right]^2 + \mu^2 \left[r^2 \rho^2 - (x\xi + y\eta + z\zeta)^2\right] \tag{267}$$

now since

$$|x\xi + y\eta + z\zeta| \leq r\rho$$

it follows that

$$\mu^2 \left[r^2 \rho^2 - (x\xi + y\eta + z\zeta)^2 \right] > 0$$

and from 267

$$\rho r_0 \geq \left| \rho^2 - \mu(x\xi + y\eta + z\zeta) \right| \tag{268}$$

In an analogous manner

$$\rho r_1 \geq \left| \rho^2 + \lambda(x\xi + y\eta + z\zeta) \right| \tag{269}$$

Now from 266, 268, 269

$$\rho \frac{d^2 \rho}{dt^2} \geq - M\left[\frac{\lambda}{r_0^2} \frac{\rho r_0}{r_0} + \frac{\mu}{r_1^2} \frac{\rho r_1}{r_1}\right]$$

and

$$\frac{d^2 \rho}{dt^2} \geq - M\left(\frac{\lambda}{r_0^2} + \frac{\mu}{r_1^2}\right)$$

Now from 244,

$$\frac{d^2 \rho}{dt^2} + \frac{M(\lambda + \mu)\sigma^2}{(\sigma - 1)^2 \rho^2} \geq 0$$

or

$$\frac{d^2 \rho}{dt^2} + \frac{C}{\rho^2} \geq 0 \tag{270}$$

where

$$c = \frac{M\sigma^2}{(\sigma - 1)^2} \tag{271}$$

Before going further, we must consider separately,

1) The case where $d\rho/dt < 0$ when t decreases from t' to $\bar{t}$.

2) The case where there exists an instant t''' between t' and $\bar{t}$ such that $d\rho/dt = 0$ for $t = t'''$ and $d\rho/dt < 0$ between t' and $\bar{t}$. These are two subcases of 258.

In the first case, between t' and $\bar{t}$, we see that $d\rho/dt < 0$ and from 270,

$$2\,\frac{d\rho}{dt}\left[\frac{d^2\rho}{dt^2} + \frac{c}{\rho^2}\right] \leq 0 \tag{272}$$

Integrating between the limits t' and $\bar{t}$

$$2\int_{t'}^{\bar{t}} \left(\frac{d\rho}{dt}\right) \frac{d^2\rho}{dt^2}\, dt + 2\int_{\rho'}^{\bar{\rho}} \frac{c}{\rho^2}\, \frac{d\rho}{dt}\, dt \leq 0$$

we have

$$\left(\frac{d\bar{\rho}}{dt}\right)^2 \leq \left(\frac{d\rho'}{dt}\right)^2 + \frac{2c}{\bar{\rho}} - \frac{2c}{\rho'}$$

and finally

$$\left(\frac{d\bar{\rho}}{dt}\right)^2 \leq \left(\frac{d\rho'}{dt}\right)^2 + \frac{2c}{\bar{\rho}} \tag{273}$$

In the second case we integrate 272 between t''' and $\bar{t}$, and noting that $d\rho/dt = 0$ for $t = t'''$ we have

$$\left(\frac{d\bar{\rho}}{dt}\right)^2 \leq \frac{2c}{\bar{\rho}} - \frac{2c}{\rho'''}$$

We can now see that the inequality 273 is valid in both the first and second cases. We shall now return to the determination of an upper limit for $|\bar{\rho}\, d\bar{\rho}/dt|$. R was, by hypothesis, a maximum for $t = t'$ so that $d^2 R^2/dt^2 \leq 0$ for $t = t'$ and after Equation 51, we see that

$$V' \leq K \tag{274}$$

where V' is the value of V when $t = t'$. From 274, we note $2V' - K \leq 2K - K = K$ and from 262

$$g\left(\frac{dr'}{dt}\right)^2 + h\left(\frac{d\rho'}{dt}\right)^2 \leq K \tag{275}$$

Since $dR'/dt = 0$ we find by differentiating Equation 50

$$gr' \frac{dr'}{dt} + h\rho' \frac{d\rho'}{dt} = 0$$

or by rearranging terms

$$g\left(\frac{dr'}{dt}\right)^2 = \frac{h^2 \rho'^2}{gr'^2} \left(\frac{d\rho'}{dt}\right)^2$$

substituting this into 275

$$\left(\frac{h^2 \rho'^2}{gr'^2} + h\right) \left(\frac{d\rho'}{dt}\right)^2 \leq K$$

or

$$\left[\frac{hgr'^2 + h^2 \rho'^2}{gr'^2}\right]\left(\frac{d\rho'}{dt}\right)^2 \leq K$$

and

$$\left(\frac{d\rho'}{dt}\right)^2 \leq \frac{Kgr'^2}{hR'^2}$$

after 238, 258, 255

$$\left(\frac{d\rho'}{dt}\right)^2 \leq \frac{Kgq^2}{h\bar{R}_0^2} \leq \frac{Kgq^2}{h^2 \bar{\rho}^2}$$

from 273

$$\left(\frac{d\bar{\rho}}{dt}\right)^2 < \frac{2c}{\bar{\rho}} + \frac{Kgq^2}{h^2\,\bar{\rho}^2}$$

multiplying both sides of this inequality by $\bar{\rho}^2$ and taking the square root, we have

$$\left|\bar{\rho}\,\frac{d\bar{\rho}}{dt}\right| < \sqrt{\frac{Kgq^2}{h^2} + 2c\,\bar{\rho}}$$

and after 255

$$\left|\bar{\rho}\,\frac{d\bar{\rho}}{dt}\right| < \sqrt{\frac{Kgq^2}{h^2} + \frac{2c\,\bar{R}_0}{\sqrt{h}}} \tag{276}$$

Now from 259 with the aid of 265 and 276

$$\left|\bar{R}\,\frac{d\bar{R}}{dt}\right| < \sqrt{\frac{2Mgq}{m_2}} + \sqrt{Kgq^2 + 2c\,\bar{R}_0\,h\sqrt{h}}$$

where it follows that

$$\bar{R}^2\left(\frac{d\bar{R}}{dt}\right)^2 < \frac{4Mgq}{m_2} + 2Kgq^2 + 4c\,\bar{R}_0\,h\sqrt{h}$$

since $\bar{R} > R_0$ (254)

$$\bar{R}\left(\frac{d\bar{R}}{dt}\right)^2 < \frac{1}{R_0}\left(\frac{4Mgq}{m_2} + 2Kgq^2 + 4c\,\bar{R}_0\,h\sqrt{h}\right)$$

Upon recalling Equation 71, and since $\bar{R} > R_0$ we have,

$$\bar{H} + K\bar{R} < S_2 \tag{277}$$

where

$$S_2 = \frac{1}{\bar{R}_0}\left(\frac{4Mgq}{m_2} + f_1^2 + 2Kgq^2 + 4c\,\bar{R}_0\,h\sqrt{h}\right) + 2K\,\bar{R}_0$$

$$f_1^2 = g^2 h^2 c^2 \qquad (278)$$

from 75b we note $\bar{H} + K\bar{R} \leq H' + KR'$ and from 220, 227

$$R > \frac{f_1^2}{S_2}$$

This is the lower limit for the third case, 258. Since the above inequality is true for all R, one sees that

$$R_0 > \frac{f_1^2}{S_2}$$

Now if we note the limits for the first and second cases (256, 257) and from the definition of S_2 (278) we see that the inequality

$$R > \frac{f_1^2}{S_2 + f_1\sqrt{K}} \qquad (279)$$

is valid for all three cases, (256, 257, 258). The next few pages will be devoted to further rearrangement of this lower limit on R.

We shall designate by "m" the smallest of the masses m_0, m_1, m_2. We note right away that

$$M > m_2 \geq m$$

$$m \leq \frac{M}{3}$$

$$m_0 + m_1 \geq 2m \qquad (280)$$

from 27, 40 we find

$$g = \frac{1}{m_0 + m_1} + \frac{1}{m_2} \leq \frac{3}{2m}$$

$$h = \frac{1}{m_0} + \frac{1}{m_1} \leq \frac{2}{m} \tag{281}$$

from 280 $1/m > 3/M$ and $1/m_0 + 1/m_1 + 1/m_2 > 9/M$ so that

$$\frac{9}{M} \leq \frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2} \leq \frac{3}{m} \tag{282}$$

from 234

$$q = \frac{2M}{K}\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right)$$

and since $9/M \leq 1/m_0 + 1/m_1 + 1/m_2$ we have $18/K \leq q$ and since $1/m_0 + 1/m_1 + 1/m_2 \leq 3/m$ we have

$$\frac{18}{K} \leq q \leq \frac{6M}{Km} \tag{283}$$

from 236

$$\frac{R_0}{q} = \sqrt{5}\sqrt{\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right)}$$

and using 282

$$3\sqrt{\frac{5}{M}} \leq \frac{R_0}{q} \leq \sqrt{\frac{15}{m}} \tag{284}$$

from 284,

$$\frac{q}{R_0} \leq \frac{1}{3}\sqrt{\frac{M}{5}}$$

and

$$\frac{1}{R_0} \leq \frac{1}{3q}\sqrt{\frac{M}{5}}$$

using 283, we find

$$\frac{1}{R_0} \leq \frac{K}{54}\sqrt{\frac{M}{5}} \tag{285}$$

Now from 284, $R_0 \leq q\sqrt{15/m}$ and 283,

$$R_0 \leq \frac{6M}{Km}\sqrt{\frac{15}{m}} \tag{286}$$

from 236, 239

$$5\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right) = g + \sigma^2 h$$

Using 40, 240

$$\sigma^2 h = \frac{4(m_0 + m_1)}{m_0 m_1} + \frac{4}{m_2} + \frac{m_0^2 + m_0 m_1 + m_1^2}{(m_0 + m_1)\, m_0 m_1}$$

$$= \frac{4(m_0 m_2 + m_1 m_2 + m_0 m_1)}{m_0 m_1 m_2} + \frac{m_0^2 + m_0 m_1 + m_1^2}{(m_0 + m_1)\, m_0 m_1}$$

so that from 249, 250, we have

$$\frac{\bar{R}_0}{R_0} = \sqrt{\frac{g + \sigma^2 h}{\sigma^2 h}} = \sqrt{\frac{5\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right)}{4\left(\frac{1}{m_0} + \frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{m_0^2 + m_0 m_1 + m_1^2}{m_0 m_1 (m_0 + m_1)}}} < \sqrt{\frac{5}{4}} \tag{287}$$

from 241, 271 we see that

$$C < 4M \tag{288}$$

Now from 278 and the inequalities 280, 281, 283, 285, 288.

$$S_2 < \left(\frac{8M}{m^2} + \frac{f_1^2 K}{54}\right)\sqrt{\frac{M}{5}} + 2\left(8\sqrt{10} + 15\sqrt{3}\right)\frac{M}{m\sqrt{m}}$$

and consequently

$$\frac{f_1^2}{S_2 + f\sqrt{K}} > \frac{m^2 f_1^2}{\left(8M + \frac{f_1^2 K m^2}{54}\right)\sqrt{\frac{M}{5}} + 2\left(8\sqrt{10} + 15\sqrt{3}\right)M\sqrt{m} + f_1\sqrt{K}\, m^2}$$

by replacing m by M/3 in the denominator and simplifying the numerical coefficients, we have

$$\frac{f_1^2}{S_2 + f_1\sqrt{K}} > \frac{f_1^2}{64 + \frac{f_1}{2}\sqrt{KM} + \frac{1}{1024} f_1^2 KM} \frac{m^2}{M\sqrt{M}}$$

and finally

$$S = \left(\frac{f_1 m}{8 + \frac{1}{32} f_1 \sqrt{KM}}\right)^2 \frac{1}{M\sqrt{M}} \tag{289}$$

While considering the maxima of the third class, we have supposed that r_2 remained smaller than q for $R \geq R_0$. However, since S is symmetrical with respect to the three masses m_0, m_1, m_2, it is evident that the result we have obtained remains true in the cases where $r_0 < q$ and $r_1 < q$ for $R > R_0$. In short we will have $R > S$ in the case where $K > 0$, $f > 0$ and R has at least one maximum for a finite value of t. Referring back to 222 and 223 we are now ready to state an important theorem of Sundman. Theorem 6: If the area integral is not zero for the three bodies one will always have

$$R \geq J \tag{290}$$

where J designates the quantity

$$\frac{f_1^2 R^0}{\left(R^0 \frac{dR^0}{dt}\right)^2 + f_1^2}$$

if $K \leq 0$ and the smaller of the quantities S, and

$$\frac{f_1^2 R^0}{\left(R^0 \frac{dR^0}{dt}\right)^2 + 2K R^{0^2} + f_1^2}$$

if $K > 0$.

The case of $K > 0$ has two limits corresponding to the cases where R passes through a maximum (S) and the case where R doesn't pass through a maximum.

The next section dealing with Sundman's exposition will treat the problem of finding a lower limit for the radius of convergence of the development into powers of $u - u_1$.

DETERMINATION OF A LOWER LIMIT FOR THE RADIUS OF CONVERGENCE FOR SERIES SOLUTIONS IN POWERS OF $u - u_1$

Theorem 7: If $f_1 > 0$, the two larger of the distances r_0, r_1, r_2 remain constantly superior to the quantity

$$\ell = \frac{1}{3}\sqrt{m}\, J \tag{291}$$

To prove the theorem we note from 52, that

$$R^2 \leq \frac{r_0^2 + r_1^2 + r_2^2}{m}$$

If the theorem were not true, at least two of the distances r_0, r_1, r_2 would assume, at a certain instant, values smaller than or equal to ℓ, (i.e., say $r_1, r_2 \leq \ell$). The third distance would be less than the sum of the other two, and hence it would be less than or equal to 2ℓ. (i.e., $r_0 \leq r_1 + r_2 \leq 2\ell$) Now

$$\frac{r_0^2 + r_1^2 + r_2^2}{m} \leq \frac{6\ell^2}{m} = \frac{2}{3} J^2$$

so that

$$R^2 \leq \frac{2}{3} J^2$$

This is a contradiction to the preceding theorem 290, and our present theorem is proved. We shall now fix, in a convenient manner, the constant k_1. Since r_0 and r_1 are the two larger distances, they will be considered greater than ℓ (291). Noting that $\rho > r_0 - r$ and $\rho > r_1 - r$, we conclude from 183 and 184 that

$$\rho > 14k_1$$

$$r < \frac{k_1}{2} \tag{292}$$

Now it is clear that $\rho > r_0 - r$ or $\rho > \ell - k_1/2$, and if we let $\ell = k_1/2 + 14k_1$, we arrive at the first inequality of 292. Therefore from 291, and fixing the value of k_0 from 132, 183 we have:

$$k_1 = \frac{2}{29}\ell = \frac{2}{87}\sqrt{m}\, J$$

and

$$k_0 = \frac{k_1}{28} = \frac{J\sqrt{m}}{1218} \tag{293}$$

Under the perturbative effect of the third body, the remaining two bodies can asymptotically approach each other as t goes toward infinity. The question arises, as to the nature of the velocity ($\dot{\rho}$) as the two bodies approach each other. It will be shown in the following subsection that the velocity $v = d\rho/dt$ remains below a finite limit when $r < k_1/2$, $r_0, r_1 > \ell$.

The proof begins by defining

$$\Lambda = \frac{m_2 (m_0 + m_1)}{2\ell} + \frac{m_0 m_1 m_2}{4M} |K| \tag{294}$$

with the assumptions being,

$$r < \frac{k_1}{2}$$

while

$$r_0, r_1 > \ell \tag{295}$$

from 45, 80, 295

$$g\left(\frac{d\vec{r}}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 \leq 2M\left(\frac{1}{m_0} + \frac{1}{m_1}\right)\frac{1}{\ell} + \frac{2M}{m_2 r} + |K|$$

or

$$g\left(\frac{d\vec{r}}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 \leq \frac{4M}{m_0 m_1 m_2}\left\{\frac{m_2 (m_0 + m_1)}{2\ell} + \frac{|K| m_0 m_1 m_2}{4M}\right\} + \frac{2M}{m_2 r}$$

from 27, 40, 294

$$g\left(\frac{d\vec{r}}{dt}\right)^2 + h\left(\frac{d\vec{\rho}}{dt}\right)^2 \leq 4gh\Lambda + \frac{2M}{m_2 r} \tag{296}$$

now

$$hv^2 \leq 4gh\Lambda + \frac{2M}{m_2 r}$$

and

$$v^2 \leq 4g\Lambda + \frac{2M}{m_2 r} \frac{m_0 m_1}{m_0 + m_1} = 2g\left[2\Lambda + \frac{M}{m_2 r} \cdot \frac{m_0 m_1}{m_0 + m_1} \cdot \frac{m_2 (m_0 + m_1)}{M}\right]$$

and

$$v^2 \leq 2g\left(\frac{m_0 m_1}{r} + 2\Lambda\right) \tag{297}$$

multiplying both sides of 297 by r^2 and noting $r < k_1/2$ we see that

$$rv < \sqrt{gk_1 (m_0 m_1 + \Lambda k_1)}$$

Starting with 296 we can easily show that

$$\left|r \frac{dx}{dt}\right|, \left|r \frac{dy}{dt}\right| \text{ and } \left|r \frac{dz}{dt}\right| < \sqrt{2hr (m_0 m_1 + 2\Lambda r)} \tag{298}$$

We now assume that when $r < k_1/2$

$$v^2 \geq D \tag{299}$$

where

$$D = 4g\left(\frac{m_0 m_1}{k_1} + \Lambda\right)$$

It will be our objective to show that the assumption 299 leads to a contradiction. We therefore assume the inequalities

$$r < \frac{k_1}{2}$$

$$rv < \sqrt{gk_1 (m_0 m_1 + \Lambda k_1)}$$

$$\left|r \frac{dx}{dt}\right|, \left|r \frac{dy}{dt}\right|$$

and

$$\left| r \frac{dz}{dt} \right| < \sqrt{hk_1 \left(m_0 m_1 + \Lambda k_1 \right)} \tag{300}$$

are verified when $v^2 \geq D$. From relations 41; 300, we note

$$(\xi\eta' - \eta\xi') = gc_0 - \frac{g}{h}\left[x \frac{dy}{dt} - y \frac{dx}{dt}\right]$$

$$|\xi\eta' - \eta\xi'| \leq g\,|c_0| + \frac{g}{h}\left[r \frac{dy}{dt} + r \frac{dx}{dt}\right] < g\left[|c_0| + 2\sqrt{\frac{k_1}{h}\left(m_0 m_1 + \Lambda k_1\right)}\right] = A \tag{301}$$

similarly

$$|\eta\zeta' + \zeta\eta'| < g\left[|c_1| + 2\sqrt{\frac{k_1}{h}\left(m_0 m_1 + \Lambda k_1\right)}\right] = B \tag{302}$$

and

$$|\zeta\xi' - \xi\zeta'| < g\left[|c_2| + 2\sqrt{\frac{k_1}{h}\, m_0 m_1 + \Lambda k_1}\right] = C \tag{303}$$

so that

$$\begin{aligned} |\xi\eta' - \eta\xi'| &< A \\ |\eta\zeta' - \zeta\eta'] &< B \\ |\zeta\xi' - \xi\zeta'| &< C \end{aligned} \tag{304}$$

It can be shown that

$$\rho^2 \left(\frac{d\rho}{dt}\right)^2 = \left(\vec{\rho} \cdot \frac{d\vec{\rho}}{dt}\right)^2 + \left(\vec{\rho} \times \frac{d\vec{\rho}}{dt}\right)^2$$

or

$$\rho^2\left(\frac{d\rho}{dt}\right)^2 = (\rho\cdot\dot{\rho})^2 + (\xi\dot{\eta}-\eta\dot{\xi})^2 + (\eta\dot{\zeta}-\zeta\dot{\eta})^2 + (\zeta\dot{\xi}-\xi\dot{\zeta})^2$$

now since $\dot{\rho} = v$ and $\rho > \ell - k_1/2$

$$\rho^2\left(\frac{d\rho}{dt}\right)^2 > v^2\left(\ell - \frac{k_1}{2}\right)^2 - A^2 - B^2 - C^2$$

so that

$$\left|\rho\frac{d\rho}{dt}\right| > W > 0 \qquad (305)$$

where

$$W = \sqrt{v^2\left(\ell - \frac{k_1}{2}\right)^2 - A^2 - B^2 - C^2} \qquad (306)$$

is verified if v satisfies the conditions $v^2 \geq D$

$$v^2\left(\ell - \frac{k_1}{2}\right)^2 - A^2 - B^2 - C^2 > 0 \qquad (307)$$

Employing Equations 154 we see that

$$\frac{d}{dt}\left(\rho\frac{d\rho}{dt}\right) = \rho\frac{d^2\rho}{dt^2} + \left(\frac{d\rho}{dt}\right)^2 = v^2 - M\left[\frac{\lambda}{r_0^2}\,\frac{\rho^2 - \mu(x\xi + y\eta + z\zeta)}{r_0} + \frac{\mu}{r_1^2}\,\frac{\rho^2 + \lambda(x\xi + y\eta + z\zeta)}{r_1}\right] \qquad (308)$$

and

$$\frac{dv^2}{dt} = 2v\frac{dv}{dt} = 2\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} = 2\left(\frac{d^2\xi}{dt^2} + \frac{d^2\eta}{dt^2} + \frac{d^2\zeta}{dt^2}\right)\frac{d\rho}{dt}$$

$$\frac{dv^2}{dt} = -2M\left(\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right)\rho\frac{d\rho}{dt} + 2M\lambda\mu\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)(x\dot{\xi} + y\dot{\eta} + z\dot{\zeta}) \qquad (309)$$

from 268, 269

$$\left|\rho^2 - \mu(x\xi + y\eta + z\zeta)\right| \le \rho\, r_0$$

$$\left|\rho^2 + \lambda(x\xi + y\eta + z\zeta)\right| \le \rho\, r_1$$

now 308 gives us

$$\frac{d}{dt}\left(\rho \frac{d\rho}{dt}\right) \ge v^2 - M\left(\frac{\lambda}{r_0}\frac{\rho}{r_0} + \frac{\mu}{r_1}\frac{\rho}{r_1}\right)$$

Since $r_0 > \rho - r > \rho - k_1/2$ and $r_1 > \rho - r > \rho - k_1/2$ and remembering $\rho > 14k_1$ (292) we have ρ/r_0 and

$$\frac{\rho}{r_1} < \frac{\rho}{\rho - \frac{k_1}{2}} = \frac{1}{1 - \frac{k_1}{2\rho}} < \frac{1}{1 - \frac{1}{28}} = \frac{28}{27}$$

so that by using 295

$$\frac{d}{dt}\left(\rho \frac{d\rho}{dt}\right) > v^2 - E \tag{310}$$

where

$$E = \frac{28}{27}\frac{M}{\ell} \tag{311}$$

from 309 we see that dv^2/dt will have the same sign as $-\rho\, d\rho/dt$ if

$$\left(\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3}\right)\left|\rho \frac{d\rho}{dt}\right| > \left|\lambda\mu\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)(x\dot{\xi} + y\dot{\eta} + z\dot{\zeta})\right| \tag{312}$$

now since

$$\frac{\lambda}{r_0^3} + \frac{\mu}{r_1^3} > \lambda\mu\left(\frac{1}{r_0^3} + \frac{1}{r_1^3}\right) > \left|\lambda\mu\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\right|$$

then 312 is true if

$$\rho \frac{d\rho}{dt} > (x\dot{\xi} + y\dot{\eta} + z\dot{\zeta})$$

from 300

$$\left| x\dot{\xi} + y\dot{\eta} + z\dot{\zeta} \right| \leq rv < \sqrt{gk_1 (m_0 m_1 + \Lambda k_1)}$$

now using 305 and if we assume

$$W \geq \sqrt{gk_1 (m_0 m_1 + \Lambda k_1)} \tag{313}$$

then

$$\left| \rho \frac{d\rho}{dt} \right| > W \geq \sqrt{gk_1 (m_0 m_1 + \Lambda k_1)} \geq rv = (x\dot{\xi} + y\dot{\eta} + z\dot{\zeta})$$

In short, we can say that if 313 is true then 312 is verified. We can go further and say that dv^2/dt has the sign of the quantity $-\rho\ d\rho/dt$ if

$$v^2 \left(\ell - \frac{k_1}{2}\right)^2 - A^2 - B^2 - C^2 \equiv W^2 \geq gk_1 (m_0 m_1 + \Lambda k_1)$$

and

$$v^2 \geq D \tag{314}$$

We now let

$$G_2 = \max \left\{ \sqrt{2E}\ ,\ \sqrt{D}\ ,\ \frac{2}{2\ell - k_1} \sqrt{A^2 + B^2 + C^2 + gk_1 (m_0 m_1 + \Lambda k_1)} \right\} \tag{315}$$

If we choose

$$G_2 = \sqrt{2E}$$

and

$$v \geq G_2 \tag{316}$$

it is immediately obvious from 310 that

$$\frac{d}{dt}\left(\rho \frac{d\rho}{dt}\right) > E \tag{310'}$$

If $G_2 = \sqrt{D}$ or

$$\frac{2}{2\ell - k_1} \sqrt{A^2 + B^2 + C^2 + gk_1 \left(m_0 m_1 + \Lambda k_1\right)}$$

then since it must be the maximum of the three, it follows that if $G_2 = \sqrt{D}$ then $D > 2E$ and if

$$G_2 = \frac{2}{2\ell - k_1} \sqrt{A^2 + B^2 + C^2 + gk_1 \left(m_0 m_1 + \Lambda k_1\right)}$$

then

$$\left[\frac{2}{2\ell - k_1}\right]^2 \left[A^2 + B^2 + C^2 + gk_1 \left(m_0 m_1 + \Lambda k_1\right)\right] > 2E$$

so that 310' is valid for either of the 3 possible values of G_2.

We now try to get a contradiction to 316 to demonstrate that v always remains smaller than G_2. Now there will be an instant of time t' when v has a finite value $v'(\geq G_2)$ and from 305, we can conclude that $\rho\ d\rho/dt$ admits for $t = t'$ a finite value $\rho'\ d\rho'/dt$ which verifies one or the other of the inequalities

$$\rho' \frac{d\rho'}{dt} < -W' < 0$$

$$\rho' \frac{d\rho'}{dt} > W' > 0$$

Suppose that $\rho'\ d\rho'/dt < -W'$. We increase the time t after t'. Under our assumption 316, $v^2 \geq 0$ since G_2 is, and in this first case we have assumed $\rho\ d\rho/dt < 0$. From the discussion following 313, we know that dv^2/dt and $-\rho\ d\rho/dt$ have the same sign so that $dv^2/dt > 0$. In this first case we conclude that v^2 increases positively if $\rho\ d\rho/dt < 0$ (if $v' \geq G_2$). Since $v \geq G_2$ is assumed true we remember that 305 is also verified and the desired contradiction is reached. Now since the equations of motion are invariant with respect to a change in the sign of t, we can show that the second inequality also leads to a contradiction for t decreasing after t'. We now conclude that the assumption 316 was incorrect and

$$v < G_2$$

when

$$r \equiv r_2 < \frac{k_1}{2}$$

We now direct our attention to look for a upper limit on G_2 which doesn't change by a permutation of m_0, m_1, m_2. With this in mind, some inequalities will be developed. From 281 and some previous inequalities it can be shown that

$$g \leq \frac{3}{2m}$$

$$m_0 m_1 < \frac{M^2}{4}$$

$$\frac{1}{h} = \frac{m_0 m_1}{m_0 + m_1} < \frac{M}{4}$$

$$g m_0 m_1 = \frac{M}{m_2} \frac{m_0 m_1}{m_0 + m_1} < \frac{M^2}{4m}$$

$$g\Lambda < \frac{M}{2\ell} + \frac{M}{16}|K|$$

$$\frac{m_0 m_1 m_2}{M} \leq \frac{M^2}{27}$$

$$g\left(m_0 m_1 + \Lambda k_1\right) < M\left(\frac{1}{29} + \frac{M}{4m} + \frac{k_1}{16} + |K|\right)$$

$$D < \frac{4M}{k_1}\left(\frac{1}{29} + \frac{M}{4m} + \frac{k_1}{16}|K|\right)$$

$$A^2 + B^2 + C^2 \leq 2g^2\left[c_0^2 + c_1^2 + c_2^2 + \frac{4k_1}{h}\left(m_0 m_1 + \Lambda k_1\right)\right] < \frac{9}{2m^2}\left(c_0^2 + c_1^2 + c_2^2\right) + \frac{3M^2 k_1}{m}\left(\frac{1}{29} + \frac{M}{4m} + \frac{k_1}{16}|K|\right)$$

Now, from the above inequalities it is found that the expressions $\sqrt{2E}$, $\sqrt{D}$ and

$$\frac{2}{2\ell - k_1}\sqrt{A^2 + B^2 + C^2 + gk_1\left(m_0 m_1 + \Lambda k_1\right)}$$

are all smaller than the quantity

$$G = \frac{1}{14k_1}\sqrt{\frac{9}{2m^2}\left(c_0^2 + c_1^2 + c_2^2\right) + \left(775 + \frac{3M}{m}\right) Mk_1 \left(\frac{1}{29} + \frac{M}{4m} + \frac{k_1}{16}|K|\right)} \tag{318}$$

considering the definition 315, it follows that

$$G_2 < G \tag{319}$$

With this in mind, one notes that in order to calculate an inferior limit for the quantity Q_2', (i.e., the smallest of the quotients following 212) we take $v_1 = G$. Thus, we now have an appropriate value for v_1 in 210 and in the quotients. It can be shown that the denominators

$$4\left(m_0 + m_1 + \Lambda_1 k_1\right)$$

$$m_0 + m_1 + \lambda_1 k_1$$

$$\frac{m_2}{2} + 4\left(m_0 + m_1\right) + 3\Lambda_1 k_1$$

$$2\lambda_1 k_1 + m_2$$

of the quotients are all smaller than the quantity

$$4M + \frac{5M^2}{4m} + \frac{M}{m} G^2 k_1 + 3\frac{M}{m} G\sqrt{Mk_1} + \frac{M}{2}|K| k_1$$

now since

$$\frac{1}{2v_1 + \sqrt{\frac{M}{k_1}}} < \sqrt{\frac{k_1}{M}}$$

we see that

$$\sqrt{\frac{k_1}{M}}$$

is not the smallest quotient and is eliminated as a possible lower limit for the radius of convergence. Since

$$m_0 + m_1 + \Lambda_1 k_1 > \frac{4}{27} M$$

or

$$\sqrt{k_1 \left(m_0 + m_1 + \Lambda_1 k_1\right)} > \frac{2}{3} \sqrt{\frac{k_1 M}{3}}$$

it follows that all the quotients following 212 are greater than

$$Q = \frac{\sqrt{\frac{k_1}{3M}}}{6 + \frac{15M}{8m} + \frac{3}{2m} G^2 k_1 + \frac{9}{2m} G\sqrt{Mk_1} + \frac{3}{4} |K| k_1}$$

or

$$\frac{1}{2G + \sqrt{\frac{M}{k_1}}} \tag{320}$$

whichever is the smallest. The first expression is always the smaller of the two, and consequently

$$Q_2' > Q \tag{321}$$

We now conclude that the development of the unknowns of the Equations 172 into powers of $u - u_1$ converge if u verifies the condition

$$|u - u_1| \leq Q \tag{322}$$

Hence, we have succeeded in establishing a lower limit for the radius of convergence of the series solutions for Equations 172.

INTRODUCTION OF A NEW INDEPENDENT VARIABLE "w"

In the preceding work, we have employed in the place of t, a pseudo-time "u" which is a regularizing variable for only two particular bodies. It has been assumed up until now that m_0 and m_1 are the colliding bodies and the distance, $r_2 = r$, between the two bodies has been going towards zero. A glance at Equations 154, will point out that for a collision between m_0 and m_2 or m_1 and m_2, the differential equations of motion have singular points, namely when $r_0 = 0$ or $r_1 = 0$. Therefore "u" is only a satisfactory regularizing variable for two particular colliding bodies. An independent variable "w" will be introduced that will remove all singularities and allow any number or combinations of collisions to occur without singularities. We define;

$$dt = \Gamma \, dw$$

$$t = 0$$

for

$$w = 0 \tag{323}$$

where

$$\Gamma = \left(1 - e^{-r_0/\ell}\right)\left(1 - e^{-r_1/\ell}\right)\left(1 - e^{-r_2/\ell}\right) \tag{324}$$

Now ℓ is defined by 291, and the function Γ has a determined value for each real value of time and

$$0 \leq \Gamma < 1 \tag{325}$$

From 323, 325 it is seen that w and t increase and decrease together. Furthermore, there exists a one to one correspondence between w and t. It is clear that Γ is positive when all the distances r_0, r_1, r_2 are greater than zero, and w cannot become infinite when t tends toward a finite value, say for $t = t_1$. From 117 and 323.

$$\frac{dw}{du} = \frac{r}{\Gamma}$$

Since r, and Γ go to zero together, the right side of the equation remains finite for $r \to 0$. It is also clear that if $r \to 0$ for $t \to t_1$, w tends toward a finite value when $u \to u_1$ or $t \to t_1$. One finds the same result if the distance r_0 or r_1 goes to zero for $t = t_1$. The variable "w" will be finite when t is finite and since $|t| < |w|$, from 323 and 325, the reciprocal is also true. It can therefore be said that

$$\lim_{t=+\infty} w = +\infty \qquad \lim_{t=-\infty} w = -\infty$$

$$\lim_{w=+\infty} t = +\infty \qquad \lim_{w=-\infty} t = -\infty$$

We shall devote the next section to finding a lower limit of the radii of convergence for the coordinates of the three bodies, their mutual distances and the time expressed in power series of $w - \bar{w}$. Two cases are to be investigated.

Case 1: For $w = \bar{w}$ one of the distances r_0, r_1, r_2 is inferior to $k_1/2$; for example

$$r_2 < \frac{k_1}{2}$$

Let t_1 be the value of t for $w = \bar{w}$. We shall designate by u_1 the value of u for $t = t_1$ or $w = \bar{w}$. We can then say that the coordinates of the bodies, their mutual distances and the time will be developable into powers of $u - u_1$ if u verifies the condition

$$|u - u_1| \leq Q$$

The variables u and w are related by the equation

$$\frac{du}{dw} = \frac{\Gamma}{r} \tag{326}$$

$$u = u_1$$

for

$$w = \bar{w}$$

Since both Γ and $1/r$ are developable into a series in powers of $u - u_1$, the right side of 326 is also developable into a power series if $|u - u_1| \leq Q$. In order to apply the Cauchy-Picard Theorem to Equation 326, we must find an upper limit for $|\Gamma/r|$ when $|u - u_1| \leq Q$. Writing down 198, 201, 184

$$r_0^2 = \rho_1^2 + \underline{P}_1$$

$$\underline{P}_1 < \frac{45}{49} \rho_1^2$$

$$\rho_1 \geq 14k_1$$

we see that r_0^2 never becomes zero or negative and the real part of r_0 doesn't change sign. Since the real part of r_0 is positive for $u = u_1$, it will remain positive if $|u - u_1| \leq Q$. It therefore follows that

$$\left| e^{-r_0/\ell} \right| < 1$$

and

$$\left|1 - e^{-r_0/\ell}\right| < 2$$

In an analogous manner

$$\left|1 - e^{-r_1/\ell}\right| < 2$$

Now we observe that

$$\left|\frac{1 - e^{-r/\ell}}{r}\right| = \left|\frac{1}{\ell} - \frac{r}{2\ell^2} + \frac{r^2}{6\ell^3} - \cdots\right| \le \frac{1}{\ell} + \frac{k_1}{2\ell^2} + \frac{k_1^2}{6\ell^3} + \cdots = \frac{e^{k_1/\ell} - 1}{k_1}$$

from 293 $\ell = 29/2\ k_1$ and $e^{2/29} - 1 < 1/12$, we see that

$$\left|\frac{\Gamma}{r}\right| < \left|\frac{1 - e^{-r/\ell}}{r}\right| \left|1 - e^{-r_0/\ell}\right| \left|1 - e^{-r_1/\ell}\right| < \frac{\frac{1}{12}\cdot 2\cdot 2}{k_1} = \frac{1}{3k_1}$$

so that

$$\left|\frac{\Gamma}{r}\right| < \frac{1}{3k_1}$$

if $|u - u_1| \le Q$.

Now we can apply the Cauchy-Picard Theorem and say that the Equation 326

$$\frac{du}{dw} = \frac{\Gamma}{r}$$

has a unique analytic solution (in powers of $w - \bar{w}$) for

$$|w - \bar{w}| \le 3Qk_1 \tag{327}$$

A result of this is that the coordinates of the bodies, the distances r_0, r_1, r and the time are developable into powers of $w - \bar{w}$ if w verifies 327. Since k_1 and Q are symmetric with respect to the masses m_0, m_1 and m_2 there would be no change if, in the place of $r_2 \equiv r < k_1/2$ one had $r_0 < k_1/2$ or $r_1 < k_1/2$ for $w = \bar{w}$.

Case 2: For $w = \bar{w}$ all of the distances r_0, r_1, r are $\geq k_1/2$ or from 293, $r_0, r_1, r \geq 14\kappa_0$. We note that this is condition 132 if $\bar{t}$ is the value of t for $w = \bar{w}$. We have already found that the co-ordinates of the three bodies and the distances r_0, r_1, r_2 are developable into powers of $t - \bar{t}$ if t verifies 153. From the above reasoning

$$\left|1 - e^{-r_0/\ell}\right|, \left|1 - e^{-r_1/\ell}\right|, \left|1 - e^{-r_2/\ell}\right| < 2$$

so that $|\Gamma| < 8$ if $|t - \bar{t}| \leq T'$. We note from 323 that

$$\frac{dt}{dw} = \Gamma$$

and the Cauchy-Picard Theorem insures that this equation has a unique analytic solution in powers of $w - \bar{w}$ for

$$|w - \bar{w}| \leq \frac{1}{8} T' \tag{328}$$

That is, the coordinates of the three bodies, the distances r_0, r_1, r_2 and the time are, in case 2, developable into powers of $w - \bar{w}$ if 328 is satisfied. From 153, 293

$$\frac{T'}{8} = \frac{1}{8} \frac{k_0}{\sqrt{\frac{4M^2}{21mk_0} + M|K|}} = \frac{k_1}{\frac{1}{224}\sqrt{\frac{112M^2}{21mk_1} + M|K|}} = \frac{\frac{k_1}{\sqrt{M}}}{224\sqrt{\frac{112M}{21mk_1} + |K|}} \cdot \frac{\sqrt{3k_1}}{\sqrt{3k_1}}$$

or

$$\frac{T'}{8} = \frac{k_1\sqrt{\frac{3k_1}{M}}}{224\sqrt{16\frac{M}{m} + 3|K|k_1}}$$

From 320

$$3Qk_1 = \frac{k_1\sqrt{\frac{3k_1}{M}}}{6 + \frac{15}{8}\frac{M}{m} + \frac{3}{2m}G^2 k_1 + \frac{9}{2m}G\sqrt{Mk_1} + \frac{3}{4}|K|k_1}$$

Both $3Qk_1$ and $T'/8$ are greater than

$$\Omega = \frac{k_1 \sqrt{\frac{3k_1}{M}}}{\frac{15}{8}\frac{M}{m} + \frac{3}{2m} G^2 k_1 + \frac{9}{2m} G\sqrt{Mk_1} + \frac{3}{4}|K| k_1 + 224 \sqrt{16 \frac{M}{m} + 3|K| k_1}} \tag{329}$$

where G and k_1 are defined by 318 and 293 and m designates the smallest of the masses m_0, m_1, m_2. An important result of Sundman's work can finally be stated.

The coordinates of the three bodies, their mutual distances and the time are developable in powers of $w - \bar{w}$, such that these developments are convergent for

$$|w - \bar{w}| \leq \Omega$$

We now have a "convergent strip" with a width 2Ω and the real w axis runs symmetrically through the center of this strip. Since the domain of convergence for a Taylor series is circular and we now have a strip of convergence, a transformation would be useful. With this in mind, we introduce a new variable τ by the transformation.

$$\left.\begin{aligned} w &= \frac{2\Omega}{\pi} \log \frac{1 + \tau}{1 - \tau} \\ \tau &= \frac{e^{\pi w/2\Omega} - 1}{e^{\pi w/2\Omega} + 1} \end{aligned}\right\} \tag{330}$$

It can be shown that as $w \to \infty$, $|\tau| \to 1$ and that $|\tau| < 1$. The real values of τ between -1 and $+1$ will have a one to one correspondence with the real values of t between $-\infty$ and $+\infty$. We have effectively transformed the strip of convergence in the w plane to a unit circle in the τ plane. We can now state Sundman's final theorem.

Theorem 8: If in the problem of three bodies, the constants of area are not all zero, one can find two constants ℓ and Ω, (the coordinates and the velocities of the bodies being given for a certain finite moment) such that, if one introduces in place of "t" a variable "τ", the coordinates of the three bodies, their mutual distances and the time are developable in powers of τ. These series solutions converge for $|\tau| < 1$ and represent the movement for all time. The equations remain regular for collisions between any two bodies.

Through the introduction of the variable τ, the coordinates of the three bodies, their mutual distances and the time is developable into powers of τ if $|\tau| < 1$. In the general case considered by Sundman, the series are very complicated. The convergence of these series was investigated by Belorizky and the results are summerized below.

INVESTIGATION OF CONVERGENCE

After determining the constants ℓ and Ω, the new variable τ is substituted for t and the regularized solution series are obtained. Belorizky uses the equilateral triangle solution to the three body problem in an effort to investigate the convergence of the Sundman solutions. He takes the mutual distances between the three bodies as one astronomical unit, the sum of the masses is the unit mass and the Gaussian constant is set equal to one. Using these simplifications Γ is a constant and

$$\Gamma = \left(1 - e^{-1/\ell}\right)^3$$

consequently $t = \Gamma w$ and

$$t = \frac{2\Omega\Gamma}{\pi} \log \frac{1+\tau}{1-\tau} = A \log \frac{1+\tau}{1-\tau}$$

and

$$t = 2A\left(\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5} + \cdots + \frac{\tau^{2n+1}}{2n+1} + \cdots\right)$$

Furthermore

$$x = \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$$

$$= 1 - 2A^2\tau^2 - \left(\frac{4}{3}A^2 - \frac{2}{3}A^4\right)\tau^4 - \left(\frac{46}{45}A^2 - \frac{8}{9}A^4 + \frac{4}{45}A^6\right)\tau^6 - \cdots$$

$$y = \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

$$= 2A\tau + \left(\frac{2A}{3} - \frac{4}{3}A^3\right)\tau^3 + \left(\frac{2A}{5} - \frac{4}{3}A^3 + \frac{4}{15}A^5\right)\tau^5 + \cdots$$

We consider m_2 to be the larger of the three masses, and $m_1 = m_0$. If

1) $m_1 = \frac{1}{200}$

2) $m_1 = \frac{1}{10}$

3) $m_1 = \frac{1}{3}$

the corresponding values of Ω will be;

1) $\Omega_1 < 9 \times 10^{-8}$

2) $\Omega_2 < 4 \times 10^{-6}$

3) $\Omega_3 < 10^{-5}$

so that A has the values

1) $A_1 < 4 \times 10^{-8}$

2) $A_2 < 2 \times 10^{-6}$

3) $A_3 < 4 \times 10^{-6}$

Let "h" be the error introduced by stopping the "t" series at $2n - 1$ terms so that

$$t - h = 2A\left(\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5} + \cdots + \frac{\tau^{2n-1}}{2n-1}\right)$$

and x, y can be expressed as

$$x = 1 - \frac{1}{2!}\left[2A\left(\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5} + \cdots + \frac{\tau^{2n-1}}{2n-1}\right)\right]^2 + \frac{1}{4!}\left[2A\left(\tau + \frac{\tau^3}{3} + \cdots + \frac{\tau^{2n-1}}{2n-1}\right)\right]^4$$

$$y = 2A\left(\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5} + \cdots + \frac{\tau^{2n-1}}{2n-1}\right) - \frac{1}{3!}\left[2A\left(\tau + \frac{\tau^3}{3} + \cdots + \frac{\tau^{2n-1}}{2n-1}\right)\right]^3$$

If we wish to have the coordinates x, y for the epoch $t = 1$ with an accuracy of only one decimal place, the number of terms necessary (n) can be computed by setting

$$1 = 2A\left(\tau + \frac{\tau^3}{3} + \frac{\tau^4}{5} + \cdots + \frac{\tau^{2n-1}}{2n-1} + \cdots\right)$$

hence

$$\sum_{0}^{n} = 2A\left(\tau + \frac{\tau^3}{3} + \cdots \frac{\tau^{2n-1}}{2n-1}\right) > 0.9$$

and the sum

$$S_\tau = \tau + \frac{\tau^3}{3} + \frac{\tau^5}{5} + \cdots + \frac{\tau^{2n-1}}{2n-1} > \frac{9}{20A}$$

The variable τ is determined by the relation $\tau = e^{1/A} - 1/e^{1/A} + 1$ and for

$A < 4 \times 10^{-8}$ on finds $S_{\tau_1} > 10^7$

$A < 2 \times 10^{-6}$ on finds $S_{\tau_2} > 2 \times 10^5$

$A < 4 \times 10^{-6}$ on finds $S_{\tau_3} > 10^5$

We shall now compare the sum S_τ with the divergent series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots$$

Since $\tau < 1$ one has $\tau^{2n-1}/2p - 1 < 1/2p - 1$ and if

$$S = \sum_{1}^{p} \frac{1}{2p-1}$$

Belorizky shows that $p > e^{2s-2.4}$ and since $S_\tau < S$, $p > e^{2s_\tau - 2.4}$. Now for

$S_{\tau_1} = 10^7$ one finds $p_1 > 10^{8\times 10^6}$

$S_{\tau_2} = 2 \times 10^5$ one finds $p_2 > 10^{17\times 10^4}$

$S_{\tau_3} = 10^5$ one finds $p_3 > 10^{8\times 10^4}$

We have designated by "n" the number of terms which is necessary in the series

$$\tau + \frac{\tau^3}{3} + \cdots + \frac{\tau^{2n-1}}{2n-1} + \cdots = \frac{1}{2A}$$

in order to have $S_\tau \geq 9/20A$. This number n is greater than the number "p" of the terms which it is necessary to take in the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

in order to have the same sum. From above we can see that $n > p$.

We wish to calculate x, y in our particular case with only a single decimal accuracy by Sundman's method. For the epoch $t = 1$, in the three different cases where $M/m = 200$, $M/m = 10$, $M/m = 3$, the number of terms is respectively greater than $10^{8\times10^6}$, $10^{17\times10^4}$, $10^{8\times10^4}$.

The extreme slowness of convergence in this particular case is apparent. The Sundman solution is, in a practical sense, quite useless for ephemeris computation. The motion of a body near collision, can however be represented by the Sundman series solution. The Sundman exposition, although not generally useful for a solution to the three body problem, has allowed a rigorous investigation of the motion near collision. A few of the more important results are given below.

SUMMARY

1. In a double collision, the distance to the non-participating third body remains bounded.
2. In order for a triple collision to occur, the area integral of the system must equal zero.
3. The velocity and acceleration of the colliding body tends toward infinity at the point of double collision.
4. The unit vector along the radius joining the two colliding bodies tends to a limit.
5. Singularities in the equations of motion can be removed at one of the bodies by the introduction of the independent variable "u".
6. An independent variable "w" was found such that all the singularities of the motion were removed, allowing any number or combinations of double collisions to occur.
7. The power series solutions were found to converge, although extremely slowly.

Although this paper, up until this point, has considered only the regularization introduced by Sundman, there are several other known transformations that regularize the equations of motion. One of the best known methods for the restricted three body problem is that of Levi-Civita.

We consider two primary bodies in the cartesian plane having the coordinates $(\mu, 0)$, $(\mu - 1, 0)$. The third body's position is denoted by (x, y). A transformation is made from the x, y plane to a p, q plane by the following equations

$$x - \mu = p^2 - q^2$$

$$y = 2pq$$

$$dt = 4(p^2 + q^2)\, d\tau$$

Using these transformations, one can transform the equations of motion in the (x, y) plane

$$\frac{d^2 x}{dt^2} - 2 \frac{dy}{dx} = \Omega_x$$

$$\frac{d^2 y}{dt^2} + 2 \frac{dx}{dt} = \Omega_y$$

$$\Omega = \frac{1}{2} \left[(1 - \mu) r_1^2 + \mu r_2^2 \right] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$$

$$r_1 = \sqrt{(x - \mu)^2 + y^2}$$

$$r_2 = \sqrt{(x + 1 - \mu)^2 + y^2}$$

to the corresponding equations of motion in the (p, q) plane. These are;

$$\frac{d^2 p}{d\tau^2} - 8(p^2 + q^2) \frac{dq}{d\tau} = \left[4\left(\Omega - \frac{1}{2} C\right)(p^2 + q^2) \right]_p$$

$$\frac{d^2 q}{d\tau^2} + 8(p^2 + q^2) \frac{dp}{d\tau} = \left[4\left(\Omega - \frac{1}{2} C\right)(p^2 + q^2) \right]_q$$

$$\left(\frac{dp}{d\tau}\right)^2 + \left(\frac{dq}{d\tau}\right)^2 = 8\left(\Omega - \frac{1}{2} C\right)(p^2 + q^2)$$

Subscripts in the right side of the equations denote differentiation with respect to that variable and C is a constant of integration.

If we now consider a collision at $(\mu, 0)$, then as $r_1 \to 0$ the term $(1 - \mu)/r_1$ or $(1 - \mu)/p^2 + q^2$ in Ω, which becomes infinite, appears multiplied by a factor $p^2 + q^2$. At $p = q = 0$, the last equation shows that the square of the velocity is proportional to $8(1 - \mu)$. The curves of motion are analytic curves without singularity near the origin. Thus for a single collision at $(\mu, 0)$, the equations are regular.

Recently, a new method has been introduced into the three body problem. Originating in atomic physics, "spinors" were used to describe the process of spinning of an elementary particle. In celestial mechanics, spinors are no longer used to describe spin but are used instead as a mathematical aid in simplifying the equations of motion for the three body problem. A spinor can be thought of as a vector in the complex plane connecting a complex number $z_1 = u_1 + iv_1$ to another complex number $z_2 = u_2 + iv_2$. The spinor is then a four space vectors of the form:

$$S = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

It is these vectorial analogies of spinors which are used to regularize the differential equations of motion. Transformations of the Levi-Civita type cannot immediately be extended into three dimentions. By using spinor notation, a position vector $\vec{r}$ and its 3 scalar components (x, y, z) can be expressed in terms of z_1, z_2 and their complex conjugates.

For each position vector, there corresponds differential equations for a spinor (z_1, z_2) which are regular, when a certain pseudo-time is chosen to be the independent variable.

SELECTED BIBLIOGRAPHY

1. Belorizky, M. D., "Application Pratique Des Méthodes De M. Sundman A Un Cas Particulier Du Problème Des Trois Corps," Bulletin Astronomique, Tome 5 and 6, 1925-1930, Pg. 417.

2. Birkhoff, George D., "The Restricted Probleme of Three Bodies," Circolo Matematico di Palermo, Volume 39, 1915.

3. Pollard, H., "Sundman's Theory of the Three Body Problem," Notes of lectures delivered at the Summer Institute in Dynamical Astronomy. Purdue University, 1962.

4. Sundman, Karl, "Memoire Sur Le Problème Des Trois Corps," Acta Mathematica, Tome 36, 1912, Pg. 105.

5. Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, University Press, Cambridge; 1927, Pg. 339.