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THE INFLUENCE OF PREBUCKLING DEFORMATION ON THE BUCKLING LOAD OF TRUNCATED CONICAL SHELLS UNDER AXIAL COMPRESSION

by Shigeo Kobayashi

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CALIFORNIA INSTITUTE OF TECHNOLOGY
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LIST OF SYMBOLS

A_j $(j = 1,2)$	complex constants defined by eq. (27)
a_{im}, $\bar{a}_{im}$	coefficients, see eqs. (65), (67), (70), (72) and (74)
b_{im}	elements in eigenvalue determinant
C_j $(j = 1 \sim 4)$	arbitrary constants
C_{jR}, C_{jI}	real and imaginary parts of C_j, respectively, see eq. (35)
D	$= Et^3/12(1-\nu^2)$
E	Young's modulus
F	stress function defined by eq. (3)
F_o	dimensionless stress function for prebuckling deformation defined by eq. (19)
f_n	dimensionless stress function for buckling deformation defined by eq. (47.2)
g	function of ξ which represents $W_n(\xi)$ or $f_n(\xi)$
h	finite difference interval of ξ, see Fig. 2
k	parameter defined by eq. (56)
$\bar{k}$	parameter defined by eq. (79)
L	length of cylindrical shell
ℓ_1, ℓ_2	coordinates of the ends of a conical shell, see Fig. 1
M_x, M_θ, $M_{x\theta}$, Q_x, Q_θ	moments and shearing forces, see Fig. 1
N	number of points in finite difference calculation
n	circumferential wave number
P	axial compressive load
P_{cl}	classical compressive buckling load of conical shells, see eq. (1)
q	dimensionless buckling load defined by eq. (21)
r	radius of cylindrical shell
s	dimensionless coordinate defined by eq. (20)

$s_{\ell 1}, s_{\ell 2}$	values of s at $x = \ell_1$ and ℓ_2, respectively, see eq. (23)
s_j $(j = 1, 2)$	complex variable defined by eq. (28.2)
t	shell thickness
$N_x, N_\theta, N_{x\theta}$	membrane stresses, see Fig. 1
u, v, w	displacement components, see Fig. 1
u_n, v_n, w_n	dimensionless displacement components of buckling mode defined by eqs. (47.3), (47.2) and (47.1), respectively
u_H, u_V	horizontal (radial) and vertical components of displacement, respectively, see eq. (7) and Fig. 1
W	dimensionless prebuckling deformation defined by eq. (18)
x	generatrix coordinate, distance from vertex, see Fig. 1
y_k	function defined by eq. (63)
α	semivertex angle of cone
$\beta, \overline{\beta}$	parameters for circumferential wave number defined by eqs. (59) and (80), respectively
$\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}$	middle surface membrane strain components
θ	variable in circumferential direction defined by eq. (4)
$\overline{\theta}$	circumferential coordinate, see Fig. 1
$\xi, \overline{\xi}$	dimensionless coordinates defined by eqs. (55) and (78), respectively
η	variable defined by eq. (33)
η_1, η_2	values of η at $x = \ell_1$ and ℓ_2, respectively
ζ_j $(j = 1, 2)$	functions defined by eq. (28.1)
λ	parameter for circumferential wave number defined by eq. (49)

ν Poisson's ratio

ϕ constant defined by eq. (32)

Suffixes A and B indicate axisymmetrical prebuckling deformation and buckling mode, respectively.

THE INFLUENCE OF PREBUCKLING DEFORMATION ON THE BUCKLING LOAD OF TRUNCATED CONICAL SHELLS UNDER AXIAL COMPRESSION

by

Shigeo Kobayashi*

ABSTRACT

An analytical investigation is carried out to determine the effect of prebuckling deformation on the compressive buckling load of truncated conical shells. The shell is assumed to have clamped boundary conditions and a finite difference approach is used to obtain the solution. In order to reduce the number of parameters and to clarify the essential property of the effect of the cone angle, a semi-infinite approach is used. It is concluded that the buckling load is determined by local buckling in the region close to the smaller radius end. The decrease in the buckling load from the classical result as a result of prebuckling deformation is somewhat greater for the conical shell than that for the cylindrical shell. However, the effect of the cone angle on this decrease is very small.

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I. INTRODUCTION

There have been many reports (Refs. 1-19) that are concerned with the problem of the buckling of conical shells under uniform axial compression loading. The so-called classical buckling load was obtained by Seide (Ref. 1) and is given by

$$P_{cl} = \frac{2\pi E t^2}{\sqrt{3(1 - \nu^2)}} \cos^2 \alpha \tag{1}$$

The buckling mode corresponding to this load is axisymmetric. Singer has shown (Ref. 2), using a Galerkin method, that the buckling load for non-axisymmetric modes is only slightly lower than the one calculated by Seide. In his typical example the value of P/P_{cl} is 0.9905. As in the case of cylindrical shells under axial compression, there exists a great disagreement between theory and experiment as shown by the experimental results of Weingarten, Morgan, and Seide (Ref. 3) and Schnell and Schiffner (Ref. 4).

Recent work on the buckling of cylindrical shells shows that the buckling load is changed if the analysis includes the effect of prebuckling deformations. Stein (Ref. 20) calculated the buckling load for the "shear free simply supported" (S-3) boundary conditions including the prebuckling deformations due to the constraint at the boundaries. The value he obtained, $P \approx 0.42P_{cl}$, was chiefly due to the shear free condition. This can be concluded by comparing his results with those obtained by Ohira (Refs. 21,22) who calculated the effect of the boundary conditions alone. The effect of prebuckling deformation is more pronounced in the case of "zero displacement, simply supported" (S-1) boundary condition.

The buckling load is about $0.84P_{cl}$ in the numerical example calculated by Fischer (Ref. 23). For the more realistic boundary of "perfectly clamped" (C-1) the buckling load P is in the range from $0.908P_{cl}$ to $0.930P_{cl}$ for various shell lengths as shown by Almroth (Ref. 24).

The effect of prebuckling deformations on the axial buckling load of a conical shell having "perfectly clamped" boundary conditions is studied in this report. In order to clarify the effect of cone angle, a semi-infinite approach is used in the numerical calculations. In addition, the buckling load of a semi-infinite cylindrical shell is obtained for comparison.

II. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS FOR CONICAL SHELLS

The geometry and symbols used in this paper are illustrated in Fig. 1. If we consider a Donnell type approach, the governing nonlinear differential equations of deformation of thin conical shells having no initial imperfection are expressed as

$$\frac{1}{Et}\Delta\Delta F = \cot\alpha \frac{1}{x}\frac{\partial^2 w}{\partial x^2} + \left[\frac{1}{x^2}\left(\frac{\partial^2 w}{\partial x \partial \theta}\right)^2 - \frac{1}{x^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{2}{x^3}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial x \partial \theta} + \frac{1}{x^4}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{x}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] \quad (2.1)$$

$$D\Delta\Delta w + \cot\alpha \frac{1}{x}\frac{\partial^2 F}{\partial x^2} - \left[\left(\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \theta^2}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2}\right) - 2\left(\frac{1}{x^2}\frac{\partial F}{\partial \theta} - \frac{1}{x}\frac{\partial^2 F}{\partial x \partial \theta}\right)\left(\frac{1}{x^2}\frac{\partial w}{\partial \theta} - \frac{1}{x}\frac{\partial^2 w}{\partial x \partial \theta}\right)\right] = 0 \quad (2.2)$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}$$

In these equations F is the stress function defined as

$$N_x = \frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \theta^2}, \quad N_\theta = \frac{\partial^2 F}{\partial x^2}, \quad N_{x\theta} = -\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial \theta}\right) = N_{\theta x} \quad (3)$$

and the variable θ is given by

$$\theta = \bar{\theta} \sin\alpha \quad (4)$$

(See Fig. 1.)

In addition, the following relations exist between the midsurface strains, displacements u and v and the stress function F.

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{Et}\left[\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial x^2}\right] \tag{5.1}$$

$$\varepsilon_\theta = \frac{1}{x}\frac{\partial v}{\partial \theta} + \frac{u+w\cot\alpha}{x} + \frac{1}{2x^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{1}{Et}\left[\frac{\partial^2 F}{\partial x^2} - \nu\left(\frac{1}{x}\frac{\partial F}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F}{2\theta^2}\right)\right] \tag{5.2}$$

$$\gamma_{x\theta} = \frac{1}{x}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \frac{v}{x} + \frac{1}{x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} = \frac{2(1+\nu)}{Et}\left[\frac{1}{x^2}\frac{\partial F}{\partial \theta} - \frac{1}{x}\frac{\partial^2 F}{\partial x \partial \theta}\right] \tag{5.3}$$

Expressions for the bending and twisting moments and shearing forces in this theory are as follows:

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\left(\frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{x}\frac{\partial w}{\partial x}\right)\right] \tag{5.4}$$

$$M_\theta = -D\left[\left(\frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{x}\frac{\partial w}{\partial x}\right) + \nu\frac{\partial^2 w}{\partial x^2}\right] \tag{5.5}$$

$$M_{x\theta} = M_{\theta x} = -(1-\nu)D\left[\frac{1}{x}\frac{\partial^2 w}{\partial x \partial \theta} - \frac{1}{x^2}\frac{\partial w}{\partial \theta}\right] \tag{5.6}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{x}\frac{\partial M_{\theta x}}{\partial \theta} + \frac{1}{x}(M_x - M_\theta) \tag{5.7}$$

$$Q_\theta = \frac{\partial M_{x\theta}}{\partial x} + \frac{1}{x}\frac{2M_\theta}{\partial \theta} + \frac{1}{x}(M_{x\theta} + M_{\theta x}) \tag{5.8}$$

In the derivation of eqs. (2) the following assumptions have been made:

i) The shell is thin and truncated so that the thickness-radius of curvature ratio $t/x\tan\alpha$ is small.

ii) The circumferential wave number n is large enough so that the Donnell type approach and the large deflection expressions of eqs. (5) are valid. However, the above equations are also satisfactorily applied to the axisymmetrical deformation.

The boundary conditions are chosen so that the shell is perfectly clamped at $x = \ell_1$ and $x = \ell_2$. The axial load P is applied through rigid blocks as shown in Fig. 1. If the upper block is fixed and the lower block is allowed to move, the geometrical boundary conditions are expressed as

$$\begin{aligned} &\text{at } x = \ell_1 \ ; \quad u_H = 0, \quad u_V = 0, \quad v = 0, \quad \frac{\partial w}{\partial x} = 0 \\ &\text{at } x = \ell_2 \ ; \quad u_H = 0, \quad \frac{\partial u_V}{\partial \theta} = 0, \quad v = 0, \quad \frac{\partial w}{\partial x} = 0 \end{aligned} \tag{6}$$

where u_H and u_V are the horizontal and vertical components of the displacement respectively, defined as

$$\begin{aligned} u &= u_H \sin\alpha + u_V \cos\alpha \\ w &= u_H \cos\alpha - u_V \sin\alpha \end{aligned} \tag{7}$$

In addition there exists one traction boundary condition

$$\int_0^{2\pi\sin\alpha} \left[N_x \cos\alpha - \left(Q_x + \frac{1}{x} \frac{\partial M_{x\theta}}{\partial \theta} \right) \sin\alpha \right] x \, d\theta = -P \tag{8}$$

at $x = \ell_2$. The above differential equation (2) and this traction boundary condition (8)* have been confirmed using a variation principle.

---

* If the condition $w = 0$ is used instead of $u_H = 0$ at $x = \ell_2$, a different traction boundary condition is needed. Therefore when considering the conditions of eq. (6), the geometrical condition $w = 0$ at $x = \ell_2$ cannot be used.

III. PREBUCKLING DEFORMATION

Since we are considering an initially perfect conical shell, the prebuckling deformation is axisymmetrical. The condition of axisymmetrical stress distribution indicates that F must be a function of x alone. Then, taking $\frac{\partial}{\partial\theta} = 0$ in eqs. (2), the differential equations of prebuckling deformation are as follows:

$$\begin{cases} \frac{1}{Et}\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left(x\frac{dF}{dx}\right)\right\}\right] - \frac{\cot\alpha}{x}\frac{d^2w}{dx^2} + \frac{1}{2x}\frac{d}{dx}\left(\frac{dw}{dx}\right)^2 = 0 & (9.1) \\ D\frac{1}{x}\frac{d}{dx}\left[x\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left(x\frac{dw}{dx}\right)\right\}\right] + \frac{\cot\alpha}{x}\frac{d^2F}{dx^2} - \frac{1}{x}\frac{d}{dx}\left(\frac{dF}{dx}\frac{dw}{dx}\right) = 0 & (9.2) \end{cases}$$

These equations can be integrated once. The integration constant of eq. (9.1) is determined from the equation of compatibility of axisymmetrical deformation and the integration constant of eq. (9.2) is determined from the traction boundary condition (8). The following equations result.

$$\begin{cases} \frac{1}{Et}x\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left(x\frac{dF}{dx}\right)\right\} - \cot\alpha\frac{dw}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = 0 \\ Dx\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left(x\frac{dw}{dx}\right)\right\} + \cot\alpha\frac{dF}{dx} - \frac{dF}{dx}\frac{dw}{dx} = -\frac{P}{2\pi(\sin\alpha)^2} \end{cases} \qquad (10)$$

The condition of axisymmetrical deformation reduces the boundary conditions to

$$\text{at } x = \ell_1 \text{ and } x = \ell_2; \quad u_H = 0, \quad \frac{\partial w}{\partial x} = 0 \qquad (11)$$

$$\text{at } x = \ell_1; \quad u_V = 0 \text{ or } w = 0 \qquad (12)$$

and eq. (5.2) shows that the condition $u_H = 0$ is expressed as

$$\frac{d^2F}{dx^2} - \nu\frac{1}{x}\frac{dF}{dx} = 0 \qquad (13)$$

in terms of the stress function F. The solution for $\frac{dw}{dx}$ can be found from eqs. (10) using the boundary conditions (11). The condition (12) is used in the calculation of w. This condition determines the rigid body displacement. Therefore it does not have any effect on the buckling load.

From linear membrane theory the corresponding governing equations are

$$\frac{1}{Et} x \frac{d}{dx} \left\{ \frac{1}{x} \frac{d}{dx} \left(x \frac{dF}{dx} \right) \right\} - \cot\alpha \frac{dw}{dx} = 0 \tag{14.1}$$

$$\cot\alpha \frac{dF}{dx} = - \frac{P}{2\pi(\sin\alpha)^2} \tag{14.2}$$

Eq. (14.2) yields a membrane stress

$$N_x = - \frac{P}{\pi \sin 2\alpha} \cdot \frac{1}{x} \tag{15}$$

and eq. (14.1) gives a corresponding deflection

$$w = \frac{P}{2\pi Et(\cos\alpha)^2} \log \frac{x}{\ell_1} + \text{constant} \tag{16}$$

Eq. (16) shows that even if there is no constraint of deformation at the boundary, there exists a non-constant prebuckling deformation from the membrane theory in the case of conical shells.

Then, introducing a new stress function

$$\frac{dF_A}{dx} = \frac{dF}{dx} + \frac{P}{\pi \sin 2\alpha} \tag{17}$$

and using a symbol w_A instead of w to express prebuckling deformation and using the nondimensional symbols

$$W = \frac{\sqrt{12(1-\nu^2)}\, w_A}{t} \tag{18}$$

$$F_o = \frac{12(1-\nu^2) F_A}{Et^3} \tag{19}$$

$$s = \frac{x \cot\alpha \sqrt{12(1-\nu^2)}}{t} \tag{20}$$

$$q = \frac{P}{P_{cl}} \tag{21}$$

where P_{cl} is the classical buckling load shown in eq. (1), the differential equations (10) are expressed as

$$\begin{aligned} & s \frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{dF_o}{ds}\right)\right\} - \frac{dW}{ds} + 2q\frac{1}{s} + \underline{\frac{1}{2}\left(\frac{dW}{ds}\right)^2} = 0 \\ & s \frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}\left(s\frac{dW}{ds}\right)\right\} + \frac{dF_o}{ds} + 2q\frac{dW}{ds} \underline{- \frac{dF_o}{ds}\frac{dW}{ds}} = 0 \end{aligned} \tag{22}$$

and the boundary conditions (11) are given by

$$s = \frac{\ell_1 \cot\alpha \sqrt{12(1-\nu^2)}}{t} \equiv s_{\ell 1}; \quad \frac{dW}{ds} = 0, \quad \frac{d^2F_o}{ds^2} - \nu\frac{1}{s}\frac{dF_o}{ds} = \underline{-\nu\frac{2q}{s}} \qquad (23.1)\ (23.2)$$

at ℓ_1, and

$$s = \frac{\ell_2 \cot\alpha \sqrt{12(1-\nu^2)}}{t} \equiv s_{\ell 2}; \quad \frac{dW}{ds} = 0, \quad \frac{d^2F_o}{ds^2} - \nu\frac{1}{s}\frac{dF_o}{ds} = \underline{-\nu\frac{2q}{s}} \qquad (23.3)\ (23.4)$$

at ℓ_2

As shown in Appendix A, the effect of the underlined nonlinear terms in eqs. (22) is very small numerically. Therefore, in the numerical computation of the present paper the linearized approximate equations

$$\begin{cases} s \dfrac{d}{ds}\left\{\dfrac{1}{s}\dfrac{d}{ds}\left(s\dfrac{dF_o}{ds}\right)\right\} - \dfrac{dW}{ds} + 2q\dfrac{1}{s} = 0 & (24.1) \\ s \dfrac{d}{ds}\left\{\dfrac{1}{s}\dfrac{d}{ds}\left(s\dfrac{dW}{ds}\right)\right\} + \dfrac{dF_o}{ds} + 2q\dfrac{dW}{ds} = 0 & (24.2) \end{cases}$$

have been used. *

Calculating (24.2) + (24.1) x A, where A is an undetermined arbitrary constant, leads to

$$s\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left\{s\frac{d}{ds}(W+AF_o)\right\}\right] + \frac{d}{ds}\left[(2q-A)W+F_o\right] = -\frac{2q}{s}A \qquad (25)$$

Then the value of A is selected as

$$A = \frac{1}{2q-A} \qquad (26)$$

The roots of eq. (26) are

$$A_{1,2} = q \pm i\sqrt{1-q^2} \qquad (27)$$

Therefore, by using the symbols

$$\zeta_j = W + A_j F_o\,, \quad s_j = \frac{s}{A_j} \qquad (j = 1,2) \qquad (28)$$

eq. (25) can be written as

$$s_j\frac{d}{ds_j}\left\{\frac{1}{s_j}\frac{d}{ds_j}\left(s_j\frac{d\zeta_j}{ds_j}\right)\right\} + \frac{d\zeta_j}{ds_j} = -\frac{2qA_j^2}{s_j} \qquad (j = 1,2) \qquad (29)$$

The general solutions of eqs. (29) are

$$\frac{d\zeta_j}{ds_j} = \overline{C}_{j1}\,H_2^{(1)}(2\sqrt{s_j}\,) + \overline{C}_{j2}\,H_2^{(2)}(2\sqrt{s_j}\,) - \frac{2qA_j^2}{s_j} \quad (j = 1,2) \qquad (30)$$

* In order to derive Seide's differential equations and boundary conditions of axisymmetrical buckling, the underlined terms in eqs. (23) and the last term in eq. (24.1) must be dropped.

where $\overline{C}_{j1}$ and $\overline{C}_{j2}$ are arbitrary constants and the H's are Hankel functions. From the above solutions the following result for $\frac{dW}{ds}$ and $\frac{dF_o}{ds}$ is obtained.

$$\frac{dW}{ds} = \frac{1}{A_2 - A_1}\left[\frac{A_2}{A_1}\frac{d\zeta_1}{ds_1} - \frac{A_1}{A_2}\frac{d\zeta_2}{ds_2}\right]$$

$$= -C_1 H_2^{(1)}(2\sqrt{s_1}) - C_2 H_2^{(2)}(2\sqrt{s_1}) - C_3 H_2^{(1)}(2\sqrt{s_2}) - C_4 H_2^{(2)}(2\sqrt{s_2}) + \frac{2q}{s} \tag{31.1}$$

$$\frac{dF_o}{ds} = \frac{1}{A_2 - A_1}\left[-\frac{1}{A_1}\frac{d\zeta_1}{ds_1} + \frac{1}{A_2}\frac{d\zeta_2}{ds_2}\right]$$

$$= C_1 A_1 H_2^{(1)}(2\sqrt{s_1}) + C_2 A_1 H_2^{(2)}(2\sqrt{s_1}) + C_3 A_2 H_2^{(1)}(2\sqrt{s_2}) +$$

$$+ C_4 A_2 H_2^{(2)}(2\sqrt{s_2}) - \frac{4q^2}{s} \tag{31.2}$$

where C_j ($j = 1 \rightarrow 4$) are arbitrary constants.

Considering the case $q < 1$ the following symbols ϕ and η are introduced.

$$A_1 = q + i\sqrt{1-q^2} = e^{i\phi}; \quad A_2 = q - i\sqrt{1-q^2} = e^{-i\phi} \tag{32}$$

$$\eta = 2\sqrt{s} \tag{33}$$

Separating the Hankel function $H_\gamma^j(2\sqrt{s_1})$ into real and imaginary parts

$$H_\gamma^{(j)}(2\sqrt{s_1}) = H_\gamma^{(j)}(\eta e^{-i\frac{\phi}{2}}) \equiv H_{\gamma R}^{(j)} + i H_{\gamma I}^{(j)} \tag{34.1}$$

the following relation is obtained.

$$H_{\gamma}^{(j)}(2\sqrt{s_2}) = H_{\gamma}^{(j)}(\eta e^{i\frac{\phi}{2}}) = H_{\gamma R}^{(j)} - i\,H_{\gamma I}^{(j)} \tag{34.2}$$

If the complex arbitrary constants are written as

$$C_j = C_{jR} + iC_{jI} \qquad (j = 1 \sim 4) \tag{35}$$

the condition that imaginary parts in eqs. (31) must vanish yields

$$C_{3R} = C_{1R},\quad C_{3I} = -C_{1I},\quad C_{4R} = C_{2R},\quad C_{4I} = -C_{2I} \tag{36}$$

Then the real expressions of $\frac{dW}{ds}$ etc. are as follows

$$\frac{dW}{ds} = -2\left[C_{1R}H_{2R}^{(1)} - C_{1I}H_{2I}^{(1)} + C_{2R}H_{2R}^{(2)} - C_{2I}H_{2I}^{(2)}\right] + \frac{2q}{s} \tag{37.1}$$

$$\frac{dF_o}{ds} = 2\left[C_{1R}(H_{2R}^{(1)}\cos\phi - H_{2I}^{(1)}\sin\phi) + C_{1I}(-H_{2R}^{(1)}\sin\phi - H_{2I}^{(1)}\cos\phi)\right.$$
$$\left. + C_{2R}(H_{2R}^{(2)}\cos\phi - H_{2I}^{(2)}\sin\phi) + C_{2I}(-H_{2R}^{(2)}\sin\phi - H_{2I}^{(1)}\cos\phi)\right] - \frac{4q^2}{s} \tag{37.2}$$

$$\frac{d^2W}{ds^2} = -2\Big[C_{1R}\{(H_{0R}^{(1)}+H_{2R}^{(1)})\cos\phi + (H_{0I}^{(1)}+H_{2I}^{(1)})\sin\phi\} + C_{1I}\{(H_{0R}^{(1)}+H_{2R}^{(1)})\sin\phi$$
$$-(H_{0I}^{(1)}+H_{2I}^{(1)})\cos\phi\} + C_{2R}\{(H_{0R}^{(2)}+H_{2R}^{(2)})\cos\phi + (H_{0I}^{(2)}+H_{2I}^{(2)})\sin\phi\}$$
$$+C_{2I}\{(H_{0R}^{(2)}+H_{2R}^{(2)})\sin\phi - (H_{0I}^{(2)}+H_{2I}^{(2)})\cos\phi\}\Big] - \frac{1}{s}\frac{dW}{ds} \tag{37.3}$$

$$\frac{d^2F_o}{ds^2} = 2\Big[C_{1R}(H_{0R}^{(1)}+H_{2R}^{(1)}) - C_{1I}(H_{0I}^{(1)}+H_{2I}^{(1)}) + C_{2R}(H_{0R}^{(2)}+H_{2R}^{(2)})$$
$$-C_{2I}(H_{0I}^{(2)}+H_{2I}^{(2)})\Big] - \frac{1}{s}\frac{dF}{ds} \tag{37.4}$$

Substituting eqs. (37) into the boundary conditions (23), four simultaneous equations to determine C_{jR} and C_{jI} are derived. Solving these equations, the final expression for the prebuckling deformations is obtained.

In order to determine the order of the value η, the following numerical example is examined.

$$\alpha = 30^{o}, \quad \frac{\ell_1}{t} = 800, \quad \frac{\ell_2}{t} = 1600, \quad \nu = 0.3 \tag{38}$$

For this example

$$\eta_1 = [\eta]_{at\, x=\ell_1} = 135.3 \qquad \eta_2 = [\eta]_{at\, x=\ell_2} = 191.4$$

Therefore, the absolute value of the argument in the Hankel function is very large and the value of the Hankel function can be satisfactorily evaluated by the asymptotic expansions

$$H^{(1)}_{\gamma R} = \sqrt{\frac{2}{\pi\eta}}\, e^{\eta \sin\frac{\phi}{2}} [\cos(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi + \frac{\phi}{4}) + \ldots.] \tag{39.1}$$

$$H^{(1)}_{\gamma I} = \sqrt{\frac{2}{\pi\eta}}\, e^{\eta \sin\frac{\phi}{2}} [\sin(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi + \frac{\phi}{4}) + \ldots.] \tag{39.2}$$

$$H^{(2)}_{\gamma R} = \sqrt{\frac{2}{\pi\eta}}\, e^{-\eta \sin\frac{\phi}{2}} [\cos(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{\phi}{4}) - \frac{(4\gamma^2-1)}{8\eta}\sin(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{3\phi}{4})$$
$$- \frac{(4\gamma^2-1)(4\gamma^2-9)}{128\eta^2}\cos(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{5\phi}{4}) + \ldots.] \tag{39.3}$$

$$H^{(2)}_{\gamma I} = \sqrt{\frac{2}{\pi\eta}}\, e^{-\eta \sin\frac{\phi}{2}} [-\sin(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{\phi}{4})$$
$$- \frac{(4\gamma^2-1)}{8\eta}\cos(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{3\phi}{4})$$
$$+ \frac{(4\gamma^2-1)(4\gamma^2-9)}{128\eta^2}\sin(\eta\cos\frac{\phi}{2} - \frac{2\gamma+1}{4}\pi - \frac{5\phi}{4}) + \ldots.] \tag{39.4}$$

For example, in the case of $q = 0.93$, i.e., $\phi = 21.56^{o}$,

$$e^{-(\eta_2-\eta_1)\sin\frac{\phi}{2}} = e^{-10.485} = 2.75 \times 10^{-5}$$

This value is quite small. Therefore, if $(\ell_2 - \ell_1)/t$ is large and q is not close to one, such that $\exp[-(\eta_2 - \eta_1)\sin\frac{\phi}{2}]$ is small, the prebuckling deformation in the region close to the upper end is approximately expressed by taking $C_{1R} = C_{1I} = 0$ and using the boundary conditions (23.1) and (23.2). The prebuckling deformation in the region close to the lower end is approximately expressed by taking $C_{2R} = C_{2I} = 0$ and using the boundary conditions (23.3) and (23.4). Since the value η is large, three terms of the asymptotic expansion have been used in the present numerical computation.

IV. BUCKLING PROBLEM

In our definition, the buckling load is found by determining the bifurcation points of the symmetric solution. The following expressions are substituted into eqs. (2)

$$
\begin{aligned}
w &= w_A(x) + w_B(x,\theta) \\
F &= \frac{-P}{\pi \sin 2\alpha}\, x + F_A(x) + F_B(x,\theta)
\end{aligned}
\tag{40}
$$

where $(w_B F_B)$ is the buckling mode and the corresponding displacements are

$$
\begin{aligned}
u &= u_A(x) + u_B(x,\theta) \\
v &= v_B(x,\theta)
\end{aligned}
\tag{41}
$$

Taking into account the equation of prebuckling deformation (9) for w_A and F_A and dropping nonlinear terms of w_B and F_B, which are taken as infinitesimally small quantities, the differential equations of buckling are obtained.

$$
\frac{1}{Et}\Delta\Delta F_B = \frac{\cot\alpha}{x}\,\frac{\partial^2 w_B}{\partial x^2} + \left[-\frac{1}{x^2}\frac{d^2 w_A}{dx^2}\frac{\partial^2 w_B}{\partial\theta^2} - \frac{1}{x}\frac{dw_A}{dx}\frac{\partial^2 w_B}{\partial x^2} - \frac{1}{x}\frac{d^2 w_A}{dx^2}\frac{\partial w_B}{\partial x}\right]
$$

$$
\begin{aligned}
D\Delta\Delta w_B &+ \frac{\cos\alpha}{x}\,\frac{\partial^2 F_B}{\partial x^2} - \left[\frac{1}{x}\frac{dF_A}{dx}\frac{\partial^2 w_B}{\partial x^2} + \frac{d^2 w_A}{dx^2}\left(\frac{1}{x}\frac{\partial F_B}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F_B}{\partial\theta^2}\right)\right. \\
&\left. + \frac{d^2 F_A}{dx^2}\left(\frac{1}{x}\frac{\partial w_B}{\partial x} + \frac{1}{x^2}\frac{\partial^2 w_B}{\partial\theta^2}\right) + \frac{1}{x}\frac{dw_A}{dx}\frac{\partial^2 F_B}{\partial x^2}\right] + \frac{P}{\pi\sin 2\alpha}\,\frac{1}{x}\,\frac{\partial^2 w_B}{\partial x^2} = 0
\end{aligned}
\tag{42}
$$

The corresponding boundary conditions

$$
\text{at } x = \ell_1;\quad u_{HB} = 0,\quad u_{VB} = 0,\quad v_B = 0,\quad \frac{\partial w_B}{\partial x} = 0 \tag{43}
$$

$$\text{at } x = \ell_2;\ u_{HB} = 0,\ \frac{\partial u_{VB}}{\partial\theta} = 0,\ v_B = 0,\ \frac{\partial w_B}{\partial x} = 0$$

$$\int_0^{2\pi\sin\alpha} [T_{xB}\cos\alpha - (Q_{xB} + \frac{1}{x}\frac{\partial M_{x\theta B}}{\partial\theta})\sin\alpha]\ x d\theta = 0 \tag{44}$$

where the suffix B designates the buckling mode, are derived from eqs. (6) and (8). Eqs. (5) show that we have the following relations

$$\frac{du_A}{dx} + \frac{1}{2}\left(\frac{dw_A}{dx}\right)^2 = \frac{1}{Et}\left(\frac{1}{x}\frac{dF_A}{dx} - \nu\frac{d^2F_A}{dx^2}\right)$$

$$\frac{u_A + w_A\cot\alpha}{x} = \frac{1}{Et}\left(\frac{d^2F_A}{dx^2} - \nu\frac{1}{x}\frac{dF}{dx}\right) \tag{45}$$

for the axisymmetrical prebuckling deformation. Substituting eqs. (40) and (41) into eqs. (5), taking into account eqs. (45) and dropping the nonlinear terms of w_B, the following equations result

$$\frac{\partial u_B}{\partial x} + \frac{dw_A}{dx}\frac{\partial w_B}{\partial x} = \frac{1}{Et}\left[\left(\frac{1}{x}\frac{\partial F_B}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F_B}{\partial\theta^2}\right) - \nu\frac{\partial^2 F_B}{\partial x^2}\right]$$

$$\frac{1}{x}\frac{\partial v_B}{\partial\theta} + \frac{u_B + w_B\cot\alpha}{x} = \frac{1}{Et}\left[\frac{\partial^2 F_B}{\partial x^2} - \nu\left(\frac{1}{x}\frac{\partial F_B}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F_B}{\partial\theta^2}\right)\right] \tag{46}$$

$$\frac{1}{x}\frac{\partial u_B}{\partial\theta} + \frac{\partial v_B}{\partial x} - \frac{v_B}{x} + \frac{1}{x}\frac{dw_A}{dx}\frac{\partial w_B}{\partial\theta} = \frac{2(1+\nu)}{Et}\left[\frac{1}{x^2}\frac{\partial F}{\partial\theta} - \frac{1}{x}\frac{\partial^2 F}{\partial x\partial\theta}\right]$$

The buckling deformation is expressed in a Fourier series

$$w_B = \frac{t}{\sqrt{12(1-\nu^2)}}\sum_{n=2}^{\infty} w_n(x)\sin n\overline{\theta} \tag{47.1}$$

where n is the circumferential wave number and should be an integer. Eqs. (42) and (46) suggest that the corresponding series for $F_B u_B$ and v_B are

$$F_B = \frac{Et^3}{12(1-\nu^2)} \sum_{n=2}^{\infty} f_n(x) \cdot \sin n\overline{\theta} \tag{47.2}$$

$$u_B = \frac{t}{\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} u_n(x) \cdot \sin n\overline{\theta} \tag{47.3}$$

$$v_B = \frac{t}{\sqrt{12(1-\nu^2)}} \sum_{n=2}^{\infty} v_n(x) \cdot \cos n\overline{\theta} \tag{47.4}$$

The differential equation (42) and the boundary conditions (43) and (44) lead to the conclusion that there are independent eigenvalues for each n. Substituting eqs. (47) into eqs. (42) and changing the variable from x to s, the equations take on the following form

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{\lambda^2}{s^2}\right)\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{\lambda^2}{s^2}\right)f_n = \frac{1}{s}\frac{d^2 w_n}{ds^2} + \frac{\lambda^2}{s^2}\frac{d^2 W}{ds^2} w_n$$
$$- \frac{1}{s}\frac{dW}{ds}\frac{d^2 w_n}{ds^2} - \frac{1}{s}\frac{d^2 W}{ds^2}\frac{dw_n}{ds}$$

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{\lambda^2}{s^2}\right)\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \frac{\lambda^2}{s^2}\right)w_n + \frac{1}{s}\frac{d^2 f_n}{ds^2} + \left(\frac{2q}{s} - \frac{1}{s}\frac{dF_o}{ds}\right)\frac{d^2 w_n}{ds^2}$$
$$- \frac{d^2 W}{ds^2}\left(\frac{1}{s}\frac{df_n}{ds} - \frac{\lambda^2}{s^2} f_n\right)$$
$$- \frac{d^2 F_o}{ds^2}\left(\frac{1}{s}\frac{dw_n}{ds} - \frac{\lambda^2}{s^2} w_n\right) - \frac{1}{s}\frac{dW}{ds}\frac{d^2 f_n}{ds^2} = 0 \tag{48}$$

where

$$\lambda = \frac{n}{\sin\alpha} \tag{49}$$

Substitution of eqs. (47) into eqs. (46) yields

$$\frac{1}{\cot\alpha}\frac{du_n}{ds}=\left(\frac{1}{s}\frac{df_n}{ds}-\frac{\lambda^2}{s^2}f_n\right)-\nu\frac{d^2f_n}{ds^2}-\frac{dW}{ds}\frac{dw_n}{ds} \tag{50.1}$$

$$\frac{1}{\cot\alpha}(u_n-\lambda v_n)=s\frac{d^2f_n}{ds^2}-\nu\left(\frac{df_n}{ds}-\frac{\lambda^2}{s}f_n\right)-w_n \tag{50.2}$$

$$\frac{1}{\cot\alpha}\left(\lambda u_n-v_n+s\frac{dv_n}{ds}\right)=-2(1+\nu)\lambda\left(\frac{df_n}{ds}-\frac{1}{s}f_n\right)-\lambda\frac{dW}{ds}w_n \tag{50.3}$$

Calculating

$$(50.2)-s\times\left\{\frac{d}{ds}(50.2)-(50.1)\right\}-(50.3)\times\lambda$$

and $$(50.2)\times\lambda-\frac{s}{\lambda}\left\{\frac{d}{ds}(50.2)-(50.1)\right\}-(50.3)$$

expressions for u_n and v_n are obtained.

$$\frac{1-\lambda^2}{\cot\alpha}u_n=-s^2\frac{d^3f_n}{ds^3}+(1-\nu+2\lambda^2+\nu\lambda^2)\frac{df_n}{ds}-\frac{3\lambda^2}{s}f_n+s\frac{dw_n}{ds}\left(1-\frac{dW}{ds}\right)+w_n\left(-1+\lambda^2\frac{dW}{ds}\right) \tag{51.1}$$

$$\frac{1-\lambda^2}{\cot\alpha}v_n=-\frac{s^2}{\lambda}\frac{d^3f_n}{ds^3}+\frac{(\lambda^2-1)}{\lambda}s\frac{d^2f_n}{ds^2}+\frac{2\lambda^2+1}{\lambda}\frac{df_n}{ds}+\frac{\lambda(-3-\nu+\nu\lambda^2)}{s}f_n$$

$$+s\frac{dw_n}{ds}\frac{1}{\lambda}\left(1-\frac{dW}{ds}\right)+w_n\lambda\left(-1+\frac{dW}{ds}\right) \tag{51.2}$$

Substitution of eqs. (47) into eqs. (43), where eq. (7) is also used, yields

$$\text{at } s=s_{\ell 1} \text{ and } s=s_{\ell 2};\quad u_n=0,\ v_n=0,\ w_n=0,\ \frac{dw_n}{ds}=0 \tag{52}$$

The traction boundary condition (44) is identically satisfied by substitution of eqs. (47). The boundary conditions, which are the same at both ends, are the so-called "clamped C-1" conditions. The boundary conditions $u_n = 0$ and $v_n = 0$ are expressed in terms of f_n using eqs. (51), where w_n and $\frac{dw_n}{ds}$ disappear because $w_n = \frac{dw_n}{ds} = 0$ at the boundary. Furthermore calculations of $[(51.2)\times\lambda-(51.1)]\div(\lambda^2-1)\,s$ yields

$$\frac{d^2 f_n}{ds^2} - \nu\left(\frac{1}{s}\frac{df_n}{ds} - \frac{\lambda^2}{s^2} f_n\right) = 0 \tag{53}$$

This equation is equal to the condition $\varepsilon_{\theta B} = 0$ derived from $u_{HB} = 0$ and $v_B = 0$. Accordingly we arrive at the following expression of the boundary condition:

$$\text{at } s = s_{\ell 1} \text{ and } s = s_{\ell 2}; \quad \left\{ \begin{aligned} & w_n = 0 \\ & \frac{dw_n}{ds} = 0 \\ & \frac{d^2 f_n}{ds^2} - \nu\left(\frac{1}{s}\frac{df_n}{ds} - \frac{n^2}{s^2} f_n\right) = 0 \\ & \frac{d^3 f_n}{ds^3} - \frac{(1-\nu+2\lambda^2+\nu\lambda^2)}{s^2}\frac{df_n}{ds} + \frac{3\lambda^2}{s^3} f_n = 0 \end{aligned} \right. \tag{54}$$

A new variable ξ is now introduced

$$\frac{x}{\ell_1} = \frac{s}{s_{\ell 1}} = e^{k\xi} \tag{55}$$

where

$$k = \sqrt{\frac{t}{\ell_1 \cot\alpha \sqrt{12(1-\nu^2)}}} = \sqrt{\left(\frac{t}{\ell_1 \tan\alpha}\right)\frac{1}{\sqrt{12(1-\nu^2)}}} \times \tan\alpha \tag{56}$$

$$k = \frac{1}{\sqrt{s_{\ell 1}}} = \frac{2}{\eta_1} \tag{57}$$

By changing the variable from s to ξ, the differential equations (48) are written as

$$\frac{d^4 f_n}{d\xi^4} - 4k\frac{d^3 f_n}{d\xi^3} + (-2\beta^2 + 4k^2)\frac{d^2 f_n}{d\xi^2} + 4\beta^2 k\frac{df_n}{d\xi} + (\beta^4 - 4\beta^2 k^2) f_n$$

$$= (e^{k\xi} - k\frac{dW}{d\xi})(\frac{d^2 w_n}{d\xi^2} - k\frac{dw_n}{d\xi}) + (\frac{d^2 W}{d\xi^2} - k\frac{dW}{d\xi})(\beta^2 w - k\frac{dw}{d\xi}) \tag{58.1}$$

$$\frac{d^4 w_n}{d\xi^4} - 4k\frac{d^3 w_n}{d\xi^3} + (-2\beta^2 + 4k^2)\frac{d^2 w_n}{d\xi^2} + 4\beta^2 k\frac{dw_n}{d\xi} + (\beta^4 - 4\beta^2 k^2) w_n$$

$$+ (e^{k\xi} - k\frac{dW}{d\xi})(\frac{d^2 f_n}{d\xi^2} - k\frac{df_n}{d\xi}) + (2qe^{k\xi} - k\frac{dF_o}{d\xi})(\frac{d^2 w_n}{d\xi^2} - k\frac{dw_n}{d\xi})$$

$$+ (\frac{d^2 F_o}{d\xi^2} - k\frac{dF_o}{d\xi})(\beta^2 w_n - k\frac{dw_n}{d\xi}) + (\frac{d^2 W}{d\xi^2} - k\frac{dW}{d\xi})(\beta^2 f_n - k\frac{df_n}{d\xi}) = 0 \tag{58.2}$$

where

$$\beta = \lambda k \tag{59}$$

and the boundary conditions (54) at $s = s_{\ell_1}$ are written as

$$\text{at } \xi = 0; \begin{cases} w_n = 0 & (60.1) \\ \dfrac{dw_n}{d\xi} = 0 & (60.2) \\ \dfrac{d^2 f_n}{d\xi^2} - (1+\nu)k\dfrac{df_n}{d\xi} + \nu\beta^2 f_n = 0 & (60.3) \\ \dfrac{d^3 f_n}{d\xi^3} - 3k\dfrac{d^2 f_n}{d\xi^2} + \left[-\beta^2(2+\nu) + (1+\nu)k^2\right]\dfrac{df_n}{d\xi} + 3\beta^2 k f_n = 0 & (60.4) \end{cases}$$

In the previous section $\frac{dW}{ds}$ etc. have been obtained as functions of s. $\frac{dW}{d\xi}$ and $\frac{dF_o}{d\xi}$ etc. in eqs. (59) are expressed in terms of $\frac{dW}{ds}$ and $\frac{dF_o}{ds}$ etc. as follows

$$\frac{dW}{d\xi} = \left(\frac{1}{k} e^{k\xi}\right) \frac{dW}{ds} \, , \quad \frac{d^2W}{d\xi^2} - k \frac{dW}{d\xi} = \left(\frac{1}{k^2} e^{2k\xi}\right) \frac{d^2W}{ds^2} \tag{61}$$

The numerical example given by eq. (38) yields $k = 0.01478$. In eq. (56) the term $(t/\ell_1 \tan\alpha)$ is a thickness to radius of curvature ratio, which is a finite small value. The effect of the cone angle is represented in the factor $\tan\alpha$. Therefore, the value of k is small for problems where α is not close to 90°.

V. FINITE DIFFERENCE EXPRESSION

The differential equations (58) are linear but do not have constant coefficients. Therefore, the solution of this eigenvalue problem has been obtained by using a finite difference approach. The following finite difference expressions have been used for the derivatives at the point j; (see Fig. 2)

$$\left\{\begin{aligned} \left(\frac{dg}{d\xi}\right)_j &= \frac{1}{2h}\,(g_{j+1}-g_{j-1}) \\ \left(\frac{d^2g}{d\xi^2}\right)_j &= \frac{1}{h^2}\,(g_{j+1}-2g_j+g_{j-1}) \\ \left(\frac{d^3g}{d\xi^3}\right)_j &= \frac{1}{2h^3}\,(g_{j+2}-2g_{j+1}+2g_{j-1}-g_{j-2}) \\ \left(\frac{d^4g}{d\xi^4}\right)_j &= \frac{1}{h^4}\,(g_{j+2}-4g_{j+1}+6g_j-4g_{j-1}+g_{j-2}) \end{aligned}\right. \tag{62}$$

where g is a function of ξ and h is an interval of equal distance as shown in Fig. 2.

By using this definition and a unified new symbol y_k

$$w_j = y_{2j}\,, \quad f_j = y_{2j+1}\,, \tag{63}$$

finite difference expressions for the differential equations (58.2) and (58.1) at a point $\xi = \xi_j$ are written as

$$(58.2);\ \sum_{m=1}^{9} a_{2j,m}\, y_{2j-5+m} = 0 \tag{64}$$

where $a_{2j,1} = 1 + 2kh, \quad a_{2j,2} = 0$

$$a_{2j,3} = \left\{-4-4kh+(-2\beta^2+4k^2)h^2-2\beta^2kh^3\right\} + e^{d\xi_j}\left(2q - \left(\frac{dF_o}{ds}\right)_j\right)\left(1 + \frac{kh}{2}\right)h^2$$

$$+ e^{2k\xi_j}\left(\frac{d^2F}{ds^2}\right)_j \frac{h^3}{2k}$$

$$a_{2j,4} = e^{k\xi_j}\left(1 - \left(\frac{dW}{ds}\right)_j\right)\left(1 + \frac{kh}{2}\right)h^2 + e^{2k\xi_j}\left(\frac{d^2W}{ds^2}\right)_j \frac{h^3}{2k}$$

$$a_{2j,5} = \left\{6-2(-2\beta^2+4k^2)h^2+(\beta^4-4\beta^2k^2)h^4\right\} - 2e^{k\xi_j}\left(2q - \left(\frac{dF_o}{ds}\right)_j\right)h^2$$

$$+ e^{2k\xi_j}\left(\frac{d^2F_o}{ds^2}\right)_j \frac{\beta^2h^4}{k^2}$$

$$a_{2j,6} = e^{k\xi_j}\left(1 - \left(\frac{dW}{ds}\right)_j\right)(-2h^2) + e^{2k\xi_j}\left(\frac{d^2W}{ds^2}\right)_j \cdot \frac{\beta^2h^4}{k^2}$$

$$a_{2j,7} = \left\{-4+4kh+(-2\beta^2+4k^2)h^2+2\beta^2kh^3\right\} + e^{k\xi_j}\left(2q-\left(\frac{dF_o}{ds}\right)_j\right)\left(\left(1-\frac{kh}{2}\right)h^2\right)$$

$$-e^{2k\xi_j}\left(\frac{d^2F_o}{ds^2}\right)_j \frac{h^3}{2k}$$

$$a_{2j,8} = e^{k\xi_j}\left(1-\left(\frac{dW}{ds}\right)_j\right)\left(1-\frac{kh}{2}\right)h^2 - e^{2k\xi_j}\left(\frac{d^2W}{ds^2}\right)_j \frac{h^3}{2k}$$

$$a_{2j,9} = 1 - 2kh$$

(65)

$$(58.1); \quad \sum_{m=1}^{9} a_{2j+1,m}\, y_{2j+1-5+m} = 0 \tag{66}$$

where

$$a_{2j+1,1} = 1 + 2kh$$

$$a_{2j+1,2} = -e^{k\xi_j}\left(1-\left(\frac{dW}{ds}\right)_j\right)\left(1 + \frac{kh}{2}\right)h^2 - e^{2k\xi_j}\left(\frac{d^2W}{ds^2}\right)_j \frac{h^3}{2k}$$

$$a_{2j+1,3} = -4-4kh+(-2\beta^2+4k^2)h^2-2\beta^2kh^3$$

$$a_{2j+1,4} = 2e^{k\xi_j}\left(1-(\frac{dW}{ds})_j\right)h^2-e^{2k\xi_j}(\frac{d^2W}{ds^2})_j\ \frac{\beta^2h^4}{k^2}$$

$$a_{2j+1,5} = 6-2(-2\beta^2+4k^2)h^2 + (\beta^4-4\beta^2k^2)h^4$$

$$a_{2j+1,6} = -e^{k\xi_j}\left(1-(\frac{dW}{ds})_j\right)(1-\frac{kh}{2})h^2+e^{2k\xi_j}(\frac{d^2W}{ds^2})_j\ \frac{h^3}{2k}$$

$$a_{2j+1,7} = -4+4kh+(-2\beta^2+4k^2)h^2 + 2\beta^2kh^3$$

$$a_{2j+1,8} = 0$$

$$a_{2j+1,9} = 1 - 2kh \qquad (67)$$

The coefficients $a_{i,m}$ for the case of a cylindrical shell are given in appendix B. In the following work the finite difference equations (64) and (66) are numbered by the suffix 2j and 2j+1 of a. If the boundary $\xi = 0$ is taken as the point j = 0, finite difference expressions of the boundary conditions (60) at $\xi = 0$ are written as

$$y_o = 0 \qquad (68.1)$$

$$y_{-2} = y_2 \qquad (68.2)$$

$$y_{-1}\left[1+\frac{(1+\nu)kh}{2}\right]+y_1\left[-2+\nu\beta^2h^2\right]+y_3\left[1-\frac{(1+\nu)kh}{2}\right] = 0 \qquad (68.3)$$

$$-y_{-3}+y_{-1}\left[2-6kh-\left\{-\beta^2(2+\nu)+(1+\nu)k^2\right\}h^2\right]+y_1\left[12kh+6\beta^2kh^3\right]$$
$$+y_3\left[-2-6kh+\left\{-\beta^2(2+\nu)+(1+\nu)k^2\right\}h^2\right]+y_5 = 0 \qquad (68.4)$$

Since $y_o = 0$, equation (64) is used for $2j \geqslant 2$, and equation (66) is used for $2j+1 \geqslant 1$. In the first, second and third equations, functions at outside fictitious points, i.e., y_{-3}, y_{-2} and y_{-1}, appear. However, the function at outside points can be expressed by the function at inside points using the boundary conditions (68.2), (68.3) and (68.4). Consequently the first three equations are expressed in terms of functions at inside points alone as follows

First eq.;

$$\sum_{m=5}^{9} \bar{a}_{1,m}\, y_{1-5+m} = 0 \tag{69}$$

$$\bar{a}_{1,5} = a_{1,5} + a_{1,1} \times 6kh(2+\beta^2h^2) + \left[a_{1,1}\left\{2-6kh+(2+\nu)\beta^2h^2-(1+\nu)k^2h^2\right\}+a_{1,3}\right] \times$$
$$\times \frac{(4-2\nu\beta^2h^2)}{\{2+(1+\nu)kh\}}$$

$$\bar{a}_{1,6} = a_{1,6} + a_{1,2}$$

$$\bar{a}_{1,7} = a_{1,7} - \frac{4a_{1,1}\left[2+\{(2+\nu)\beta^2+2(1+\nu)k^2\}h^2\right]-a_{1,3}\left[2-(1-\nu)kh\right]}{\left[2+(1+\nu)kh\right]}$$

$$\bar{a}_{1,8} = a_{1,8}$$

$$\bar{a}_{1,9} = a_{1,9} + a_{1,1} \tag{70}$$

Second eq.;

$$\sum_{m=4}^{9} \bar{a}_{2,m}\, y_{2-5+m} = 0 \tag{71}$$

$$\bar{a}_{2,4} = a_{2,4}, \quad \bar{a}_{2,5} = a_{2,5} + a_{2,1}, \quad \bar{a}_{2,6} = a_{2,6}, \quad \bar{a}_{2,7} = a_{2,7}$$

$$\bar{a}_{2,8} = a_{2,8}, \quad \bar{a}_{2,9} = a_{2,9} \tag{72}$$

Third eq.;

$$\sum_{m=3}^{9} \bar{a}_{3,m}\, y_{3-5+m} = 0 \tag{73}$$

$$\bar{a}_{3,3} = a_{3,3} - a_{3,1} \frac{(-4+2\nu\beta^2 h^2)}{2+(1+\nu)kh}$$

$$\bar{a}_{3,4} = a_{3,4}$$

$$\bar{a}_{3,5} = a_{3,5} - a_{3,1} \times \frac{2-(1+\nu)kh}{2+(1+\nu)kh}$$

$$\bar{a}_{3,6} = a_{3,6}, \quad \bar{a}_{3,7} = a_{3,7}, \quad \bar{a}_{3,8} = a_{3,8}, \quad \bar{a}_{3,9} = a_{3,9} \tag{74}$$

Now let us consider a semi-infinite conical shell. The equations (69), (71), (73) and (64) and (66) for $j \geqslant 2$ construct an infinite number of linear equations for infinite number of y_k $(k \geqslant 1)$. In order for these equations to have a nontrivial solution the determinant of the coefficients of y_k must vanish, i.e.,

$$\text{Det} = \begin{vmatrix} \bar{a}_{15} & \bar{a}_{16} & \bar{a}_{17} & \bar{a}_{18} & \bar{a}_{19} & & & & & & \\ \bar{a}_{24} & \bar{a}_{25} & \bar{a}_{26} & \bar{a}_{27} & \bar{a}_{23} & \bar{a}_{29} & & & 0 & & \\ \bar{a}_{33} & \bar{a}_{34} & \bar{a}_{35} & \bar{a}_{36} & \bar{a}_{37} & \bar{a}_{38} & \bar{a}_{39} & & & & \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} & & \\ & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & \\ 0 & & a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\ & & & \ddots & & & & & & & \ddots \end{vmatrix} = 0 \tag{75}$$

Equation (75) determines the eigenvalues of the problem. The value of this determinant can be calculated as follows. First, the values of q and β^2 are assumed. This determines the numerical value of all elements. Next the determinant is converted to a lower triangular determinant by the following operations.

$$\left\{\begin{array}{l}
a_{i,m} - \dfrac{a_{i,5}}{a_{i,5}} \times a_{i,m} \rightarrow \text{new value of element } (i,m) = 0 \text{ for } m = 6,7,8,9 \\[2ex]
a_{i+1,m} - \dfrac{a_{i+1,4}}{a_{i,5}} \times a_{i,m+1} \rightarrow \text{new value of element } (i+1,m) \text{ for } m = 5,6,7,8 \\[2ex]
a_{i+2,m} - \dfrac{a_{i+2,3}}{a_{i,5}} \times a_{i,m+2} \rightarrow \text{new value of element } (i+2,m) \text{ for } m = 4,5,6,7 \\[2ex]
a_{i+3,m} - \dfrac{a_{i+3,2}}{a_{i,5}} \times a_{i,m+3} \rightarrow \text{new value of element } (i+3,m) \text{ for } m = 3,4,5,6 \\[2ex]
a_{i+4,m} - \dfrac{a_{i+4,1}}{a_{i,5}} \times a_{i,m+4} \rightarrow \text{new value of element } (i+4,m) \text{ for } m = 2,3,4,5
\end{array}\right.$$

In this operation $a_{i,m}$ means $\bar{a}_{i,m}$ for i = 1,2,3. As the values of the elements (1,m), where m = 6,7,8,9, have become zero, we can apply the above calculations to the new elements for i = 2. Carrying out such calculations successively, we finally arrive at

$$\text{Det} = \begin{vmatrix}
b_{15} & & & & & & \\
b_{24} & b_{25} & & & 0 & & \\
b_{33} & b_{34} & b_{35} & & & & \\
b_{42} & b_{43} & b_{44} & b_{45} & & & \\
b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & & \\
 & b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & \\
0 & & b_{71} & b_{72} & b_{73} & b_{74} & b_{75} \ddots
\end{vmatrix} = \prod_{i=1}^{\infty} b_{i5} \qquad (77)$$

where the b_{ij}'s are the final values of the elements.

Since the above calculation does not change the value of the determinant, the correct value is given by infinite products of the diagonal elements. If the buckling does not occur over the whole region, the diagonal term b_{i5} sufficiently far from the boundary will converge to two values alternately, (i.e., $b_{501,5} \approx B_f$, $b_{502,5} \approx B_w$, $b_{503,5} \approx B_f$, $b_{504,5} \approx B_w$ and so on). Therefore, it can be decided if the value of the determinant is positive or negative by the products of a suitable finite number of diagonal terms.

By calculating the value of Det for various values of q with a constant value β^2, the eigenvalue q can be found. Subsequently by changing the value β^2, the smallest value of the eigenvalue q is obtained. This smallest value is the buckling load q_{cr}.

VI. RESULTS

The above calculations have been carried out using an IBM 7094 at the California Institute of Technology. In the present paper the minimum value of q over a range of β^2 has been obtained regarding β^2 as a continuous variable. The results q_{cr} and β^2_{min} for several values of the parameter k are shown in Table 1. In the case of local buckling in the region close to the smaller radius end of semi-infinite conical shells, the differential equations and the boundary conditions for the prebuckling deformation and the buckling deformation show that k is the only parameter related to shell dimensions and cone angle. The case k = 0 indicates a cylindrical shell.

On the other hand, if a new variable $\bar{\xi}$ is introduced

$$\frac{x}{\ell_2} = \frac{s}{s_{\ell 2}} = e^{\overline{k\xi}} \tag{78}$$

where

$$\bar{k} = \sqrt{\frac{t}{\ell_2 \cot\alpha \sqrt{12(1-\nu^2)}}} \tag{79}$$

and using

$$\bar{\beta} = \lambda \bar{k} \tag{80}$$

the same differential equations as eqs. (58) and the same boundary conditions as eqs. (60) for buckling deformation result having the following correspondence

$$\bar{\xi} \rightarrow \xi\,, \quad \bar{k} \rightarrow k, \quad \bar{\beta} \rightarrow \beta$$

The same is true for the differential equations, eqs. (22), and the boundary conditions, eqs. (23), for the prebuckling deformation. The value $\bar{\xi}$ at the smaller end $x = \ell_1$ is usually quite large. For the numerical example shown in eq. (38),

$$\bar{\xi}_{\ell 1} = (\bar{\xi})_{at\ x=\ell_1} = \frac{1}{\bar{k}} \log \frac{\ell_1}{\ell_2} = \frac{1}{-0.01045} \log \frac{1}{2} = 66.33$$

If $\ell_1 \rightarrow 0$, then $\bar{\xi} \rightarrow \infty$. Therefore, a semi-infinite approach for $\bar{\xi}$ can also be applied to a finite length cone having the boundary at the larger radius end. Then, it can be concluded that the expressions given in Section V can also be applied to local buckling in the region close to the larger radius end, if the cone is long enough that the value $\bar{\xi}_{\ell 1}$ is large. Therefore, if we extend the value of k to a negative region by the definition $k = \bar{k}$, the local buckling load in the region close to the larger radius end can be shown in the same figure.

The results in Table 1 are shown in Figs. 3 and 4. For the numerical example of eq. (38), k at the smaller radius end is 0.0478 and k at the larger radius end is -0.0045. Therefore, it is concluded that buckling at the smaller radius boundary determines the critical load. However, the effect of the cone angle represented by the parameter k on the buckling load q_{cr} is quite small since the factor of a square root of thickness to radius of the curvature ratio makes the value of k small.

The results in Table 1 have been obtained using the interval $h = 0.2$ and the number of points $N = 150$. Therefore, the maximum value of ξ is $0.2 \times 150 = 30$. If a smaller value of h and a greater value of N is used so that $h \times N$ is larger, the numerical values of q_{cr} and β^2_{min} are improved. A numerical example for accuracy is

$h = 0.2, \quad N = 100; \quad q_{cr} = 0.9260 \quad \beta^2_{min} = 0.228$

$h = 0.2, \quad N = 150; \quad q_{cr} = 0.9268 \quad \beta^2_{min} = 0.224$

in the case of the cylindrical shell.

An example of computed prebuckling deformation of a semi-infinite cylindrical shell is shown in Fig. 5. The above results for the effect of the number of points on the accuracy of q_{cr} and Fig. 5 suggest that, if the dimensionless half length of a cylindrical shell $\xi^*_{L/2}$ is larger than about 25, the semi-infinite approach can be applied to the finite length cylinder. The requirement $\xi_{L/2} > 25$ means

$$\frac{L}{\sqrt{tr}} = \frac{2}{\sqrt[4]{12(1-\nu^2)}} \xi_{L/2} > 1.10 \times 25 = 27.5 \tag{81}$$

For example, if $r/t = 800$, then $L/r > 0.972$. If the same results are applied to a conical shell, reliable results should be obtained using a semi-infinite approach if

$$\ell_2 e^{25\bar{k}} > \ell_1 e^{25k} \qquad \text{i.e.,} \quad \frac{\ell_2}{\ell_1} > e^{25(k-\bar{k})} \tag{82}$$

For the numerical example of eq. (38),

$$\frac{\ell_2}{\ell_1} = 2, \qquad e^{25(k-\bar{k})} = 1.879$$

The value obtained for the buckling load of a semi-infinite cylindrical shell, $q_{cr} = 0.9268$, is reasonable when compared with

* The explanation of variable ξ for cylindrical shells is shown in eq. (A-2).

Almroth's results (Ref. 24). His results, for the "C-1" boundary condition, are in the range $q_{cr} = 0.907 \rightarrow 0.930$ for various radius-thickness ratios and length-radius ratios. The values $L/\sqrt{tr}$ in his numerical examples are 7, 16, 24, and 32.

VII. CONCLUSIONS

The analysis presented in this paper shows that the smallest axial buckling load of a truncated conical shell is associated with local buckling in the region close to the smaller radius end. The analysis, which includes the effect of prebuckling deformation, shows that the decrease of the buckling load from the classical load for a conical shell is quite close to the decrease previously shown for a cylindrical shell. The numerical calculations show that the effect of the cone angle on this decrease is very small and the cylindrical shell results can be used with very little error. The reason for the small difference in the conical and cylindrical results is due to the fact that the parameter k, that expresses the cone angle, includes the square root of the thickness to radius of curvature ratio as well as tan α. Therefore, as long as the cone angle α is not close to 90 degrees, the parameter k is small and the effect of the cone angle is negligible. If α is close to 90 degrees, the neglect of the nonlinear terms in the prebuckling solution is not valid and the above conclusion will not hold.

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APPENDIX A NEGLECTION OF THE NONLINEAR TERMS IN THE PREBUCKLING EQUATIONS

The nonlinear terms in equation (22) have been dropped in the present analysis. The approximation will be supported as follows. First the equations governing the prebuckling deformation (22) are written using the variable ξ defined in equation (55).

$$\frac{d^3F_o}{d\xi^3} - 2k\frac{d^2F_o}{d\xi^2} + 2kqe^{k\xi} - e^{k\xi}\frac{dW}{d\xi}\left(1 - \frac{k}{2}e^{-k\xi}\frac{dW}{d\xi}\right) = 0 \qquad \text{(A-1.1)}$$

$$\frac{d^3W}{d\xi^3} - 2k\frac{d^2W}{d\xi^2} + 2qe^{k\xi}\frac{dW}{d\xi} + e^{k\xi}\frac{dF_o}{d\xi}\left(1 - ke^{-k\xi}\frac{dW}{d\xi}\right) = 0 \qquad \text{(A-1.2)}$$

If it can be shown that $\left|ke^{-k\xi}\frac{dW}{d\xi}\right| << 1$ then the nonlinear terms in the above equations can be neglected without introducing any appreciable error into the analysis. In order to look at the order of magnitude of $\left|\frac{dW}{d\xi}\right|$, the prebuckling deformation for a cylindrical shell (Appendix B, eq. B-5) will be used since this deformation is similar to that for a conical shell but very much simpler. From this expression it is found that for $q < 1$

$$\left|\frac{dW}{d\xi}\right| < 2\sqrt{2}\,\nu \qquad \text{(A-2)}$$

Therefore the following result is found

$$\left|k\frac{dW}{d\xi}e^{-k\xi}\right| < k\,2\sqrt{2}\,\nu << 1 \qquad \text{(A-3)}$$

since $k << 1$. The linearization performed in the prebuckling analysis is a valid approximation as long as k is a small number. This means that the present analysis will not be valid if the cone angle α is close to 90^o.

APPENDIX B EXPRESSIONS FOR CYLINDRICAL SHELLS

If the limit $k \to 0$ is taken, expressions for a cylindrical shell are obtained. In this limit $\alpha \to 0$ and $\ell_1 \to \infty$ so that the product $\ell_1 \sin\alpha = r$ which is the radius of the cylinder. The limit $\alpha \to 0$ gives $k \to 0$ but

$$k\ell_1 = \sqrt{\frac{t \cdot \ell_1 \tan\alpha}{\sqrt{12(1-\nu^2)}}} \to = \sqrt{\frac{tr}{\sqrt{12(1-\nu^2)}}} \tag{B-1}$$

is finite. Using equation (55) the real axial coordinate $x-\ell_1$ is found in terms of the nondimensional coordinate ξ.

$$x-\ell_1 = \sqrt{\frac{tr}{\sqrt{12(1-\nu^2)}}}\,\xi \tag{B-2}$$

Although $\alpha \to 0$ gives $\lambda \to \infty$, a finite value for β is obtained.

$$\beta = \lambda k = \frac{n}{\sin\alpha} x \sqrt{\left(\frac{t}{\ell_1 \tan\alpha}\right)\frac{1}{\sqrt{12(1-\nu^2)}}}\;\tan\alpha \to = n\sqrt{\frac{t}{r\sqrt{12(1-\nu^2)}}} \tag{B-3}$$

Equation (61) shows the following relation

$$\frac{d}{ds} = k\frac{d}{d\xi} \tag{B-4}$$

in the limit $k \to 0$.

The prebuckling deformation of a semi-infinite cylindrical shell having the "clamped" boundary condition is

$$W = 2\nu q\left[1 - e^{-\sqrt{\frac{1-q}{2}}\,\xi}\left(\cos\sqrt{\frac{1+q}{2}}\,\xi + \sqrt{\frac{1-q}{1+q}}\sin\sqrt{\frac{1+q}{2}}\,\xi\right)\right] \tag{B-5}$$

In the case of cylindrical shells the effect of nonlinear terms disappears. Therefore, eq. (B-5) is the exact solution. The stress function is expressed as

$$\frac{d^2F_o}{d\xi^2} = W - 2\nu q \tag{B-6}$$

Substituting eqs. (B-3), (B-4) and (B-5) into eqs. (58) and (60), and taking the limit process $k \to 0$, the differential equations and the boundary conditions for the buckling of a cylindrical shell results. Eqs. (65) and (69) are as follows for the cylindrical shell.

$$
\begin{aligned}
a_{2j,1} &= 1 \\
a_{2j,2} &= 2 \\
a_{2j,3} &= -4-2\beta^2h^2 + 2qh^2 \\
a_{2j,4} &= h^2 \\
a_{2j,5} &= 6+4\beta^2h^2+\beta^4h^4-4qh^2+\beta^2h^4(W_j-2\nu q) \\
a_{2j,6} &= -2h^2 + \beta^2h^2 \left(\frac{d^2W}{d\xi^2}\right)_j \\
a_{2j,7} &= -4-2\beta^2h^2 + 2qh^2 \\
a_{2j,8} &= h^2 \\
a_{aj,9} &= 1 \qquad\qquad (B\text{-}7) \\
a_{2j+1,1} &= 1 \\
a_{2j+1,2} &= -h^2 \\
a_{2j+1,3} &= -4-2\beta^2h^2 \\
a_{2j+1,4} &= 2h^2-\beta^2h^4 \left(\frac{d^2W}{d\xi^2}\right)_j \\
a_{2j+1,5} &= 6+4\beta^2h^2+\beta^4h^4 \\
a_{2j+1,6} &= -h^2 \\
a_{2j+1,7} &= -4-2\beta^2h^2 \\
a_{2j+1,8} &= 0 \\
a_{2j+1,9} &= 1
\end{aligned}
$$

Table 1 Computed Results, $h = 0.2$, $N = 150$

k	-0.01	0	0.01	0.02
q_{cr}	0.9283	0.9268	0.9255	0.9244
β^2_{min}	0.209	0.224	0.239	0.254

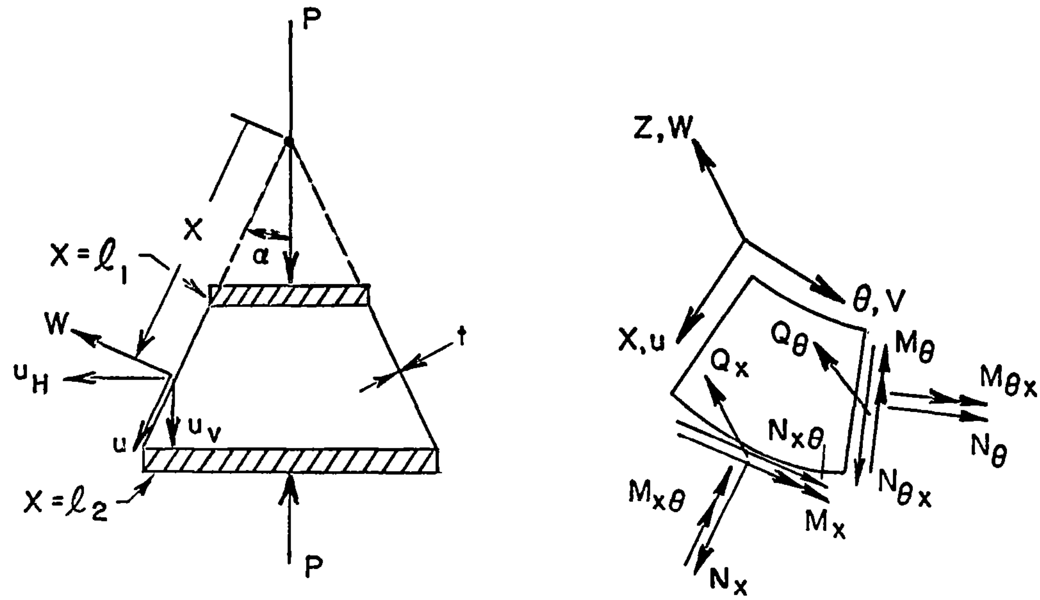

FIG. 1 COORDINATES AND SYMBOLS

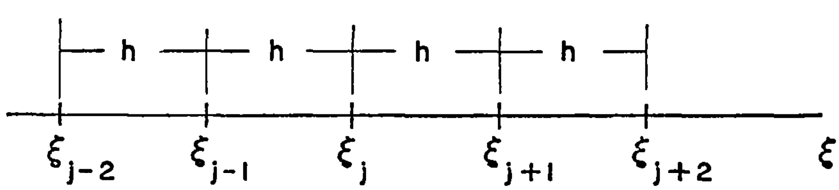

FIG. 2 FINITE DIFFERENCE INTERVALS

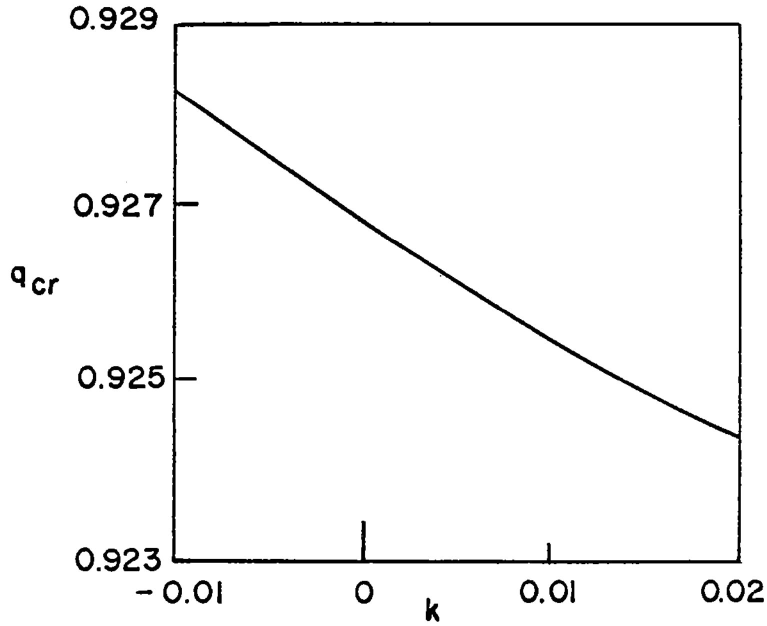

FIG. 3 BUCKLING LOAD q_{cr} VS. CONE ANGLE PARAMETER k

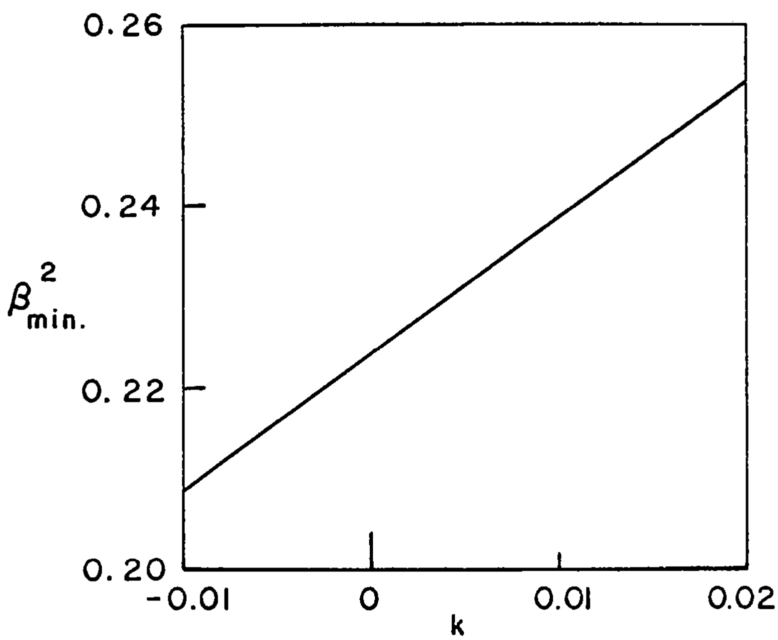

FIG. 4 CIRCUMFERENTIAL WAVE NUMBER PARAMETER $\beta^2_{min.}$ VS. CONE ANGLE PARAMETER k

NASA-Langley, 1967 CR-707

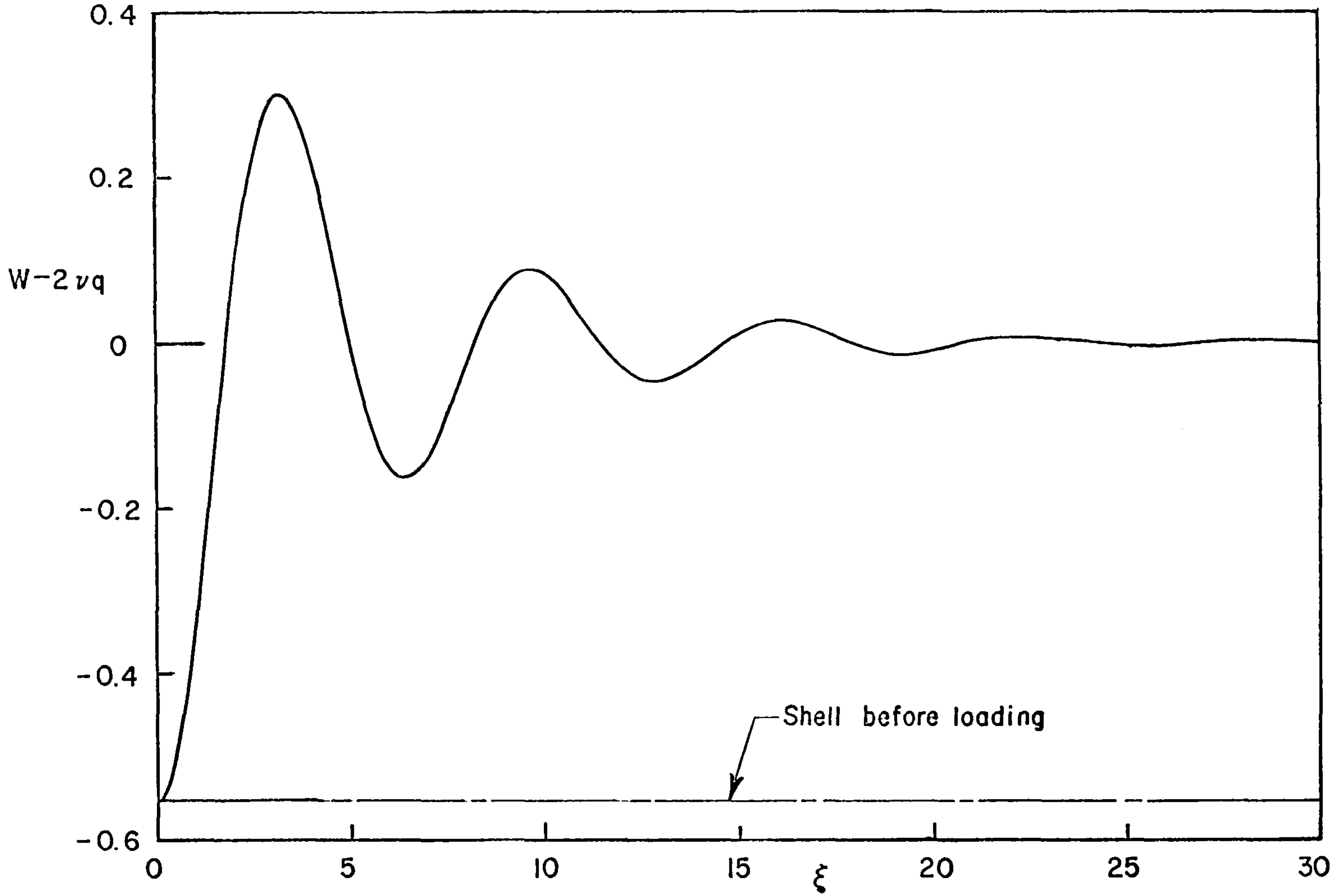

FIG. 5 PREBUCKLING DEFORMATION OF SEMI-INFINITE CYLINDRICAL SHELL
$q = 0.927$, $\nu = 0.3$