NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1041

Analytic Linear Systems of Differential Equations in Implicit Form

Richard J. Hanson

JET PROPULSION LABORATORY
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Approved by:

R. J. Jirka

R. J. Jirka, Manager
Scientific Programming Section

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Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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N67-15149

Abstract

A local theory is developed for the system $B(z)(dy/dz) = A(z)y + f(z)$ (1) for square holomorphic matrices A and B and a holomorphic vector f with the assumption that $\det B(z) \equiv 0$. Necessary and sufficient conditions for the existence of a solution to (1) and an algorithm for calculating this solution (provided it exists) are given.

Author

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I. Introduction

In the study of certain systems of ordinary differential equations, particularly in turning-point theory and other linear systems associated with the singular perturbation of small parameters, there appears the following problem.

Suppose $A(z)$ and $B(z)$ are $n \times n$ matrices holomorphic in a neighborhood U of $z = 0$. What form do solutions of the system

$$B(z)\frac{dy}{dz} = A(z)\,y + f(z) \tag{1}$$

possess if

$$det\,B(z) \equiv 0? \tag{2}$$

In Section II we prove two lemmas concerned with a decomposition of holomorphic matrices. This decomposition becomes the keystone of a reduction of the system (1), obtained in Section III, to a linear system of differential equations with a singularity of finite rank at $z = 0$ whose solution forms are known, (see Ref. 1, pp. 8–116 for an excellent treatment of this problem), and a linear system of algebraic equations and (possibly) a linear integral equation. A solution to this last-mentioned integral equation is a necessary and sufficient condition of compatibility which the system (1), (2) must satisfy. In Section IV, a procedure is given whereby one may either solve the integral equation in Section III or determine that no solutions exist.

It is of interest to note that systems of the form (1), (2) have been studied using the theory of pencils of matrices, *provided A and B are constant* (Ref. 2, Vol. II, p. 45).

Lettenmeyer (Ref. 3, pp. 87–91) has obtained results regarding the number of holomorphic solutions of a homogeneous system of the form (1) with the restriction that $det\,B(z) \not\equiv 0$.

It is convenient to introduce the following notation: Throughout this paper 0_p and I_q will designate the $p \times p$ matrix of zeros and the $q \times q$ identity matrix, respectively.

Suppose that x is a vector with n components and that p integers $n_i \geqq 1$ are chosen so that $n_1 + \cdots + n_p = n$. Then

$$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{bmatrix}, \qquad \{(x_1, n_1), \cdots, (x_p, n_p)\}$$

will denote a segmentation of the vector x into p vectors $x_1, \cdots, x_p$ of respective lengths $n_1, \cdots, n_p$.

Suppose C and D are square matrices of respective dimensions n_1 and m_1. Then $C \oplus D$ will denote the direct sum of C and D, the $(n_1 + m_1) \times (n_1 + m_1)$ matrix

$$\begin{bmatrix} C & O \\ O & D \end{bmatrix}$$

II. Two Lemmas on Matrix Theory

Lemma 1

Suppose that $B(z) \not\equiv 0$ is an $n \times n$ matrix holomorphic in U. (a) Then the rank of $B(z)$ is constant in $U - \{0\}$ if U is sufficiently restricted. (b) If the nonnegative integer k is the rank of $B(z)$ on $U - \{0\}$, then a fixed minor of $B(z)$ of order k does not vanish in $U - \{0\}$.

Proof. We first prove part (a). The set of points in U where the rank of $B(z)$ changes is a subset of the set of points in U where some minor, not identically zero, of $B(z)$ has a zero. Hence it is a finite set and every point of U has a punctured neighborhood about it in which the rank of $B(z)$ does not change.

We now prove part (b). If (b) were false then there would be points $z \in U - \{0\}$ such that rank $B(z) \neq k$ no matter how small the neighborhood U of $z = 0$, which would contradict (a). This completes the proof of Lemma 1. With the aid of this lemma, we can now prove the following.

Lemma 2

Suppose $B(z)$ is an $n \times n$ matrix holomorphic in U. Suppose U is sufficiently restricted so that $B(z)$ has constant rank k in $U - \{0\}$. Then there exist $n \times n$ matrices $P(z)$ and $Q(z)$, both of which are holomorphic in U and nonsingular for $z \neq 0$ such that

$$P(z)\,B(z)\,Q(z) = 0_{n-k} \oplus C(z) \tag{3}$$

The holomorphic matrix of dimension $C(z)$ is $k \times k$ with $\det C(z) \not\equiv 0$. If $B(z)$ is of constant rank k on all of U, then the matrices P and Q of Eq. (3) are nonsingular at $z = 0$ also.

Proof. It follows from Lemma 1(b) that we may find a constant permutation matrix S such that

$$S^{-1} B(z) S = \begin{bmatrix} C_{11}(z) & C_{12}(z) \\ C_{21}(z) & C_{22}(z) \end{bmatrix} \tag{4}$$

where $C_{22}(z)$ is a $k \times k$ matrix with $\det C_{22}(z) \neq 0$ for $z \in U - \{0\}$.

Suppressing the dependence upon z, the identity

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I_{n-k} & 0 \\ -C_{22}^{-1}C_{21} & I_k \end{bmatrix} = \begin{bmatrix} C_{11} - C_{12}C_{22}^{-1}C_{21} & C_{12} \\ 0 & C_{22} \end{bmatrix} \tag{5}$$

valid on $U - \{0\}$, implies that

$$C_{11}(z) - C_{12}(z)\,C_{22}^{-1}(z)\,C_{21}(z) \equiv 0, \qquad (z \in U) \tag{6}$$

For if the left member of expression (6) were nonzero at some point of $U - \{0\}$ it would follow that the rank of the right member of identity (5) would exceed k at some point of $U - \{0\}$. But the rank of the left member of (5) is exactly k on $U - \{0\}$. (Ref. 2, Vol. I, p. 66.) Thus expression (6) follows.

Let nonnegative integers μ_0 and ν_0 be chosen large enough so that both of the matrices

$$Q_1(z) = z^{\mu_0} \begin{bmatrix} I_{n-k} & 0 \\ -C_{22}^{-1}(z)\,C_{21}(z) & I_k \end{bmatrix} \tag{7}$$

and

$$P_1(z) = z^{\nu_0}\begin{bmatrix} I_{n-k} & -C_{12}(z)\,C_{22}^{-1}(z) \\ 0 & I_k \end{bmatrix} \tag{8}$$

are holomorphic in U. Observe that P_1 and Q_1 are both nonsingular for $z \neq 0$.

Noting that

$$P_1(z)\begin{bmatrix} C_{11}(z) & C_{12}(z) \\ C_{21}(z) & C_{22}(z) \end{bmatrix} Q_1(z) = \begin{bmatrix} 0_{n-k} & 0 \\ 0 & z^{\mu_0+\nu_0}C_{22}(z) \end{bmatrix}$$

$$= 0_{n-k} \oplus z^{\mu_0+\nu_0}C_{22}(z) \tag{9}$$

we define

$$P(z) = P_1(z)\,S^{-1} \tag{10}$$

$$Q(z) = SQ_1(z) \tag{11}$$

$$C(z) = z^{\mu_0+\nu_0}C_{22}(z) \tag{12}$$

The first part of Lemma 2 follows from Eq. (9) and (12).

The last statement of Lemma 2 follows from the observation that if $B(z)$ is of constant rank k on U, the permutation matrix S in Eq. (4) may be so chosen that $\det C_{22}(z) \neq 0, (z \in U)$. Thus the integers μ_0 and ν_0 in the matrices in the right-hand members of (7) and (8) may both be taken to be zero.

The matrices P, Q, and C defined by Eq. (10)–(12) are holomorphic in U. This completes the proof of Lemma 2.

Lemma 2 bears a resemblance to a classical theorem found in Ref. 3, p. 75. In this theorem it is proved that if $\det B(z) \not\equiv 0$ then there exist matrices P and Q such that

$$P(z)\,B(z)\,Q(z) = z^D, \qquad (z \in U) \tag{13}$$

where P is a polynomial matrix with $\det P(z) \equiv 1$, $Q(z)$ is holomorphic and nonsingular in U, and D is a diagonal matrix of nonnegative integers.

From Lemma 2 and expression (3) it is clear that a decomposition of the form (13) may now be obtained for $C(z)$.

We will discuss this no further since the theorem proving the decomposition (13) is not used directly in the remainder of this paper.

III. Reduction of the System (1)

Let the matrix $B(z)$ satisfy the hypotheses of Lemma 2. We then introduce the following definition.

Definition

The integer $n - k$, which appears in Eq. (3) of Lemma 2, is the *analytic nullity* of $B(z), (z \in U)$.

Let the integer m_0 designate the analytic nullity of the matrix $B(z)$ of the system (1). By identity (2) $m_0 > 0$. Note that if $B(z)$ is of rank k on $U - \{0\}$, then $m_0 = n - k$.

Using Lemma 2, there exist $n \times n$ matrices $P_0(z)$ and $Q_0(z)$ holomorphic in U and nonsingular for $z \neq 0$ such that

$$P_0(z)\,B(z)\,Q_0(z) = 0_{m_0} \oplus z^{\tau} I_{n-m_0} \tag{14}$$

where $\tau \geqq 0$ is an integer.

The change of the dependent variable

$$y = Q_0(z)\,w \tag{15}$$

introduced into the system (1) results in the system

$$B(z)\,Q_0(z)\frac{dw}{dz} = \left[\, A(z)\,Q_0(z) - B(z)\frac{dQ_0(z)}{dz}\right] w + f(z) \tag{16}$$

Multiplying the system (16) by the holomorphic matrix $P_0(z)$ and using Eq. (14) we obtain

$$(0_{m_0} \oplus z^{\tau} I_{n-m_0})\frac{dw}{dz} = D(z)\,w + g(z) \tag{17}$$

The denfinitions of the $n \times n$ matrix $D(z)$ and the n-dimensional vector $g(z)$ are obvious.

We now partition the $n \times n$ matrix $D(z)$ by setting

$$D(z) = \begin{bmatrix} D_{11}(z) & D_{12}(z) \\ D_{21}(z) & D_{22}(z) \end{bmatrix} \tag{18}$$

where $D_{11}(z)$ is an $m_0 \times m_0$ matrix holomorphic in U.

Again applying Lemma 2, there exist $m_0 \times m_0$ matrices $P_1(z)$ and $Q_1(z)$ holomorphic in (a possibly reduced neighborhood of $z = 0$) U and nonsingular for $z \neq 0$ such that

$$P_1(z) D_{11}(z) Q_1(z) = 0_{m_1} \oplus z^{\sigma} I_{m_0 - m_1} \tag{19}$$

where $m_1 \geqq 0$ and $\sigma \geqq 0$ are integers.

We introduce the change of variable

$$w = [Q_1(z) \oplus I_{n-m_0}]\, x \tag{20}$$

in the system (17) and obtain

$$(0_{m_0} \oplus z^{\tau} I_{n-m_0}) \frac{dx}{dz} = \begin{bmatrix} D_{11}(z) Q_1(z) & D_{12}(z) \\ D_{21}(z) Q_1(z) & D_{22}(z) \end{bmatrix} x + g(z) \tag{21}$$

Let us define

$$h(z) = [P_1(z) \oplus I_{n-m_0}]\, g(z) \tag{22}$$

With

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \{(x_1, m_1), (x_2, m_0 - m_1), (x_3, n - m_0)\}$$

let

$$\begin{bmatrix} h_1(z) \\ h_2(z) \\ h_3(z) \end{bmatrix}$$

be a segmentation of the vector function $h(z)$ of Eq. (22). Each of the vectors h_i has the same length as the corresponding x_i for $i = 1, 2, 3$.

Define

$$M(z) = \begin{bmatrix} P_1(z) D_{11}(z) Q_1(z) & P_1(z) D_{12}(z) \\ D_{21}(z) Q_1(z) & D_{22}(z) \end{bmatrix} \tag{23}$$

Multiplication of the system (21) by the matrix $P_1(z) \oplus I_{n-m_0}$ and using Eq. (22) and (23) result in the system

$$(0_{m_0} \oplus z^{\tau} I_{n-m_0}) \frac{dx}{dz} = M(z)\, x + h(z) \tag{24}$$

From Eq. (19) and (23), the $n \times n$ matrix $M(z)$ may be written as the following partitioned matrix:

$$M(z) = \begin{array}{r} \\ m_1 \{ \\ \\ n - m_0 \{ \end{array} \overbrace{\phantom{0_{m_1} \quad z^{\sigma} I_{m_0 - m_1}}}^{m_0} \left[\begin{array}{c|c|c} 0_{m_1} & 0 & M_{13}(z) \\ \hline 0 & z^{\sigma} I_{m_0 - m_1} & M_{23}(z) \\ \hline M_{31}(z) & M_{32}(z) & M_{33}(z) \end{array}\right] \begin{array}{l} \\ \} \; m_0 - m_1 \\ \\ \end{array} \tag{25}$$

From the form of $M(z)$ displayed in Eq. (25), the system (24) is equivalent to the following equations for the segments x_i, $(i = 1, 2, 3)$, of the vector x:

$$M_{13}(z) x_3 + h_1(z) = 0 \tag{26}$$

$$z^{\sigma} x_2 + M_{23}(z) x_3 + h_2(z) = 0 \tag{27}$$

$$M_{31}(z) x_1 + M_{32}(z) x_2 + M_{33}(z) x_3 + h_3(z) = z^{\tau} \frac{dx_3}{dz} \tag{28}$$

We put $t = \tau + \sigma$ and insert Eq. (27) into (28) to obtain

$$z^{t} \frac{dx_3}{dz} = \{ z^{\sigma} M_{33}(z) - M_{32}(z) M_{23}(z) \} x_3 + z^{\sigma} h_3(z) - M_{32}(z) h_2(z) + z^{\sigma} M_{31}(z) x_1 \tag{29}$$

Let us define

$$R(z) = z^{\sigma} M_{33}(z) - M_{32}(z) M_{23}(z) \tag{30}$$

Note that $R(z)$ is an $(n - m_0) \times (n - m_0)$ matrix holomorphic in U.

Let a fundamental matrix $\Phi(z)$ for the homogeneous equation

$$z^{t} \frac{dx_3}{dz} = R(z)\, x_3 \tag{31}$$

be obtained on a Riemann surface Σ with branch point at zero (Ref. 1, pp. 7–8).

We now make a critical hypothesis.

The equation (26) has at least one solution on Σ. (32)

Let z_0 be a point of Σ where Eq. (26) has a solution $x_3(z_0)$. Then there exists a unique $(n - m_0)$-dimensional vector $\phi_0 = \Phi^{-1}(z_0)\, x_3(z_0)$ such that $\Phi(z_0)\,\phi_0 = x_3(z_0)$ solves Eq. (26) for $z = z_0$.

By the use of the formula for the variation of parameters (Ref. 1, pp. 8–9), together with Eq. (29), Eq. (28) has a solution

$$x_3(z) = \Phi(z)\,\phi_0 + \int_{z_0}^{z} \Phi(z)\,\Phi^{-1}(s)\,[s^{\sigma} h_3(s) - M_{32}(s)\,h_2(s)]\,ds + \int_{z_0}^{z} \Phi(z)\,\Phi^{-1}(s)\,s^{\sigma} M_{31}(s)\,x_1(s)\,ds\,, \qquad (z \in \Sigma) \tag{33}$$

Notice that $x_3(z)$ solves Eq. (26) at $z = z_0$.

To simplify the notation let us define

$$q(z) = M_{13}(z)\left\{\Phi(z)\,\phi_0 + \int_{z_0}^{z} \Phi(z)\,\Phi^{-1}(s)\,[s^{\sigma} h_3(s) - M_{32}(s)\,h_2(s)]\,ds\right\} \tag{34}$$

Then from Eq. (33) and (34), Eq. (26) will have a solution on Σ if and only if

$$M_{13}(z)\,x_3(z) + h_1(z) = M_{13}(z)\,\Phi(z)\int_{z_0}^{z} \Phi^{-1}(s)\,s^{\sigma} M_{31}(s)\,x_1(s)\,ds + q(z) + h_1(z) = 0\,, \qquad (z \in \Sigma) \tag{35}$$

A solution to the integral equation (35) for $x_1(z)$, $(z \in \Sigma)$, together with the hypothesis (32) are necessary and sufficient conditions for the Eq. (26)–(28) to possess solutions on Σ.

We refer now to the matrix in the right member of Eq. (18). From Eq. (19) it is clear that if the analytic nullity of $D_{11}(z)$ is zero $(m_1 = 0)$, Eq. (26) and hence the integral equation (35) are absent. The function f of the system (1) can be any arbitrary vector holomorphic on Σ.

In other words, Eq. (27) and (28) reduce to a trivial linear system of algebraic equations and a linear analytic nonhomogeneous system with a singularity of finite rank at $z = 0$.

IV. On Solutions of the Integral Equation of Compatibility

Suppose that Σ is the Riemann surface of Section II on which a fundamental matrix for Eq. (31) has been obtained. Consider the integral equation

$$M_1(z)\int_{z_0}^{z} N_1(s)\,x_1(s)\,ds = k_1(z)\,, \qquad (z \in \Sigma) \tag{36}$$

Here x_1 and k_1 are m-dimensional, while M_1 and N_1 are $m \times m$ matrices holomorphic and of respective constant ranks $m_1 \geqq 0$ and $n_1 \geqq 0$ on Σ. The integer m_1 is used again here for notational convenience and is, in general, different from the integer m_1 of Section III.

Notice that the integral equation (35) is of the form (36) and satisfies each of the above requirements except that M_1 and N_1, as reconstrued from Eq. (35), may be rectangular matrices. This inconvenience can be avoided by adding blocks of zeros to M_1 and N_1 and (possibly) introducing a dummy segment onto x_1 while adding a segment of zeros onto k_1. The new integral equation so obtained will have the form (36), with M_1 and N_1 both square matrices, and will be equivalent to Eq. (35).

To illustrate this, suppose

$$M_1 = \overbrace{\begin{bmatrix} * \end{bmatrix}}^{m} \Big\} m - p$$

and

$$N_1 = \overbrace{\begin{bmatrix} * \end{bmatrix}}^{m-p} \Big\} m$$

designate respective dimensions of the matrices M_1 and N_1 appearing in Eq. (36), where the integer $p > 0$.

Define matrices

$$\widetilde{M}_1(z) = \begin{bmatrix} 0 \\ \hline M_1(z) \end{bmatrix} \}p$$

$$\widetilde{N}_1(z) = \begin{bmatrix} \overbrace{0}^{p} & \vdots & N_1(z) \end{bmatrix} \tag{37}$$

and vectors

$$\widetilde{x}_1(z) = \begin{bmatrix} \widetilde{x}_1(z) \\ \hline x_1(z) \end{bmatrix} \}p$$

$$\widetilde{k}_1(z) = \begin{bmatrix} 0 \\ \hline k_1(z) \end{bmatrix} \}p \tag{38}$$

Then from Eq. (37) and (38) the equation

$$\widetilde{M}_1(z) \int_{z_0}^{z} \widetilde{N}_1(s)\widetilde{x}_1(s)\,ds = \begin{bmatrix} 0 \\ \hline M_1(z) \int_{z_0}^{z} N_1(s)\,x_1(s)\,ds \end{bmatrix} \}p$$

$$= \widetilde{k}_1(z)$$

is equivalent to Eq. (36), but now the matrices $\widetilde{M}_1$ and $\widetilde{N}_1$ are both of dimension $m \times m$.

A completely analogous reconstruction of Eq. (36) can be made if the dimensions of M_1 and N_1 are

$$M_1 = \Big[\overbrace{\;\ast\;}^{m-p}\Big]\Big\}m$$

and

$$N_1 = \Big[\overbrace{\;\ast\;}^{m}\Big]\Big\}m-p$$

where again the integer $p > 0$.

We now proceed to solve the integral equation (36).

From Lemma 2 there exist four $m \times m$ matrices R_1, S_1, T_1, and V_1, all of which are holomorphic and nonsingular on Σ, such that

$$R_1(z)\,M_1(z)\,S_1(z) = 0_{m-m_1} \oplus I_{m_1} \tag{39}$$

and

$$T_1(z)\,N_1(z)\,V_1(z) = 0_{m-n_1} \oplus I_{n_1}, \qquad (z \in \Sigma) \tag{40}$$

From Eq. (39), the integral equation (36) is equivalent to

$$(0_{m-m_1} \oplus I_{m_1})\,S_1^{-1}(z) \int_{z_0}^{z} N_1(s)\,x_1(s)\,ds = R_1(z)\,k_1(z) \tag{41}$$

Thus a *necessary* condition that Eq. (36) have a solution is that

$$R_1(z)\,k_1(z) = \begin{bmatrix} 0 \\ k_{22}(z) \end{bmatrix}, \qquad \{(0, m - m_1), (k_{22}, m_1)\} \tag{42}$$

where $k_{22}(z)$ is holomorphic on Σ.

Suppose, then, that Eq. (42) is satisfied.

Let

$$k_2(z) = \begin{bmatrix} k_{21}(z) \\ k_{22}(z) \end{bmatrix} \tag{43}$$

where k_{21} is an as yet unspecified $(m - m_1)$-dimensional vector holomorphic on Σ. Then Eq. (41) has a solution, provided

$$\int_{z_0}^{z} N_1(s)\,x_1(s)\,ds = S_1(z)\,k_2(z), \qquad (z \in \Sigma) \tag{44}$$

has a solution.

From Eq. (40), Eq. (44) is equivalent to

$$(0_{m-n_1} \oplus I_{n_1})\,V_1^{-1}(z)\,x_1(z) =$$

$$T_1(z)\frac{d}{dz}\{S_1(z)\,k_2(z)\}, \qquad (z \in \Sigma) \tag{45}$$

So let

$$k_3(z) = T_1(z)\frac{d}{dz}\{S_1(z)\,k_2(z)\} \tag{46}$$

Then a further necessary condition that Eq. (45) have a solution for $V_1^{-1}(z)\,x_1(z)$ is that

$$k_3(z) = \begin{bmatrix} 0 \\ k_{32}(z) \end{bmatrix}, \qquad \{(0, m - n_1), (k_{32}, n_1)\} \tag{47}$$

Suppose that Eq. (47) is satisfied. Then Eq. (45) has a solution

$$x_1(z) = V_1(z)\begin{bmatrix} k_{31}(z) \\ k_{32}(z) \end{bmatrix}, \qquad (z \in \Sigma) \tag{48}$$

where $k_{31}(z)$ is any arbitrary $(m - n_1)$-dimensional vector function holomorphic on Σ.

Recall, however, that if Eq. (42) is satified, the segment k_{21} of k_2 defined in Eq. (43) is still a free holomorphic vector function.

Thus the condition (47), or

$$T_1(z)\frac{d}{dz}\{S_1(z)\,k_2(z)\} = \begin{bmatrix} 0 \\ k_{32}(z) \end{bmatrix} \tag{49}$$

may involve a linear system for k_{21} on Σ possibly of the same type as considered in the system (1), (2). *But this new system will be of an order lower by at least one than that of the system (1), (2).*

It is clear that a decision will ultimately be reached regarding the solvability of the original system (1), (2). If no solution exists, the procedures and methods given in Sections III and IV will so indicate. If a solution does exist, it may be obtained with the classical theory of systems of linear analytic differential equations with singularities of finite rank and the application of a finite number of rational operations to elements of fundamental matrices of these systems and the elements of other given holomorphic matrices.

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