AN INITIAL VALUE PROBLEM FOR OSCILLATIONS OF THE INTERSTELLAR GAS*

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ABSTRACT

For a simple model of the interstellar gas-magnetic field system, in which the disk population of stars provides most of the gravity, it is demonstrated that a dispersion relation approach to the problem of small perturbations of the equilibrium state leads to unstable modes which possess a constant energy density at an infinite distance from the disk. Since one normally treats perturbations in which the energy density vanishes at infinity, rather than merely being bounded, we investigate an initial value problem in which the energy density does vanish at infinity. In one simple case it is shown that an unstable situation can still develop but that the perturbed quantities grow linearly with time, rather than exponentially as a normal mode analysis would indicate. It is further shown that one can still characterize the time scale for instability by the free-fall time so that conclusions drawn elsewhere (Parker 1966, 1967a) remain valid.

I. Introduction

It has recently been demonstrated (Parker, 1966) that the interstellar gas, whose weight holds down the large scale galactic magnetic field threaded through the gas, is unstable to a Rayleigh-Taylor perturbation in such a manner that the gas tends to accumulate in the low regions of magnetic field. However, in order to describe the 'clumping' of an initially uniform atmosphere consideration should be given to finite amplitude disturbances since these may lead to radical departure from the intuitive picutre suggested by a normal mode analysis of the linearized problem. As an example of such a departure the reader may find interest in the problem of finite amplitude oscillations of the Maclaurin spheroids (Rossner, 1967).

Further, consideration of individual normal modes may only represent a rather special situation which is difficult to realize in nature. Thus it is hard for solitary normal modes to exist and satisfy reasonable boundary conditions (but see § II). We are indebted to Professor S. Chandrasekhar for bringing this point, and Rossner's calculations, to our attention.

In view of these facts it seems worthwhile to discuss an initial value problem which contains the essential physics of the interstellar gas-magnetic field system, but which is considerably simpler than the real state. The main reason for considering such a problem is essentially self-educative since the interstellar gas is a highly complex phenomenon. We hope that by considering simple geometries and various facets of several idealized situations, we will obtain some insight into the processes which influence the dynamical behaviour of the interstellar gas.

We acknowledge that there are many complications which we tacitly ignore in this paper. For example we could take into account warm gas, cosmic rays, non-zero rotation, variable Alfven speed, the fact that the system is not really two dimensional, finite streaming velocity of the gas, etc. Despite these complications we believe that the essential physics is captured in our simple model system and that the results will have some bearing on the response of the interstellar gas to small perturbations. Although the quantitative predictions of our model may be incorrect we believe that the qualitative predictions are properties primarily of the laws of physics and not of our model and, as such, they will be mirrored in the dynamical state of the interstellar gas.

Further we concern ourselves only with the linearized equations which describe small departures from equilibrium, the more difficult problem of non-linear oscillations is considered elsewhere (Parker, 1967b).

The plan of this paper is as follows. In § II we discuss a single normal mode and show that, in at least one instance, it can satisfy the appropriate boundary conditions. It is then demonstrated that such a mode has associated with it a constant (in space) energy density. This demonstrates that the point raised by Chandrasekhar is a query of general character rather than just a manifestation of the geometry. In § III a general solution of an initial value problem for the linearized equations of § II is presented by means of a van Kampen 'normal mode' analysis-the perturbation quantities being subject to the boundary conditions of § II. It is shown that, in general, there are two classes of perturbation which influence the interstellar gas in strikingly different ways. Attention is then restricted to that class of perturbation which gives rise to a spatial structure in the perturbed quantities-we feel that this is probably the most commonly occurring case in nature. In § IV a particularly simple illustration is presented of the way in which these 'normal modes' are excited by an initial perturbation. Further the 'long' time behaviour of the perturbations is developed and it is demonstrated that, even though there no longer exists an exponential growth with time, the solution is unstable. We also present a semi-quantitative criterion under which the linear analysis might reasonably be expected to hold. In § V we discuss the general problem in the light of results derived in preceding sections of the paper.

II. A Single Normal Mode and Instability

Consider the equilibrium of a cold gas of density $\rho(z)$ supported in a uniform gravitational field--e.g. by a horizontal magnetic field $\hat{y}\, B(z)$. Assume that the gas is distributed so that the Alfven speed $V_A = B(z)/\sqrt{[4\pi\rho(z)]}$ is independent of height z. It follows from the usual barometric equation that

$$\frac{B^2(z)}{B^2(0)} = \frac{\rho(z)}{\rho(0)} = \exp\left(-\frac{2z}{\Lambda}\right), \tag{1}$$

where Λ is the characteristic length V_A^2/g.

Suppose that this equilibrium is perturbed by the small gas velocity $\underset{\sim}{v} = \hat{y}\, v_y(y, z, t) + \hat{z}\, V_z(y, z, t)$ and the associated vector potential $\hat{x}\, \delta A(y, z, t)$ and density fluctuation $\delta\rho(y, z, t)$. The linearized perturbation equations for conservation of mass and momentum are

$$\frac{1}{\rho}\frac{\partial \delta\rho}{\partial t} = \frac{\alpha v_z}{\Lambda} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z} , \tag{2}$$

$$\frac{\partial v_y}{\partial t} = \frac{-V_A^2}{\Lambda B} b_z , \tag{3}$$

$$\frac{\partial v_z}{\partial t} = \frac{V_A^2}{B}\left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z} + \frac{b_y}{\Lambda}\right) - g\frac{\delta\rho}{\rho} . \tag{4}$$

The hydromagnetic equations yield

$$\frac{\partial \delta A}{\partial t} + v_z B = 0 , \tag{5}$$

neglecting resistivity, ambipolar diffusion, etc.

Differentiating the equation for v_z three times with respect to t and using the other equations to eliminate δA, $\delta\rho$, and finally v_y, one obtains

$$\left\{\left[\frac{\partial^2}{\partial t^2} - V_A^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \alpha\frac{V_A^2}{\Lambda}\frac{\partial}{\partial z}\right]\frac{\partial^2}{\partial t^2} + \frac{V_A^4}{\Lambda^2}\frac{\partial^2}{\partial y^2}\right\} v_y = 0 . \tag{6}$$

The solution of this partial differential equation can be written in the form

$$v_z = \epsilon V_A \exp\left(\frac{t}{\tau}\right) \exp\left(i k_y y + i k_z z + z \frac{n}{\Lambda}\right) , \tag{7}$$

where ϵ and n are constants and n is real. The dispersion relation follows as

$$\frac{1}{\tau^4} + \frac{1}{\tau^2}\left[V_A^2(k_y^2+k_z^2) + 2i\frac{k_z}{\Lambda}V_A^2(1-n) + n(2-n)\frac{V_A^2}{\Lambda^2}\right] - \frac{V_A^4}{\Lambda^2}k_y^2 = 0 . \tag{8}$$

Now we are interested in sinusoidal variation in the y-direction, so k_y is considered real. We make $z = 0$ the base of the atmosphere, requiring that $v_z = 0$ there. Inspection of the dispersion relation makes it clear that τ depends upon both the magnitude and the sign of k_z. Thus it is not possible to satisfy the boundary conditions by the common device of combining the modes $\exp(ik_z z)$ and $\exp(-ik_z z)$ to give $\sin(k_z z)$, because the amplitudes of the two modes grow at different rates, giving $\sin(k_z z)$ at no more than one instant of time. If however we consider only those modes for which $n = 1$ this difficulty can be avoided, for then τ depends only upon k_z^2. The modes may then be paired to give $\sin(k_z z)$, which automatically satisfies the boundary condition that $v_z = 0$ at $z = 0$. Suitable choice of k_z then permits v_z to vanish at any other height $z = h$ too. The set of normal modes so obtained forms a complete set obviously.

If we do not wish to terminate the perturbation at some height h, it is evident that the individual modes (with $n = 1$) each give a kinetic energy density $\frac{1}{2}\rho v^{2}$ which is constant over z and does not vanish at $z = +\infty$. Generally speaking, however, one usually treats perturbations which are more localized, with the energy density going to zero as $z \to +\infty$ instead of merely being bounded. Thus one would prefer to redefine the modes to be $F_m(y, z, t)$ where

$$F_m(y,z,t) = \exp\left(ik_y y + \frac{z}{\Lambda}\right)\int_{-\infty}^{\infty} dk_z\, f_m(k_z) \times$$
$$\exp\left[ik_z z + \frac{t}{\tau(k_z^2)}\right], \tag{9}$$

where the f_m are chosen so that $F_m \exp(-z/\Lambda)$ vanishes at $z = +\infty$. One such function is

$$f_m = \frac{i}{(ik_z + \Lambda^{-1} + i\varphi)} - \frac{i}{(ik_z + \Lambda^{-1} - i\varphi)}, \tag{10}$$

with

$$F_m(y, z, 0) = 4\pi \exp(ik_y y) \sin\varphi z \tag{11}$$

so that the velocity is bounded and the energy density vanishes as $z \to \infty$.

The modes are then

$$F_m(y,z,t) = i \exp\left(ik_y y + \frac{z}{\Lambda}\right)\int_{-\infty}^{\infty} dk_z \, \exp\left[ik_z z + \frac{t}{\tau(k_z^2)}\right] \times$$

$$\left[\frac{1}{(ik_z + \Lambda^{-1} + i\varphi)} - \frac{1}{(ik_z + \Lambda^{-1} - i\varphi)}\right] , \qquad (12)$$

in place of $\exp(ik_y y + ik_z z + z/\Lambda)\exp(t/\tau)$.

In our other papers we have avoided working with the modes F_m because of their mathematical complexity, involving integrals over $\tau(k_z^2)$ under circumstances when the dispersion relation is much more complicated than here. Instead we have explored the properties of the individual modes $\exp(\pm ik_z z)$ with their bounded energy density under a variety of circumstances ($k_x \neq 0$, $u \neq 0$, nonvanishing cosmic ray pressure). In the present paper we inquire into the more proper initial value problem in which the energy density vanishes at $z = +\infty$.

III. An Initial Value Approach

With the perturbation equations (2) through (5) and the equilibrium state of equation (1) we consider the following initial value problem. We suppose that at some initial time, which without loss of generality we can take to be $t = 0$, we specify the perturbed quantities. We can then ask for the subsequent time development of v_y, v_z, δA and $\delta\rho$.

There are two strikingly different methods for solving such an initial value problem, however the solutions obtained are, of course, completely identical.

The first method is to make use of a Laplace transform in time and to carry the initial values throughout the analysis. Such an approach has been used in many problems in widely differing disciplines of physics. In particular in the realm of plasma physics it has been used with considerable success by Landau (1946).

The second method is based on some essentially very simple properties of complex function theory. This method was first applied to a plasma physics problem by van Kampen (1955). In this "van Kampen normal mode" analysis no statement about the initial values of the perturbed quantities need to be made until near the end of the analysis. The price that is paid for this simplification is that one, or more, unknown functions have to be carried throughout the analysis and can be specified only by declaring the initial values. In practice it seems very convenient to use a normal mode analysis. However it appears to be largely a matter of taste as to whether one does this or uses some other standard method like the Laplace transform.

In the initial value problem characterized by equations (1) through (5) we choose to use the van Kampen normal mode analysis since some general properties of the solution are easily exhibited.

Since the equilibrium state of § II is a function only of z we Fourier transform the perturbed quantities as

$$Q(y,z,t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \, Q(k,z,\omega)\, e^{i(ky-\omega t)} , \tag{13}$$

where the contours of integration in equation (13) are taken to be along the real ω and real k axes respectively.

It then follows that equations (2) through (5) may be written

$$\omega\rho(z)v_y = \frac{k}{4\pi}\frac{dB(z)}{dz}\delta A, \tag{14}$$

$$i\omega\rho(z)v_z = g\delta\rho + \frac{1}{4\pi}\frac{dB(z)}{dz}\frac{\partial\delta A}{\partial z} + \frac{B(z)}{4\pi}\left(-k^2\delta A + \frac{\partial^2\delta A}{\partial z^2}\right), \tag{15}$$

$$-i\omega\delta\rho + v_z\frac{d\rho(z)}{dz} + \rho(z)\left(ikv_y + \frac{\partial v_z}{\partial z}\right) = 0, \tag{16}$$

and

$$i\omega\delta A + v_z B(z) = 0 \tag{17}$$

We now set $v_y = U(k,\omega,z)\exp(z/\Lambda)$, $v_z = W(k,\omega,z)\exp(z/\Lambda)$ and $\delta\rho = \sigma(k,\omega,z)\exp(-z/\Lambda)$ and Fourier transform U , W , σ and δA in z -space as

$$(U, W, \sigma, \delta A) = \int_{-\infty}^{\infty} e^{i\mu z}(U, W, \sigma, \delta A)\,d\mu , \tag{18}$$

where the contour of integration in equation (18) is chosen to be the real μ axis. It is a simple matter to show that equations (14) through (17) may then be written

$$\omega \rho(0) U = -\frac{k B(0)}{4\pi \Lambda} \delta A, \tag{19}$$

$$i\omega \rho(0) W = g\sigma - \frac{i\mu B(0)}{4\pi\Lambda} \delta A - \frac{B(0)}{4\pi}(k^2+\mu^2)\delta A, \tag{20}$$

$$-i\omega\sigma - \frac{\rho(0)}{\Lambda} W + i\rho(0)(\mu W + kU) = 0 , \tag{21}$$

$$i\omega \delta A = B(0) W . \tag{22}$$

We see from equation (19) that, in general, δA can be written

$$\delta A = -4\pi \frac{\Lambda \rho(0)}{k\, B(0)} \omega U + \Phi(\omega, \mu)\, \delta(k) , \tag{23}$$

where Φ is a function which has to be determined in the light of the initial conditions. The reader who is interested in the theory of generalized distributions, of which equation (23) is an example, is referred to Muskhelishvili (1953). Use of equation (23) in equation (22) shows that

$$W = -i\,4\pi \frac{\rho(0)}{\beta(0)\Lambda}\,\frac{\omega^2}{k}\,U + \frac{i\omega}{\beta(0)}\,\Phi(\omega,\mu)\,\delta(k)\,. \tag{24}$$

Then we see from equation (20) that

$$\sigma = \rho(0)\,U\left[\frac{k}{\omega} + \frac{\omega}{kg}\left(\frac{1}{\Lambda} - i\mu\right)\right] - \frac{\rho(0)}{\beta(0)}\left(\frac{1}{\Lambda} - i\mu\right)\Phi(\omega,\mu)\,\delta(k) + \Psi(k,\mu)\,\delta(\omega)\,, \tag{25}$$

where Ψ is another unknown function which is also to be specified by the initial conditions. Use of equations (24) and (25) in equation (21) enables us to write

$$\begin{aligned} U\left[\omega^4 - \Lambda g\,\omega^2\left(k^2+\mu^2+1/\Lambda^2\right) - k^2 g^2\right] = {} & \\ \frac{g\,\omega}{\beta(0)}\left[\omega^2 - \Lambda g\left(k^2+\mu^2+\frac{1}{\Lambda^2}\right)\right]&\Phi(\omega,\mu)\,k\,\delta(k) \\ + \frac{k g^2}{\rho(0)}\,\Psi(k,\mu)\,\omega\,\delta(\omega)\,. & \end{aligned} \tag{26}$$

Since Φ is not a function of k and since Ψ is not a function of ω we can write $\omega\,\delta(\omega) = k\,\delta(k) = 0$ in equation (26). Thus

$$U = \Gamma(k,\omega,\mu)\,\delta\left[\omega^4 - \Lambda g\,\omega^2(k^2+\mu^2) - g\,\omega^2/\Lambda - k^2 g^2\right], \tag{27}$$

where $\Gamma(k,\omega,\mu)$ is an unknown function which is to be determined by the initial conditions.

Even without invoking the initial values of the perturbed quantities some restrictions can be placed on the functions Ψ , Φ , and Γ by use of the boundary conditions.

As in § II we make the base of the atmosphere $z = 0$, requiring

$$v_z(y, t, z=0) = 0 \quad , \tag{28}$$

for all y and t .

Further we assume that there is no magnetic flux through the base of the atmosphere so that

$$\left. \frac{\partial \delta A(y,z,t)}{\partial y} \right|_{z=0} = 0 \quad , \tag{29}$$

for all y and t . Finally we insist that v_y , v_z , δA and $\delta \rho$ are all bounded as $z \rightarrow +\infty$ so that the perturbed energy density vanishes as $z \rightarrow +\infty$. Note that the above conditions must be satisfied by all perturbations <u>including the initial values.</u>

Before making use of the boundary conditions it is worthwhile considering the interpretation of the δ -function which occurs in equation (27). Since we wish to make statements concerning the values of the perturbed quantities on $z = 0$ we choose to interpret this δ -function as a function of μ .

Thus

$$\delta\left[\omega^4 - \Lambda g\omega^2(k^2+\mu^2) - g\omega^2/\Lambda - k^2g^2\right] \equiv$$

$$\frac{1}{\Lambda g\omega^2}\,\delta\left[\mu^2 - \frac{1}{\Lambda g\omega^2}\left(\omega^4 - k^2g^2 - \Lambda g\omega^2k^2 - \frac{g\omega^2}{\Lambda}\right)\right] \quad . \tag{30}$$

Suppose that we have an integral

$$I = \int_{-\infty}^{\infty} f(\mu)\,\delta(\mu^2 - B^2)\,d\mu\,, \tag{31}$$

to evaluate along the real μ axis. (Note also that ω and k are both real since the Fourier integrals were chosen along the real ω and real k axes respectively).

Then

$$I = \frac{1}{2B}\left[f(B) + f(-B)\right]\,, \qquad B > 0\,, \tag{32}$$

provided $B^2 > 0$. For $B^2 < 0$ we have $I = 0$.

Thus, for example, we can write

$$v_z(y,z,t) = \frac{-i}{\Lambda g^2}\exp\left(\frac{z}{\Lambda}\right)\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}\frac{dk}{2\mu_1 k}\,e^{i(ky-\omega t)}\left[e^{i\mu_1 z}\,\Gamma(k,\omega,\mu_1) + e^{-i\mu_1 z}\,\Gamma(k,\omega,-\mu_1)\right]$$
$$+\frac{i}{B(0)}\exp\left(\frac{z}{\Lambda}\right)\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\mu\,\omega\,\Phi(\omega,\mu)\,e^{i(\mu z-\omega t)}\,, \tag{33}$$

where

$$\mu_1^2 = \frac{1}{\Lambda g \omega^2}\left(\omega^4 - \frac{g}{\Lambda}\omega^2 - \Lambda g \omega^2 k^2 - k^2 g^2\right) , \tag{34}$$

and, although we have written the range of both ω and k integrations in equation (33) as running between $-\infty$ and $+\infty$, we make the proviso that this is subject to the condition : $\mu_1^2 > 0$. We denote this condition by an asterisk on, say, the k integral.

Since the boundary condition (28) is to be valid for all y and t we see that we must have

$$\Gamma(k, \omega, \mu_1) = -\Gamma(k, \omega, -\mu_1) , \tag{35}$$

and

$$\Phi(\omega, \mu) = -\Phi(\omega, -\mu) . \tag{36}$$

Thus $v_z(y, z, t)$ can be written

$$v_z(y, z, t) = \frac{2}{\Lambda g^2} \exp\left(\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty *} dk \, e^{i(ky - \omega t)} \frac{\sin(\mu_1 z)}{2k\mu_1} \Gamma(k, \omega, \mu_1)$$
$$- \frac{2}{\beta(0)} \exp\left(\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\mu \, \omega \sin(\mu z) e^{-i\omega t} \Phi(\omega, \mu) . \tag{37}$$

Use of equations (35) and (36) enables us to write

$$v_y(y,z,t) = \frac{i}{\Lambda g} \exp\left(\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \overset{*}{\int_{-\infty}^{\infty}} dk \; e^{i(ky-\omega t)} \frac{\sin(\mu_1 z)}{\omega^2 \mu_1} \Gamma(k,\omega,\mu_1), \tag{38}$$

$$\delta A = 2i \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\mu \; \sin(\mu z)\, e^{-i\omega t}\, \Phi(\omega,\mu)$$

$$-\frac{iB(0)}{\Lambda g^2} \int_{-\infty}^{\infty} d\omega \overset{*}{\int_{-\infty}^{\infty}} dk \; e^{i(ky-\omega t)} \frac{\sin(\mu_1 z)}{\omega k \mu_1} \Gamma(k,\omega,\mu_1) \,. \tag{39}$$

Further since equations (2) through (5) hold true for arbitrary y, z, and t we see from equation (4) that

$$\Psi(k,\mu) \equiv 0 \,, \tag{40}$$

so that we have

$$\delta\rho(y,z,t) = -2i \frac{\rho(0)}{B(0)} \exp\left(-\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\mu \left[\frac{\sin(\mu z)}{\Lambda} - \mu\cos(\mu z)\right] \times$$

$$e^{-i\omega t}\Phi(\omega,\mu) \qquad + \frac{i\,\rho(0)}{\Lambda g} \exp\left(-\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \overset{*}{\int_{-\infty}^{\infty}} dk \; e^{i(ky-\omega t)} \Gamma(k,\omega,\mu_1) \times \tag{41}$$

$$\left[\left(\frac{k}{\omega^3} + \frac{1}{\omega k \Lambda g}\right) \frac{\sin(\mu_1 z)}{\mu_1} - \frac{1}{\omega k g} \cos(\mu_1 z) \right] .$$

That $\underline{\Psi}$ is identically zero should not be surprising. Consider what would result if $\underline{\Psi}$ were finite. We would then obtain a contribution to $\delta\rho$ which would be completely time independent. As a consequence it would exist before the system is perturbed at $t = 0$. Thus it would be a part of the equilibrium density and, since we are searching for the deviation from equilibrium as a result of a perturbation being made at some time, it follows that require $\underline{\Psi} \equiv 0$.

Equations (37), (38), (39) and (41) complete the formal solution to the initial value problem and no further information is obtained about $\underline{\Phi}$ and Γ without specifying the initial values. It is a simple matter to verify directly that the perturbed quantities given by equations (37) through (41) do satisfy equations (2) through (5) and also satisfy the boundary conditions imposed.

Before proceeding to illustrate the formal solution by an extremely simple example of an initial perturbation we notice that the response of the system can be divided into two fundamentally different patterns of behaviour.

Class A Perturbations: If we insist that the perturbed quantities be completely independent of y it follows that $\Gamma = 0$ and then

$$v_y = 0, \tag{42}$$

$$\delta A = 2i \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\mu \, \sin(\mu z)\, e^{-i\omega t}\, \underline{\Phi}(\omega,\mu) \,, \tag{43}$$

$$v_z = -\frac{2}{B(0)} \exp\left(\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\mu \; \omega e^{-i\omega t} \sin(\mu z) \, \Phi(\omega, \mu) , \tag{44}$$

and

$$\delta\rho = -2i \frac{\rho(0)}{B(0)} \exp\left(-\frac{z}{\Lambda}\right) \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\mu \, e^{-i\omega t} \left[\frac{\sin(\mu z)}{\Lambda} - \mu \cos(\mu z) \right] \Phi(\omega, \mu) . \tag{45}$$

In this case we see that the interstellar gas undergoes pure compression or expansion in the z-direction. This situation obviously does not give rise to any tendency for the gas to drain into the low regions of field since there are none. Thus the gas is not unstable in the sense described by Parker (1966). However the gas may still be unstable in the sense that $\delta\rho$ increases continually with time i.e. the compression is monotonic. We feel that this type of perturbation is probably difficult to realize in nature, on its own, unless the only perturbation to the gas is the uniform (in y) accumulation of intergalactic gas. It seems unreasonable to believe that this is the only perturbation allowed since a supernova will perturb the gas (Kahn and Woltjer, 1966) and this certainly depends on y.

Class B Perturbations: If we insist that the perturbations depend on both y and z and that they do not have a y-independent part we see that $\Phi = 0$. Then we certainly obtain some structure since y is explicitly included. We feel that this class of perturbation is probably more easily realized in nature, on its own, than the purely compressive class A perturbations.

In principle if we are given any pertrubation at time $t = 0$ we can split it into a y -independent part and a y -dependent part and thus compute both Φ and Γ. In practice this may be a difficult task.

IV. An Illustrative Example

In order to demonstrate that one can obtain an unstable situation without meeting some of the problems mentioned in § II we will concern ourselves with an extremely simple perturbation which is of class B.

In particular we suppose that the y dependence of v_z is $\sin(\alpha y/\Lambda)$ so that

$$\Gamma(k,\omega,\mu_1) = \Gamma(\omega)\left[\delta(k-\alpha\Lambda^{-1}) + \delta(k+\alpha\Lambda^{-1})\right]. \tag{46}$$

Then

$$v_Y = i\sqrt{\left(\frac{\Lambda}{g_3}\right)}\, e^{Z}\cos(\alpha Y)\int_{-\infty}^{*\infty} dx\, e^{-ixT}\frac{\Gamma(x)}{\nu x^2}\sin(\nu Z), \tag{47}$$

$$v_Z = \frac{2i}{\alpha}\sqrt{\left(\frac{\Lambda}{g_3}\right)}\, e^{Z}\sin(\alpha Y)\int_{-\infty}^{*\infty} dx\, e^{-ixT}\frac{\Gamma(x)}{\nu}\sin(\nu Z), \tag{48}$$

$$\delta A = 2\frac{\Lambda B(0)}{\alpha g^2}\sin(\alpha Y)\int_{-\infty}^{*\infty} dx\, e^{-ixT}\frac{\Gamma(x)}{x\nu}\sin(\nu Z), \tag{49}$$

$$\delta p = -2\frac{p(0)}{g^2}e^{-Z}\sin(\alpha Y)\int_{-\infty}^{*\infty}dx\, e^{-ixT}\frac{\Gamma(x)}{x^2}\Big[\Big(\alpha x+\frac{1}{\alpha x}\Big)\frac{\sin(\nu Z)}{\nu} - \frac{x}{\alpha}\cos(\nu Z)\Big], \tag{50}$$

where $Z = z/\Lambda$, $Y = y/\Lambda$, $T = t\sqrt{(g/\Lambda)}$, $x = \omega\sqrt{(\Lambda/g)}$ and $x^2\nu^2 = x^4 - x^2(1+\alpha^2) - \alpha^2$, $\nu > 0$.

The asterisk on the intergral over x denotes the fact that the region of integration along the real x -axis is to be confined to the domain $\nu^2 > 0$.

In order to demonstrate instability with a minimum amount of algebra we shall now assume that the variations in the y direction occur on a scale which is extremely short compared to the gravitational scale height in the z -direction, i.e. $\alpha >> 1$.

In such a case we have

$$\nu^2 \simeq \alpha^2(p^2-1), \tag{51}$$

where $\alpha p = x$.

We further assume that $\Gamma(-x) = \Gamma(x)$, so that we can write equations (47) through (50) in the form

$$v_y = \frac{2i}{\alpha^2}\sqrt{\Big(\frac{\Lambda}{g^3}\Big)}e^{Z}\cos(\alpha Y)\int_1^{\infty}dp\cos(\alpha T p)\sin[\alpha Z\sqrt{(p^2-1)}]\frac{\Gamma(p)}{p^2\sqrt{(p^2-1)}}, \tag{52}$$

$$v_z = \frac{4i}{\alpha}\sqrt{\left(\frac{\Lambda}{g^3}\right)} e^{Z} \sin(\alpha Y) \int_1^{\infty} dp \cos(\alpha T p) \sin\left[\alpha Z \sqrt{(p^2-1)}\right] \frac{\Gamma(p)}{\sqrt{(p^2-1)}} , \tag{53}$$

$$\delta A = -4i \frac{\Lambda B(0)}{g^2 \alpha^2} \sin(\alpha Y) \int_1^{\infty} dp \sin(\alpha T p) \sin\left[\alpha Z \sqrt{(p^2-1)}\right] \frac{\Gamma(p)}{p\sqrt{(p^2-1)}} , \tag{54}$$

and

$$\delta\rho = 4i \frac{\rho(0)}{g^2 \alpha^2} e^{-Z} \sin(\alpha Y) \int_1^{\infty} dp \sin(\alpha T p) \frac{\Gamma(p)}{p^2\sqrt{(p^2-1)}} \left\{ \left(\alpha^2 p + \frac{1}{\alpha^2 p}\right) \sin\left[\alpha Z \sqrt{(p^2-1)}\right] - p\sqrt{(p^2-1)} \cos\left[\alpha Z \sqrt{(p^2-1)}\right] \right\} . \tag{55}$$

As an extremely simple example of the response of the interstellar gas to an initial perturbation we consider the case where, at $t = 0$, we have

$$\delta A = \delta \rho = 0 , \tag{56a}$$

and

$$v_z = V_0 \sin(\alpha Y) Z e^{-\beta Z} , \quad (\beta > 0) . \tag{56b}$$

It then follows that

$$\Gamma(p) = -i\frac{V_0\,(\beta+1)}{\pi\,\alpha}\sqrt{\left(\frac{g^3}{\Lambda}\right)}\frac{p\sqrt{(p^2-1)}}{[p^2-1+(\beta+1)^2\alpha^{-2}]^2} \quad . \tag{57}$$

By direct substitution of equation (57) into equation (52) it can be shown that

$$v_y(Y, Z, T=0) = \frac{\alpha V_0 \cos(\alpha Y)}{[\alpha^2-(\beta+1)^2]}\left\{\frac{1}{2}Z e^{-\beta Z} - \frac{(\beta+1)}{[\alpha^2-(\beta+1)^2]}\times [e^{-\beta Z} - e^{-Z(\alpha-1)}]\right\}. \tag{58}$$

It is clear by inspection of equations (56) and (58) that both $v_y\,(T=0)$ and $v_Z\,(T=0)$ satisfy the boundary conditions at $Z = 0$ and $Z \to +\infty$ provided only that $\beta > 0$ and $\alpha > 1$.

For arbitrary values of T (but $T > 0$) we have been unable to evaluate analytically exactly the integrals occurring in equations (52) through (55) when $\Gamma(p)$ is given by equation (57). However we can obtain approximate expressions for v_y, v_Z, δH and $\delta\rho$ when T is sufficiently large and $\beta << \alpha$. What is meant by T sufficiently large will become clear in the process of calculation.

Consider, for example, v_y which can be written

$$v_y = \frac{(\beta+1)}{\pi\alpha^3} V_0 e^{Z}\cos(\alpha Y)\, \mathrm{Im}\int_{-\infty}^{\infty}\frac{\sinh\theta\, e^{i\alpha Z\sinh\theta}\cos(\alpha T\cosh\theta)\,d\theta}{\cosh\theta\,[\sinh^2\theta + (\beta+1)^2\alpha^{-2}]^2}\,, \tag{59}$$

where $p = \cosh\theta$

For $T \gtrsim O(\alpha^{-1})$ we see that the phase factor $\alpha T \cosh\theta$ is extremely rapidly varying as θ increases. Further consider

$$\frac{\sinh\theta}{\left[\sinh^2\theta + (\beta+1)^2\alpha^{-2}\right]^2} .$$

This function has a peak value of $\sqrt{3}\,\alpha^3/[16(\beta+1)^3]$ when $\sqrt{3}\,\alpha \sinh\theta = (\beta+1)$ and it vanishes when $\theta = 0$, and is small when $\sinh\theta \gtrsim 2$.

For $\sinh\theta = O(\alpha^{-1})$ we see that the phase factor $\alpha Z \sinh\theta = O(Z)$ which is slowly varying compared to $\alpha T \cosh\theta$ provided only that $\alpha T > O(Z)$ which condition is easily realizable.

Thus we can evaluate approximately the integral in equation (59) by the method of stationary phases. Upon so doing we find that

$$v_y \simeq \frac{(\beta+1)^2 e^2}{8\sqrt{\pi}\,\alpha^3} V_0 T\, e^{-\beta Z} \cos(\alpha Y) \cos\left\{\alpha\left[T - \frac{Z}{T(\beta+1)}\right]\right\}, \tag{60}$$

provided

$$T > O\left\{\max\left[\frac{Z}{\sqrt{\alpha}}, \frac{Z}{(\beta+1)}\right]\right\}, \tag{61}$$

and also provided that $Z > O(\alpha^{-1})$.

Under these conditions it can also be shown that

$$v_z \simeq \frac{(\beta+1)^2 e^2}{2\sqrt{\pi}\,\alpha^2} V_0 T e^{-\beta z} \sin(\alpha Y) \cos\left\{\alpha\left[T - \frac{z}{T(\beta+1)}\right]\right\} , \qquad (62)$$

$$\frac{\delta A}{B(0)} \simeq -\frac{(\beta+1)^2 e^2}{2\sqrt{\pi}\,\alpha^3} \sqrt{\left(\frac{\Lambda}{g}\right)} V_0 T e^{-(\beta+1)z} \sin(\alpha Y) \sin\left\{\alpha\left[T - \frac{z}{T(\beta+1)}\right]\right\} , \qquad (63)$$

and

$$\frac{\delta\rho}{\rho(0)} \simeq -\frac{(\beta+1)^2 e^2}{2\sqrt{\pi}\,\alpha} \sqrt{\left(\frac{1}{\Lambda g}\right)} V_0 T e^{-z(\beta+2)} \sin(\alpha Y) \sin\left\{\alpha\left[T - \frac{z}{T(\beta+1)}\right]\right\} . \qquad (64)$$

We see by inspection that the system is unstable in the sense that all the perturbed quantities grow with time. There are two essentially different time scales connected with the perturbed quantities. The first has a characteristic time, τ_1 given by

$$\tau_1 = \frac{1}{\alpha}\sqrt{\left(\frac{\Lambda}{g}\right)} , \qquad (65)$$

and is connected with the fine corrugation in time of the perturbed quantities due to the fine corrugation in y. The second time scale is characterized by say, τ_2, given by

$$\tau_2 \simeq \alpha \tau_1 , \qquad (66)$$

and represents the rate of growth of the root mean square perturbed quantities and is of the order of the free-fall time.

Finally, in this section, we remark that the approximate linear solution given by equations (61) through (64) will fail when non-linear terms need to be taken into account. Thus the solution is only valid for times such that

$$\left| \frac{\partial \underset{\sim}{v}}{\partial t} \right| \gg \left| (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} \right| \quad . \tag{67}$$

Use of equations (61) and (62) enables condition (67) to be written

$$V_0 T \ll \alpha^2 \sqrt{(\Lambda g)} \quad . \tag{68}$$

For example, if we take V_0 to be the typical expansion velocity of a supernova ($\sim 10^3$ km/sec) and take α to be the ratio of interstellar cloud separation to cloud size (~ 10) then we expect the linear solution to break down after a time of the order of 10^7 years (assuming a galactic scale height of the order of 100 parsecs) when the non-linear terms are about 10% the size of the linear terms.

However by this time we see that the condensations in density are typically of the order of 10% of the background density and non-linear calculations indicate that the condensations will still grow for longer times.

V. Discussion

The main point established by the analysis is that it is possible to find a set of "normal modes" of the van Kampen type which describes the system without

running into the problem of finite energy density in the modes as $z \rightarrow +\infty$.
It has also been possible to demonstrate, in one particularly simple case, that the system is unstable after "long" times.

We point out that the particular example chosen for illustrative purposes is rather restrictive. Despite this fact, and despite the extremely simple model chosen to describe the interstellar gas, we believe that this case is sufficient to demonstrate the inherently unstable nature of the interstellar gas-magnetic field system. Further we note that, under typical conditions, the instability progresses in a time of the order of the galactic free-fall time (10^6 - 10^7 yrs.).

Finally we state that no attempt has been made in this paper to consider finite amplitude waves. Thus we offer no guarantee that the linear calculations presented here are a definitive statement of instability. However they do indicate instability for times short compared to, say, 10^7-10^8 years and non-linear calculations (Parker, 1967b) confirm this indication.

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