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TECHNICAL REPORT NO. 94

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DEPARTMENT OF ELECTRICAL ENGINEERING
THE UNIVERSITY OF MICHIGAN, ANN ARBOR

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THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN

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ABSTRACT

A system of ordinary nonlinear differential equations, governing the scalar and vector potentials associated with a one-dimensional electromagnetic wave in a hot plasma, has been derived using a properly constructed stationary solution of the nonlinear Boltzmann-Vlasov equation in a moving reference frame. The propagation of the transverse electromagnetic wave is considered for three cases: no applied static electromagnetic field, a static magnetic field in the direction of wave propagation, and static electric and magnetic fields in the direction of propagation.

In the static field-free case, assuming electrical neutrality and considering an electron temperature anisotropy in the plasma, the derived dispersion relation indicates that the wavelength of the transverse electromagnetic wave is amplitude dependent. In the second case, the transverse electromagnetic wave appears as a circularly polarized sinusoidal wave in a laboratory frame of reference. For an electrically neutral plasma with a small-temperature anisotropy and whose mean velocity in the direction of wave propagation vanishes, the derived dispersion relation reduces to the commonly quoted dispersion relation for Alfvén waves.

The influence of a static electric field along the direction of propagation is studied and it is found that under small-amplitude and weak static electric field conditions, the transverse electromagnetic wave appears as an elliptically polarized plane wave in the laboratory frame. The effect of the static electric field on the wavelength and the Faraday rotation is investigated and discussed.

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I. INTRODUCTION

The study of electromagnetic wave propagation in a plasma has application in many diverse fields of physics such as, for example, the interpretation of microwave diagnostic data obtained from laboratory plasmas[1,2,3], astrophysical problems such as the generation of cosmic r-f radiation[4], and the entire field of radio wave propagation in the ionosphere[5,6]. This wide range of interests in the basic problem has led in recent years to many theoretical studies of plasma oscillations[7-13]. The only dynamical plasma phenomena that have been treated in a satisfying way are those describable in terms of small-amplitude departures from uniform equilibria. Many, if not most, plasmas--both laboratory and astrophysical--do not fit such a description. The number of nonlinear problems which have been solved to date is rather limited. In particular, two kinds of nonlinear plasma configurations have been investigated. In the first, which is known as a "constant profile" description, there exists a "wave" coordinate system in which all quantities appear to be time-independent. A "laboratory" observer, in general, would not view the phenomena from this particular frame, but all macroscopic variables would appear to have the form of $(\vec{r} - \vec{v}_o t)$, where the velocity $\vec{v}_o$ is a constant, and $\vec{r}$ and t are the position and time variables respectively. The more usual approach is to study the problem in a coordinate frame which moves with the velocity $\vec{v}_o$ and refer the result back to laboratory coordinates at the end. The more general nonlinear problems involve situations in which no such preferred frame exists; a simple example would be the steepening of nonlinear sound waves according to the Euler equation[14].

The former approach has been used to study magnetosonic waves in a cold plasma[15] as well as the nonlinear Alfvén waves[16,17,18]. Nekrasov[19] has studied the steady-state nonlinear motion of an electron-ion plasma by a similar approach. In the present paper an attempt is made to study the interaction of plasma with a propagating electromagnetic plane wave in a wave frame, using the one-dimensional Boltzmann-Vlasov equation and Maxwell's equations.

II. BASIC EQUATIONS

Consider a two-component plasma (positive ions and electrons) in which the effects of collisions are assumed to be negligible. The electron distribution function $f(\vec{r},\vec{v},t)$ and the ion distribution function $F(\vec{r},\vec{v},t)$ for this plasma are governed by the Boltzmann-Vlasov equations written as follows:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \frac{e}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0 \tag{1a}$$

and

$$\frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F + \frac{e}{M} (\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0 \quad , \tag{1b}$$

where m and M denote, respectively, the mass of the electron and ion, and e is the electronic charge which is taken as a positive quantity.

The electromagnetic fields in the plasma are governed by the Maxwell equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} , \tag{2a}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} , \tag{2b}$$

$$\nabla \cdot \vec{D} = \rho \tag{2c}$$

and

$$\nabla \cdot \vec{B} = 0 . \tag{2d}$$

The electric displacement vector $\vec{D}$ and the magnetic flux density $\vec{B}$ are, respectively, related to the electric field intensity $\vec{E}$ and the magnetic field intensity $\vec{H}$ in the usual manner:

$$\vec{D} = \epsilon_o \vec{E} \tag{3a}$$

and

$$\vec{B} = \mu_o \vec{H} , \tag{3b}$$

where ϵ_o and μ_o denote the dielectric constant and the permeability of vacuum respectively. The convection current density $\vec{J}$ and the charge density ρ may be given in terms of the distribution functions as

$$\vec{J} = e \int \vec{v}(F - f) d^3v \tag{4a}$$

and

$$\rho = e \int (F - f) d^3v . \tag{4b}$$

It is well known that the analysis of electromagnetic fields is often facilitated by the use of auxiliary potential functions. A general solution of the inhomogeneous system (Eqs. 2) can be given as follows[20]:

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} - \frac{1}{\epsilon_o} \nabla \times \vec{A}_o \tag{5a}$$

and

$$\vec{B} = \nabla \times \vec{A} - \mu_o \frac{\partial \vec{A}_o}{\partial t} - \mu_o \nabla \Phi_o , \tag{5b}$$

where Φ and $\vec{A}$ are the potentials of the source distribution which is internal to the region under consideration, and Φ_o and $\vec{A}_o$ are potentials of the source distribution which is entirely external to the region under consideration. These potentials are subject to the following conditions:

$$\Box \vec{A} = -\mu_o \vec{J} \quad , \tag{6a}$$

$$\Box \Phi = -\frac{1}{\epsilon_o} \rho \quad , \tag{6b}$$

$$\nabla \cdot \vec{A} + \mu_o \epsilon_o \frac{\partial \Phi}{\partial t} = 0 \quad , \tag{6c}$$

$$\Box \vec{A}_o = 0 \quad , \tag{7a}$$

$$\Box \Phi_o = 0 \tag{7b}$$

and

$$\nabla \cdot \vec{A}_o + \mu_o \epsilon_o \frac{\partial \Phi_o}{\partial t} = 0 \quad , \tag{7c}$$

where the symbol $\Box$ denotes the D'Alembertian operator defined by

$$\Box \equiv \nabla^2 - \mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \quad . \tag{7d}$$

Define an equivalent potential function "$\vec{a}$" by the following differential equations:

$$\frac{\partial \vec{a}}{\partial t} = \frac{1}{\epsilon_o} \nabla \times A_o \tag{8a}$$

and

$$\nabla \times \vec{a} = -\mu_o \left(\frac{\partial \vec{A}_o}{\partial t} + \nabla \Phi_o \right) \quad , \tag{8b}$$

so that Eqs. 5 can be written as

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{V}}{\partial t} \tag{9a}$$

and

$$\vec{B} = \nabla \times \vec{V} \quad , \tag{9b}$$

where

$$\vec{V} = \vec{A} + \vec{a} \quad . \tag{9c}$$

It should be noted that the set of Eqs. 8, which is equivalent to the set of Eqs. 7a-c, can also be written as

$$\Box \vec{a} = 0 \quad \text{and} \quad \nabla \cdot \vec{a} = 0 \quad . \tag{10}$$

Postulate the existence of a moving frame of reference in which all quantities of interest appear to be stationary, i.e., a transformation $\xi \equiv (z - v_o t)$ is made to a moving coordinate system where v_o is a constant independent of t and z, and thus ξ is the distance measured in this moving frame of reference. In the present one-dimensional analysis, it is assumed that macroscopic quantities such as the electromagnetic fields and potentials depend only upon ξ, while the density distribution functions f and F are functions of ξ as well as the particle velocities v_x, v_y and v_z. Thus for

$$\xi = (z - v_o t) \quad , \tag{11}$$

Eq. 9a gives

$$E_x = v_o \frac{dV_x}{d\xi} \quad , \quad E_y = v_o \frac{dV_y}{d\xi} \quad \text{and} \quad E_z = - \frac{d\Phi}{d\xi} + v_o \frac{dV_z}{d\xi} \quad , \tag{12a}$$

and Eq. 9b becomes

$$B_x = - \frac{dV_y}{d\xi} \quad , \quad B_y = \frac{dV_x}{d\xi} \quad \text{and} \quad B_z = 0 \quad . \tag{12b}$$

It is to be noted that the time-dependent electromagnetic field components are related in the following manner:

$$E_x B_x + E_y B_y = 0 \quad , \tag{12c}$$

which implies that $\vec{E}$ is perpendicular to $\vec{B}$ spatially.

With the aid of Eqs. 4, Eq. 6a yields

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2A_x}{d\xi^2} = -\mu_o e \iiint_{-\infty}^{\infty} v_x(F - f)dv_x dv_y dv_z \quad , \tag{13a}$$

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2A_y}{d\xi^2} = -\mu_o e \iiint_{-\infty}^{\infty} v_y(F - f)dv_x dv_y dv_z \tag{13b}$$

and

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2A_z}{d\xi^2} = -\mu_o e \iiint_{-\infty}^{\infty} v_z(F - f)dv_x dv_y dv_z \quad , \tag{13c}$$

whereas Eq. 6b becomes

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2\Phi}{d\xi^2} = -\frac{e}{\epsilon_o}\iiint_{-\infty}^{\infty} (F - f)dv_x dv_y dv_z \tag{13d}$$

and Eq. 6c gives

$$\frac{dA_z}{d\xi} - \frac{v_o}{c^2}\frac{d\Phi}{d\xi} = 0 \quad , \tag{13e}$$

where $c = 1/\sqrt{\mu_o \epsilon_o}$ is the speed of light in vacuum.

On the other hand Eq. 1a may be written as follows with the aid of Eqs. 12a and 12b:

$$u_z\left(\frac{\partial f}{\partial \xi} + \frac{e}{m}\frac{dV_x}{d\xi}\frac{\partial f}{\partial v_x} + \frac{e}{m}\frac{dV_y}{d\xi}\frac{\partial f}{\partial v_y}\right)$$

$$+ \frac{e}{m}\frac{d}{d\xi}(\Phi - v_x V_x - v_y V_y - v_o V_z)\frac{\partial f}{\partial u_z} = 0 \tag{14a}$$

and Eq. 1b becomes

$$u_z\left(\frac{\partial F}{\partial \xi} - \frac{e}{M}\frac{dV_x}{d\xi}\frac{\partial F}{\partial v_x} - \frac{e}{M}\frac{dV_y}{d\xi}\frac{\partial F}{\partial v_y}\right)$$

$$- \frac{e}{M}\frac{d}{d\xi}(\Phi - v_xV_x - v_yV_y - v_oV_z)\frac{\partial F}{\partial u_z} = 0 \quad , \qquad (14b)$$

where $u_z = (v_z - u_o)$.

It is not difficult to show that the general solution of Eq. 14a has the following form:

$$f(\xi,\ v_x,\ v_y,\ v_z) = \bar{f}(U_x,\ U_y,\ W) \quad , \qquad (15a)$$

where $U_x = [v_x - (e/m)V_x]$,

$U_y = [v_y - (e/m)V_y]$,

$W = 1/2m[v_x^2 + v_y^2 + u_z^2 - (U_x^2 + U_y^2)] + e(v_oV_z - \Phi)$ and

$\bar{f}$ is an arbitrary differentiable function of its arguments. Similarly, the general solution of Eq. 14b has the form

$$F(\xi,\ v_x,\ v_y,\ v_z) = \bar{F}(U_{xi},\ U_{yi},\ W_i) \quad , \qquad (15b)$$

where $U_{xi} = [v_x + (e/M)V_x]$,

$U_{yi} = [v_y + (e/M)V_y]$ and

$W_i = 1/2M[v_x^2 + v_y^2 + u_z^2 - (U_{xi}^2 + U_{yi}^2) - e(v_oV_z - \Phi)]$.

It is obvious that once the forms of the distribution functions $\bar{F}$ and $\bar{f}$ are known, then the integration in Eqs. 13a-d can be carried out. Thus a set of differential equations governing the potentials $\vec{A}$ and Φ can be derived.

Suppose that the distribution functions are of the form

$$\overline{f}(U_x, U_y, W) = N_e\left[\left(\frac{2\pi}{m}\right)^{3/2}\sqrt{\alpha_e\beta_e\gamma_e}\right]^{-1} \exp\left[-\frac{m}{2}\left(\alpha_e U_x^2 + \beta_e U_y^2\right) - \gamma_e W\right] \tag{16a}$$

and

$$\overline{F}(U_{xi}, U_{yi}, W_i)$$

$$= N_i\left[\left(\frac{2\pi}{M}\right)^{3/2}\sqrt{\alpha_i\beta_i\gamma_i}\right]^{-1} \exp\left[-\frac{M}{2}\left(\alpha_i U_{xi}^2 + \beta_i U_{yi}^2\right) - \gamma_i W_i\right] , \tag{16b}$$

where

$$\alpha_e = \frac{1}{KT_{ex}} , \quad \beta_e = \frac{1}{KT_{ey}} , \quad \gamma_e = \frac{1}{KT_{ez}} ,$$

$$\alpha_i = \frac{1}{KT_{ix}} , \quad \beta_i = \frac{1}{KT_{iy}} , \quad \gamma_i = \frac{1}{KT_{iz}} ,$$

with K denoting the Boltzmann constant, and T_x, T_y, and T_z are the temperatures corresponding to the directions along the three coordinate axes. Then upon evaluating the integrals of Eqs. 13a-d, with the aid of Eq. 10, the following set of differential equations is obtained:

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2V_x}{d\xi^2} = \mu_o e G_x V_x , \tag{17a}$$

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2V_y}{d\xi^2} = \mu_o e G_y V_y , \tag{17b}$$

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2V_z}{d\xi^2} = -\mu_o e v_o\left(N_i e^{-\eta_i} - N_e e^{-\eta_e}\right) \qquad (17c)$$

and

$$\left(1 - \frac{v_o^2}{c^2}\right)\frac{d^2\Phi}{d\xi^2} = -\frac{e}{\epsilon_o}\left(N_i e^{-\eta_i} - N_e e^{-\eta_e}\right) , \qquad (17d)$$

where

$$G_x = \left[\frac{eN_i}{M}\left(1 - \frac{T_{ix}}{T_{iz}}\right) e^{-\eta_i} + \frac{eN_e}{m}\left(1 - \frac{T_{ex}}{T_{ez}}\right) e^{-\eta_e}\right] ,$$

$$G_y = \left[\frac{eN_i}{M}\left(1 - \frac{T_{iy}}{T_{iz}}\right) e^{-\eta_i} + \frac{eN_e}{m}\left(1 - \frac{T_{ey}}{T_{ez}}\right) e^{-\eta_e}\right] ,$$

$$\eta_i = \frac{1}{KT_{iz}}\left\{\frac{M}{2}\left[\left(1 - \frac{T_{ix}}{T_{iz}}\right)\left(\frac{e}{M} V_x\right)^2 + \left(1 - \frac{T_{iy}}{T_{iz}}\right)\left(\frac{e}{M} V_y\right)\right]^2 - e(v_o V_z - \Phi)\right\}$$

and

$$\eta_e = \frac{1}{KT_{ez}}\left\{\frac{m}{2}\left[\left(1 - \frac{T_{ex}}{T_{ez}}\right)\left(\frac{e}{m} V_x\right)^2 + \left(1 - \frac{T_{ey}}{T_{ez}}\right)\left(\frac{e}{m} V_y\right)^2\right] + e(v_o V_z - \Phi)\right\} ,$$

where N_i and N_e are the constant number densities of ions and electrons respectively in the plasma at some reference point $\xi = \xi_o$.

The above set of nonlinear ordinary differential equations can be solved in principle once the values of V_x, V_y, V_z and Φ and their derivatives with respect to ξ are specified at $\xi = \xi_o$. It is of interest to note that the vector potential may be denoted by $\vec{V} = \vec{A} + \vec{a}$ where $\vec{a}$ is that part associated with the incident electromagnetic wave and $\vec{A}$ is that due to the motion of the charged particles in the plasma. Once $\overline{V}$ and Φ are known, then the electromagnetic field in the plasma can be obtained from Eqs. 12a and 12b.

For convenience a quantity $\varphi(\xi)$ is defined by

$$\varphi(\xi) = \Phi(\xi) - v_o V_z(\xi) \quad . \tag{18a}$$

Then

$$E_z(\xi) = -\frac{d\varphi}{d\xi} \tag{18b}$$

and Eqs. 17c and 17d can be combined to give

$$\frac{d^2\varphi}{d\xi^2} = -\frac{e}{\epsilon_o}\left(N_i e^{-\eta_i} - N_e e^{-\eta_e}\right) \quad . \tag{18c}$$

Thus Eqs. 17a, 17b and 18c form a set of nonlinear equations which must be solved for the potential functions. It is of interest to note that when

$$T_{ex} = T_{ey} = T_{e\perp} \quad \text{and} \quad T_{ix} = T_{iy} = T_{i\perp} \quad , \tag{19}$$

G_x is equal to G_y, and Eqs. 17a and 17b become respectively

$$\frac{d^2V_x}{d\xi^2} = R_o V_x \quad \text{and} \quad \frac{d^2V_y}{d\xi^2} = R_o V_y \quad , \tag{20}$$

where

$$R_o = \frac{1}{(c^2 - v_o^2)}\left[\Omega_p^2\left(1 - \frac{T_{i\perp}}{T_{iz}}\right)e^{-\eta_i} + \omega_p^2\left(1 - \frac{T_{e\perp}}{T_{ez}}\right)e^{-\eta_e}\right] \quad ,$$

$$\eta_i = \frac{\epsilon_o}{2N_i KT_{iz}}\left[\Omega_p^2\left(1 - \frac{T_{i\perp}}{T_{iz}}\right)\left(V_x^2 + V_y^2\right)\right] + \frac{e\varphi}{KT_{iz}}$$

and

$$\eta_e = \frac{\epsilon_o}{2N_e KT_{ez}}\left[\omega_p^2\left(1 - \frac{T_{e\perp}}{T_{ez}}\right)\left(V_x^2 + V_y^2\right)\right] - \frac{e\varphi}{KT_{ez}}$$

with $\omega_p^2 \equiv (N_e e^2/m\epsilon_o)$ and $\Omega_p^2 \equiv (N_i e^2/M\epsilon_o)$.

Let $p(\xi)$ be the amplitude and $\Theta(\xi)$ the spatial angle between the x- and y-components of the transverse magnetic field vector in the system, i.e.,

$$p(\xi) = \sqrt{B_x^2 + B_y^2} \quad \text{and} \quad \Theta(\xi) = \tan^{-1}\left(\frac{B_y}{B_x}\right) \; ; \tag{21}$$

then, with the aid of Eqs. 20,

$$\frac{dp}{d\xi} = \frac{R_o(\xi)}{2p(\xi)} \frac{d}{d\xi} (V_x^2 + V_y^2) \tag{22a}$$

and

$$\frac{d\Theta}{d\xi} = \frac{R_o}{p^2}\left(V_y \frac{dV_x}{d\xi} - V_x \frac{dV_y}{d\xi}\right) . \tag{22b}$$

On the other hand, from Eqs. 20,

$$\frac{d}{d\xi}\left(V_y \frac{dV_x}{d\xi} - V_x \frac{dV_y}{d\xi}\right) = 0 \quad , \tag{23}$$

which suggests that

$$\frac{d\Theta}{d\xi} = \frac{R_o}{p^2} K_1 \quad , \tag{24}$$

where K_1 is independent of ξ, equal to $[V_y(dV_x/d\xi)-V_x(dV_y/d\xi)]$, and can be determined from the values of V_x, V_y, $dV_x/d\xi$ and $dV_y/d\xi$ at $\xi = \xi_o$.

Suppose that the condition of electrical neutrality is satisfied, i.e.,

$$N_i e^{-\eta_i} = N_e e^{-\eta_e} \; , \quad N_i = N_e \quad \text{and} \quad \eta_i = \eta_e \quad . \tag{25}$$

Then the right-hand side of Eq. 18c vanishes so that E_z must be independent of ξ. Although E_z may still contain the electrostatic field, it is assumed to be zero in the present discussion. Consequently φ is independent of ξ, i.e., $\varphi = \varphi_o$, a constant, which is taken to be zero for convenience. Thus, under the conditions (Eqs. 25), R_o can be expressed as

$$R_o = -LQ \exp\left[\frac{1}{2} Q(V_x^2 + V_y^2)\right] , \tag{26}$$

where

$$L = \frac{\frac{N_e}{\epsilon_o}(KT_{ez})}{(c^2 - v_o^2)}\left(1 + \frac{T_{iz}}{T_{ez}}\right)$$

and

$$Q = \frac{e^2}{m(KT_{ez})}\left(\frac{T_{e\perp}}{T_{ez}} - 1\right) .$$

It should be noted that a possible solution of Eq. 23 is the periodic function of ξ, given in the form:

$$V_x = V_o \sin k(\xi - \xi_o) \quad \text{and} \quad V_y = V_o \cos k(\xi - \xi_o) , \tag{27}$$

where k and V_o are constant and independent of ξ.

The transverse magnetic field then is obtained from Eq. 12b as

$$B_x = kV_o \sin k(\xi - \xi_o) \quad \text{and} \quad B_y = kV_o \cos k(\xi - \xi_o) . \tag{28}$$

Since $\xi = (z - v_o t)$ it is easily recognized that this form of solution represents a propagating wave with a propagation constant k and angular frequency $\omega = kv_o$. Furthermore $(V_x^2 + V_y^2) = V_o^2$ is a constant so that R_o is independent of ξ. From Eq. 22a $dp/d\xi = 0$, and from Eq. 22b

$d\Theta/d\xi$ is constant, which implies that the electromagnetic wave propagating in the plasma is a circularly polarized wave. The propagation constant k of the wave must be so chosen that Eqs. 20 are satisfied. Consequently k must satisfy the following relationship:

$$k^2 = LQe^{(1/2)QV_o^2} , \tag{29a}$$

which can be written as

$$(c^2k^2 - \omega^2) = \omega_o^2 \exp\left(\frac{1}{2} QV_o^2\right) , \tag{29b}$$

where

$$\omega_o^2 = \omega_p^2\left(1 + \frac{T_{iz}}{T_{ez}}\right)\left(\frac{T_{e\perp}}{T_{ez}} - 1\right) . \tag{29c}$$

It should be observed that Eq. 29b is simply the dispersion equation for the transverse electromagnetic wave propagating in the plasma.

III. PLASMA IN COMBINED ELECTROSTATIC AND MAGNETOSTATIC FIELDS

Suppose that the externally applied electrostatic and magnetostatic fields are directed along the z-direction. For this case Eqs. 14 must be modified as follows:

$$u_z\left(\frac{\partial f}{\partial \xi} + \frac{e}{m}\frac{dV_x}{d\xi}\frac{\partial f}{\partial v_x} + \frac{e}{m}\frac{dV_y}{d\xi}\frac{\partial f}{\partial v_y}\right)$$

$$+ \frac{e}{m}\frac{d}{d\xi}(\Phi - v_xV_x - v_yV_y - v_oV_z)\frac{\partial f}{\partial u_z} = \frac{e}{m}B_o\left(v_x\frac{\partial f}{\partial v_y} - v_y\frac{\partial f}{\partial v_x}\right) \tag{30a}$$

and

$$u_z\left(\frac{\partial F}{\partial \xi} - \frac{e}{M}\frac{dV_x}{d\xi}\frac{\partial F}{\partial v_x} - \frac{e}{M}\frac{dV_y}{d\xi}\frac{\partial F}{\partial v_y}\right) - \frac{e}{M}\frac{d}{d\xi}(\Phi - v_xV_x - v_yV_y - v_oV_z)\frac{\partial F}{\partial u_z}$$

$$= -\frac{e}{M}B_o\left(v_x\frac{\partial F}{\partial v_y} - v_y\frac{\partial F}{\partial v_x}\right), \qquad (30b)$$

where B_o denotes a constant applied static magnetic field.

Suppose that a solution of Eq. 30a is looked for in the form:

$$f(v_x,v_y,v_z,\xi) = g(u_z)h(v_x,v_y,\xi), \qquad (31)$$

where

$$g(u_z) = g_o \exp\left(-\frac{m}{2KT_{ez}}(u_z - u_o)^2\right)$$

in which g_o is an arbitrary constant determined by the normalization of the distribution function. T_{ez} and u_o are constants which correspond respectively to the temperature and directed velocity (or drift velocity) along the z-axis. Upon substitution of Eq. 31 into Eq. 30a the following set of equations is obtained:

$$\frac{e}{m}B_o\left(v_x\frac{\partial h}{\partial v_y} - v_y\frac{\partial h}{\partial v_x}\right) = \frac{eu_o}{KT_{ez}}h\frac{d}{d\xi}(\Phi - v_oV_z - v_xV_x - v_yV_y) \qquad (32a)$$

and

$$\frac{\partial h}{\partial \xi} + \frac{e}{m}\frac{dV_x}{d\xi}\frac{\partial h}{\partial v_x} + \frac{e}{m}\frac{dV_y}{d\xi}\frac{\partial h}{\partial v_y} = \frac{eh}{(KT_{ez})}\frac{d}{d\xi}(\Phi - v_oV_z - v_xV_x - v_yV_y), \qquad (32b)$$

which can also be written in terms of the electromagnetic fields as follows (using Eqs. 12a and 12b):

$$B_o\left(v_y\frac{\partial h}{\partial v_x} - v_x\frac{\partial h}{\partial v_y}\right) = \frac{-mu_o}{KT_{ez}}h(E_z + v_xB_y - v_yB_x) \qquad (33a)$$

and

$$\frac{\partial h}{\partial \xi} + \frac{e}{m} B_y \frac{\partial h}{\partial v_x} - \frac{e}{m} B_x \frac{\partial h}{\partial v_y} = \frac{-eh}{(KT_{ez})}\left(E_z + v_x B_y - v_y B_x\right) \quad . \tag{33b}$$

A possible general solution of Eq. 33a can be written as follows:

$$h(\xi, v_x, v_y) = h_o(\xi, v_x^2 + v_y^2)\Psi(v_x, v_y, \xi) \quad , \tag{34}$$

where

$$\Psi(v_x, v_y, \xi) = \exp\left[\frac{mu_o}{(KT_{ez})B_o}\left(E_z\theta + v_x B_x + v_y B_y\right)\right]$$

and

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \quad ,$$

in which h_o is to be determined by substituting Eq. 34 into Eq. 33b. For convenience of calculation Eqs. 33 and 34 are converted into cylindrical coordinates in velocity space by letting

$$v_x = v_r \cos\theta \quad \text{and} \quad v_y = v_r \sin\theta \quad . \tag{35}$$

In order that Eq. 33b be satisfied identically with respect to θ for h given by Eq. 34, the following conditions must be satisfied:

$$\frac{dE_z}{d\xi} = 0 \quad , \tag{36a}$$

$$\frac{\partial h_o}{\partial \xi} + \frac{eE_z}{(KT_{ez})} h_o = 0 \quad , \tag{36b}$$

$$\omega_c\left(\frac{B_y}{B_o}\right)\frac{\partial h_o}{\partial v_r} - \frac{e}{KT_{ez}}\left[\left(\frac{B_x}{B_o}\right)\left(\frac{u_o}{v_r}\right)E_z - \frac{u_o v_r}{\omega_c}\frac{dB_x}{d\xi} - v_r B_y\right] h_o = 0 \tag{36c}$$

and

$$-\omega_c\left(\frac{B_x}{B_o}\right)\frac{\partial h_o}{\partial v_r} - \frac{e}{KT_{ez}}\left[\left(\frac{B_y}{B_o}\right)\left(\frac{u_o}{v_r}\right)E_z - \frac{u_o v_r}{\omega_c}\ \frac{dB_y}{d\xi} + v_r B_x\right] h_o = 0 \ , \tag{36d}$$

where $\omega_c \stackrel{\Delta}{=} eB_o/m$ is the electron cyclotron frequency.

If h_o is chosen as

$$h_o(\xi,\ v_x^2 + v_y^2) = \exp\left(\frac{e\varphi(\xi)}{KT_{ez}} - \frac{mv_r^2}{2KT_{e\perp}}\right) \ , \tag{37}$$

where $\varphi(\xi)$ is related to E_z by Eq. 18b, then the function h given by Eq. 34 satisfies Eqs. 33. Thus the distribution function for electrons can be written as follows:

$$f(v_x,v_y,v_z,\xi) = n_e \exp\left[\frac{-m}{2KT_{ez}}(u_z - u_{oe})^2 - \frac{m}{2KT_{e\perp}}(v_x^2 + v_y^2) + \frac{e\varphi}{(KT_{ez})} + \frac{e}{KT_{ex}}\left(\frac{u_{oe}}{\omega_c}\right)(E_z\theta + v_x B_x + v_y B_y)\right] \ , \tag{38}$$

where $u_z = (v_z - v_o)$ and n_e is an undetermined constant of normalization.

The distribution function for ions can be obtained by replacing e, m, ω_c, T_e and n_e by -e, M, $-\Omega_c$, T_i and n_i respectively in Eq. 38:

$$F(v_x,v_y,v_z,\xi) = n_i \exp\left[\frac{-M}{2KT_{iz}}(u_z - u_{oi})^2 - \frac{M}{2KT_{i\perp}}(v_x^2 + v_y^2) - \frac{e\varphi}{KT_{iz}} + \frac{e}{KT_{iz}}\left(\frac{u_{oi}}{\Omega_c}\right)(E_z\theta + v_x B_x + v_y B_y)\right] . \tag{39}$$

Since the form of the distribution functions has been determined, the integrals of Eqs. 13a-d can, in principle, be evaluated. The calculation of these integrals involves error functions which can be treated approximately under the conditions illustrated below. For the appropriate approximation these integrals can be evaluated analytically.

For convenience, suppose that a factor δ is defined as

$$\delta \equiv \sqrt{\frac{mu_o^2}{2KT_z}} \sqrt{\frac{T_\perp}{T_z}} \left(\frac{B_\perp}{B_o}\right) , \tag{40}$$

where $B_\perp$ denotes the magnitude of the transverse magnetic field and B_o denotes the longitudinal static magnetic field. The first factor represents the ratio of the directed velocity (or drift velocity) in the z-direction to the thermal velocity in the same direction, and the second factor is the ratio of the thermal velocity in the transverse direction to that in the longitudinal direction. Then for

$$\delta^3 \ll 1 \tag{41}$$

the components of the electronic current density are given as follows (see Appendix A for details):

$$j_x = K_\perp (p_o + p_1 b_x + p_2 b_y + p_3 b_x^2 + p_4 b_x b_y + p_5 b_y^2) ,$$

$$j_y = K_\perp (q_o + q_1 b_x + q_2 b_y + q_3 b_x^2 + q_4 b_x b_y + q_5 b_y^2) ,$$

$$j_z = K_{\parallel} (\ell_o + \ell_1 b_x + \ell_2 b_y + \ell_3 b_x^2 + \ell_4 b_x b_y + \ell_5 b_y^2) , \tag{42}$$

where

$$b_x \equiv \frac{B_x}{B_o} \; , \quad b_y \equiv \frac{B_y}{B_o} \; ,$$

$$K_\perp = \frac{eN_e\sqrt{\pi}}{2a}\left(\frac{1 - e^{2\pi s}}{2\pi}\right) \exp\left(\frac{e\varphi(\xi)}{KT_{ez}}\right) \; ,$$

$$K_{||} = eN_e\bar{v}\left(\frac{1 - e^{2\pi s}}{2\pi}\right) \exp\left(\frac{e\varphi(\xi)}{KT_{ez}}\right) \; ,$$

$$p_o = \frac{s}{(s^2+1)} \; , \quad p_1 = \frac{4}{\sqrt{\pi}} \frac{(s^2+2)\sigma}{s(s^2+4)} \; , \quad p_2 = -\frac{4}{\sqrt{\pi}} \frac{\sigma}{(s^2+4)} \; ,$$

$$p_3 = \frac{3(s^2+7)s\sigma^2}{(s^2+1)(s^2+9)} \; , \quad p_4 = \frac{-6(s^2+3)\sigma^2}{(s^2+1)(s^2+9)} \; , \quad p_5 = \frac{6s\sigma^2}{(s^2+1)(s^2+9)} \; ,$$

$$q_o = \frac{-1}{(s^2+1)} \; , \quad q_1 = -\frac{4}{\sqrt{\pi}} \frac{\sigma}{(s^2+4)} \; , \quad q_2 = \frac{8}{\sqrt{\pi}} \frac{\sigma}{s(s^2+4)} \; ,$$

$$q_3 = \frac{-3(s^2+3)\sigma^2}{(s^2+1)(s^2+9)} \; , \quad q_4 = \frac{12s\sigma^2}{(s^2+1)(s^2+9)} \; , \quad q_5 = \frac{-18\sigma^2}{(s^2+1)(s^2+9)} \; ,$$

$$\ell_o = \frac{1}{s} \; , \quad \ell_1 = \frac{\sqrt{\pi}s\sigma}{(s^2+1)} \; , \quad \ell_2 = -\frac{\sqrt{\pi}\sigma}{(s^2+1)} \; ,$$

$$\ell_3 = \frac{-2(s^2+2)\sigma^2}{s(s^2+4)} \; , \quad \ell_4 = \frac{4\sigma^2}{(s^2+4)} \; , \quad \ell_5 = \frac{-4\sigma^2}{s(s^2+4)} \; ,$$

$$a^2 \equiv \frac{m}{2KT_{e\perp}} \; , \quad s \equiv \frac{m}{KT_{ez}}\left(\frac{u_{oe}}{B_o}\right) E_z \; ,$$

$$\sigma \equiv \sqrt{\frac{mu_{oe}^2}{2KT_{ez}}} \sqrt{\frac{T_{e\perp}}{T_{ez}}} \; , \quad \bar{v} = (v_o + u_{oe}) \tag{43}$$

in which E_z is the z-directed electric field, and N_e is the electron concentration at the reference point $\xi = \xi_o$ with $E_z = 0$.

On the other hand, the components of the ion current density have the same form as the electron current density, namely Eqs. 42, and the coefficients now take the following form (the subscript i is introduced to denote the fact that the quantity is associated with ions):

$$K_{\perp i} = -\frac{eN_i\sqrt{\pi}}{2a_i}\left(\frac{1-e^{2\pi s_i}}{2\pi}\right)\exp\left(-\frac{e\varphi(\xi)}{KT_{iz}}\right) ,$$

$$K_{\parallel i} = -eN_i\bar{v}_i\left(\frac{1-e^{2\pi s_i}}{2\pi}\right)\exp\left(-\frac{e\varphi(\xi)}{KT_{iz}}\right) ,$$

$$a_i^2 \equiv \frac{M}{2KT_{i\perp}} , \quad s_i \equiv \frac{M}{KT_{iz}}\left(\frac{u_{oi}}{B_o}\right)E_z ,$$

$$\sigma_i \equiv \sqrt{\frac{Mu_{oi}^2}{2KT_{iz}}}\sqrt{\frac{T_{i\perp}}{T_{iz}}} , \quad \bar{v}_i = (v_o + u_{oi}) . \tag{44}$$

Since the current densities of electrons and ions have been determined, with the aid of Eqs. 12a and 12b, Eqs. 13a-d can be written as

$$\left(1-\frac{v_o^2}{c^2}\right)\frac{dB_y}{d\xi} = -\mu_o(j_{xi}+j_{xe}) , \tag{45a}$$

$$\left(1-\frac{v_o^2}{c^2}\right)\frac{dB_x}{d\xi} = \mu_o(j_{yi}+j_{ye}) \tag{45b}$$

and

$$\left(1-\frac{v_o^2}{c^2}\right)\frac{dE_z}{d\xi} = \frac{j_{zi}}{\epsilon_o\bar{v}_i}\left(1-\frac{v_o\bar{v}_i}{c^2}\right)+\frac{j_{ze}}{\epsilon_o\bar{v}_e}\left(1-\frac{v_o\bar{v}_e}{c^2}\right) . \tag{45c}$$

In view of the fact that $dE_z/d\xi$ must be zero as suggested by Eq. 36a, the right-hand side of Eq. 45c must vanish. In other words, the following condition must be imposed on the parameters of both ions and electrons:

$$\frac{j_{zi}}{\bar{\bar{v}}_i}\left(1 - \frac{v_o\bar{v}_i}{c^2}\right) + \frac{j_{ze}}{\bar{\bar{v}}_e}\left(1 - \frac{v_o\bar{v}_e}{c^2}\right) = 0 \quad . \tag{46}$$

Furthermore, since E_z is independent of ξ, the presence of a uniform static electric field in the z-direction is permitted in the present analysis.

The electromagnetic fields in the plasma as a function of ξ can, in principle, be obtained by solving Eqs. 45 with the aid of Eqs. 43 and 44 for properly specified boundary conditions. However, Eqs. 45a and 45b can also be conveniently used to study the effect of the longitudinal static electromagnetic fields on the transverse dynamic magnetic field as illustrated in the following section.

IV. BEHAVIOR OF TRANSVERSE ELECTROMAGNETIC FIELDS

Equations 45a and 45b can be written as follows with the aid of Eqs. 43 and 44:

$$\frac{dY}{d\xi} = P_o + P_1X + P_2Y + P_3X^2 + P_4XY + P_5Y^2 \tag{47}$$

and

$$-\frac{dX}{d\xi} = Q_o + Q_1X + Q_2Y + Q_3X^2 + Q_4XY + Q_5Y^2 \quad , \tag{48}$$

where

$$P_n \equiv (C_e p_{n,e} - C_i p_{n,i}) \quad , \tag{49a}$$

$$Q_n \equiv (C_e q_{n,e} - C_i q_{n,i}) \quad ; \quad n = 0,\ 1,\ 2,\ 3,\ 4 \text{ and } 5, \tag{49b}$$

$$C_i \equiv \frac{-1}{(c^2 - v_o^2)} \frac{\Omega_p^2}{\Omega_c} \frac{\sqrt{\pi}}{2a_i}\left(\frac{1 - e^{2\pi s_i}}{2\pi}\right) \exp\left[\left(\frac{eE_o}{KT_{iz}}\right)\xi\right] \quad , \tag{49c}$$

$$C_e \equiv \frac{-1}{(c^2 - v_o^2)} \frac{\omega_p^2}{\omega_c} \frac{\sqrt{\pi}}{2a_e} \left(\frac{1 - e^{2\pi s_e}}{2\pi} \right) \exp \left[- \left(\frac{eE_o}{KT_{ez}} \right) \xi \right] , \tag{49d}$$

$$\omega_p^2 \equiv \frac{e^2 N_e}{m\epsilon_o} , \quad \omega_c \equiv \frac{eB_o}{m} , \quad \Omega_p^2 \equiv \frac{e^2 N_i}{M\epsilon_o} , \quad \Omega_c \equiv \frac{eB_o}{M} , \tag{49e}$$

in which E_o is a constant longitudinal static field present in the system.

<u>Case I. Static Electric Field-Free Case ($E_o = 0$).</u>

In this case the coefficients P and Q in Eqs. 47 and 48 all vanish except for P_1 and Q_2 which become equal to one another, so that

$$\frac{dY}{d\xi} = P_{1,o} X \quad \text{and} \quad \frac{dX}{d\xi} = -P_{1,o} Y , \tag{50a}$$

where

$$P_{1,o} = \frac{1}{(c^2 - v_o^2)} \left(\frac{\Omega_p^2}{\Omega_c} \frac{\sigma_i}{a_i} - \frac{\omega_p^2}{\omega_c} \frac{\sigma_e}{a_e} \right) . \tag{50b}$$

Since the coefficients $P_{1,o}$ are independent of ξ, the solution of Eq. 50a obviously is a periodic function of ξ and can be written as

$$X = M_o \cos k_o(\xi - \xi_o) \quad \text{and} \quad Y = M_o \sin k_o(\xi - \xi_o) , \tag{51a}$$

where M_o and ξ_o are arbitrary constants and the constant k_o, which determines the spatial period in the wave frame, yet to be determined, is given by

$$k_o = \pm P_{1,o} . \tag{51b}$$

On the other hand, from Eqs. 36c and 36d (using Eq. 37) one has

$$\frac{dB_y}{d\xi} + \frac{\omega_c}{u_{oe}} \left(1 - \frac{T_{ez}}{T_{el}} \right) B_x = 0 ,$$

$$\frac{dB_x}{d\xi} - \frac{\omega_c}{u_{oe}} \left(1 - \frac{T_{ez}}{T_{el}} \right) B_y = 0 , \tag{52a}$$

which relate the electron parameters to the transverse magnetic field. The corresponding set relating the ion parameters to the transverse magnetic field can be obtained by replacing ω_c, T_{ez}, and $T_{e\perp}$ with $-\Omega_c$, T_{iz} and $T_{i\perp}$ in Eq. 52a. Then in order that the fields obtained from these two sets of equations agree, it is required that

$$\frac{1}{mu_{oe}}\left(1 - \frac{T_{ez}}{T_{e\perp}}\right) = \frac{1}{Mu_{oi}}\left(\frac{T_{iz}}{T_{i\perp}} - 1\right) . \tag{52b}$$

Furthermore, since the transverse magnetic field components B_x and B_y must satisfy both set (50a) and set (52a), the following relationship is established:

$$P_{1,o} = \frac{eB_o}{mu_{oe}}\left(\frac{T_{ez}}{T_{e\perp}} - 1\right) . \tag{52c}$$

Suppose that

$$N_i = N_e \text{ and } u_{oi} = u_{oe} ; \tag{52d}$$

then by using Eqs. 50b, 51b and 52b, Eq. 52c can be written as

$$(c^2 - v_o^2)\, B_o^2 \epsilon_o = u_{oe}^2\left(\frac{T_{i\perp}}{T_{iz}}(MN_i) + \frac{T_{e\perp}}{T_{ez}}(mN_e)\right) , \tag{52e}$$

which can also be obtained by equating $P_{1,o}$ to $[eB_o/Mu_{oi}(1 - T_{iz}/T_{i\perp})]$.

It is of interest to observe that since $\xi = (z - v_o t)$, the form of solution given by Eq. 51a appears to an observer in a laboratory frame of reference as a circularly polarized plane wave with a propagation constant k_o and angular frequency $\omega = k_o v_o$. The dispersion equation for this mode of propagation is given by Eq. 52e, which can also be written as

$$(c^2k_o^2 - \omega^2) = c^2k_o^2u_{oe}^2\left(\frac{\mu_o\rho_o}{B_o^2}\right) , \tag{52f}$$

where

$$\rho_o \equiv \left[MN_i\left(1 + \frac{\Delta T_i}{T_{iz}}\right) + mN_e\left(1 + \frac{\Delta T_e}{T_{ez}}\right)\right] , \tag{52g}$$

in which

$$\Delta T_i \equiv (T_{i\perp} - T_{iz}) \text{ and } \Delta T_e \equiv (T_{e\perp} - T_{ez}) .$$

On the other hand, Eqs. 51 being a parametric representation of a circle in the X-Y plane suggests that the tip of the transverse magnetic field vector denoted by the point W(X,Y) describes a circle as the parameter ξ increases. This picture represents the rotation of the transverse field vector about the z-axis as it propagates along the z-axis. The rate at which the transverse magnetic field vector (or the transverse electric field) rotates per unit distance in ξ (in the wave frame) is given by k_o. For example, given ω, k_o can be determined from Eq. 52f in terms of the system parameters.

The algebraic signs associated with Eq. 51b denote the fact that the plasma is capable of supporting both right-hand and left-hand circularly polarized plane waves. Consequently the electromagnetic wave propagating in a magnetoactive plasma suffers a Faraday rotation, which is to be expected. The angle of rotation φ_o can be determined by[2,3]

$$\varphi_o = \frac{1}{2}(k_{o\ell} - k_{or})d , \tag{53}$$

where $k_{o\ell}$ and k_{or} denote respectively the wave number of the left-hand and right-hand circularly polarized waves, and the distance d is measured in a laboratory frame.

Case II. Longitudinal Uniform Static Field ($E_o \neq 0$).

Although the set of nonlinear ordinary differential equations (47) and (48) can be solved numerically for $X(\xi)$ and $Y(\xi)$ by a standard technique such as the Runge-Kutta method, once the values of X and Y are specified at some reference point $\xi = \xi_o$, it is of interest to investigate the following differential equation:

$$-\frac{dY}{dX} = \frac{P_o + P_1 X + P_2 Y + P_3 X^2 + P_4 XY + P_5 Y^2}{Q_o + Q_1 X + Q_2 Y + Q_3 X^2 + Q_4 XY + Q_5 Y^2} , \tag{54}$$

which is obtained by combining Eqs. 47 and 48.

Suppose that Eq. 46 is satisfied (i.e., $dE_z/d\xi = 0$), which is the case if

$$s_i = s_e = s , \quad \sigma_i = \sigma_e = \sigma \tag{55a}$$

and

$$N_i\left(1 - \frac{v_o \bar{v}_i}{c^2}\right) \exp\left[\left(\frac{eE_o}{KT_{iz}}\right)\xi\right] = N_e\left(1 - \frac{v_o \bar{v}_e}{c^2}\right) \exp\left[-\left(\frac{eE_o}{KT_{ez}}\right)\xi\right] . \tag{55b}$$

Then from Eq. 47

$$P_n = (C_e - C_i)\, p_{n,e} \quad \text{and} \quad Q_n = (C_e - C_i)\, q_{n,e} \tag{55c}$$

so that Eq. 54 becomes

$$-\frac{dY}{dX} = \frac{p(X,Y)}{q(X,Y)} , \tag{56a}$$

where

$$p(X,Y) \equiv p_o + p_1 X + p_2 X + p_3 X^2 + p_4 XY + p_5 Y^2$$

$$q(X,Y) \equiv q_o + q_1 X + q_2 Y + q_3 X^2 + q_4 XY + q_5 Y^2 , \tag{56b}$$

where $p_n \equiv p_{n,e}$ and $q_n \equiv q_{n,e}$ are given in Eqs. 43 and are independent of ξ. The plot of the solution of differential equation (56a) in the X-Y plane gives the desired information with regard to the variation of magnitude and polarization of the transverse magnetic field with the variation of static electric and magnetic fields.

It should be noted that from Eqs. 43

$$\frac{\partial p(X,Y)}{\partial Y} = \frac{\partial q(X,Y)}{\partial X} , \tag{56c}$$

which is the necessary and sufficient condition for Eq. 56a to be an exact differential equation. Therefore the solution of Eq. 56a can be given as

$$p_o X \left(1 + \frac{p_3}{3p_o} X^2 + \frac{p_5}{p_o} Y^2\right) + q_o Y \left(1 + \frac{q_3}{q_o} X^2 + \frac{q_5}{3q_o} Y^2\right) + \frac{p_1}{2} X^2 + p_2 XY + \frac{q_2}{2} Y^2 = C , \tag{57}$$

where C is a constant of integration which is to be determined by the value of X and Y at some reference point $\xi = \xi_o$. For example, if $X = 0$ and $Y = Y_o$, then C can be given by

$$C = q_o Y_o + \frac{q_2}{2} Y_o^2 + \frac{q_5}{3} Y_o^3 . \tag{58}$$

As an illustration Eq. 57 is plotted for a few selected sets of parameters s and σ and shown in Figs. 1 through 6.

In view of the fact that $(p_3/3p_o)$, (p_5/p_o), (q_3/q_o) and $(q_5/3q_o)$ are all less than σ^2 for an arbitrary value of s, if the condition

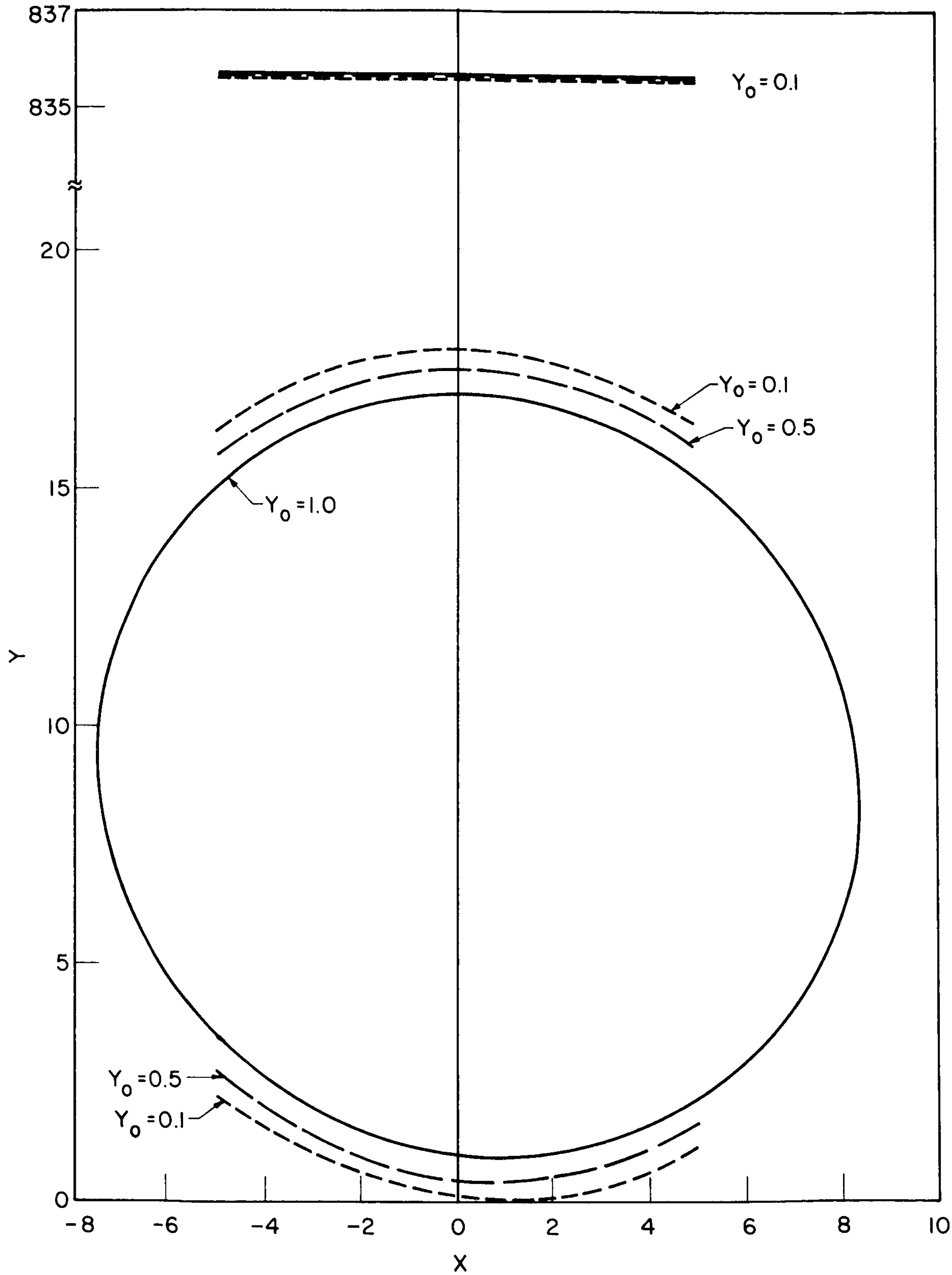

FIG. 1 PLOT OF Y VS. X FOR σ = 0.01, s = 0.1, AND Y_o = 0.1, 0.5, 1.0.

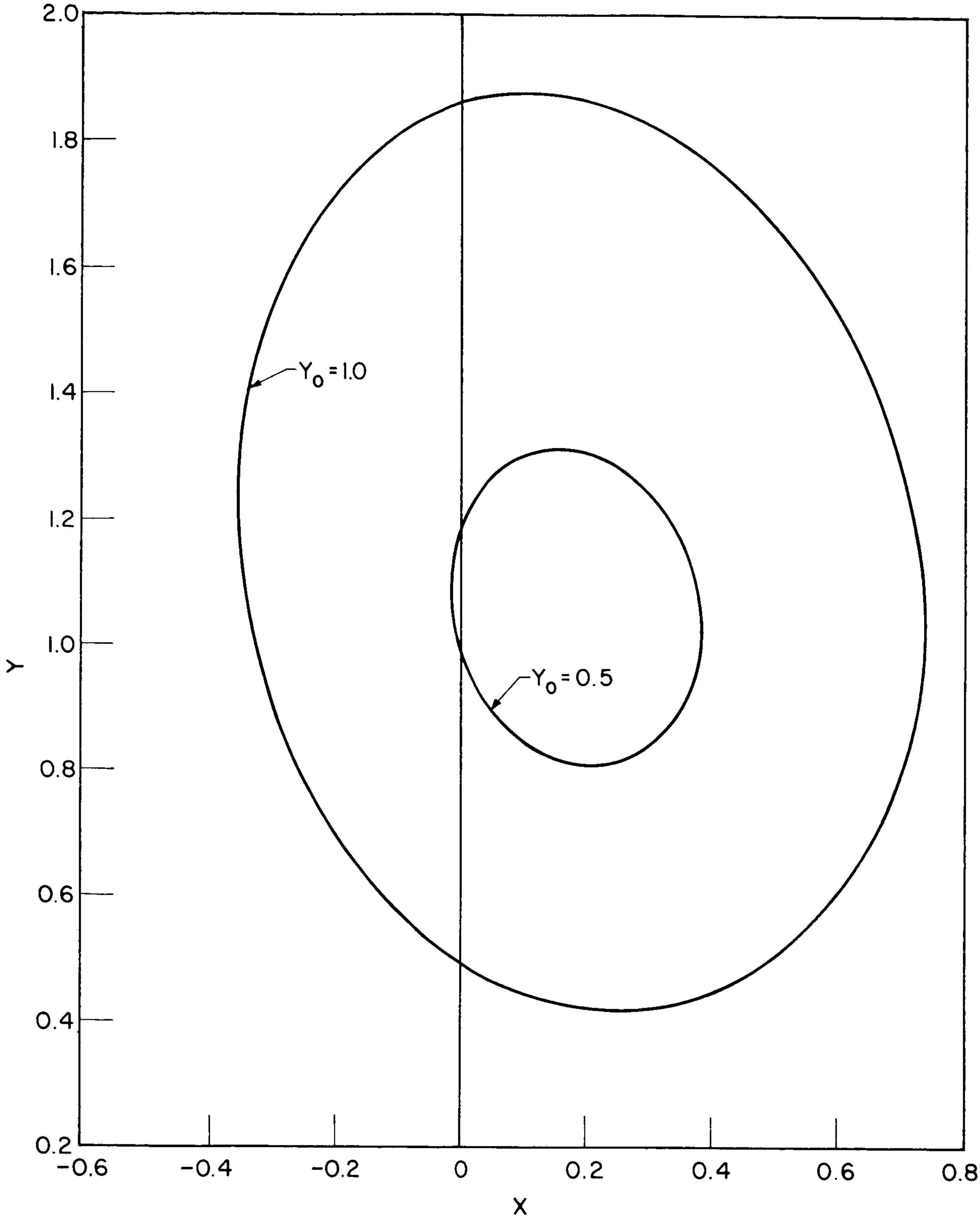

FIG. 2 PLOT OF Y VS. X FOR $\sigma = 0.4$, $s = 0.4$, AND $Y_o = 0.5, 1.0$.

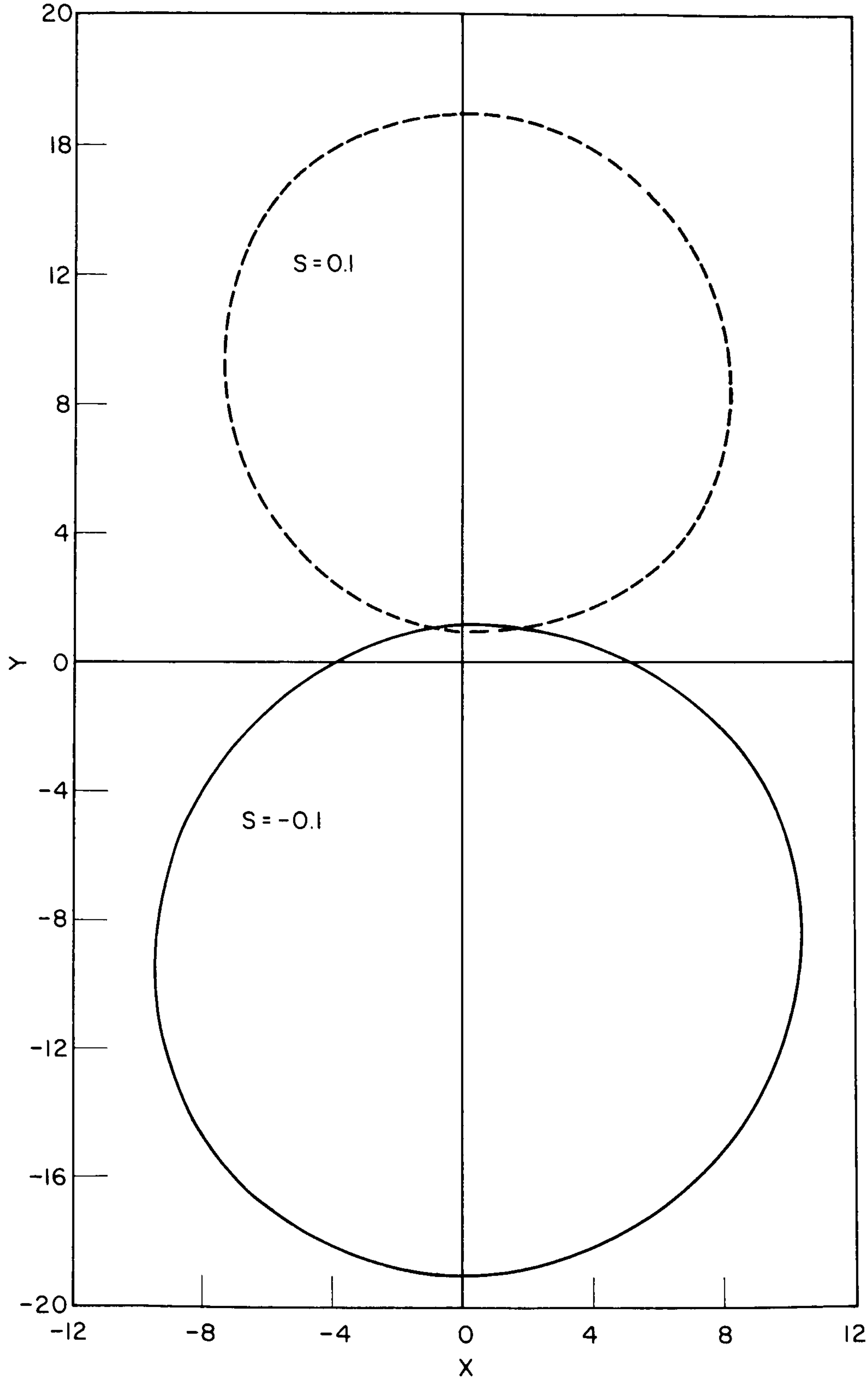

FIG. 3 PLOT OF Y VS. X FOR σ = 0.01, Y_o = 1.0, AND S = 0.1, -0.1.

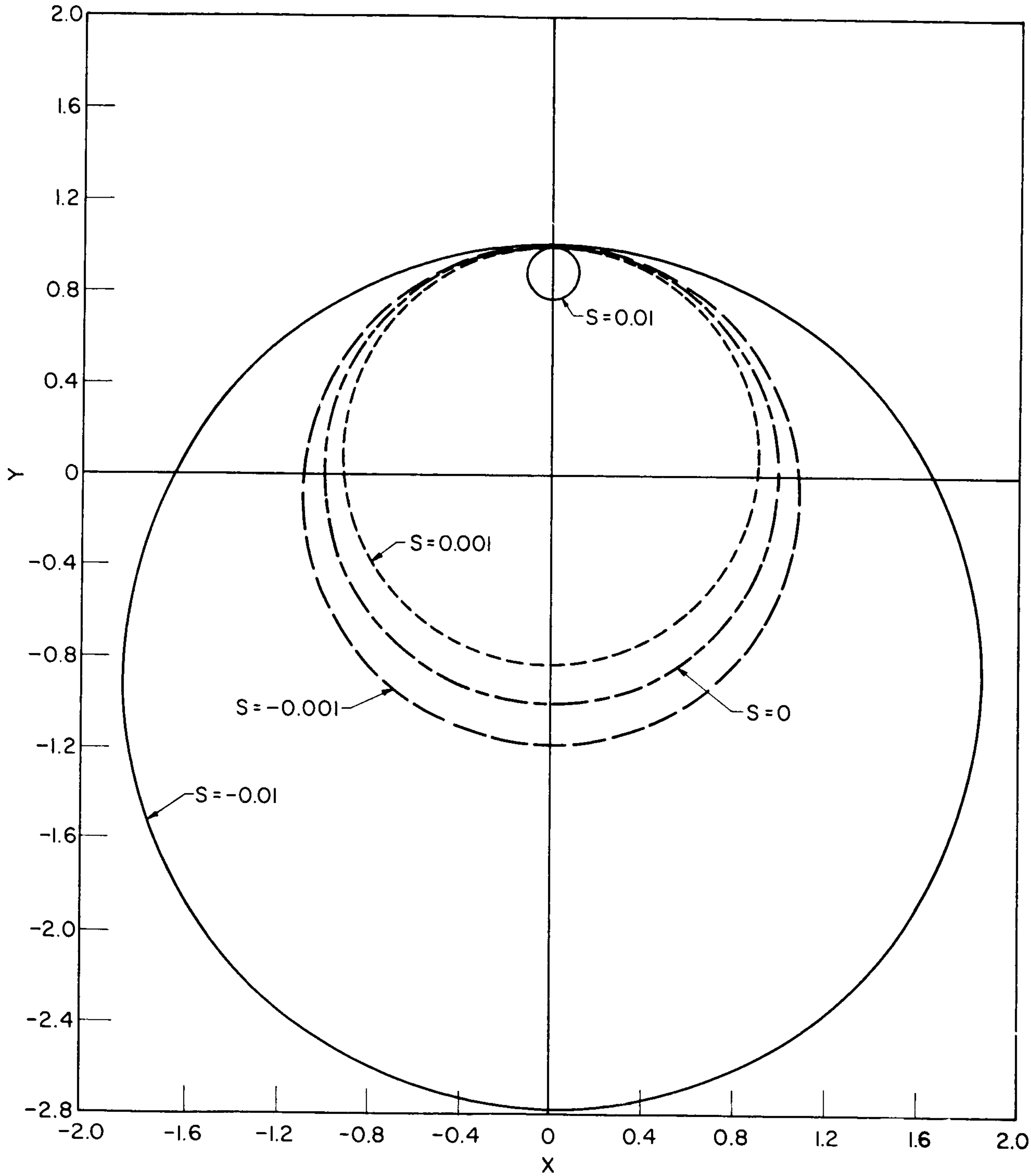

FIG. 4 PLOT OF Y VS. X FOR σ = 0.01, Y_o = 1.0, AND s = 0, ±0.001, ±0.01.

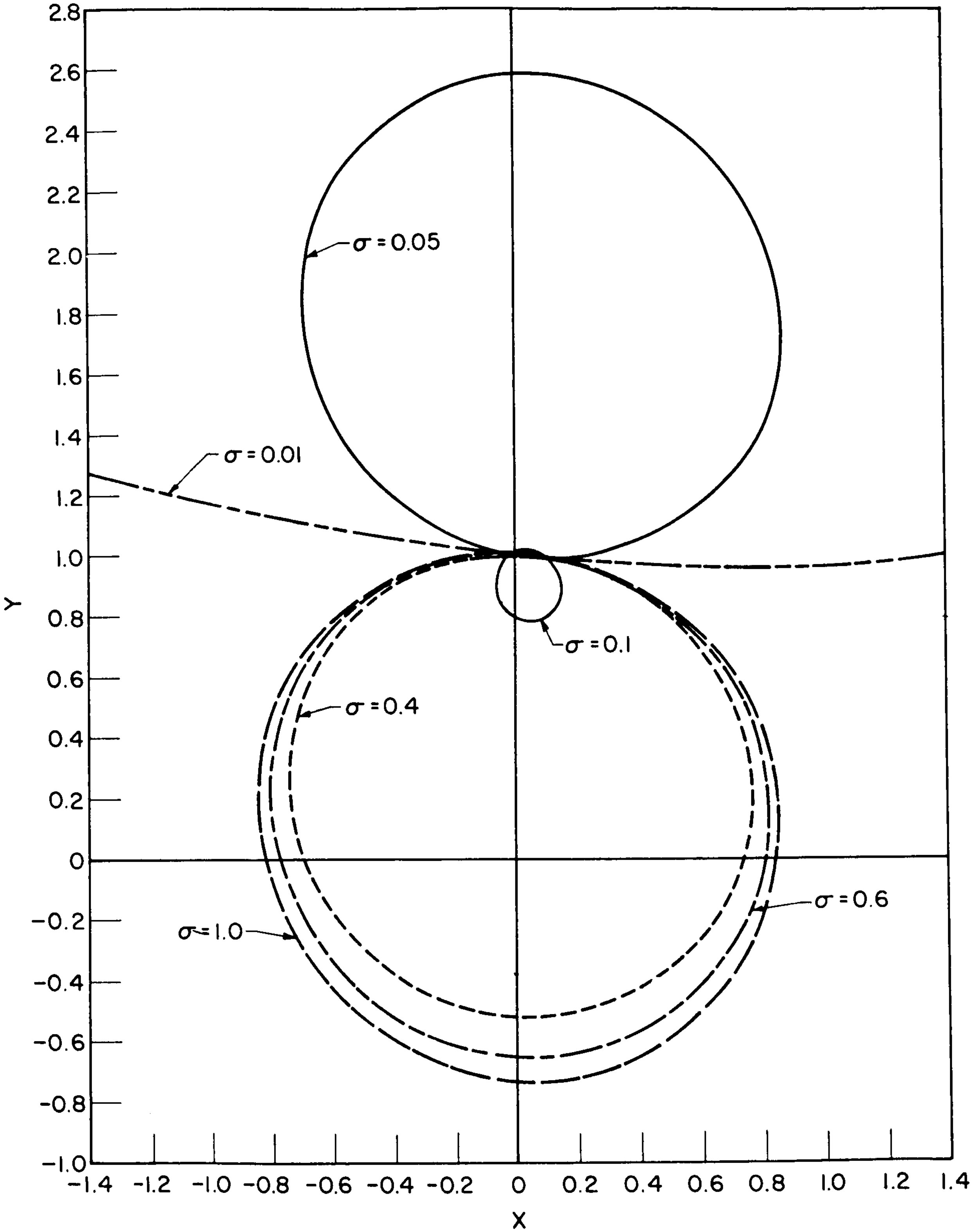

FIG. 5 PLOT OF Y VS. X FOR s = 0.1 AND Y_o = 1.0, WITH σ AS A PARAMETER.

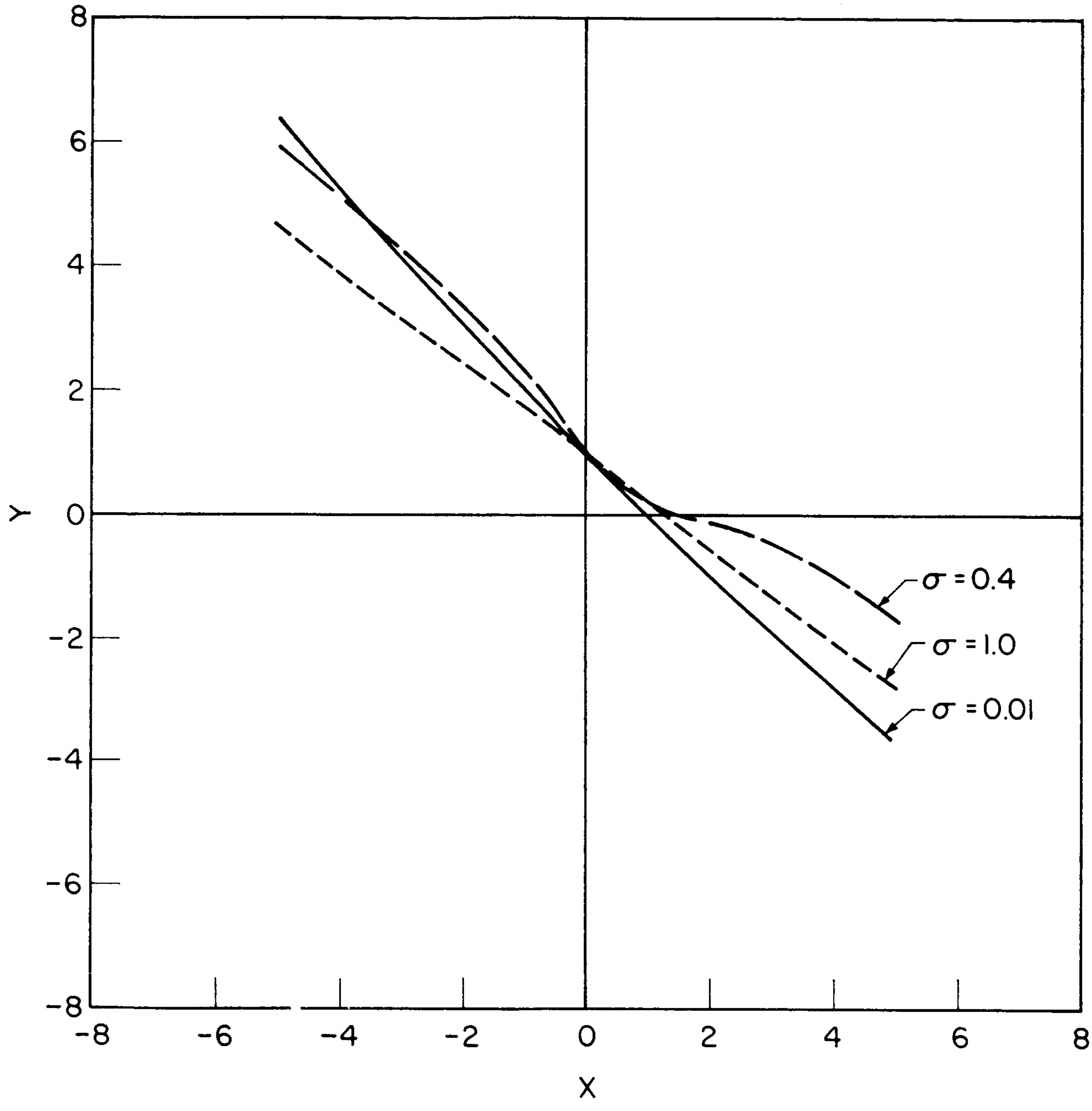

FIG. 6 PLOT OF Y VS. X FOR s = 1.0, Y_o = 1.0, AND σ = 0.01, 0.4, 1.0.

$$3\sigma^2(X^2 + Y^2) \ll 1 \tag{59}$$

is satisfied, then Eq. 57 can be reduced to the following second-degree equation:

$$p_1 X^2 + 2p_2 XY + q_2 Y^2 + 2p_o X + 2q_o Y = 2C \quad , \tag{60a}$$

which represents a conic section in the X-Y plane. Since

$$(p_2^2 - p_1 q_2) = -\frac{16}{\pi} \frac{\sigma^2}{s^2} \frac{1}{(s^2 + 4)} \quad , \tag{60b}$$

which is a negative quantity, Eq. 60a represents a family of ellipses. The term in (XY) can be made to vanish in Eq. 60a by a rotation of coordinate axes through an angle τ, such that

$$\tan 2\tau = \frac{2p_2}{(p_1 - q_2)} = \left(\frac{-2}{s}\right) . \tag{60c}$$

Upon performing this rotation,

$$X = X' \cos \tau - Y' \sin \tau \quad ,$$

$$Y = X' \sin \tau + Y' \cos \tau \quad , \tag{60d}$$

and Eq. 60a can be arranged into the following standard form for an ellipse:

$$\frac{(X' - X_o')^2}{a_o^2} + \frac{(Y' - Y_o')^2}{b_o^2} = 1 \quad , \tag{60e}$$

in which

$$X_o' = \frac{-D'}{2A'} , \quad Y_o' = \frac{-E'}{2C'} ,$$

$$a_o^2 = \frac{1}{A'}\left(\frac{D'^2}{4A'} + \frac{E'^2}{4C'} + F' \right) ,$$

$$b_o^2 = \frac{1}{C'}\left(\frac{D'^2}{4A'} + \frac{E'^2}{4C'} + F' \right) ,$$

where

$$A' = \frac{2}{\sqrt{\pi}}\left(\frac{\sigma}{s}\right)\left(1 - \frac{s}{s^2+4}\,(2 \sin 2\tau + s \cos \tau)\right) ,$$

$$C' = \frac{2}{\sqrt{\pi}}\left(\frac{\sigma}{s}\right)\left(1 - \frac{s}{s^2+4}\,(2 \sin 2\tau + s \cos \tau)\right) ,$$

$$D' = \frac{-2}{(s^2+1)}\,(\sin \tau + s \cos \tau) ,$$

$$E' = \frac{2}{(s^2+1)}\,(s \sin \tau - \cos \tau) ,$$

$$F' = -\frac{2Y_o}{s^2+1} + \frac{8}{\sqrt{\pi}}\;\frac{\sigma}{s}\;\frac{Y_o^2}{(s^2+4)} .$$

The center of the ellipse is located at the point $(X' = X_o', Y' = Y_o')$ and the lengths of the axes of the ellipses are $2a_o$ and $2b_o$. A typical plot of Eq. 60a is illustrated in Fig. 7.

The desired information in regard to the transverse magnetic field vector can be obtained from Fig. 7; the magnitude and the angle between the x- and y-components, which specifies the spatial orientation of the transverse magnetic field vector, are given respectively by (RB_o) and Θ, where

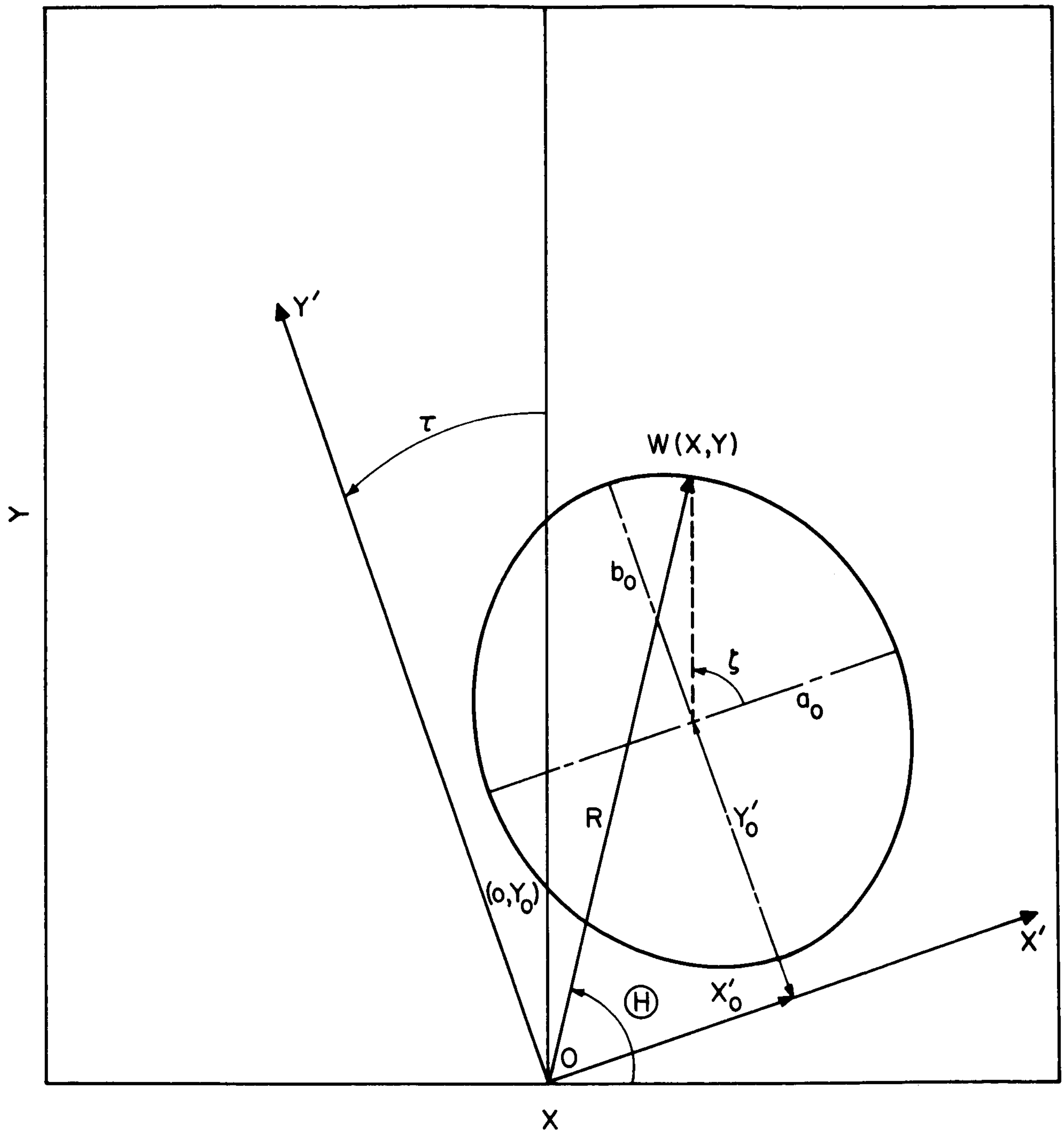

FIG. 7 PLOT OF Y VS. X BASED ON EQ. 60a FOR $s \neq 0$ AND $3\delta^2 \ll 1$.

$$R = \sqrt{X^2 + Y^2} \quad \text{and} \quad \Theta = \tan^{-1}\left(\frac{Y}{X}\right) , \tag{61}$$

in which X and Y are the coordinates of a point W(X,Y) on the ellipse. In view of the fact that the ellipse of Eq. 60c can be written parametrically as

$$X' = X'_o + a_o \cos \zeta ,$$

$$Y' = Y'_o + b_o \sin \zeta , \tag{62}$$

where the parameter ζ is the angle which is to be measured as indicated in Fig. 7, $R(\zeta)$ and $\Theta(\zeta)$ can be expressed as

$$R(\zeta) = \sqrt{(X'_o + a_o \cos \zeta)^2 + (Y'_o + b_o \sin \zeta)^2}$$

$$\Theta(\zeta) = \tau + \tan^{-1}\left(\frac{Y'_o + b_o \sin \zeta}{X'_o + a_o \cos \zeta}\right) . \tag{63}$$

$R(\zeta)$ is a periodic function of ζ and has its critical values (maximum or minimum) when $dR/d\zeta = 0$, which occurs at $\zeta = \zeta_c$, such that

$$\frac{a_o \sin \zeta_c}{b_o \cos \zeta_c} = \frac{Y'_o + b_o \sin \zeta_c}{X'_o + a_o \cos \zeta_c} . \tag{64}$$

Once σ, s and Y_o are specified, the quantities X'_o, Y'_o, a_o and b_o are all determined so that Eq. 64 can be solved for ζ_c, from which the maximum and minimum values of $R(\zeta)$ can be obtained. Furthermore it should be noted that as the parameter s approaches zero, the angle of rotation of the coordinate axes $\tau \to -\pi/4$, so that $A' \to (2/\sqrt{\pi})\sigma/s$, $C' \to (2/\sqrt{\pi})\sigma/s$, $D' \to -\sqrt{2}$, $E' \to -\sqrt{2}$ and $F' \to (2/\sqrt{\pi})(\sigma/s)Y_o^2$. Consequently both X'_o and Y'_o approach zero as of $[(-\sqrt{\pi}/8)(s/\sigma)]$ and both a_o and b_o approach Y_o. Therefore Eqs. 63 give

$$R(\zeta) = Y_o = \text{constant} \quad \text{and} \quad \Theta(\zeta) = \zeta + \ell\pi + \pi/4 \quad , \tag{65}$$

where ℓ is an integer, and X and Y can be given as

$$X(\zeta) = Y_o \cos [\zeta + (\ell + 1/4)\pi] \quad ,$$

$$Y(\zeta) = Y_o \sin [\zeta + (\ell + 1/4)\pi] \quad . \tag{66}$$

Thus it is observed that as $s \to 0$, the ellipse is gradually deformed into a circle whose center is located at the origin $(X = 0,\ Y = 0)$ and whose radius is equal to $a_o = Y_o$.

A comparison of Eqs. 51 and 66 suggests that

$$\zeta = k_o\xi \quad , \text{ for } E_o = 0 \quad . \tag{67}$$

It should be pointed out that Eq. 67 is valid only if $E_o = 0$. On the other hand, if E_o is different from zero, but sufficiently small so that the coefficients P and Q in Eqs. 47 and 48 are very slowly varying functions of ξ, then it would not be unreasonable to expect that $X(\xi)$ and $Y(\xi)$ will be almost periodic, and to expect the W-point in Fig. 7 to move along the ellipse as ξ varies. However, the spatial period in the wave frame is expected to be different from that in the case of $E_o = 0$. It should be noted that under the conditions of Eq. 59, Eqs. 47 and 48 are reduced to the following set of linear equations:

$$\frac{dY}{d\xi} = P_o + P_1 X + P_2 Y \quad ,$$

$$-\frac{dX}{d\xi} = Q_o + Q_1 X + Q_2 Y \quad , \tag{68}$$

in which the coefficients P and Q depend upon ξ in the form of an exponential function: $\exp\,[\mp(eE_o/KT_z)\xi]$. For the case where E_o is sufficiently small, the solution of Eqs. 68 may be expected to be almost periodic and its period can be estimated approximately by solving Eqs. 68 as if the coefficients P and Q are independent of ξ. In other words, $|(eE_o/KT_z)\xi| \ll 1$ and $\exp[\mp(eE_o/KT_z)\xi] \simeq 1$. Then through a transformation of dependent variables:

$$X = X'' - \frac{\sqrt{\pi}}{4}\,\frac{s^2}{\sigma}\,\frac{1}{(s^2+1)} ,$$

$$Y = Y'' + \frac{\sqrt{\pi}}{2}\,\frac{s}{\sigma}\,\frac{1}{(s^2+1)} , \tag{69}$$

and the set of Eqs. 68 is transformed into the set,

$$\frac{dY''}{d\xi} = A_1 X'' + B_1 Y'' \tag{70a}$$

and

$$-\frac{dX''}{d\xi} = C_1 X'' + D_1 Y'' , \tag{70b}$$

where

$$A_1 = \left(\frac{s^2+2}{2}\right) Q_2 , \quad B_1 = \frac{-s}{2} Q_2 , \quad C_1 = \frac{-s}{2} Q_2 , \quad D_1 = Q_2 ,$$

$$Q_2 = \frac{4}{(s^2+4)}\left(\frac{1 - e^{2\pi s}}{-2\pi s}\right) P_{1,o} , \tag{70c}$$

in which condition (55a) has been used and $P_{1,o}$ is given in Eq. 50b.

Elimination of Y" from Eqs. 70a and 70b yields

$$\frac{d^2X''}{d\xi^2} = -K^2 X'' , \tag{71a}$$

and elimination of X" gives

$$\frac{d^2Y''}{d\xi^2} = -K^2 Y'' , \tag{71b}$$

where

$$K^2 = \frac{(s^2 + 4)}{4} Q_2^2 \quad . \tag{71c}$$

In view of the fact that K^2 is a positive quantity, the solutions of Eqs. 71a and 71b are periodic functions of ξ in the wave frame and can be written as

$$X''(\xi) = M_1 \cos (K\xi + L_1) \tag{72a}$$

and

$$Y''(\xi) = M_2 \cos (K\xi + L_2) \quad , \tag{72b}$$

where M_1, L_1, M_2 and L_2 are arbitrary constants. The constant K is given by

$$K = \pm\mu(s)P_{1,o} \quad , \tag{72c}$$

where

$$\mu(s) \equiv \sqrt{\frac{4}{(s^2 + 4)}} \left(\frac{1 - e^{2\pi s}}{-2\pi s} \right) . \tag{72d}$$

It should be noted that when $(L_1 - L_2) = \pi/2$, Eqs. 72a and 72b are the parametric equations of an ellipse. Thus the result is an elliptically polarized plane wave in a laboratory frame. Furthermore, if K_o and Δ_o denote, respectively, the values of k_o and ψ_o which are quantities defined for the case $E_o = 0$ under the condition stated by Eq. 55a, then the wave number K, appearing in Eqs. 72, and the Faraday rotation Δ for the case $E_o \simeq 0$ can be expressed as follows:

$$\frac{K}{K_o} = \mu(s) \quad \text{and} \quad \frac{\Delta}{\Delta_o} = \mu(s) \quad , \tag{73}$$

where the factor $\mu(s)$ is defined in Eq. 72d. Since $\mu(0)$ equals unity and for $s < 0$, it decreases as $|s|$ increases, Eqs. 73 suggest that the Faraday rotation angle decreases while the wavelength $\Lambda = 2\pi/K$ increases with an increase of $|s|$.

V. DISCUSSION OF RESULTS

For a properly constructed solution of the nonlinear Boltzmann-Vlasov equation in a moving frame of reference, sets of ordinary nonlinear differential equations governing the components of the vector and scalar potentials have been derived; Eqs. 17, for the case where the static electric and magnetic fields are absent, and Eqs. 45 for the case where static electric and magnetic fields are present in the plasma. The numerical analysis of these sets of differential equations is in progress and will be discussed in a future report.

It is of interest, however, to consider a few special cases. For example, in the case of no static fields, with $T_{ex} = T_{ey} = T_{e\perp}$ and $T_{ix} = T_{iy} = T_{i\perp}$ and under the condition of electrical neutrality, Eqs. 17 are simplified considerably and could lead to a circularly polarized plane wave solution (Eqs. 27) with a dispersion relationship given by Eq. 29b. It should be observed that for a real k, ω can be real or complex, depending upon whether $(ck)^2$ is greater or less than $\omega_o^2 \exp(QV_o^2/2)$, which suggests the possibility of instabilities in the system. It should also be noted that if $T_{e\perp} \neq T_{ez}$, the wavelength of the transverse electromagnetic wave, $\lambda = 2\pi/k$, does depend upon the amplitude V_o of the wave and is obvious from Eq. 29b. On the other hand, if $QV_o^2/2 \ll 1$, then λ becomes independent of the wave amplitude, and Eq. 29b becomes

$$c^2k^2 - \omega^2 = \omega_o^2 \quad . \tag{74}$$

Moreover if $T_{i\perp} \ll T_{iz}$ and $T_{e\perp} \ll T_{ez}$, then from Eqs. 25 $T_{iz}/T_{ez} = \Omega_p^2/\omega_p^2$ so that $\omega_o^2 = -(\omega_p^2+\Omega_p^2)$. Furthermore if the ion motion can also be neglected, i.e., $T_{iz} \ll T_{ez}$, then $\omega_o^2 = -\omega_p^2$. Thus Eq. 74 is reduced to a familiar linear dispersion equation[9] for transverse plasma oscillations when no static magnetic field is present.

It should be noted that the set of nonlinear differential equations (47) and (48), governing the behavior of the transverse magnetic field, when there exists longitudinal magnetostatic and electrostatic fields, is derived under the condition that $\delta^3 \ll 1$, where δ is defined in Eq. 40. δ can also be written as $\delta = \sigma R$, where $\sigma \equiv \sqrt{mu_o^2/2KT_z}\ \sqrt{T_\perp/T_z}$ and $R \equiv (B_\perp/B_o)$. The condition $\delta^3 \ll 1$ is not a severe restriction and permits consideration of a wide range of system parameters. However, it should be pointed out that this condition was considered mainly because of mathematical convenience in illustrating the method of analysis. If it were not imposed then the higher-order terms in b_x and b_y would have appeared in the current density expressions of Eqs. 42 as well as in Eqs. 47 and 48.

It has been shown that in the absence of a longitudinal static electric field the plasma can support circularly polarized plane waves, whose dispersion relation is given by Eq. 52f under the condition of electrical neutrality. For a plasma whose mean velocity along the z-axis vanishes (i.e., $\bar{v} = 0$, or $u_o = -v_o$) and which exhibits a small temperature anisotropy, $\Delta T_e \ll T_{ez}$ and $\Delta T_i \ll T_{iz}$, ρ_o, given in Eq. 52g, becomes $(MN_i + mN_e)$, which is the mass density of the plasma. Consequently Eq. 52f can be written as

$$\omega^2 = \frac{c^2 k_o^2}{\left(1 + \frac{c^2}{v_A^2}\right)} \quad ; \quad v_A \equiv \sqrt{\frac{B_o^2}{\mu_o \rho_o}} \ , \tag{75}$$

where v_A is the Alfvén velocity and Eq. 75 is recognized as the dispersion relation for the Alfvén waves[21].

On the other hand, from Eqs. 50b and 51b,

$$c^2 k_o^2 - \omega^2 = \pm \frac{\omega}{\omega_c} \left(\frac{u_{oe}}{v_o} \right) \omega_p^2 \left[\left(\frac{N_i u_{oi}}{N_e u_{oe}} \right) \frac{T_{i\perp}}{T_{iz}} - \frac{T_{e\perp}}{T_{ez}} \right] . \tag{76a}$$

For a plasma under the condition of quasi-electrical neutrality, i.e., $N_i u_{oi}$ $N_e u_{oe}$, and vanishing mean velocity along the z-axis (i.e., $\overline{v} = 0$), the refractive index n can be expressed as

$$n^2 = 1 \mp \frac{\omega_p^2}{\omega \omega_c} \left(\frac{T_{i\perp}}{T_{iz}} - \frac{T_{e\perp}}{T_{ez}} \right) , \tag{76b}$$

which is recognizable as the dispersion relation from magnetoionic theory for $\omega \ll \Omega_c < \omega_c$. Thus it appears that Eq. 52f or Eq. 76a may be profitably applied to the investigation of some ionospheric phenomena, such as VLF emissions and whistler mode propagation in ionospheric plasmas.

It has also been shown that for the case where a longitudinal static electric field is present (i.e., $E_o \neq 0$), the magnitude of the transverse magnetic field no longer remains invariant, as in the case of $E_o = 0$, but varies with distance in the wave frame. Under the condition $\delta^3 \ll 1$, the x- and y-components of the magnetic field vector are related by Eq. 57 and the tip of the magnetic field vector describes the curves as illustrated in Figs. 1 through 6. However, under the small-amplitude condition of Eq. 59, except for extremely small values of σ, the locus of the tip of the magnetic field vector describes an ellipse as shown in Fig. 7. It is observed that the magnitude of the normalized transverse magnetic field vector $R = (B_\perp/B_o)$ varies between its minimum and maximum values in a wave frame as ξ varies, and that as $s \to 0$, the ellipse is gradually deformed into a circle. Furthermore, under condition (59) for the region $|eE_o\xi/kT_z| \ll 1$, it is shown that the solution of Eqs. 47 and 48 is a periodic function of ξ, as given by Eqs. 72 and the tip of the magnetic field vector describes an ellipse in a wave frame. Thus it is observed that the transverse electromagnetic wave in the presence of a weak static longitudinal electric field can propagate along the static longitudinal magnetic field as an elliptically polarized plane wave with the magnitude of the rotating magnetic field vector varying periodically with distance z. Therefore the amplitude and phase of a circularly polarized plane wave propagating in a magnetoactive plasma is modified by the presence of a longitudinal static electric field. The modification of the wavelength Λ and the Faraday rotation angle, Δ, due to a weak static electric field E_o is given in Eqs. 73 and is valid for an arbitrary value of the parameter s. An examination of Eqs. 73 reveals that the Faraday rotation angle Δ tends to decrease, while the wavelength Λ increases, with an increase of $|s|$. It is of interest to note that since s is defined as $[(mu_o/KT_z)(E_o/B_o)]$, an increase in B_o will cause $|s|$ to decrease, Δ to increase and Λ to decrease, which is considered reasonable.

APPENDIX A. DERIVATION OF EQS. 42 AND 43

For the electron distribution function given by Eq. 38, the electron current density components may be given as

$$j_x = L_o \int_0^{2\pi} I_2(\theta) \cos\theta \, e^{s\theta} d\theta \quad ,$$

$$j_y = L_o \int_0^{2\pi} I_2(\theta) \sin\theta \, e^{s\theta} d\theta \quad ,$$

$$j_z = L_1 \int_0^{2\pi} I_1(\theta) \, e^{s\theta} d\theta \quad , \qquad \text{(A.1)}$$

where

$$L_o \equiv -n_e e \exp\left(\frac{e\varphi}{KT_{ez}}\right) \int_{-\infty}^{\infty} e^{-\frac{m}{2KT_{ez}}(v_z-\bar{v})^2} dv_z = -\sqrt{\pi}\sqrt{\frac{2KT_{ez}}{m}}\, n_e e \exp\left(\frac{e\varphi}{KT_{ez}}\right) \quad ,$$

$$L_1 \equiv -n_e e \exp\left(\frac{e\varphi}{KT_{ez}}\right) \int_{-\infty}^{\infty} v_z e^{-\frac{m}{2KT_{ez}}(v_z-\bar{v})^2} dv_z = \bar{v} L_o \quad ,$$

$$I_1(\theta) \equiv \int_0^{\infty} v_r \exp\left[-\frac{m}{2KT_{e\perp}}\left((v_r-\beta)^2 - \beta^2\right)\right] dv_r \quad ,$$

$$I_2(\theta) \equiv \int_0^{\infty} v_r^2 \exp\left[-\frac{m}{2KT_{e\perp}}\left((v_r-\beta)^2 - \beta^2\right)\right] dv_r \quad ,$$

$$s \equiv \frac{m}{KT_{ez}}\left(\frac{u_o}{B_o}\right) E_z$$

and

$$\beta(\theta) \equiv \left(\frac{T_{e\perp}}{T_{ez}}\right)\left(\frac{u_o}{B_o}\right)(B_x \cos\theta + B_y \sin\theta)$$

$$= u_o \left(\frac{T_{e\perp}}{T_{ez}}\right)\left(\frac{B_\perp}{B_o}\right)\cos(\Theta - \theta) \quad ,$$

where

$$B_\perp = \sqrt{B_x^2 + B_y^2} \quad \text{and} \quad \Theta = \tan^{-1}\left(\frac{B_y}{B_x}\right) . \tag{A.2}$$

$I_1(\theta)$ and $I_2(\theta)$ can be evaluated to give

$$I_1(\theta) = \frac{1}{2a^2}\left[1 + \sqrt{\pi}\gamma e^{\gamma^2} \operatorname{erfc}(\gamma)\right] \quad ,$$

$$I_2(\theta) = \frac{1}{2a^3}\left[3\gamma + \left(\frac{1}{2} + \gamma^2\right)\sqrt{\pi}\, e^{\gamma^2} \operatorname{erfc}(\gamma)\right] \quad , \tag{A.3}$$

where

$$a^2 \equiv \left(\frac{m}{2KT_{e\perp}}\right) \quad , \quad \gamma \equiv (a\beta) \quad ,$$

$$\operatorname{erfc}(\gamma) \equiv \frac{2}{\sqrt{\pi}}\int_\gamma^\infty e^{-t^2} dt = [1 - \operatorname{erf}(\gamma)] \quad , \tag{A.4}$$

in which $\operatorname{erf}(\gamma)$ is the usual error function.

It is well known that the function $\operatorname{erf}(\gamma)$ can be expanded into a power series in γ (e.g., see Dwight[22], p.129):

$$\operatorname{erf}(\gamma) = \frac{2\gamma}{\sqrt{\pi}}\left(1 - \frac{\gamma^2}{1!3} + \frac{\gamma^4}{2!5} - \frac{\gamma^6}{3!7} + \ldots\right) \quad , \quad \gamma^2 < \infty \quad .$$

For cases in which $\gamma^3 \ll 1$ (e.g., $\gamma_o^3 = 0.01$, which is equivalent to $\gamma_o \simeq 0.217$), for $\gamma < \gamma_o$,

$$\operatorname{erfc}(\gamma) \simeq \left[1 - \frac{2\gamma}{\sqrt{\pi}}\left(1 - \frac{\gamma^2}{3}\right)\right] \tag{A.5}$$

so that $I_1(\theta)$ and $I_2(\theta)$ can be approximated as

$$I_1(\theta) \simeq \frac{1}{2a^2}\left(1 + \sqrt{\pi}\gamma - 2\gamma^2\right) ,$$

$$I_2(\theta) \simeq \frac{\sqrt{\pi}}{4a^3}\left(1 + \frac{4\gamma}{\sqrt{\pi}} + 3\gamma^2\right) . \tag{A.6}$$

Since γ can also be written conveniently as

$$\gamma = \delta \cos(\Theta - \sigma) , \tag{A.7}$$

where

$$\delta \equiv \sqrt{\frac{mu_{oe}^2}{2KT_{ez}}}\sqrt{\frac{T_{e\perp}}{T_{ez}}}\left(\frac{B_\perp}{B_o}\right) ,$$

the condition $\gamma^3 \ll 1$ implies that

$$\delta^3 \ll 1 . \tag{A.8}$$

On the other hand γ can also be written as

$$\gamma = (a\tau)(B_x \cos\theta + B_y \sin\theta) , \tag{A.9}$$

where

$$\tau \equiv \left(\frac{T_{e\perp}}{T_{ez}}\right)\left(\frac{u_o}{B_o}\right) .$$

Substituting $I_1(\theta)$ and $I_2(\theta)$, given by Eq. A.6, into Eqs. A.1 and carrying out the integration yields

$$j_x = \frac{\sqrt{\pi}L_o}{4a^3}\left[C_1^1 + \left(\frac{4a\tau}{\sqrt{\pi}}\,C_1^2\right)B_x + \left(\frac{4a\tau}{\sqrt{\pi}}\,\frac{S_2^1}{2}\right)B_y\right.$$

$$\left. + 3(a^2\tau^2C_1^3)\;B_x^2 + 6a^2\tau^2(S_1^1 - S_1^3)\;B_xB_y + 3a^2\tau^2(C_1^1 - C_1^3)\;B_y^2\right],$$

$$j_y = \frac{\sqrt{\pi}L_o}{4a^3}\left[S_1^1 + \left(\frac{4a\tau}{\sqrt{\pi}}\,\frac{S_2^1}{2}\right)B_x + \left(\frac{4a\tau}{\sqrt{\pi}}\,S_1^2\right)B_y\right.$$

$$\left. + 3a^2\tau^2(S_1^1 - S_1^3)\;B_x^2 + 6a^2\tau^2(C_1^1 - C_1^3)\;B_xB_y + (3a^2\tau^2S_1^3)\;B_y^2\right],$$

$$j_z = \frac{L_1}{2a^2}\left[\left(\frac{1 - e^{2\pi s}}{s}\right) + (\sqrt{\pi}a\tau C_1^1)\;B_x + (\sqrt{\pi}a\tau S_1^1)\;B_y\right.$$

$$\left. - (2a^2\tau^2C_1^2)\;B_x^2 - (2a^2\tau^2S_2^1)\;B_xB_y - (2a^2\tau^2S_1^2)\;B_y^2\right], \qquad (A.10)$$

where

$$C_1^1 \equiv \int_0^{2\pi} \cos\theta e^{s\theta}d\theta = \frac{-s}{(s^2+1)}(1 - e^{2\pi s}), \qquad (A.11a)$$

$$C_1^2 \equiv \int_0^{2\pi} \cos^2\theta e^{s\theta}d\theta = \frac{-\left(s + \frac{2}{s}\right)}{(s^2+4)}(1 - e^{2\pi s}), \qquad (A.11b)$$

$$C_1^3 \equiv \int_0^{2\pi} \cos^3\theta e^{s\theta}d\theta = -\frac{(1 - e^{2\pi s})}{(s^2+9)}\left(s + \frac{6s}{s^2+1}\right), \qquad (A.11c)$$

$$S_1^1 \equiv \int_0^{2\pi} \sin\theta e^{s\theta}d\theta = \frac{1}{(s^2+1)}(1 - e^{2\pi s}), \qquad (A.11d)$$

$$S_1^2 \equiv \int_0^{2\pi} \sin^2\theta e^{s\theta}d\theta = \frac{-2}{s(s^2+4)}(1 - e^{2\pi s}), \qquad (A.11e)$$

$$S_1^3 \equiv \int_0^{2\pi} \sin^3 \theta e^{s\theta} d\theta = \frac{(1 - e^{2\pi s})}{(s^2 + 9)} \frac{6}{(s^2 + 1)} , \tag{A.11f}$$

$$S_2^1 \equiv \int_0^{2\pi} \sin 2\theta e^{s\theta} d\theta = \frac{2}{(s^2 + 4)} (1 - e^{2\pi s}) . \tag{A.11g}$$

Defining

$$b_x \equiv \frac{B_x}{B_o} , \quad b_y \equiv \frac{B_y}{B_o} \quad \text{and} \quad \sigma \equiv \sqrt{\frac{mu_o^2}{2KT_{ez}}} \sqrt{\frac{T_{e\perp}}{T_{ez}}} , \tag{A.12}$$

the factors $(a\tau B_x)$ and $(a\tau B_y)$ appearing in Eq. A.10 can be respectively replaced by (σb_x) and (σb_y), so that

$$j_x = K_\perp (p_o + p_1 b_x + p_2 b_y + p_3 b_x^2 + p_4 b_x b_y + p_5 b_y^2) ,$$

$$j_y = K_\perp (q_o + q_1 b_x + q_2 b_y + q_3 b_x^2 + q_4 b_x b_y + q_5 b_y^2) ,$$

$$j_z = K_{||} (\ell_o + \ell_1 b_x + \ell_2 b_y + \ell_3 b_x^2 + \ell_4 b_x b_y + \ell_5 b_y^2) , \tag{A.13}$$

where

$$K_\perp \equiv \frac{\sqrt{\pi}}{4a^3} L_o (e^{2\pi s} - 1) ,$$

$$K_{||} \equiv \frac{L_1}{2a^2} (e^{2\pi s} - 1) , \tag{A.14a}$$

$$p_o = \frac{s}{(s^2+1)} , \quad p_1 = \frac{4}{\sqrt{\pi}} \frac{(s^2+2)}{(s^2+4)} \frac{\sigma}{s} , \quad p_2 = -\frac{4}{\sqrt{\pi}} \frac{\sigma}{(s^2+4)} ,$$

$$p_3 = \frac{(s^2+7)}{(s^2+9)} \frac{3s\sigma^2}{(s^2+1)} , \quad p_4 = \frac{-(s^2+3)}{(s^2+9)} \frac{6\sigma^2}{(s^2+1)} , \quad p_5 = \frac{6}{(s^2+9)} \frac{s\sigma^2}{(s^2+1)} , \tag{A.14b}$$

$$q_0 = \frac{-1}{(s^2+1)} , \quad q_1 = \frac{-4}{\sqrt{\pi}} \frac{\sigma}{(s^2+4)} , \quad q_2 = \frac{4}{\sqrt{\pi}} \frac{2\sigma}{s(s^2+4)} ,$$

$$q_3 = \frac{-(s^2+3)}{(s^2+9)} \frac{3\sigma^2}{(s^2+1)} , \quad q_4 = \frac{12}{(s^2+9)} \frac{s\sigma^2}{(s^2+1)} , \quad q_5 = \frac{-18\sigma^2}{(s^2+9)(s^2+1)} , \tag{A.14c}$$

$$\ell_0 = \frac{1}{s} , \quad \ell_1 = \sqrt{\pi} \frac{s}{(s^2+1)} \sigma , \quad \ell_2 = -\sqrt{\pi} \frac{1}{(s^2+1)} \sigma ,$$

$$\ell_3 = -\frac{2(s^2+2)}{s(s^2+4)} \sigma^2 , \quad \ell_4 = \frac{4}{(s^2+4)} \sigma^2 , \quad \ell_5 = \frac{-4}{s(s^2+4)} \sigma^2 . \tag{A.14d}$$

By substituting the values of L_0 and L_1 into the expressions for $K_\perp$ and $K_{||}$ and if the distribution function f is normalized to a constant density N_e, with $E_0 = 0$, $K_\perp$ and $K_{||}$ can be expressed as

$$K_\perp = \frac{eN_e \sqrt{\pi}}{2a} \left(\frac{1 - e^{2\pi s}}{2\pi} \right) \exp \left(\frac{e\varphi}{KT_{ez}} \right) ,$$

$$K_{||} = eN_e \bar{v} \left(\frac{1 - e^{2\pi s}}{2\pi} \right) \exp \left(\frac{e\varphi}{KT_{ez}} \right) . \tag{A.15}$$

Thus Eqs. 42 and 43 are obtained.

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