MODIFICATIONS OF
THE GODDARD MINIMUM VARIANCE PROGRAM
FOR THE PROCESSING OF REAL DATA

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SUMMARY

This report contains results of the studies carried out by Analytical Mechanics Associates, Inc. in the continued development of the Goddard Minimum Variance Program under Contract NAS 5-2535. The report contains a universal solution of the two body problem, based on a formulation by Stumpff, specifically designed for the modified Encke method. The report contains a modified set of variational parameters derived to eliminate the singularity existing for parabolic and near-parabolic orbits. The report also contains a new development accounting for the effects of bias errors in the equations of motion of the state as well as biases in the observations on the estimate of the state, and the covariance matrix of the errors in the state. In addition, the report contains the development of the effect of machine process noise both in the solution of the equations of motion as well as in the prediction of the observations. This bias sets a lower limit to which the uncertainty in the state may be reduced by means of the orbit determination program. Finally, the report contains a derivation of a set of finite rotations designed to produce changes in the state of the orbit when the changes in the variational parameters may no longer be considered small.

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INTRODUCTION

The development of an operational orbit determination program requires the ability to weight real data properly in the effort to produce a meaningful estimate of the dynamic state of the vehicle. Many programs exist which apparently function well with simulated data, but which prove incapable of reducing real data without the use of irrational smoothing techniques, arbitrary weighting factors, and other personalized interventions into the field of orbit determination. The Kalman filter, used in a sequential manner, compels the analyst to estimate the expected residual in the observations from point to point. Any infraction of the physical laws used to describe the propagation of the covariance errors in the state or the expected stochastic noise in the observations will result in an unrealistic estimate of the lower bound to which we hope to reduce the uncertainty of the state, or an optimistic estimate of the subsequent residuals yet to be encountered. This report contains a rational method for including the effect of known biases in the equations of motion and in the observations. Basing these expected errors on known physical models, the report derives a realistic weighting of the errors in the state, as well as the rate of propagation of the errors.

The introduction of the NASA variational parameters, in Reference 1, has proven successful for typical elliptical and hyperbolic orbits. However, for an important class of parabolic and near-parabolic orbits, such as may be expected to occur for lunar trajectories, it has become evident that some modification to the existing parameters is required. Since the semi-major axis, a, is known to become infinite as the orbit approaches the parabolic condition, the terms in the transition matrix in which the semi-major axis occurs in the numerator become cumbersome in the near-parabolic case. A modification was undertaken which removed this ambiguity. The specification of insuring that only one of the variational parameters could affect the energy, in order to restrict the secular terms to only one variable, was adhered to. A detailed development of this study is contained in this report.

Perhaps the most significant contribution of this study lies in the discovery of the effect of computing machine process noise on the covariance matrix of the estimated errors in the state. It has been known for some time that the Kalman filter tends to lose the ability to reduce the norm of the covariance matrix of the estimated errors in the state, due to the loss of positive definiteness in this covariance matrix at some point along the orbit time arc. Theoretically, the state error covariance matrix may be shown to be always positive definite and monotonically decreasing so long as data is continually being processed. However, in practice, there exists a finite amount of computing machine process noise which effectively places a lower limit on the norm of the covariance matrix. By neglecting this important realistic source of error, it is possible to produce a nonpositive definite covariance matrix using the Kalman filter on a finite digit computing machine. This report derives a method for accounting for this effect and enables the Kalman filter to produce realistic estimates of the state and realistic estimates of the covariance of the estimated errors in the state.

In order to diminish the destabilizing effect of nonlinearities on the linear filter theory described herein, it has been found necessary to derive a procedure for finite displacements or rotations. Using linearized theory, it is often possible to derive a correction to the state which may reintroduce errors. In particular, due to the important role played by the energy, it is of utmost importance to insure that, once an estimate of the change in energy has been arrived at, the additional variables to be corrected for do not disturb this altered energy. A method of producing finite rotations consistent with the modified NASA variational parameters is described herein.

1. Two Body Problem

It is convenient to have a solution of the two body problem which holds for all conic sections. A solution, obtained by K. Stumpff and first published in Reference 2, is used as a basis for the form of the solution described below.

The solution of the two body problem, in Cartesian coordinates, is given as a function of the initial conditions as follows:

$$\begin{aligned} R &= f R_o + g \dot{R}_o \\ \dot{R} &= \dot{f} R_o + \dot{g} \dot{R}_o \end{aligned} \tag{1.1}$$

The functions f, g, $\dot{f}$ and $\dot{g}$ are given in terms of the initial conditions and the increment in time from the initial time, $t - t_o$, as follows:

$$\begin{aligned} f &= 1 - \frac{\beta^2}{r_o} F_3(\alpha) \\ g &= \frac{r_o}{\sqrt{\mu}} \beta F_2(\alpha) + \frac{d_o}{\mu} \beta^2 F_3(\alpha) \\ \dot{f} &= - \frac{\sqrt{\mu}}{r\, r_o} \beta F_2(\alpha) \\ \dot{g} &= 1 - \frac{\beta^2}{r} F_3(\alpha) \end{aligned} \tag{1.2}$$

The functions r_o, d_o, v_o and r are defined as

$$
\begin{aligned}
r_o &= [R(t_o) \cdot R(t_o)]^{1/2} \\
d_o &= R(t_o) \cdot \dot{R}(t_o) \\
v_o &= [\dot{R}(t_o) \cdot \dot{R}(t_o)]^{1/2} \\
r &= [R(t) \cdot R(t)]^{1/2}
\end{aligned}
\tag{1.3}
$$

The variable, β, is the regularization parameter used to unify the hyperbolic, elliptic, parabolic and rectilinear cases. β^2 is given by

$$
\beta^2 = a\,\alpha \tag{1.4}
$$

where the semi-major axis, a, and α are defined as

$$
\begin{aligned}
\frac{1}{a} &= \frac{2}{r_o} - \frac{v_o^2}{\mu} \\
\alpha &= [E - E(t_o)]^2 = \frac{\beta^2}{a}
\end{aligned}
\tag{1.5}
$$

$E - E_o$ is the increment in the eccentric anomaly measured from the initial eccentric anomaly. It is noted that β is always real since the eccentric anomaly becomes imaginary whenever the semi-major axis becomes negative.

The functions F_i are in reality the sine and cosine series with a finite number of initial terms removed. The general formula for F_i is given by

$$F_i(\alpha) = \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{(2k+i-1)!} \tag{1.6}$$

To obtain the universal anomaly, β, from the increment in time, it is necessary to solve Kepler's equation given below

$$\sqrt{\mu}\,(t - t_o) = \beta^3 F_4 + r_o \beta F_2 + \frac{d_o}{\sqrt{\mu}} \beta^2 F_3 \tag{1.7}$$

Equation (1.7) may be solved by Newton's method in an iterative manner for a given $t - t_o$ as follows:

$$\beta_{i+1} = \beta_i - \frac{\beta_i^3 F_4(\alpha_i) + r_o \beta_i F_2(\alpha_i) + \frac{d_o}{\sqrt{\mu}} \beta_i^2 F_3(\alpha_i) - \sqrt{\mu}\,(t-t_o)}{r(\beta_i)} \tag{1.8}$$

The denominator, $r(\beta_i)$, represents the partial derivative of equation (1.7) with respect to β and is given by

$$r = \beta^2 F_3(\alpha) + r_o F_1(\alpha) + \frac{d_o}{\sqrt{\mu}} \beta F_2(\alpha) \tag{1.9}$$

This formulation is presently in use in many different forms (References 3, 4, and 5). The purpose of describing it herein is to bring attention to the earliest derivation known to the authors, as contained in Reference 2.

It is convenient to obtain a reduction formula for the functions $F_i(\alpha)$ in order to reduce the number of terms required for the summation of the series for large α's. The highest function that will be required is $F_6(\alpha)$. Reduction formulae are given which express $F_i(\alpha)$ as functions of $F_j(\frac{\alpha}{4})$ as follows:

$$F_5(\alpha) = \frac{1}{4} F_4\left(\frac{\alpha}{4}\right)\left\{1 + F_2\left(\frac{\alpha}{4}\right)\right\}$$

$$F_6(\alpha) = \frac{1}{16}\left\{F_6\left(\frac{\alpha}{4}\right) + \frac{1}{2} F_4\left(\frac{\alpha}{4}\right) + F_5\left(\frac{\alpha}{4}\right) F_1\left(\frac{\alpha}{4}\right)\right\} \tag{1.10}$$

To obtain the lower order functions, we have the recursion formula

$$F_i(\alpha) = \frac{1}{(i-1)!} - \alpha F_{i+2}(\alpha) \tag{1.11}$$

2. The Modified NASA Variational Parameters

Reference 1 contains a description of a set of variational parameters. These parameters consist of three rigid rotations in addition to three other variations. The three rotations may be described as follows:

(a) A rotation of the vector $R(t)$ about the vector $\dot{R}(t)$ through a small angle α_1.

(b) A rotation of the vector $\dot{R}(t)$ about the vector $R(t)$ through a small angle α_2.

(c) A rigid rotation of both $R(t)$ and $\dot{R}(t)$ about the angular momentum vector, $H = R \times \dot{R}$, through a small angle α_3.

The remaining three variables may be described as follows:

(d) A variation in the scalar function $\frac{R \cdot \dot{R}}{\sqrt{\mu\ |a|}}$ accomplished by rotating $\dot{R}(t)$ about H through a small angle, leaving $R(t)$ and the magnitude of the velocity vector invariant.

(e) A variation in the scalar function $\frac{1}{a}$ accomplished by stretching the vectors $R(t)$ and $\dot{R}(t)$ along their respective directions, leaving the angle between them unchanged.

(f) A variation in the scalar function $\frac{rv^2}{\mu} - 1$ accomplished by stretching the vectors $R(t)$ and $\dot{R}(t)$ along their respective directions in such a proportion as to leave the magnitude of $\frac{1}{a}$ and the angle between them invariant.

The defect in the above formulation accrues from the fact that the semi-major axis, a, occurs in the numerator of several terms in the parameter state transition matrix as well as in the point transformation matrix relating the variational parameters with the components of R and $\dot{R}$. As the orbit approaches the parabolic case, the terms become unbounded and numerical inaccuracy results.

It is possible to remove this difficulty by a new choice of parameters without eliminating $\frac{1}{a}$ as one of the variables. The significance of retaining $\frac{1}{a}$ as a parameter is to insure that the remaining five variables remain independent of the energy so that no secular terms will occur in the state transition matrix due to variations in these parameters. In this fashion the secular terms may be restricted to only one variable, namely the semi-major axis, $\frac{1}{a}$.

The formulation carried out here is similar to that contained in Ref. 6, although the derivation is somewhat simpler. Let there be two sets of variables α_i and β_i. The point transformation matrix relating R and $\dot{R}$ with α_i may be expressed in terms of the β_i as follows:

$$S(x, \alpha) = \left(\frac{\partial x}{\partial \alpha}\right) = \left(\frac{\partial x}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial \alpha}\right) = S(x, \beta)\, J(\beta, \alpha) \qquad (2.1)$$

Similarly, the inverse is given by

$$S^{-1}(\alpha, x) = (\frac{\partial\alpha}{\partial x}) = (\frac{\partial\alpha}{\partial\beta})(\frac{\partial\beta}{\partial x}) = J^{-1}(\beta, \alpha)\, S^{-1}(\beta, x) \quad (2.2)$$

The state transition matrix may be altered as follows:

$$\begin{aligned} (\frac{\partial\alpha(t)}{\partial\alpha(t_o)}) &= \Omega(\alpha, \alpha_o) \\ &= (\frac{\partial\alpha(t)}{\partial\beta(t)})(\frac{\partial\beta(t)}{\partial\beta(t_o)})(\frac{\partial\beta(t_o)}{\partial\alpha(t_o)}) \\ &= J^{-1}(\beta, \alpha)\, \Omega(\beta, \beta_o)\, J(\beta_o, \alpha_o) \end{aligned} \quad (2.3)$$

Let the old variables be β_i and the new variables be α_i. We choose

$$\begin{aligned} \alpha_1 &= \beta_1 \\ \alpha_2 &= \beta_2 \\ \alpha_3 &= \beta_3 \\ \alpha_4 &= R \cdot \dot{R} = \beta_4 \sqrt{\mu}\ (\beta_5^{\,2})^{-1/4} \\ \alpha_5 &= \frac{1}{a} = \beta_5 \\ \alpha_6 &= r = \frac{(1 - \beta_6)}{\beta_5} \end{aligned} \quad (2.4)$$

The matrix of the partial derivatives is given by

$$J(\beta,\alpha) = \left(\frac{\partial\beta}{\partial\alpha}\right) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\sqrt{\mu\,|a|}} & \dfrac{1}{2}\dfrac{R\cdot\dot{R}}{\sqrt{\mu}}\sqrt{|a|} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -r & -\dfrac{1}{a} \end{bmatrix} \tag{2.5}$$

The inverse matrix is given by

$$J^{-1}(\beta,\alpha) = \left(\frac{\partial\alpha}{\partial\beta}\right) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\mu\,|a|} & -\dfrac{1}{2}R\cdot\dot{R}\;a & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -ra & -a \end{bmatrix} \tag{2.6}$$

The new point transformation matrix may be obtained from $S(x,\beta)$, given in Reference 1, and Eq. (2.1)

$$S(x,\alpha) = \left(\frac{\partial x(t)}{\partial \alpha(t)}\right) = S(x,\beta)\, J(\beta,\alpha)$$

$$S = \begin{bmatrix} -\frac{1}{v}H & 0 & \frac{1}{h}H\times R & \frac{1}{h^2}H\times R & \frac{\mu d}{2v^2h^2}H\times R & \frac{R}{r} - \frac{\mu d}{r^2v^2h^2}(1-\frac{r}{a})H\times R \\ 0 & \frac{1}{r}H & \frac{1}{h}H\times\dot{R} & 0 & -\frac{\mu}{2v^2}\dot{R} & -\frac{u}{r^2v^2}\dot{R} \end{bmatrix}$$

(2.7)

The new inverse of S may be obtained from $S^{-1}(x,\beta)$, given in Reference 1, and Eq. (2.2)

$$S^{-1}(x,\alpha) = \left(\frac{\partial \alpha(t)}{\partial x(t)}\right) = J^{-1}(\beta,\alpha)\, S^{-1}(x,\beta)$$

$$S^{-1} = \begin{bmatrix} -\frac{v}{h^2}H & 0 \\ 0 & \frac{r}{h^2}H \\ 0 & \frac{1}{hv^2}H\times\dot{R} \\ \dot{R} & R \\ -\frac{2}{r^3}R & -\frac{2}{\mu}\dot{R} \\ \frac{1}{r}R & 0 \end{bmatrix} \qquad (2.8)$$

It should be noted that the semi-major axis does not appear in the numerator of either S or S^{-1}.

The new state transition matrix may be obtained from $\Omega(\beta, \beta_o)$, given in Reference 1, and Eq. (2.3).

$$\Omega(\alpha,\alpha_o) = J^{-1}(\beta,\alpha)\,\Omega(\beta,\beta_o)\,J(\beta_o,\alpha_o)$$

$$\Omega(\alpha,\alpha_o) = \begin{bmatrix} \frac{v}{v_o} f & -\frac{v}{r_o} g & 0 & 0 & 0 & 0 \\ -\frac{r}{v_o}\dot{f} & \frac{r}{r_o}\dot{g} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \Omega_{3,4} & \Omega_{3,5} & \Omega_{3,6} \\ 0 & 0 & 0 & \dot{g} & \Omega_{4,5} & r_o \dot{f} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{r} g & \Omega_{6,5} & \frac{r_o}{r} f \end{bmatrix} \tag{2.9}$$

where

$$\Omega_{3,4} = \frac{h}{v^2}\left[\frac{\dot{g}-1}{r^2} - \frac{r_o \dot{f}^2}{\mu} - \frac{\dot{f}}{h^2}\left(\frac{\mu}{r} g - d_o\right)\right] \tag{2.9a}$$

$$\Omega_{3,5} = -\frac{\mu \dot{f} h}{2 v^2 v_o^2}\left[\dot{g} + \frac{d_o}{h^2}\left(\frac{\mu}{r} g - d_o\right)\right] + \frac{\sqrt{\mu}\, h \beta^3}{2 r^3 v^2}\left[r_o F_2^3 + \frac{d_o}{\sqrt{\mu}} \beta F_2 (F_2^2 + F_3)\right.$$

$$\left. + \beta^2(-3F_6 + F_5 - F_4 F_3 + F_3^2 + F_4 F_2^2)\right] \tag{2.9b}$$

$$\Omega_{3,6} = \frac{h\dot{f}}{r_o v^2}\left[\frac{r_o}{r} F_1 + \frac{r_o^2}{r^2} - \frac{\mu \dot{g}}{r_o v_o^2} + \frac{\mu d_o}{h^2 r_o v_o^2}(1 - \frac{r_o}{a})(\frac{\mu}{r} g - d_o)\right] \tag{2.9c}$$

$$\Omega_{4,5} = \frac{\sqrt{\mu}}{r}\left[- r_o^2 \beta F_1 F_2 - \frac{d_o r_o}{\sqrt{\mu}} \beta^2 (F_3 + 2 F_3 F_1) - \frac{d_o^2}{\mu} \beta^3 F_3 F_2\right.$$

$$+ r_o \beta^3 (\frac{3}{2} F_3 - \frac{3}{2} F_4 - 2 F_2 F_3) + \frac{d_o}{\sqrt{\mu}} \beta^4 (\frac{3}{2} F_4 - 3 F_5 - 2 F_3^2)$$

$$\left. + \beta^5 (\frac{1}{2} F_5 - \frac{3}{2} F_6 - F_3 F_4)\right] \tag{2.9d}$$

$$\Omega_{6,5} = \frac{\beta^2}{r}\left[- \frac{r_o^2}{2} F_2^2 - \frac{d_o r_o}{\sqrt{\mu}} \beta F_2 F_3 + r_o \beta^2 (\frac{1}{2} F_4 - F_5 - F_4 F_2)\right.$$

$$+ \frac{d_o^2}{\mu} \beta^2 (F_5 - \frac{1}{2} F_4 - \frac{1}{2} F_4 F_2) + \frac{d_o}{\sqrt{\mu}} \beta^3 (\frac{1}{2} F_5 - \frac{3}{2} F_6 - F_4 F_3)$$

$$\left. + \beta^4 (\frac{1}{2} F_6 - 2 F_7 - \frac{1}{2} F_4^2)\right] \tag{2.9e}$$

3. The Modified Kalman Filter With Bias Errors

Reference 1 contains a description of the modified Kalman filter in which errors in the initial conditions and stochastic noise in the observations are the main sources of uncertainties in the knowledge of the state. This report proposes to extend this analysis for the effects of bias errors both in the equations of motion and in the observations, as well as the effect of computing machine process noise on the uncertainty in the state.

A. Machine Process Noise

In finite digit arithmetic, every number, α, may be defined as a decimal followed by p digits (where p is a fixed integer for a given computing machine) followed by an exponent, n, which fixes the relative magnitude of the number in the arithmetic base (say 10). Thus, every number may be written as

$$\alpha = .a_1 a_2 --- a_p 10^n \tag{3.1}$$

Two numbers are considered equal if they agree both in the digits following the decimal as well as the exponent. Furthermore, any number, β, smaller than the least significant part of α, may be considered zero with respect to the number α. This is necessarily so since $\alpha + \beta$ will produce α in the machine. Since the number beyond the final digit in the machine is unknown, it may be considered a random variable. Furthermore, if we assume that round-off in addition, in the machine, is accomplished by simply dropping the number beyond the least significant digit, it is apparent that the sign of the number dropped is in reality the sign of α and that we are dealing with a biased stochastic process. An estimate of the machine noise associated with a given number, α, is given by

$$\eta(\alpha) = \frac{\alpha}{|\alpha|} 10^{n-p} \tag{3.2}$$

Since the process of obtaining the exponent of a number in the computing machine is time-consuming, a sufficiently accurate estimate of the machine noise associated with a given number, α, is given by

$$\eta(\alpha) \sim \alpha\ 10^{-p} \tag{3.3}$$

Furthermore, an estimate of the variance of the machine noise is given by

$$E(\eta, \eta^{*}) = \alpha^{2}\ 10^{-2p} \tag{3.4}$$

In the event that α is a vector, x, an estimate of the machine noise for the vector is given by

$$\eta(x) = \begin{Bmatrix} x_{11} \\ x_{12} \\ \cdot \\ \cdot \\ x_{1n} \end{Bmatrix} 10^{-2p} \tag{3.5}$$

The associated covariance matrix for the expected errors in the vector, x, is given by

$$E[\eta(x), \eta(x)^*] = \begin{bmatrix} x_{11}^2\, 10^{-2p} & 0 & 0 & 0 & 0 \\ & x_{22}^2\, 10^{-2p} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & 0 & 0 & 0 & x_{nn}^2\, 10^{-2p} \end{bmatrix} \tag{3.6}$$

It is noted that this matrix is a diagonal matrix since we are dealing with a random process and no correlation exists between the various components of the machine noise.

B. Bias Errors in the Equations of Motion

Let the state variables be described by a system of differential equations given below.

$$\dot{x} = f(x, u, t) \tag{3.7}$$

The variables, u, refer to biases in the equations of motion. These may be constant parameters, such as the gravitational constants, or they may be variables themselves governed by differential equations, such as thrust, atmospheric drag, etc. In any case, they may be described by differential equations of the form

$$\dot{u} = g(x, u, t) \tag{3.8}$$

The propagation of errors in the solution of equations (3.7) and (3.8) is given by the conventional variational equations

$$\delta\dot{x} = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u + \eta(x)$$
$$= F\,\delta x + G\,\delta u + \eta(x)$$

$$\delta\dot{u} = \frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial u}\delta u + \eta(u) \qquad (3.9)$$
$$= H\,\delta x + J\,\delta u + \eta(u)$$

Let the covariance matrices of the expected correlation between the variables be given as follows:

$$E(\delta x, \delta x^*) = P \quad , \quad E(\delta x, \delta u^*) = C$$
$$E(\delta u, \delta u^*) = B \quad , \quad E[\eta(x), \eta(x)^*] = Q_x \qquad (3.10)$$

The differential equation describing the time rate of change of the covariance matrix is given by

$$\frac{d}{dt}\begin{bmatrix} P & C \\ C^* & B \end{bmatrix} = \begin{bmatrix} F & G \\ H & J \end{bmatrix}\begin{bmatrix} P & C \\ C^* & B \end{bmatrix} + \begin{bmatrix} P & C \\ C^* & B \end{bmatrix}\begin{bmatrix} F^* & C \\ C^* & B^* \end{bmatrix}$$
$$+ \begin{bmatrix} Q_x & 0 \\ 0 & Q_u \end{bmatrix} \qquad (3.11)$$

A solution of this differential equation may be obtained by numerical integration. This would entail a considerable amount of computation. Since we are interested only in an approximation to the covariance matrix, it will be sufficient to follow the method outlined in Reference 1 utilizing the partial derivatives of a closed form approximate solution of the equations of motion.

Let equations (3.7) and (3.8) be approximated by

$$\dot{s} = \ell(s, v, t)$$
$$\dot{v} = m(s, v, t) \tag{3.12}$$

where s and v are known in closed form as follows:

$$s = s(s_o, v_o, t)$$
$$v = v(s_o, v_o, t) \tag{3.13}$$

The variation in the state, s, is given by

$$\begin{aligned} \delta s &= \frac{\partial s}{\partial s_o} \delta s_o + \frac{\partial s}{\partial v_o} \delta v_o + \eta(s) \\ &= \Phi\, \delta s_o + U\, \delta v_o + \eta(s) \end{aligned} \tag{3.14}$$

The covariance matrix for the deviation in the state, s, may be given by

$$E(\delta s, \delta s^*) = P = \Phi P_o \Phi^* + U B_o U^* + Q_s \tag{3.15}$$

The covariance matrix, P, will grow in a manner described by equations (3.15). Under such conditions, the covariance matrix, Q_s, of the machine process noise would become a negligible part of P. However, in the normal procedure, observations are included to decrease the uncertainty in the state and the matrix P will then become small again. Under such conditions, the matrix Q_s will act as a lower bound on the covariance matrix P beyond which the uncertainty cannot be reduced even with continued observations.

Since the covariance matrix P is subject to numerical inaccuracy due to the nature of the secular terms in the state transition matrix Φ, it is necessary to carry out the analysis in terms of the variation in the parameters α_i described in Section 2.

$$E(\delta\alpha, \delta\alpha^*) = Q = SPS^* \tag{3.16}$$

The covariance matrix Q may be propagated in a manner similar to P given in Eq. (3.15)

$$Q(t) = \Omega Q(t_0)\Omega^* + SUB_0U^*S^* + SQ_sS^* \tag{3.17}$$

C. Bias Errors in the Observations

In the original work on the modified Kalman filter (Ref. 1), the only errors accounted for in the expected residuals from the observations were those due to errors in the state and the stochastic noise in the observations. The present modification will account for errors in the observations due to biases as well as computing machine process noise. The method outlined here accounting for observation bias is taken from the method described in Reference 7.

Let an observation be given as a function of the state, x, and certain bias errors ν_i

$$y = y(x, \nu_i) \tag{3.18}$$

The linear estimate of the true observation in terms of its nominal predicted value is given by

$$y = y(x) + \sum \frac{\partial y}{\partial x} \frac{\partial x}{\partial \alpha} \delta\alpha + \sum \frac{\partial y}{\partial \nu} \delta\nu + \epsilon(y) + \eta(y) \tag{3.19}$$

The expected value of the observation residual is given by

$$y - y(x) = \delta y = N\delta\alpha + F\,\delta\nu + \epsilon(y) + \eta(y) \tag{3.20}$$

Let the expected value of the various covariances and correlation matrices be given as follows:

$$\begin{aligned}
&E(\delta\alpha, \delta\alpha^*) = Q \quad , \quad && E(\eta(y), \eta(y)^*) = Q_y \\
&E(\delta\nu, \delta\nu^*) = D \quad , \quad && E(\epsilon(y), \epsilon(y)^*) = W \\
&E(\delta\alpha, \delta\nu^*) = C \quad , \quad && E(\delta\alpha, \eta(y)^*) = G \\
&E(\delta\alpha, \epsilon(y)^*) = E(\eta(y), \epsilon(y)^*) = E(\delta\nu, \eta(y)^*) = F(\delta\nu, \epsilon(y)^*) = 0
\end{aligned} \tag{3.21}$$

The covariance matrix of the observation residual is given by

$$\begin{aligned}
E(\delta y, \delta y^*) = Y = NQN^* + FDF^* + W + Q_y + NCF^* \\
+ FC^*N^* + NG + G^*N^*
\end{aligned} \tag{3.22}$$

D. The Kalman Filter With Biases and Machine Noise

Let the correction in the variational parameters be given as a linear function of the residual in the observation as follows:

$$\Delta\alpha = L\,\{y_{obs} - y_{nom}\} \tag{3.23}$$

After making the correction, the expected error in the function is given by

$$\delta\alpha^{+} = \delta\alpha^{-} - L\,\delta y \tag{3.24}$$

The optimal filter, L, is chosen so as to obtain the smallest co-variance matrix of the expected remaining error, $\delta\alpha^{+}$

$$\begin{aligned} E(\delta\alpha^{+},(\delta\alpha^{+})^{*}) &= Q^{+} \\ &= Q - (QN^{*} + CF^{*} + G)L^{*} - L(NQ + FC^{*} + G^{*}) \\ &\quad + LYL^{*} \end{aligned} \tag{3.25}$$

The optimal L is given by

$$L = (QN^{*} + CF^{*} + G)Y^{-1} \tag{3.26}$$

The derived value of the Kalman filter also yields updated values for the covariance matrices Q^{+}, C^{+} and G^{+} following each observation.

$$Q^{+} = Q - L(NQ + FC^{*} + G^{*}) \tag{3.27a}$$

$$C^{+} = C - L(NC + FD) \tag{3.27b}$$

$$G^{+} = G - L(NG + Q_{y}) \tag{3.27c}$$

It is necessary to propagate the various covariance matrices from one observation to the next. Let the observation biases be described as follows:

$$\delta\nu(t) = \theta(t, t_o)\,\delta\nu(t_o) \tag{3.28}$$

where $\theta(t, t_o)$ represents the transition matrix of the $\left(\frac{\partial\nu(t)}{\partial\nu(t_o)}\right)$ for the bias.

The propagation of the variation in the $\alpha_i(t)$ may be obtained from Eq. (3.14) as follows:

$$\delta\alpha(t) = \Omega\,\delta\alpha(t_o) + SU\,\delta\nu(t_o) + S\eta(s) \tag{3.29}$$

The propagation of the covariance matrices $E(\delta\alpha, \delta\nu^*)$ and $E(\delta\alpha, \delta\eta(y)^*)$ between observations is given by

$$\begin{aligned} C(t) &= \Omega(t, t_o)\,C(t_o)\,\theta^*(t, t_o) \\ G(t) &= \Omega(t, t_o)\,G(t_o) \end{aligned} \tag{3.30}$$

4. Finite Rotations

The introduction of the variational parameters necessitates a method of effecting changes in the state x, given a corrected change in the parameters $\Delta\alpha$. A simple procedure would be to take the linear estimate of the correction from the point transformation matrix

$$\Delta x = S(x, \alpha)\, \Delta\alpha \tag{4.1}$$

This will usually suffice for small changes in the parameters. For larger changes, it may be necessary to obtain a nonlinear transformation. This is especially important whenever changes are required which restrict the variation in the energy to a given amount. In such a case, the formula for a rigid rotation which leaves the lengths of the vectors invariant is required rather than the approximate infinitesimal rotation given in the S matrix. This section contains a solution of the change in the state due to a non-infinitesimal change in the α_i.

Let there be three finite changes in α, α_1, α_2, and α_3, and let the original value of the state be $R(t)$ and $\dot{R}(t)$. The new state may be obtained from the old by executing three finite rotations corresponding to $\Delta\alpha_1$, $\Delta\alpha_2$, and $\Delta\alpha_3$ successively as follows:

For $\Delta\alpha_1$

$$\begin{aligned} R_1 &= \frac{\dot{R} \cdot R}{v^2}(1 - \cos \Delta\alpha_1) + \cos \Delta\alpha_1 R + \frac{\sin \Delta\alpha_1}{v} \dot{R} \times R \\ \dot{R}_1 &= \dot{R}(t) \end{aligned} \tag{4.2}$$

For $\Delta\alpha_2$

$$R_2 = R_1$$

$$\dot{R}_2 = \frac{R_1 \cdot \dot{R}_1}{r^2}(1 - \cos \Delta\alpha_2) R_1 + \cos \Delta\alpha_2 \dot{R}_1 + \frac{\sin \Delta\alpha_2}{r} R_1 x \dot{R}_1 \tag{4.3}$$

These two rotations establish the inclination of the new orbit. If $H = R x \dot{R}$, we have for the altered orbit

$$H' = R_2 x \dot{R}_2 \tag{4.4}$$

The rigid rotation corresponding to $\Delta\alpha_3$ is accomplished about this new angular momentum vector, H',

$$R_3 = \cos \Delta\alpha_3 R_2 + \frac{\sin \Delta\alpha_3}{h'} H' x R_2$$

$$\dot{R}_3 = \cos \Delta\alpha_3 \dot{R}_2 + \frac{\sin \Delta\alpha_3}{h'} H' x \dot{R}_2 \tag{4.5}$$

To obtain the altered state due to changes in $\Delta\alpha_4$, $\Delta\alpha_5$, and $\Delta\alpha_6$, we proceed as detailed below.

$$\left(\frac{1}{a}\right)' = \frac{1}{a} + \Delta\alpha_5$$

$$r' = r + \Delta\alpha_6 \tag{4.6}$$

From these, we may solve for the altered speed v',

$$v' = \{\mu[\frac{2}{r'} - (\frac{1}{a})']\}^{1/2} \tag{4.7}$$

There remains the problem of changing the angle between R_3 and $\dot{R}_3$ in the plane perpendicular to H' to accommodate $\Delta\alpha_4$. We will adjust the elevation angle between $R_3(t)$ and $\dot{R}_3(t)$ by rotating $R_3(t)$ rigidly about H', leaving $\dot{R}_3(t)$ unchanged. For such a rotation, the elevation angle, γ, is measured negatively.

$$\begin{aligned} R_4 &= R_3 \cos \Delta\gamma - \frac{H' \times R_3}{h'} \sin \Delta\gamma \\ \dot{R}_4 &= \dot{R}_3 \end{aligned} \tag{4.8}$$

The relationship between $\Delta\alpha_4$ and $\Delta\gamma$ is given by

$$R_4 \cdot \dot{R}_4 - R \cdot \dot{R} = \Delta\alpha_4 \tag{4.9}$$

where

$$R_4 \cdot \dot{R}_4 = r'v' \cos(\gamma + \Delta\gamma)$$

The required values of $\cos \Delta\gamma$ and $\sin \Delta\gamma$ are given by

$$\begin{aligned} \cos \Delta\gamma &= \cos\gamma \cos\gamma' + \sin\gamma' \sin\gamma \\ &= \frac{1}{r v r' v'}[R \cdot \dot{R}(1 + \Delta\alpha_4) + hh'] \\ \sin \Delta\gamma &= \sqrt{1 - \cos^2 \Delta\gamma} \end{aligned} \tag{4.10}$$

The final corrected vectors are adjusted for the proper lengths r' and v' as follows:

$$R_5 = \frac{r'}{r} R_4$$

$$\dot{R}_5 = \frac{v'}{v} \dot{R}_4 \tag{4.11}$$

5. Analytical Partial Derivatives of the Biases

This section contains the analytical partial derivatives for several of the biases which affect either the equations of motion or the observations.

A. The Gravitational Bias

The uncertainty in the determination of the gravitational constants used in the equations of motion of an orbiting vehicle gives rise to a corresponding uncertainty in the determination of the position and velocity of the vehicle. An approximation of the variation in R and $\dot{R}$ due to a variation in μ can be obtained in closed analytical form from the solution of the two body problem. We have

$$U(x, \mu) = \begin{bmatrix} \frac{\partial R}{\partial \mu} \\ \\ \frac{\partial \dot{R}}{\partial \mu} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \mu} & \frac{\partial g}{\partial \mu} \\ \\ \frac{\partial \dot{f}}{\partial \mu} & \frac{\partial \dot{g}}{\partial \mu} \end{bmatrix} \begin{bmatrix} R_o \\ \\ \dot{R}_o \end{bmatrix} \tag{5.1}$$

The expressions for $\frac{\partial f}{\partial \mu}, \frac{\partial \dot{f}}{\partial \mu}, \frac{\partial g}{\partial \mu}, \frac{\partial \dot{g}}{\partial \mu}$ are given below in terms of the $F_i(\alpha)$ series.

$$\sigma = \sqrt{\mu}\,(t - t_o) + \frac{d_o}{\sqrt{\mu}}\,\beta^2 F_3$$

$$\frac{\partial f}{\partial \mu} = \frac{\dot{f}}{2\mu^{3/2}}\,\sigma + \frac{v_o^2}{\mu^2 r}\left[-\frac{1}{2}\beta^4 F_3^{\,2} + \frac{\beta^6}{r_o}\left(\frac{3}{2}F_6 + \frac{3}{2}F_4^{\,2} - \frac{1}{2}F_5 - F_3 F_5\right)\right] \tag{5.2a}$$

$$\frac{\partial g}{\partial \mu} = \frac{(\dot{g} - 1)}{2\mu^{3/2}}\sigma + \frac{\beta^3 F_4}{2\mu^{3/2}} + \frac{v_o^2}{\mu^{5/2}}\Big[r_o \beta^5 (F_5 - \frac{3}{2}F_6 - \frac{1}{2}F_4 F_3)$$

$$- \frac{d_o \beta^6}{\sqrt{\mu}}(\frac{3}{2}F_6 + \frac{3}{2}F_4^2 - \frac{1}{2}F_5 - F_3 F_5)\Big] \qquad (5.2b)$$

$$\frac{\partial \dot{f}}{\partial \mu} = \dot{f}\left(\frac{1}{2\mu} - \frac{R \cdot \frac{\partial R}{\partial \mu}}{r^2}\right) - \frac{F_1}{2\sqrt{\mu}\, r_o r^2}\sigma$$

$$+ \frac{v_o^2}{r_o r^2 \mu^{3/2}}\Big[- \beta^5 (F_5 - \frac{3}{2}F_6 - \frac{1}{2}F_4 F_3) + \frac{d_o \beta^4}{\sqrt{\mu}}\frac{1}{2}F_3^2\Big] \qquad (5.2c)$$

$$\frac{\partial \dot{g}}{\partial \mu} = \frac{\beta^2}{r^3}F_3 R \cdot \frac{\partial R}{\partial \mu} - \frac{\beta F_2}{2\mu r^2}\sigma + \frac{v_o^2}{\mu^2 r^2}\Big[\beta^6 (\frac{3}{2}F_6 + \frac{3}{2}F_4^2 - \frac{1}{2}F_5 - F_3 F_5)$$

$$- \frac{r_o \beta^4}{2}F_3^2\Big] \qquad (5.2d)$$

B. The Station Location Bias

The station location uncertainty is a constant bias in an earth-fixed reference frame. However, in an inertial coordinate system, the bias becomes an oscillatory, time-varying bias. In order to express the varying bias as a constant bias, the time-varying factors may be included in the matrix $F(y, \nu)$, leaving the covariance matrix $D(\nu)$ to account for the constant uncertainty in the station location.

Let G be the right ascension of the station. Furthermore, let the Cartesian components of the station location bias in an earth-fixed system be Δu, Δv, Δw. The partial derivatives of the station location biases with respect to Δu, Δv, Δw are given below for range (ρ), range rate ($\dot{\rho}$), azimuth (A), elevation (E), and the mini-track components ℓ and m.

$$F(\rho, \Delta u_i) = -\frac{1}{\rho}\,[(x-x_s)\cos G - (y-y_s)\sin G,\ (x-x_s)\sin G + (y-y_s)\cos G,$$

$$(z-z_s),\ 0,\ 0,\ 0\,] \tag{5.3}$$

(1 x 6 row vector)

$$F(\dot{\rho}, \Delta u_i) = \Big\{[\frac{\dot{\rho}}{\rho^2}(x-x_s) - \frac{1}{\rho}(\dot{x}+\omega_e y)]\cos G + [\frac{\dot{\rho}}{\rho^2}(y-y_s) - \frac{1}{\rho}(\dot{y}-\omega_e x)]\sin G,$$

$$-[\frac{\dot{\rho}}{\rho^2}(x-x_s) - \frac{1}{\rho}(\dot{x}+\omega_e y)]\sin G + [\frac{\dot{\rho}}{\rho^2}(y-y_s) - \frac{1}{\rho}(\dot{y}-\omega_e x)]\cos G,$$

$$[\frac{\dot{\rho}}{\rho^2}(z-z_s) - \frac{1}{\rho}\dot{z}],\ 0,\ 0,\ 0\Big\} \tag{5.4}$$

(1 x 6 row vector)

$$F(A, \Delta u_i) = \frac{1}{\rho^2 - (z''')^2}\,[-y'''\sin\varphi\cos\theta' - x'''\sin\theta',\ -y'''\sin\varphi\sin\theta' + x'''\cos\theta',$$

$$y'''\cos\varphi,\ 0,\ 0,\ 0\,] \tag{5.5}$$

(1 x 6 row vector)

$$F(E, \Delta u_i) = \frac{1}{\rho^2[\rho^2 - (z''')^2]^{1/2}} [-\rho^2 \cos\varphi \cos\theta' + z'''(x - x_s)\cos G + z'''(y - y_s)\sin G,$$

$$-\rho^2 \cos\varphi \sin\theta' - z'''(x - x_s)\sin G + z'''(y - y_s)\cos G,$$

$$-\rho^2 \sin\varphi + z'''(z - z_s),\ 0,\ 0,\ 0] \tag{5.6}$$

(1 x 6 row vector)

$$F(\ell, \Delta u_i) = \frac{1}{\rho^3} [\rho^2 \sin\varphi \cos\theta' - x'''(x - x_s)\cos G - x'''(y - y_s)\sin G,$$

$$\rho^2 \sin\varphi \sin\theta' + x'''(x - x_s)\sin G - x'''(y - y_s)\cos G,$$

$$-\rho^2 \cos\varphi - x'''(z - z_s),\ 0,\ 0,\ 0] \tag{5.7}$$

(1 x 6 row vector)

$$F(m, \Delta u_i) = \frac{1}{\rho^3} [\rho^2 \sin\theta' + y'''(x - x_s)\cos G + y'''(y - y_s)\sin G,$$

$$-\rho^2 \sin\theta' - y'''(x - x_s)\sin G + y'''(y - y_s)\cos G,$$

$$y'''(z - z_s),\ 0,\ 0,\ 0] \tag{5.8}$$

(1 x 6 row vector)

The observation variables ρ, x''', y''', z''', φ, θ' are defined in Ref. 1.

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