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THEORETICAL AND EXPERIMENTAL RESEARCH ON DIGITAL ADAPTIVE CONTROL SYSTEM

by John Zaborszky, R. G. Marsh, R. E. Janitch, M. R. Chidambara, and E. E. Buder

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EMERSON ELECTRIC CO.
St. Louis, Mo.
for Langley Research Center

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FOREWORD

This research was sponsored by the National Aeronautics and Space Administration, Langley Research Center, under Contract No. NAS1-5127. Mr. Paul Rempfer was the technical monitor for the Research Center.

These studies were conducted by the Electronics and Space Division of the Emerson Electric Co. at St. Louis, Mo. in the period May 26, 1965 through May 25, 1966. The Principal Investigator was Dr. John Zaborszky, Emerson Consultant. Mr. Richard Janitch was the Project Engineer.

Simulation studies were conducted by Emerson personnel using the facilities of the McDonnell Automation Center under subcontract. Mrs. Diane Bitsch of the Emerson TEEM Center contributed in writing some of the simulation programs.

The authors gratefully acknowledge the general contributions of two of Dr. Zaborszky's graduate students, Mr. David M. Ostfeld and Mr. Charles H. Wells, who performed independent research in areas cognate to portions of this study.

This final report summarizes the conclusions and concludes the work on Contract No. NAS1-5127.

ABSTRACT

A new digital adaptive control system is developed for the effective control of a priori unknown plants. Only the desired and actual plant output states are assumed to be measurable. A flyable digital computer of conventional capabilities is the central control agent. The primary control criterion is the minimization of a weighted norm of the output state vector predicted one control interval into the future. Methods appropriate to linear stationary, linear nonstationary, and non-linear plant classes are derived and tested by simulation.

Three alternate methods for the representation of unknown linear stationary plants are investigated. More than 1500 control efficacy simulations of a representative plant spectrum through order nine are analyzed. A primary conclusion is that plant representation based on linear interpolation over measured prior responses is of widest applicability. Control of high order plants to lower than actual order is demonstrated, as is control in the presence of control force saturation. Systematic trends in the experimental data are correlated with analysis.

Two alternate methods for interpolation representation of nonstationary linear plants are investigated. Periodic updating of a linear interpolation representation is recommended on the basis of low order simulatory results.

Non-linear plant representations by second order Volterra series and by interpolation over quadratic forms are developed and compared. Control of several non-linear plants treated as linear is experimentally demonstrated.

An analytic basis for start-up based on the method of matrix pseudo-inversion is determined. Performance criteria and learning procedures are postulated.

TABLE OF CONTENTS

Section	Title	Page

LIST OF ILLUSTRATIONS

Figure	Title	Page

LIST OF TABLES

SECTION 1

INTRODUCTION AND SUMMARY

The research here described is an analytical and experimental investigation of a particular adaptive control concept. The method is conveniently designated DACS (Digital Adaptive Control System).

The approach is characterized by:

> The assumption that the particular plant under control is a priori unknown, except by its membership in one of several broad plant classifications
>
> That the control actions be derived from computation by an on-line digital control computer of conventional capabilities.

Before elaborating the specifics of this research, identification of its context in adaptive control methods is pertinent.

DEFINITION

"An adaptive control system is here defined as a control system which is capable of monitoring its own performance with respect to a given index of performance and modifying its behavior by closed-loop action in such a manner as to optimize the index of performance or approach the optimum condition," (reference 1).

APPLICABILITY

The impetus towards the evolution and use of adaptive systems comes from the existence of a class of control problems which are a priori undescribable by reason of:

Unpredictability - e.g. the unforeseen failure of a component in a space mission

Excessive complexity of description - e.g. certain chemical processes

Analytical intractability - e.g. many problems in fluid dynamics

Extreme variance - e.g. the control of high speed aircraft.

The inadequacy of conventional control systems to these problems is predictable to the extent that conventional design is customized to a postulated a priori description.

1.1 DACS CONCEPT

The following principles are innate to the DACS concept:

The system is to be adaptive in the following sense. It is to permit effective control of a variety of physical plants without a priori knowledge of the usual plant descriptors (pole zero configurations, describing functions, etc.). It is assumed that the only knowledge of the plant under control is what can be inferred from measurements made during the sequence of control actions. *

The primary control agent is an on-line digital control computer of conventional capabilities. The research consists primarily in the determination of analytic methods resulting in reasonably simple algorithms for such computer centered control.

Digital computer control implies a sample-and-hold process. The sampling period is one of two primary DACS parameters. It is designated the "decision interval" and symbolized by T.

Using state space notation, the state vector components are restricted to the plant output variables and their real time derivatives. This choice reflects the data accessability of an unknown plant.

The primary control criterion in the DACS concept is the minimization of a weighted norm of the output error state predicted one decision interval into the future. The second primary DACS parameter controls the relative weighting of error components in the norm. It is designated the "weighting coefficient" and symbolized by h.

---

* While this research has been conducted with the stated objective of unknown plant control, many of the methods are applicable to the more usual practical case of partial and/or inexact plant descriptions. They do not preclude and indeed profit by the use of any available plant descriptions.

The following assumptions have been made in the present studies, but are not necessarily inherent in the concept:

> The single input - single output plant has been exclusively investigated. This is primarily a matter of analytic convenience, and the nonlinear methods can be extended to multivariate control.
>
> A multistate controller has been postulated. No necessity for the continuum of control forces has been established, and selection from a quantized set is not excluded.

DACS FUNCTIONAL FLOW DIAGRAM

Figure 1-1 is a flow diagram illustrating the DACS functional operations. Note that with the exception of control force application and possibly data conversion, all of the indicated functions are performed by an on-line digital control computer.

1.2 RESULTS OF PREVIOUS INVESTIGATIONS

Prior to beginning the current research the DACS control concept had been investigated in some detail, particularly under sponsorship of National Aeronautics and Space Administration Contract No. NASW-599, February 1, 1963 - January 31, 1964 (references 2, 3, 4, 5, and 6). The following summary of major conclusions establishes the background for the current research:

> An equational basis was established for the digital computer control of an unknown plant. The analytic methods were partially empirical, and are substantially those described as "Taylor Prediction" and "Mixed Prediction" in paragraph 2.1.
>
> The control methods were tested by hybrid simulation on a representative set of linear stationary plants containing four or less poles only, and on a few higher order plants. Effective control was established for all plants tested including some unstable plants. The investigations were limited to solution of the regulator problem, although under a wide range of initial conditions.
>
> The correlation of Liapunov stability with conventional criteria of effective plant control was established. The practical importance of this result is that Liapunov stability is relatively

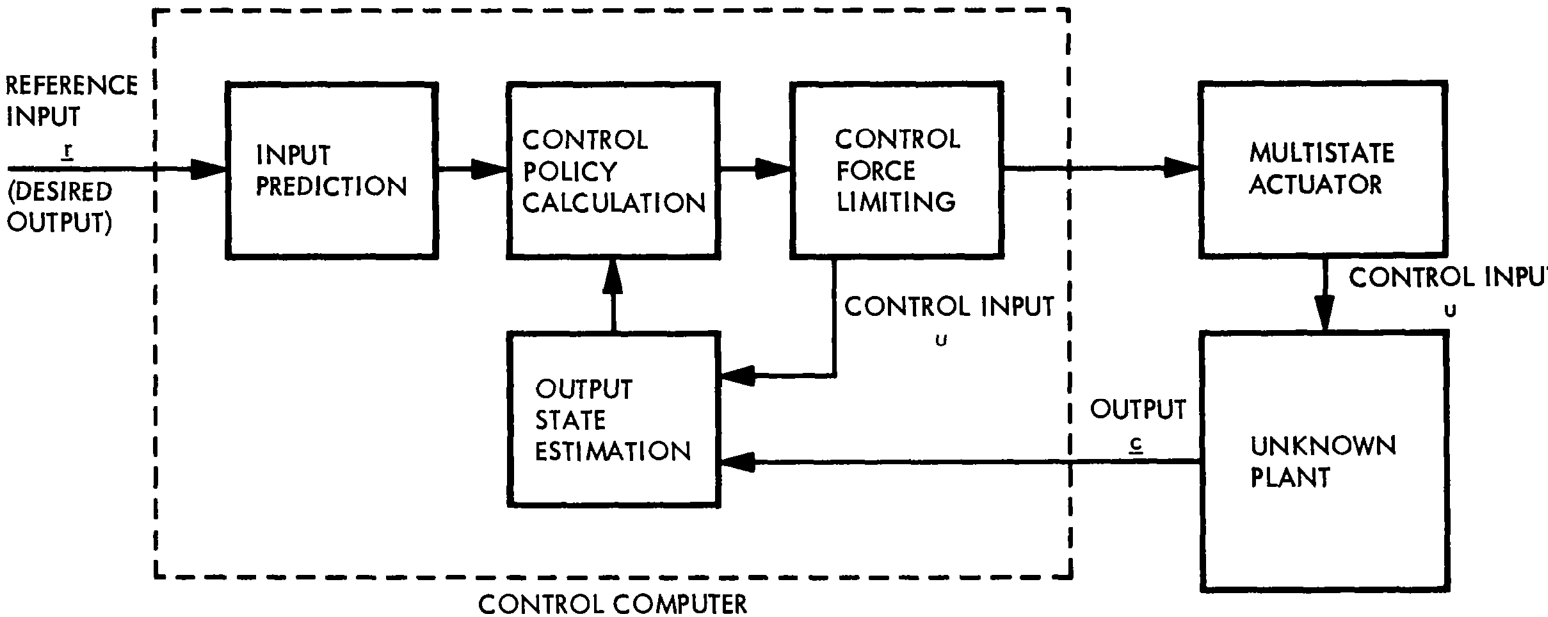


FIGURE 1-1 DACS FUNCTIONAL FLOW DIAGRAM

easy to calculate. Thus, the number of actual control simulations can be minimized.

Adaptivity was established in the sense that a limited common range of the DACS parameters (decision time T and weighting coefficient h) permitted adequate control of most of the plants studied. A learning process for optimization of these parameters appeared possible.

A new theoretical method based on Volterra series representation was developed and is described in references 5 and 7. The investigation showed that the sample-and-hold property characteristic of digital control introduced sufficient regularity into the Volterra description as to permit its application to the unknown plant problem. This development indicated a basis for generalization to non-linear and/or nonstationary plants.

Following the conclusion of Contract NASW-599, Emerson demonstrated control of linear stationary plants with significantly time varying reference inputs by hybrid simulation. In the same period, Dr. John Zaborszky and his colleagues at Washington University developed and tested a method for the representation of unknown plants by interpolation over values of a measured basis vector (references 8 and 9). Again this method is applicable to the control of non-linear and nonstationary plants as well as linear stationary plants.

1.3 OBJECTIVES

At the initiation of the NAS1-5127 research, the following four tasks were made the primary objectives of the research effort.

"Task I - Extend studies of linear plants to plants of order through nine and to linear nonautonomous plants...

Task II - Investigate the application of learning procedures to the startup of the control system to improve initial control...

Task III - Extend the study of nonlinear plants to autonomous and nonautonomous plants which are basically non-linear and/or non-linear due to state variable constraints...

Task IV - Investigate methods for the improvement of the control system..."

Tasks I and III define the objects of study. Tasks II and IV indicate technique investigations for the fulfillment or betterment of the other objectives.

1.4 METHODS OF INVESTIGATION

The methods of investigation were reflexive combinations of:

- Problem identification and definition
- Preliminary theoretical studies
- Reduction of theoretical methods to working forms
- Computer programming and control simulation
- Analysis of simulation results and correlation with theoretical method
- Validation or modification of theoretical method on basis of simulation results.

In a study of this sort, one must be constantly mindful that the methods have ultimate practicality. We have tried to do this by:

- Including practical physical plants in the simulations
- Investigating only those methods with reasonably simple control algorithms
- Attempting control of plants of higher complexity using methods rigorously appropriate only to less complex plants.

At the same time, one should not be overrestrictive of otherwise attractive methods on the basis of here-and-now means of implementation. This is particularly true of computational requirements, where current growth in computer capability suggests some optimism.

SIMULATION METHODS

All simulation results of this research were obtained by digital computation on an IBM 7094 computer. Hybrid simulation was originally proposed for the portion of the studies involving simulation of actual plant control. Late delivery of the intended equipment continued the commitment to pure digital techniques.

An anticipated economic advantage of hybrid computation for these studies may be illusory. In any event, digital simulations actually in excess of the forecast quantity were accomplished within the intended computer budget.

1.5 SUMMARY OF THEORETICAL EXTENSIONS

The primary theoretical extensions made under this contract were:

A method for including plants with pole-zero configurations in a state vector description was devised, compatible with the restriction of the state vector to the plant output components only. This is considered an original contribution to the theory. A summary of this method appears in paragraph 2.1 with detailed development in Appendix A.

The Volterra series representation of unknown non-linear plants of reference 7 has been further developed, and reduced to working equational form for the second order truncation case. A summary appears in Appendix G.

The method of interpolative representation of non-linear plants of reference 8 has been reduced to working equation form in the quadratic case. This study is summarized in paragraph 4.1 with more extensive analysis in Appendix B.

A linear form of the interpolative method was devised, and reduced to practice in a form including pole-zero plant configurations. It is synopsized in paragraph 2.1.

A method for start-up of non-linear plants based on the technique of matrix pseudoinversion was studied. It is described and illustrated in paragraph 5.1 with further analytic treatment in Appendix H.

Two methods of "learning" in the form of DACS parameter * optimization were postulated and given preliminary investigation, as summarized in paragraph 5.2.

Theoretical studies in diagnosis and rectification of several experimentally observed anomalies in the control system performance were made. They are summarized in Appendixes E and F.

---

* The optimized parameters are the DACS decision interval (T), and weighting coefficient (h). The more usual method of plant parameter adjustment should not be inferred.

1.6 SUMMARY OF EXPERIMENTAL RESULTS

The existence of over 150 graphs in this report, many synoptic of extensive data sets, indicates the extent of the experimental investigations. The unusually large amount of experimentally derived data validates conclusions by generality of observation.

LINEAR STATIONARY PLANTS

The greatest bulk of simulation data was obtained on approximately 150 linear plants of orders two through nine. Appendix D identifies the most extensively studied plants, their data origins, and their importance to these investigations.

Prediction Methods.-A primary objective of the experimental investigation was evaluation of the relative efficacy of three alternate methods for the representation of unknown linear stationary plants. Titled by their analytic origins, the underlying principles of each method are respectively:

Taylor Prediction - The predicted free response output state components are assumed to be individually representable by truncated Taylor series. The zero order free response component is represented by a Taylor series truncated at full system order. The corresponding series of higher order response components are successively truncated with linearly diminishing order. The forced response (sensitivity) vector develops as a selected subset of the free response Taylor coefficients.

This representation is analytically exact for poles-only configuration plants with all poles concentrated at the origin. It is a usable approximation for other plants of pole configuration.

Mixed Prediction - This method identically utilizes the Taylor series representation of the free response output state components as just described. However, the forced response (sensitivity) vector is estimated by averaging over the sequence of past control actions.

Again the method is restricted to plants of poles-only configuration. It is an approximation for all such plants.

Interpolation Prediction - This representation is based on the set of linear finite difference equations relating the terminal output state components of a linear stationary plant to its initial output state components and to applied step forcing functions.

Initially the transitions of a sufficient set of consecutive control actions are recorded. Subsequent predicted responses are then obtained by linear interpolation among the measured set.

The method is of general applicability to all linear stationary plants including pole-zero configurations. With exact measured data, it yields an arbitrarily precise approximation to the analytic state transition equation set.

Stability Investigations.-The first experiments were over 260 determinations of Liapunov stability with respect to the DACS (T-h) parameters. The results are summarized in Table 1.1.

The first column identifies one reference method and the three foredescribed practical methods of plant representation. Their salient properties are summarized in the second column. The third column identifies plant configuration.

The interpretation of the fourth column is that all tested plants of the maximum stated order and all lower orders are stable under control with any of certain connected sets of (T-h) points. This result implies a possibility for control without knowledge of plant order.

The data of the fifth column presupposes only that the plant order is known. It bounds a maximal order, such that all tested plants of that order exhibit control stability with any of certain connected sets of (T-h) points.

Paragraph 2.2 gives a discussion of the stability investigation and contains experimental data.

Control Simulations.-With the range of stability established, the major experimentation began in the simulation of the actual control process. The method of Mixed Prediction was discarded on the basis of the results summarized in Table 1.1. This left the following variants to be investigated:

Plant order and pole-zero configurations

Types of response prediction (Taylor, Interpolation)

TABLE 1.1 STABILITY RESULTS

PREDICTIVE METHOD	CHARACTERISTIC PROPERTIES	PLANT CONFIGURATION	MAXIMUM ORDER SHARING COMMON (T - h) BOUNDARY WITH ALL LOWER ORDER PLANTS	MAXIMUM ORDER WITH COMMON (T - h) BOUNDARY OF ALL PLANTS $\in$ ORDER	COMMENTS
Exact	Reference only. Assumes full plant knowledge. Tests control policy.	Poles only Poles-zeroes	6 7	6 7	Includes representative self unstable plants.
Mixed	Unknown plant. Simple computation. Limited averaging over past plant forced response.	Poles only Poles-zeroes	2 2	3 3	Stability of higher order plants poor or totally lacking.
Taylor	Unknown plant. Simplest computation. Completely non-learning.	Poles only Poles-zeroes	6 *	8 *	* denotes no known method of applicability. High order stability often marginal.
Interpolation	Unknown plant. Simple running computation. Independent startup procedure required. Acquires plant knowledge in startup.	Poles only Poles-zeroes	(6) (7)	(6) (7)	() denotes inferred from 'exact' values. Generally closely emulates 'exact' plant description. Known startup procedures. Known learning procedures.

(T - h) values

Reference input function

Constrained variables or control forces

Control with representation to lower than actual plant order.

An obvious prolifigation occurs, and more than 1,500 control simulation runs were made and analyzed.

Some of the most important results of one investigation are described in Table 1.2. The goal was to compare the methods of "Taylor Prediction" and "Interpolation Prediction" for simple reference input functions. Prediction was to actual order, and there were no constraints on the output variables or applied control force inputs.

The following conclusions can be inferred from the experimental studies summarized in Table 1.2:

> The "Interpolation Prediction" method in its present form shows good applicability to all poles-only configurations tested. For the more general case of pole-zero configurations with a pole at the origin, it shows good performance except for a tracking offset. It fails to track in the absence of a pole at the origin.
>
> These experimental results are in accord with theoretical analysis (see Appendix E), and a possibility for improvement exists.
>
> The "Taylor Prediction" in its present form is of limited applicability, due to its restriction to poles-only plant configurations and to limited tracking capabilities.

The foredescribed investigations have some primacy, particularly in that they exhibit innate behavior which conditions the following studies.

Practical Variants.-The remaining experimental studies of linear stationary plants deal with a number of practical variants. Some summary conclusions were:

> Control to Lower than Actual System Order-The stability studies exhibited an expected diminution of the stability boundaries. However, small boundaries of at least marginal stability exist for fourth through eighth order systems alternately assumed to be one less than actual order, or third order.

TABLE 1.2 COMPARISON OF "INTERPOLATION" AND "TAYLOR" PREDICTION RESULTS

VARIANTS				METHOD OF PLANT REPRESENTATION	
PLANT CONFIGURATION	POLE AT 0	ORDER	REFERENCE INPUT	INTERPOLATION PREDICTION	TAYLOR PREDICTION
Poles only	Yes No	1-4	Regulator	Very good convergence	Similar to interpolation
	Yes No	5-7	Regulator	Good convergence	Generally overdamped convergence
	Yes	1-5	Step	Very good convergence	Similar but slower
	No	1-5	Step	Good convergence	Converges to fixed offset
	Yes	1-5	Ramp	Very good following	Follows with fixed offset
	No	1-5	Ramp	Good following	Total divergence
Poles and zeroes	Yes No	1-5	Regulator	Very good convergence	No known applicability
	Yes No	6-7	Regulator	Good convergence	

TABLE 1.2 COMPARISON OF "INTERPOLATION" AND "TAYLOR" PREDICTION RESULTS

VARIANTS				METHOD OF PLANT REPRESENTATION	
PLANT CONFIGURATION	POLE AT O	ORDER	REFERENCE INPUT	INTERPOLATION PREDICTION	TAYLOR PREDICTION
Poles and zeroes	Yes	1-5	Step	Very good convergence	No known applicability
	No	1-5	Step	Converges to fixed offset	
	Yes	1-5	Ramp	Follows with fixed offset	
	No	1-5	Ramp	Total divergence	

Actual control simulations showed that for control to one less than actual order, the performance of low order systems (< 4) shows marked deterioration in the form of slowly oscillatory convergence. In contrast, high order system response is relatively unaffected. High order systems controlled as third order run the gamut from slowly oscillatory convergence to actually improved performance. Perhaps the remarkable fact is that they do converge.

Control Force Saturation-While not strictly in the realm of linear studies, control force saturation is a common practical departure from linearity. In a study of general plant configurations of order three through six, the effects of control force saturation was found to be sensitive to the weighting coefficient, h. Small h values combined with control force limiting produced oscillatory responses and in some cases limit cycles occurred. Conversely, large h values gave rise to an overdamped convergence.

Updating the Interpolation Prediction-In principle, for linear stationary systems full plant knowledge has been acquired at the end of start-up. However, in keeping with the general DACS principle of operating with highly immediate data, a running updating was studied. Purely periodic updating was not found desirable.

LINEAR NONSTATIONARY PLANTS

Approximately 75 control simulations on selected second and third order pole configuration plants were made. In each case all plant coefficients were constant except one, which was made significantly time variant. The Interpolation Prediction method was applied in two forms specifically appropriate to the linear nonstationary case.

One form was the inclusion of the decision interval (T) in the basis of interpolation as representative of explicit time variability. Typically this method converged to an offset for regulator or step reference inputs. It was concluded that this method was not adequate for plants with fast time variation.

The second basic device for incorporating time variation into the linear Interpolation Prediction method is periodic updating of the plant response determination. Experiments using this technique were generally successful in achieving a fast and stable convergence to the desired output state. The requisite frequency of updating predictably increases with the rate of the time varied parameter. Some performance anomalies were observed, and correlated with 'ill conditioning' of the updated matrix of basis vectors. The effect is avoidable.

Finally, a combination of the two previously described methods was found to produce somewhat improved performance over either method applied individually.

NON-LINEAR PLANTS

The non-linear plant experimentation was limited to about 25 simulations conducted on parameter variants of two basic plants. The Van der Pol non-linear oscillator was chosen as a well investigated representative of continuous non-linearities. The other plant was a third order linear system with velocity saturation, which is a common example of non-linearity due to state variable constraint.

These investigations were conducted with the interpolation method linear control policy, based on program availability. Thus, the preferred and more appropriate quadratic interpolative method has not been tested.

It was found possible to convert the free response limit cycle of the Van der Pol system to single overshoot convergence by highly frequent updating of the linear interpolation representation.

This result seemed to imply that precise describability in time could be exchanged for inadequate non-linear representability. Accordingly, the combination of frequent updating with inclusion of the decision interval (T) in the basis was tried. A less successful convergence was observed under this expedient.

Similar but less dramatic control results were obtained for the velocity limited third order plant.

1.7 COMPARISON OF METHODS

From its inception this research has characteristically given rise to a plethora of alternate methods of procedure. At this juncture, the following preferred methods have been established:

LINEAR STATIONARY PLANTS

Originally three alternate methods of plant representation were entertained:

"Mixed Prediction"
"Taylor Prediction"
"Interpolation Prediction".

The experimental stability studies (see Table 1.1) eliminated the "Mixed Prediction" method on the basis of its restriction to low order plants. The experimental actual plant control studies (see Table 1.2) backed by the theoretical analyses of Appendix E eliminated the "Taylor Prediction" method, both by its restriction to plants containing zeroes only and by observed (and apparently innate) poor performance with time varying inputs.

The "Interpolation Method" is preferred, not only by default of the other methods, but also because of its innate flexibility, partially unexploited.

LINEAR NONSTATIONARY PLANTS

Here the choice lies between the first order truncation of the Volterra series method of plant representation and a time variant form of Interpolation representation. The former has never been specifically exhibited in detail. Examination of its origins suggests that the Volterra method has deeper roots in non-linear plant representation than the time variability of linear plants.

Accordingly we have chosen the Interpolation method as primary and have successfully demonstrated its applicability by experimentation.

NON-LINEAR PLANTS

Again the choice is between Volterra series and Interpolation methods, this time in their non-linear forms. Here the applicable second order Volterra representation has been developed as summarized in Appendix G.

The choice is less clean cut, but we have continued to favor the Interpolation representation for reasons stated in paragraph 4.2.

1.8 ORGANIZATION OF REPORT

The following sections of this report have been organized so as to make possible eclectic sampling on the basis of reader interest. Table 1.3 is a guide to such reading.

TABLE 1.3 READER'S GUIDE

AREA OF PRIMARY INTEREST	PERTINENT PARTS OF REPORT
Linear Stationary Plants	Section 2 and Appendixes A, B, C, D, E, and F
Linear Nonstationary Plants	Section 3 and Appendixes A, B, and F
Non-linear Plants	Section 4 and Appendixes B and G
Control Theory	Paragraphs 2.1, 3.1, and 4.1, Section 5, and Appendixes A, B, C, E, F, G, and H
Nonanalytic Survey	Section 1, Paragraphs 2.3, 3.2, and 4.3, and Section 6
Learning Procedures	Section 5 and Appendix H
Computing Requirements	Paragraph 4.2 and Appendix I

REFERENCES

1. Gibson, J. E. and Meditch, J. S., "Adaptive Control Review," Technical Document Report No. ASD-TDR-61-28, Vol. I, September 1965.

2. "Digital Adaptive Control System Research, Quarterly Report No. 1," Emerson Electric Report No. 1544-1, May 10, 1963.

3. "Digital Adaptive Control System Research, Quarterly Report No. 2," Emerson Electric Report No. 1544-2, August 20, 1963.

4. "Digital Adaptive Control System Research, Quarterly Report No. 3," Emerson Electric Report No. 1544-3, November 15, 1963.

5. "Digital Adaptive Control System Research, Quarterly Report No. 4," Emerson Electric Report No. 1544-4, February 21, 1964.

6. "Digital Adaptive Control System Research, Summary Report," Emerson Emerson Electric Report No. 1544-5, March 31, 1964.

7. Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identification," Proceedings of the JACC, 1964, pp. 366-375.

8. Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpolation and Lagrange Expansion for Functionals," Proceedings of the JACC, 1965, pp. 169-177.

9. Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm Through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Mo., 1965.

SECTION 2

CONTROL OF LINEAR STATIONARY PLANTS

2.1 DERIVATION OF PLANT AND SYSTEM EQUATIONS

A study of the control of linear, stationary plants is a convenient starting point for investigating the general feasibility of the control method. Although the assumption of a linear stationary plant may appear to be rather specific and therefore restrictive in the sense that the plant is assumed to be unknown, it seems logical to structure the research effort so as to study the type of plant most amenable to analysis first. The more difficult problems associated with the inclusion of time-varying and non-linear plants in the class of admissible plants are then studied in a logical step by step manner. In this way, the various attributes and shortcomings of the control method may be singled out and modifications and refinements may be made as necessary as the study progresses from the simpler to the more difficult areas of investigation.

METHOD OF ANALYSIS

In order to establish a more concrete basis from which to work, the physical plant is assumed to be describable by a linear differential equation with constant coefficients of the general form indicated in equation 2-1:

$$L(p)\ c(t) = M(p)\ m(t) \qquad (2\text{-}1)$$

The plant is assumed to possess a single input, $m(t)$, and a single output, $c(t)$. $L(p)$ and $M(p)$ are linear differential operators of orders n and m, respectively, where n and m are positive integers with the restriction that n is greater than m.

Although equation 2-1 suffices to describe the type of plant to be considered, it is advantageous to rewrite the mathematical description of the plant in state-space notation:

$$\dot{\underline{x}}(t) = \underline{H}\,\underline{x}(t) + \underline{G}\,\underline{u}(t) \qquad (2\text{-}2)$$

$$\underline{y}(t) = \underline{D}\,\underline{x}(t) \qquad (2\text{-}3)$$

where $\underline{x}$, $\underline{u}$, and $\underline{y}$ are n, p, and q vectors, respectively, and $\underline{H}$, $\underline{G}$, and $\underline{D}$ are constant matrices. Equations 2-2 and 2-3 are termed the dynamical equations of the plant (reference 1). Equation 2-1 and the equation pair 2-2 and 2-3 become equivalent if $\underline{u}(t)$ and $\underline{y}(t)$ are identified as*:

$$\underline{u}'(t) = \lfloor m(t)\ \dot{m}(t) \ldots\ldots \overset{m}{m}(t) \rfloor \qquad (2\text{-}4)$$

$$\underline{y}'(t) = \lfloor c(t)\ \dot{c}(t) \ldots\ldots \overset{n-1}{c}(t) \rfloor \qquad (2\text{-}5)$$

in which case p=m+1 and q=n. The quantity $\underline{x}(t)$ is identified as the state variable of the plant.

Knowledge of the state variable $\underline{x}(t)$ at any instant of time, t_o, specifies the state of the plant at that time. Generally speaking, the state of a system (plant) is the minimal set of numbers (amount of information) given at $t = t_o$ from which, with the knowledge of the input m(t) for $t \geq t_o$, the response of the system is uniquely determined. The state variable is not unique and any minimal set of numbers which span the state space of the system (plant) would suffice. Any choice of state variable, $\underline{x}(t)$, for the plant is relatable to any other choice, $\underline{z}(t)$, by a linear transformation of the form:

$$\underline{z}(t) = \underline{E}\,\underline{x}(t)$$

---

* Vectors will be denoted by small Roman or Greek letters, matrices by Roman or Greek capitals, and transposed vectors and matrices are denoted by primes.

where $\underline{E}$ is a non-singular constant matrix (reference 1). The choice of the state variable is not arbitrary in this study, however, as the plant dynamical equations are assumed to be unknown.

Since the input, $\underline{u}(t)$, and the output, $\underline{y}(t)$, are the only observable quantities, and are therefore the only measurable quantities from which to judge the plant dynamical performance, it is necessary to restrict the matrix $\underline{D}$ which relates the state variable and the plant output to be the identity matrix $\underline{I}$. The dynamic equation pair 2-2 and 2-3 therefore reduces to the single equation:

$$\dot{\underline{x}}(t) = \underline{H}\ \underline{x}(t) + \underline{G}\ \underline{u}(t) \qquad (2\text{-}6)$$

where:

$$\underline{x}'(t) \equiv \underline{y}'(t) = \lfloor c(t)\ \dot{c}(t)\ \ldots\ldots\ \overset{n-1}{c}(t) \rfloor \qquad (2\text{-}7)$$

and the specific forms of $\underline{H}$ and $\underline{G}$ are:

$$\underline{H} = \begin{bmatrix} \ddots & \overset{1}{\ddots} & \\ & {}_{0} \ddots & \ddots \\ -A_0 & -A_1\ \ldots & -A_{n-1} \end{bmatrix} \qquad (2\text{-}8)$$

$$\underline{G} = \begin{bmatrix} \text{———} & 0 & \text{———} \\ & & \\ B_0 & B_1\ \ldots\ldots & B_m \end{bmatrix} \qquad (2\text{-}9)$$

The quantities A_i and B_i are the coefficients of the i^{th} derivatives of the left and right hand sides, respectively, of the plant differential equation 2-1. $\underline{H}$ is a square matrix of order n and $\underline{G}$ is a rectangular matrix with n rows and m+1 columns. In the context of the concepts of observability and controllability, the plant is completely observable because $\underline{D}$ is restricted to be the identity matrix. Also, the plant is completely controllable, providing equation 2-1 is an adequate description of the plant. The latter is true by assumption.

THE STATE EQUATION

The general solution of the plant dynamical equation 2-6 is given by:

$$\underline{x}(t) = \underline{\varphi}(t,t_o,\underline{x}_o) = \underline{F}(t,t_o)\ \underline{x}(0) + \int_{t_o}^{t} \underline{F}(t,\tau)\underline{G}\ \underline{u}(\tau)\ d\tau \qquad (2\text{-}10)$$

where $\underline{F}(t,t_o)$ is the transition matrix of the free differential equation and $\underline{x}(0)$ is the value of the state variable at $t = t_o$. Equation 2-10 is valid for any $t \geqslant t_o$.

Because the control action is effected by an on-line digital computer, the specific control functions (plant inputs) considered are those which are piecewise constant. This makes it desirable to obtain a solution of the plant dynamical equation in a discrete time form where the output is observed only at the instants $t = kT$ (k = positive intergers, $T > 0$), and the input $m(t)$ is constant over the intervals $kT \leqslant t < (k+1)T$, where T is the length of the sampling period. The change from a continuous to a sampling solution presents no problems as a linear stationary system that is completely controllable and completely observable will retain these properties after the introduction of sampling subject to the restriction that the sampling frequency is not an integral multiple of a natural frequency of the plant. A mathematical expression due to Kalman (reference 1) of this restriction is:

$$\text{Re } s_i = \text{Re } s_j \quad \text{implies Im } (s_i - s_j) \neq \frac{q\pi}{T} \qquad (2\text{-}11)$$

where $i, j = 1, 2, \ldots, n$ and q = positive integer. Equation 2-11 means that if two complex pole pairs have equal real parts, the sampling frequency must not be an integral multiple of the difference between the imaginary parts of the pole pairs. If this condition is violated, the phenomenon referred to as "hidden oscillations" may occur in which the sampling process "resonates" with the plant dynamics.

The State Equation in Terms of $t = kT^+$ Initial Conditions.-One form of the discrete solution of the plant dynamical equation is given by (see Appendix A):

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^+) + \underline{a}\ u_k \tag{2-12}$$

Equation 2-12 is the general discrete form of the state equation of the plant in terms of kT^+ initial conditions. It is valid for plants described by differential equations whose corresponding transfer functions, W(s), contain zeroes as well as poles. $\underline{F}$ is the state transition matrix of the plant evaluated for a sampling interval T seconds in length. In the linear stationary case $\underline{F}$ is a constant matrix for constant T. The vector $\underline{a}$ is the forced response vector of the plant and it also is constant for constant T. Strictly speaking, $\underline{F}$ should be written $\underline{F}(T)$ and $\underline{a}$ as $\underline{a}(T)$, however, the argument, T, is dropped for the sake of notational brevity.

The meaning of kT^+ and kT^- is illustrated in Figure 2-1, where kT^+ is the right hand limit as t approaches kT from the right and kT^- is the corresponding left hand limit as t approaches kT from the left. The physical significance of the differences between kT^+ and kT^- is that at kT^- the control force is u_{k-1} and at kT^+ the control force is u_k. If all elements of the state vector $\underline{x}(t)$ are continuous when a discontinuity occurs in the input, no distinction need be made between kT^- and kT^+; however, such a condition is not true in general. This is discussed in detail in Appendix A. Simply stated, $\underline{x}(t)$ will be continuous if the transfer function of the plant contains no zeroes. When zeroes are present in the transfer function, certain elements of the state vector are not continuous. This fact severely limits the usefulness of equation 2-12 and another form of the state equation must be considered.

The General State Equation in Terms of $t = kT°$ Initial Conditions.-As is shown in Figure 2-1, the initial conditions at kT^+ are the value of the elements of the state variable, $\underline{x}(t)$, at the instant the control force u_k is applied. In general, it is necessary to know the magnitude of the control force discontinuity (u_k-u_{k-1}) at $t = kT$ to determine the state $\underline{x}(kT^+)$. As will be shown later in this section when the control policy is discussed,

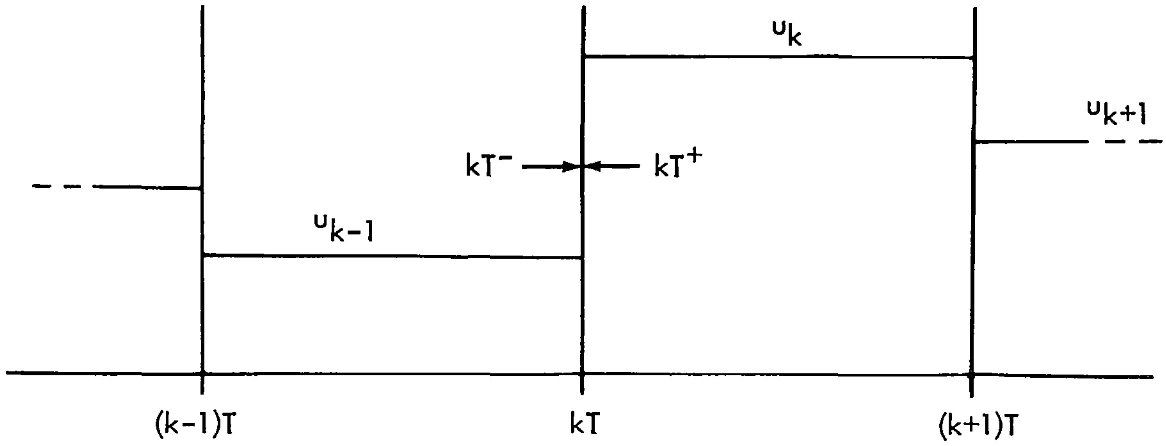


FIGURE 2-1 A SEQUENCE OF CONTROL FORCES ILLUSTRATING DIFFERENCE BETWEEN kT^- AND kT^+

the determination of the control force u_k requires the knowledge of $\underline{x}(kT^+)$ if a state equation of the form of equation 2-12 is used, and as a result an incompatible situation arises. For this reason, a state equation in terms of initial conditions at the ficticious time $kT°$ is developed in Appendix A. The instant at which $kT°$ is considered to exist is depicted in Figure 2-2. Although $kT°$ does not exist physically, it may for the purpose of analysis be considered to be that instant in time just after the removal of control force u_{k-1} but just before the application of control force u_k. A similar time $(k+1)T°$ is defined for the next sampling time and an equation relating these two states is given by equation 2-13:

$$\underline{x}((k+1)T°) = \underline{F}\ \underline{x}(kT°) + \underline{\lambda}\ u_k \qquad (2\text{-}13)$$

The vector $\underline{\lambda}$ is a composite forced response vector which may be interpreted to take into account any discontinuities in the state vector elements due to input discontinuities at the beginning and the end of the sampling interval, as well as the effect of the control force during the interval. When the plant transfer function contains no zeroes there are no discontinuities in the state vector elements in which case $\underline{\lambda} = \underline{a}$ (see Appendix A). The similarity in form of equations 2-12 and 2-13 should be noted. Equation 2-13 is the general form for the state equation used in this study and is valid for plants whose transfer functions contain zeros as well as poles.

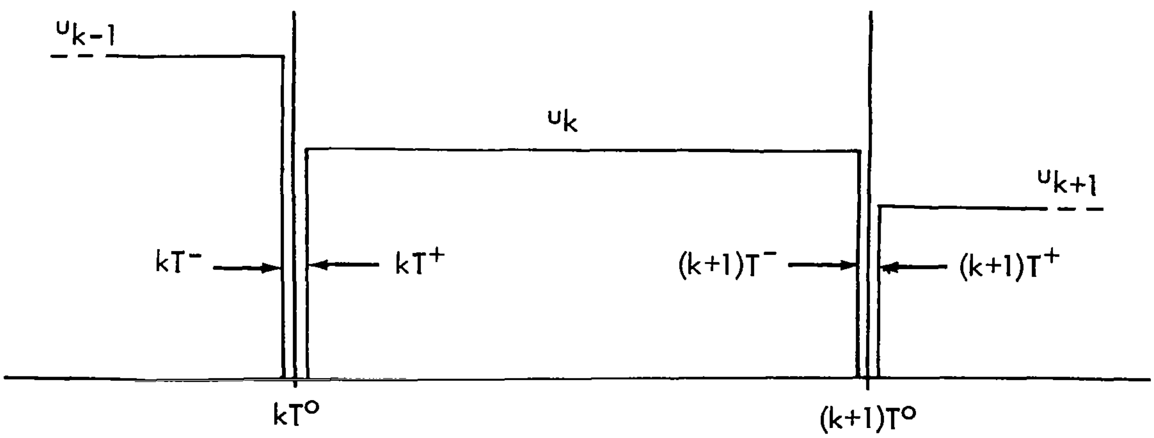


FIGURE 2-2 A SEQUENCE OF CONTROL FORCES ILLUSTRATING THE DEFINITION OF kT^o

<u>A Specific State Equation for the 'Poles Only' Case</u>.-As was pointed out previously in this section, when the plant is describable by a differential equation whose corresponding transfer function, W(s), contains no zeros, the state vector, $\underline{x}(t)$, is continuous when a discontinuity occurs in the input to the plant. For this reason, when the plant transfer function contains poles only:

$$\underline{x}(kT^-) = \underline{x}(kT^\circ) = \underline{x}(kT^+) \tag{2-14}$$

The state equation of the plant becomes simply:

$$\underline{x}((k+1)T) = \underline{F}\ \underline{x}(kT) + \underline{a}\ u_k \tag{2-15}$$

where:

$$\underline{\lambda} \equiv \underline{a} \tag{2-16}$$

<u>A Specific State Equation for the 'Pole-Zero' Case.</u>-Because the state vector, $\underline{x}(t)$, is not continuous when a discontinuity occurs in the input to the plant, the distinction between the time instants just before and just after the control force switching must be retained. A state equation which will be useful when the control law is developed, is derived in Appendix A in terms of physically existent $\underline{x}(t)$ states and is given by equation 2-17:

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{b}_1\ u_k + \underline{b}_2\ u_{k-1} \tag{2-17}$$

The two forced response vectors $\underline{b}_1$ and $\underline{b}_2$ may be interpreted as follows:

> Vector $\underline{b}_1$ is the sensitivity of the plant response to the current control force, u_k, including the effects of the discontinuities caused by applying and removing u_k at the beginning and the end of the interval $kT \leqslant t < (k+1)T$.
>
> Vector, $\underline{b}_2$, is the sensitivity of the plant to the removal of the control force, u_{k-1}, at the beginning of the interval ($t = kT$). If the plant contains no zeroes then $\underline{b}_1 \equiv \underline{a}$ and $\underline{b}_2 = \underline{0}$ (see Appendix A).

The state $\underline{x}((k+1)T^-)$ is expressed in terms of the state $\underline{x}(kT^-)$ in equation 2-17 rather that the state $\underline{x}(kT^+)$ as in equation 2-12. The incompatibility discussed previously in this section in connection with equation 2-12 does not exist when 2-17 is used as the state equation.

<u>THE CONTROL POLICY</u>

The general objective of any control policy could be stated to be to align the actual output or output state of a system with some desired output or output state. Whether this desired output is the actual input to the system or is some function of the input is considered immaterial to this discussion. It will be assumed that the function describing the desired output is analytic on some open interval (t_a, t_b) * except for at most a finite number of discontinuities and that it is accessible for measurement or is known in advance.

Figure 2-3 shows a representative time history segment of the type of control system under study where c(t) is the output of the plant which is being controlled, r(t) is the desired output, and the control force sequence is typical of that set of forces which would be applied to align c(t) with r(t).

The desired output state will be represented as:

$$\underline{r}'(t) = \lfloor r(t) \; \dot{r}(t) \ldots \overset{n-1}{r}(t) \rfloor \tag{2-18}$$

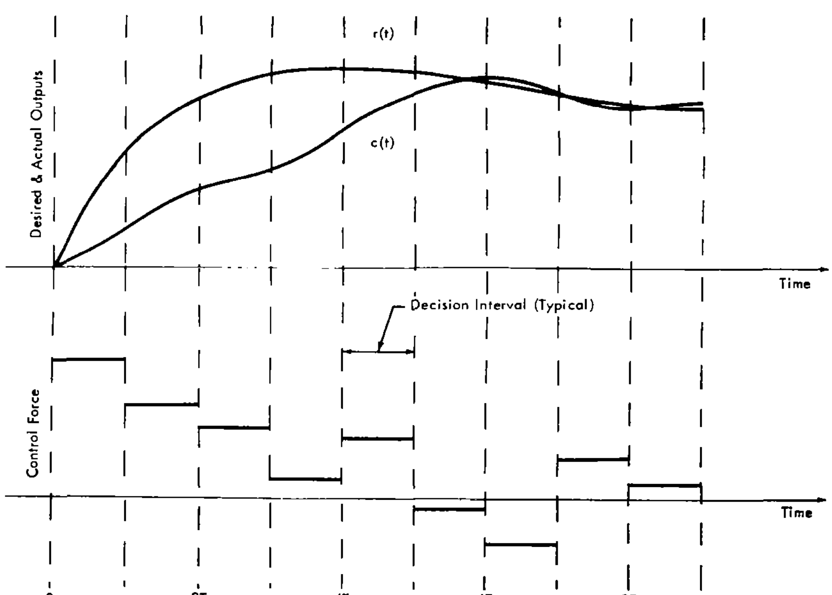


FIGURE 2-3 REPRESENTATIVE TIME HISTORY OF CONTROL SYSTEM

* The open interval (t_a, t_b) will be considered to contain that interval of time during which control of the plant is desired.

where the desired output state vector and the actual output state vector elements possess the same derivative relationship with respect to one another.

An error state vector may then be defined:

$$\underline{e}(t) = \underline{r}(t) - \underline{x}(t) \tag{2-19}$$

The General Control Policy Equation.-The control policy or control law could be identified by a variety of performance criteria, however, due to the assumption of an unknown plant, the following relatively simple criterion will be used (reference 2).

$$\underset{u_k}{\text{Min}} \left[Q_k \right] = \underset{u_k}{\text{Min}} \left[\underline{e}'((k+1)T)\ \underline{K}\ \underline{e}((k+1)T) \right] \tag{2-20}$$

where $\underline{K}$ is a positive definite, symmetric constant matrix. The specific control aim is to select a control force u_k such that the positive definite quadratic form Q_k is minimized. Essentially, this amounts to reducing to a minimum the distance measured in the n-dimensional manifold between the actual and desired states at $t = (k+1)T$. This will be the Euclidean distance if $\underline{K}$ is the unity matrix.

Substituting the general state equation 2-13 into the expression for Q_k, according to the definition equation 2-19, yields an expression for Q_k in terms of u_k:

$$Q_k = \left[\underline{r}'((k+1)T^\circ) - \underline{x}'(kT^\circ)\underline{F}' - \underline{\lambda}' u_k \right] \underline{K} \left[\underline{r}((k+1)T^\circ) - \underline{F}\ \underline{x}(kT^\circ) - \underline{\lambda}\ u_k \right] \tag{2-21}$$

The assumption is made that $\underline{r}(t)$ is continuous at $t = kT^\circ$ and the kT° notation has been included in $\underline{r}(t)$ for the sake of uniformity only. Differentiating Q_k with respect to u_k yields:

$$\frac{dQ_k}{du_k} = -2\ \underline{\lambda}'\ \underline{K} \left[\underline{r}((k+1)T^\circ - \underline{F}\ \underline{x}(kT^\circ) \right] + 2\ \underline{\lambda}'\underline{K}\ \underline{\lambda}\ u_k \tag{2-22}$$

Setting equation 2-22 equal to zero will yield the value of u_k which will minimize the quadratic form Q_k:

$$u_k = \frac{\underline{\lambda}'\underline{K}\left[\underline{r}((k+1)T°) - \underline{F}\ \underline{x}(kT°)\right]}{\underline{\lambda}'\ \underline{K}\ \underline{\lambda}} \qquad (2\text{-}23)$$

The fact that equation 2-23 yields a solution for u_k which will minimize the quadratic form Q_k is deomonstrated by the fact that the second derivative of Q_k with respect to u_k yields a quadratic form which is positive definite:

$$\frac{d^2Qk}{du_k^{\ 2}} = 2\ \underline{\lambda}'\ \underline{K}\ \underline{\lambda} > 0 \qquad (2\text{-}24)$$

Equation 2-23 is the general form of the control policy equation.

The Control Policy Equation Assuming Poles Only.-As was discussed previously, kT° is a ficticious time instant which in general precludes actual measurement of the state vector at that time. It has been shown, however, that when the plant transfer function contains poles only, there need be no distinction made between kT^- and kT° and that $\underline{\lambda} = \underline{a}$. A form of the control policy equation applicable to situations where the differential equation describing the plant possesses no derivatives of the input, m(t), may, therefore, be written as:

$$u_k = \frac{\underline{a}'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}\ \underline{x}(kT)\right]}{\underline{a}'\underline{K}\ \underline{a}} \qquad (2\text{-}25)$$

The Control Policy Equation Assuming A Pole-Zero Configuration.-The time instants kT^-, kT°, and kT^+ must be distinguished when the plant transfer function contains zeroes. In order to place the general control policy equation 2-23 in a form which includes only measurable state vector states, use is made of a relationship derived in Appendix A.

$$\underline{x}(kT°) = \underline{x}(kT^-) + \underline{F}^{-1}\underline{b}_2\ u_{k-1} \qquad (2\text{-}26)$$

Substituting equation 2-26 into equation 2-23 yields as a practical control policy equation for the case where the plant transfer function contains both poles and zeroes:

$$u_k = \frac{\underline{\lambda}' \, \underline{K} \left[\underline{r}((k+1)T) - \underline{F} \, \underline{x}(kT^-) - \underline{b}_2 \, u_{k-1} \right]}{\underline{\lambda}' \, \underline{K} \, \underline{\lambda}} \tag{2-27}$$

<u>The Weighting Matrix.</u>-The constant matrix $\underline{K}$ introduced in the quadratic form Q_k performs the function of a weighting function on the various state variable components. The particular form used in this study is that of a diagonal matrix as is shown in equation 2-28:

$$\underline{K} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & (hT)^{2i} & & \\ & & & \ddots & \\ & & & & (hT)^{2n} \end{bmatrix} \tag{2-28}$$

The matrix $\underline{K}$ is obviously symmetric and is positive definite because the discriminant and all the principle minors are greater than zero.

The matrix has as its basis previous Emerson studies (reference 3) and is largely the result of empirical observations. It does provide a large region of stable operation, particularly for plants of fifth and lower order which seems to justify its use.

The inclusion of T, the sampling interval in seconds, in the weighting matrix has the effect of normalizing the time scale to unit decision interval length because of the derivative relationship between the elements of the state vector. The effect of h is to emphasize or de-emphasize the higher state variable components depending upon whether h is greater than or less than one. It would be expected that for a given value of T, large values of h (one or greater) would tend to make the system convergence somewhat sluggish as more of the state variable components are being controlled. In contrast, for small values of h (less than one) where the higher order state variable components are progressively de-emphasized, it would intuitively appear that the control action would be somewhat faster as fewer of the state variable components are closely controlled. It remains for the experimentation to show the exact effects of the variation of T and h and their product.

Some Remarks.-All of the equational development so far has been formulated in terms of exact quantities. Specifically, the general state transition equation 2-13 with its two particular cases equations 2-15 and 2-17, and the general control policy equation 2-23 with its two particular cases, equations 2-25 and 2-27, are written in terms of the exact transition matrix, $\underline{F}$, and the exact force sensitivity vectors $\underline{\lambda}$, $\underline{a}$, $\underline{b}_1$ and $\underline{b}_2$. By assumption, however, the plant is unknown which would preclude exact knowledge of any of these quantaties. Obviously, some technique or set of techniques must be used to obtain estimates of these quantaties. A search for and study of various estimating techniques constitutes one of the major goals of this research. The exact equations do provide a basis for comparison and an intuitive deduction would be that exact knowledge of the plant would yield the 'best' control in any given situation.

ESTIMATION AND PREDICTION

The response of the plant may be broken into two parts. The first part may be termed the free response, or that response which would occur in the absence of any control input. The second part may be termed the forced response, or that part of the response which is due to the control input. In the case of a linear system, the two parts may be considered separately as the principle of superposition applies. As will be seen later when non-linear systems are considered, such a division is possible in the non-linear situation also. However, superposition of course does not apply, and the two parts are not independent of one another.

To be more specific, the general state equation 2-13 may be written in the form:

$$\underline{x}((k+1)T^\circ) = \underline{x}_k((k+1)T^\circ) + \underline{\lambda}\, u_k \tag{2-29}$$

The vector $\underline{x}_k((k+1)T°)$ represents the free response of the plant over the interval $kT° < t \leqslant (k+1)T°$ and in the absence of the control force, u_k, $\underline{x}_k((k+1)T°)$ would be the plant response at $t = (k+1)T°$. The forced response term, $\underline{\lambda}\ u_k$, may then be considered to be a "correction" which when added to the free response, aligns the total response at $(k+1)T°$ so as to be in closer agreement with the desired state $\underline{r}((k+1)T°)$.

Figure 2-4 depicts a representative situation for one of the state variable components of each of the three vectors, $\underline{x}(t)$, $\underline{x}_k(t)$, and $\underline{r}(t)$ during one sampling (decision) interval. The amount of "correction" during the sampling interval $kT° < t \leqslant (k+1)T°$ obviously depends upon the magnitude of the control force u_k. The purpose of the control policy is to determine u_k in such a way as to effect closer alignment between the state variable components $x_i((k+1)T°)$ and $r_i((k+1)T°)$ through the artifice of minimizing the quadratic form Q_k.

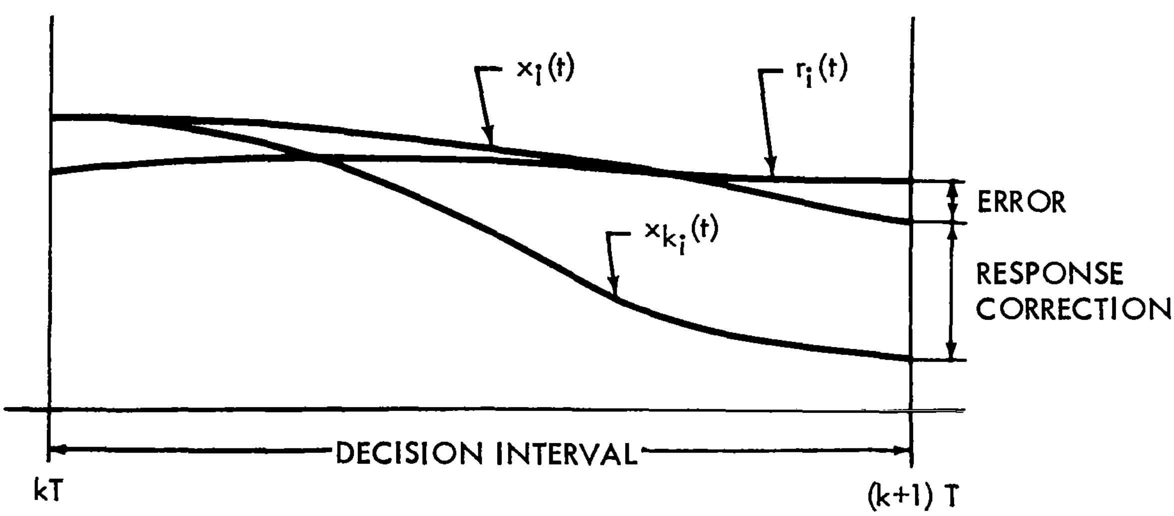


FIGURE 2-4 CORRECTIVE ACTION OF CONTROL FORCE DURING A DECISION INTERVAL

Two things must be known in order to calculate the control force u_k:

The free response $\underline{x}_k((k+1)T°)$

The current sensitivity of the plant to a control force, $\underline{\lambda}$.

The value of these quantities may be ephemeral in nature if the plant is time-varying, or constant if the plant is stationary. Considering the time sequence of events, $\underline{x}_k((k+1)T°)$ is not a measurable state at the time u_k must be calculated. Thus, estimates must be obtained for both $\underline{x}_k((k+1)T°)$ and $\underline{\lambda}$. In the case of $\underline{x}_k((k+1)T°)$, this amounts to a prediction problem as the value of $\underline{x}_k((k+1)T°)$ must be "predicted" on the basis of state variable measurements made earlier in the time history of the control process.
In the context of this study, prediction of $\underline{x}_k((k+1)T°)$ amounts to estimation of the transition matrix $\underline{F}$ and measurement of the state $\underline{x}(kT°)$. In this light, the state and control policy equations may be written in the form:

$$\underline{x}_e((k+1)T°) = \underline{F}_e\underline{x}(kT°) + \underline{\lambda}_e\, u_k \qquad (2\text{-}30)$$

$$u_k = \frac{\underline{\lambda}'_e\, \underline{K}\left[\underline{r}((k+1)T°) - \underline{F}_e\, \underline{x}(kT°)\right]}{\underline{\lambda}'_e\, \underline{K}\, \underline{\lambda}_e} \qquad (2\text{-}31)$$

where the subscript e deonotes an estimated value.

Exact Prediction-Reference Standard.-In the event the plant is known, the estimates of $\underline{F}$ and $\underline{\lambda}$ reduce to exact values and prediction is exact. This situation is termed "Exact Prediction" and serves the purpose of a reference or standard against which estimation techniques may be compared.

Taylor Predicition.-One type of prediction studied is termed "Taylor Prediction" because a truncated Taylor series is used to estimate the transition matrix $\underline{F}$. The form of the Taylor estimate $\underline{F}_T$ is given in equation 2-32:

$$\underline{F}_T = \begin{bmatrix} 1 & T & \frac{T^2}{2} & \cdots & \frac{T^{(p-1)}}{(p-1)!} \\ 0 & 1 & T & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \end{bmatrix} \qquad (2\text{-}32)$$

The order of the plant is assumed to be p which may be less than or equal to n, the actual order of the plant. The length of the sampling or decision interval is assumed to be T seconds. $\underline{F}_T$ will be identical to $\underline{F}$ for the case where the plant transfer function consists of a p^{th} order pole at the origin. $\underline{F}_T$ would then appear to be a fair approximation of those dynamic modes close to the origin of the complex plane which are the relatively more important modes to control. Study of Taylor Prediction dates back to earlier Emerson studies (reference 3) when transfer functions containing poles only were considered. The corresponding sensitivity vector analogously estimated is $\underline{a}$. The Taylor estimate of $\underline{a}$ is given by:

$$\underline{a}_T' = \left\lfloor \frac{T^p}{p!} \quad \frac{T^{(p-1)}}{(p-1)!} \quad \cdot \cdot \cdot \cdot \quad T \right\rfloor \tag{2-33}$$

The justification of the form of $\underline{a}_T$ follows from the integral relationship between the last column of $\underline{F}$ and $\underline{a}$.

Specifically, if the last column of $\underline{F}(t)$ is given by $\underline{f}(t)$, then:

$$\underline{a}(t) = \int_0^t \underline{f}(\tau)\, d\tau \tag{2-34}$$

where the gain of the transfer function is assumed to be unity. One of the great virtues of using $\underline{F}_T$ and $\underline{a}_T$ as estimates of $\underline{F}$ and $\underline{a}$, respectively, is that a minimal amount of knowledge is assumed about the plant. Also, because of the simplicity of the forms of $\underline{F}_T$ and $\underline{a}_T$, implementation is extremely simple. The "working equations" employing Taylor prediction are summarized in Table 2.1.

Mixed Prediction.-A second type of predicition studied is termed Mixed Prediction. The term "mixed" comes about from the fact that $\underline{F}_T$ as defined by equation 2-32 is used as an estimate of $\underline{F}$, however, $\underline{a}$ is estimated by a suitable averaging process, an example of which is given in equation 2-35:

TABLE 2.1 SUMMARY OF WORKING EQUATIONS

TYPE OF PREDICTION	STATE EQUATION
Exact	$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{b}_1 u_k + \underline{b}_2 u_{k-1}$
Taylor	$\underline{x}((k+1)T^-) = \underline{F}_T\ \underline{x}(kT^-) + \underline{a}_T u_k$
Mixed	$\underline{x}((k+1)T^-) = \underline{F}_T\ \underline{x}(kT^-) + \underline{a}_A u_k$
Interpolation	$\underline{x}((k+1)T^-) = \underline{F}_I\ \underline{x}(kT^-) + \underline{b}_{1_I} u_k + \underline{b}_{2_I} u_{k-1}$

TYPE OF PREDICTION	CONTROL EQUATION
Exact	$u_k = \dfrac{\underline{\lambda}'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}\ \underline{x}(kT^-) - \underline{b}_2 u_{k-1}\right]}{\underline{\lambda}'\ \underline{K}\ \underline{\lambda}}$
Taylor	$u_k = \dfrac{\underline{a}_T'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}_T\ \underline{x}(kT^-)\right]}{\underline{a}_T'\ \underline{K}\ \underline{a}_T}$
Mixed	$u_k = \dfrac{\underline{a}_A'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}_T\ \underline{x}(kT^-)\right]}{\underline{a}'_A\ \underline{K}\ \underline{a}_A}$
Interpolation	$u_k = \dfrac{\underline{\lambda}_I'\ \underline{K}\left[\underline{r}((k+1)T) - \underline{F}_I\ \underline{x}(kT^-) - \underline{b}_{2_I} u_{k-1}\right]}{\underline{\lambda}'_I\ \underline{K}\ \underline{\lambda}_I}$

TYPE OF PREDICTION	STABILITY EQUATION
Exact	$\underline{D} = \underline{F} - \dfrac{\underline{\lambda}\ \underline{\lambda}'\ \underline{K}}{\underline{\lambda}'\ \underline{K}\ \underline{\lambda}}\ \underline{F}$
Taylor	$\underline{D} = \underline{F} - \dfrac{\underline{\lambda}\ \underline{a}'_T\ \underline{K}}{\underline{a}'_T\ \underline{K}\ \underline{a}_T}\ \underline{F}_T$
Mixed	$D = \underline{F} - \dfrac{\underline{\lambda}\ \underline{a}'\ \underline{K}}{\underline{a}'\ \underline{K}\ \underline{a}}\ F_T$ (NOTE: For Stability Study $\underline{a}_A = \underline{a}$ is assumed)
Interpolation	(Exact Equation Used)

$$\underline{a}_A = \frac{\displaystyle\sum_{i=k-j}^{k} \left|\underline{x}(iT) - \underline{F}_T\underline{x}((i-1)T)\right|}{\displaystyle\sum_{i=k-j-1}^{k-1} \left|u_i\right|} \tag{2-35}$$

where j is the number of past decision intervals used in estimating $\underline{a}$. If the plant is stationary, $\underline{a}$ will be a constant vector and in the time varying case $\underline{a}$ will remain relatively constant over a series of consecutive intervals if the plant is slowly time varying. Under these circumstances, the averaging process should yield a fairly accurate estimate of $\underline{a}$. The "working equations" employing Mixed Prediction are summarized in Table 2.1.

Interpolation Prediction.-A third type of prediction studied is termed Interpolation Prediction (references 4 and 5). For a general discussion of the interpolation procedure and derivation of the basic interpolation equations, refer to Appendix B, as only that part applicable to the linear case is considered here. The interpolation method is the most generally applicable of the prediction methods studied. The first two methods discussed are tacitly restricted to the "poles-only" case. However interpolation may be employed to obtain estimates of $\underline{a}$ or $\underline{b}_1$ and $\underline{b}_2$, without a priori knowledge of whether the transfer function representing the plant contains zeroes or not.

Specifically, in the linear stationary case the vector of base functionals is chosen to be:

$$\underline{\phi}'\left[u, \underline{\eta}\right] = \lfloor \underline{\eta} \quad u \rfloor \tag{2-36}$$

where, in the notation of this section, $\underline{\phi}_k$ is given by:

$$\underline{\phi}_k' = \lfloor \underline{x}'(kT^\circ) \quad u_k \quad u_{k-1} \rfloor \tag{2-37}$$

The matrix of vector base functions, $\underline{\Phi}$, then consists of an appropriate set of the linear $\underline{\phi}$'s which need not be consecutive.

Defining a partioned matrix $\underline{B}$ derived from the base functionals:

$$\underline{B} = {}_D\underline{X}\,\underline{\Phi}^{-1} = \left[\underline{F}_I \;\vdots\; \underline{b}_{1_I} \;\vdots\; \underline{b}_{2_I} \right] \qquad (2\text{-}38)$$

where $\underline{F}_I$ is a p^{th} order square matrix and $\underline{b}_{1_I}$ and $\underline{b}_{2_I}$ are p vectors. This particular form yields as an estimate of $\underline{x}((k+1)T^-)$:

$$\underline{x}((k+1)T^-) = \underline{F}_I\underline{x}(kT^-) + \underline{b}_{1_I}u_k + \underline{b}_{2_I}u_{k-1} \qquad (2\text{-}39)$$

Equation 2-39 is identical in form to the state equation 2-17 which is repeated here for comparison purposes:

$$\underline{x}((k+1)T^-) = \underline{F}\,\underline{x}(kT^-) + \underline{b}_1 u_k + \underline{b}_2 u_{k-1} \qquad (2\text{-}17)$$

If measurement of the entire state vector is possible, then it would be expected that $\underline{F}_I \longrightarrow \underline{F}$, $\underline{b}_{1_I} \longrightarrow \underline{b}_1$, and $\underline{b}_{2_I} \longrightarrow \underline{b}_2$. It is important to point out that the interpolation procedure does not require measurement of the full state vector and therefore is not inherently a plant identification procedure. Whether the plant is sensitive to derivatives of the input will be reflected in the relative values of the sensitivity vectors $\underline{b}_{1_I}$ and $\underline{b}_{2_I}$. It would be expected that, if full state variable measurement is possible, the case of the plant whose transfer function contains no zeroes would result in $\underline{b}_{1_I} \longrightarrow \underline{a}$ and $\underline{b}_{2_I} \longrightarrow \underline{0}$. In this case, equation 2-39 would very closely approximate the state equation 2-15 which is repeated here also for comparison purposes:

$$\underline{x}((k+1)T) = \underline{F}\,\underline{x}(kT) + \underline{a}\,u_k \qquad (2\text{-}15)$$

A summary of the "working equations" employing Interpolation Prediction is given in Table 2.1.

STABILITY OF THE CONTROL POLICY

The basic purpose of the stability study is to place some bounds on the number of control simulation runs that need be made and, also, to provide a guide as to what particular runs will yield the most significant data. For a given situation, the plant must be assumed known in order to evaluate conditions under which stable operation may be achieved using various types of prediction and estimating techniques.

The particular method of control proposed in this study involves a selection of the sampling interval length, T, and the weighting factor, h, which together form the weighting matrix $\underline{K}$. The proper control of an unknown plant using this concept requires that these parameters be selected in a satisfactory manner. A stability study can be made using various T-h value combinations for individual plants, and a region can be established for each plant where stable operation of the control policy is achievable. It should be possible to select T and h so that performance will be relatively invariant for a wide variety of plants.

The particular Liapunov function utilized is formulated in Appendix C where a positive definite quadratic form is proposed. The function expressing the change in the error norm derived in general in Appendix C becomes in the linear stationary case considered in this section:

$$E_n = \left[\underline{x}'((k+1)T^\circ)\right] \underline{N} \left[\underline{x}((k+1)T^\circ)\right] - \left[\underline{x}'(kT^\circ)\right] \underline{N} \left[\underline{x}(kT^\circ)\right] \qquad (2\text{-}40)$$

Substituting the state equation 2-13 into equation 2-40 yields:

$$E_n = \left[\underline{x}'(kT^\circ)\underline{F}' + u_k\underline{\lambda}'\right] \underline{N} \left[\underline{F}\,\underline{x}(kT^\circ) + \underline{\lambda}\,u_k\right] - \left[\underline{x}'(kT^\circ)\right] \underline{N} \left[\underline{x}(kT^\circ)\right] \qquad (2\text{-}41)$$

Substituting the control policy equation 2-31, which is expressed in terms of estimates of $\underline{F}$ and $\underline{\lambda}$, into equation 2-41, assuming the desired state to be $\underline{0}$, and collecting terms yields:

$$E_n = \underline{x}'(kT^\circ) \left[\underline{F}' - \frac{\underline{F}_e'\underline{K}'\underline{\lambda}_e\,\underline{\lambda}'}{\underline{\lambda}_e\,\underline{K}'\underline{\lambda}_e'}\right] \underline{N} \left[\underline{F} - \frac{\underline{\lambda}\,\underline{\lambda}_e'\,\underline{K}\,\underline{F}_e}{\underline{\lambda}_e'\,\underline{K}\,\underline{\lambda}_e}\right] \underline{x}(kT^\circ) - \underline{x}'(kT^\circ)\,\underline{N}\,\underline{x}(kT^\circ) \qquad (2\text{-}42)$$

Making the definition:

$$\underline{D} = \underline{F} - \frac{\underline{\lambda}\,\underline{\lambda}_e'\,\underline{K}\,\underline{F}_e}{\underline{\lambda}_e'\,\underline{K}\,\underline{\lambda}_e} \qquad (2\text{-}43)$$

E_n may be expressed in form:

$$E_n = - \underline{x}'(kT°)\left[\underline{N} - \underline{D}'\underline{N}\,\underline{D}\right] \underline{x}(kT°) \tag{2-44}$$

Equation 2-44 is of the form:

$$E_n = - \underline{x}'(kT°)\ \underline{M}\ \underline{x}(kT°) \tag{2-45}$$

where if $\underline{M}$ is positive definite, global asymptotic stability is assured. Assuming $\underline{N}$ is chosen to be positive definite, then $\underline{M}$ will be positive definite if the eigenvalues of the matrix $\underline{D}$ all have absolute values less than unity. The stability study then resolves down to a determination of the eigenvalues of:

$$\underline{D} = \underline{F} - \frac{\underline{\lambda}\ \underline{\lambda}'_e\ \underline{K}\ \underline{F}_e}{\underline{\lambda}'_e\ \underline{K}\ \underline{\lambda}_e} \tag{2-46}$$

A summary of the specific equation for different types of prediction is given in Table 2.1.

2.2 EXPERIMENTAL STUDIES

This section presents the experimental response characteristics of a representative set of linear stationary plants controlled by the DACS Control Policy. Various aspects of DACS control using the three alternate types of prediction introduced in paragraph 2.1 are presented graphically to show certain limitations as well as advantages of each type of prediction.

The objectives of this experimental program were:

To determine the control performance of our control system on a representative set of plants of order through nine for several types of prediction put forth in this and previous DACS studies.

To determine the control performance of our control system on a representative set of plants controlled with less observed state variables than the true plant order for several types of prediction put forth in this and previous DACS studies.

To verify experimentally DACS control on a broad spectrum of plants of pole-zero configuration.

To investigate the effect of the weighting factor (h) on our control system response.

To investigate the effect of control force saturation on our control system response.

To investigate the effectiveness of updating with the Interpolation Prediction method on a selected number of plants of order through five.

These objectives are considered along with the appropriate experimental procedures in the following paragraphs.

LINEAR STATIONARY PLANTS

In order to accomplish these objectives, it was necessary to assemble a representative set of linear stationary plants of order through nine. Such a set of plants was assembled by utilizing all previous DACS research to select a number of low order plants as an experimental starting point, and by selecting several references (references 3, 6, and 7) to extend the set of plants to higher order. The resultant set consisted of approximately a hundred and fifty transfer functions of pole and pole-zero configuration. This large set was used for the stability investigation phase, but was considered to be too large for control simulation studies. Therefore, two smaller subsets, one of pole and the other of pole-zero configuratons, were selected for all further simulation studies. The plants composing these two subsets along with a brief discussion are listed in Appendix D.

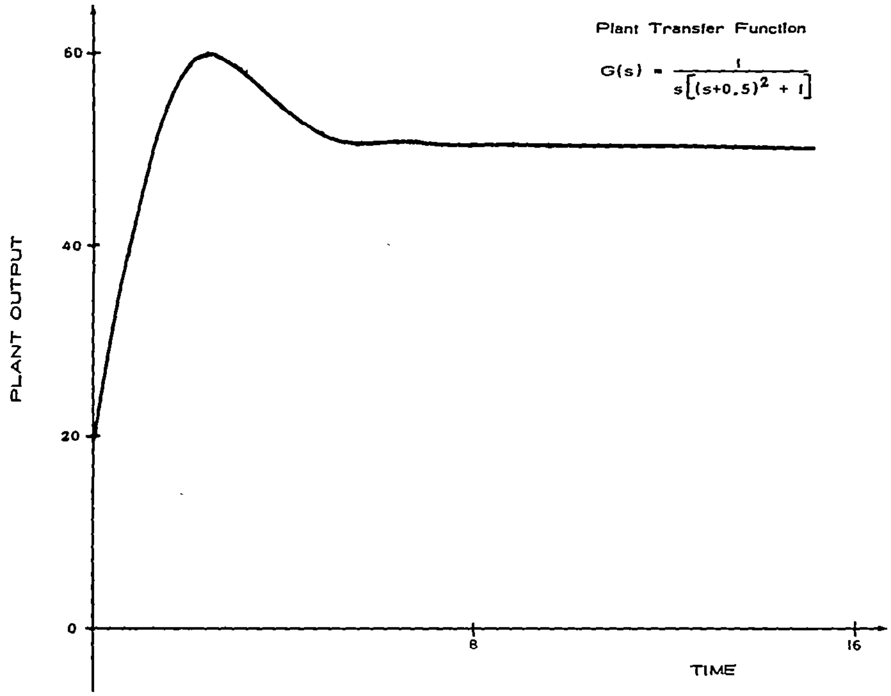
It should be noted that the plant transfer functions consist of three basic types:

Plants which contain an integration

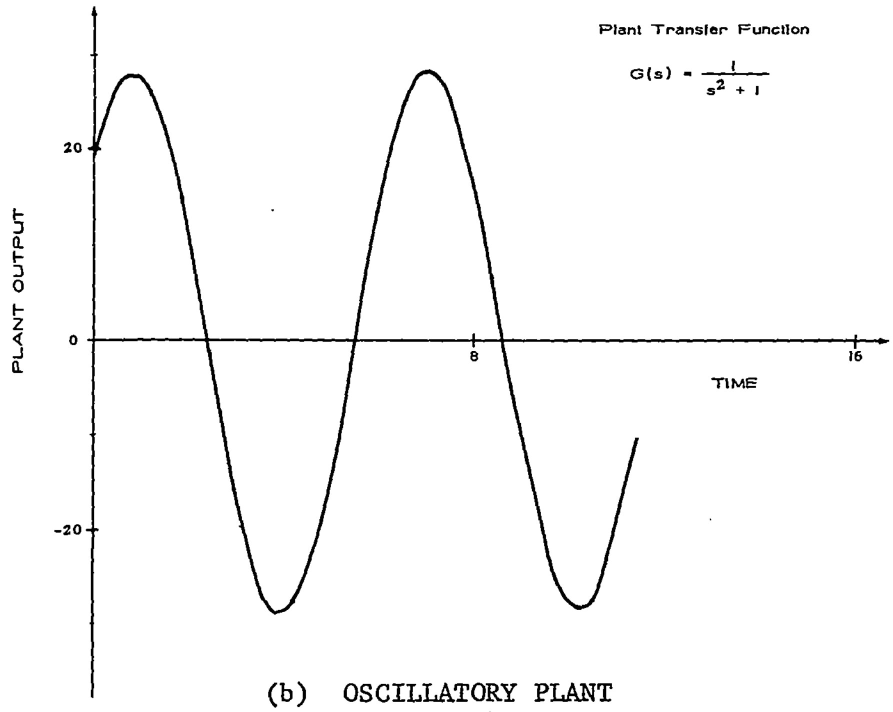
Plants which are oscillatory

Plants which do not contain an integration

The typical free (uncontrolled) response of each type of transfer function is illustrated in Figure 2-5. Also several unstable plants of order through five were included in the control simulation studies.

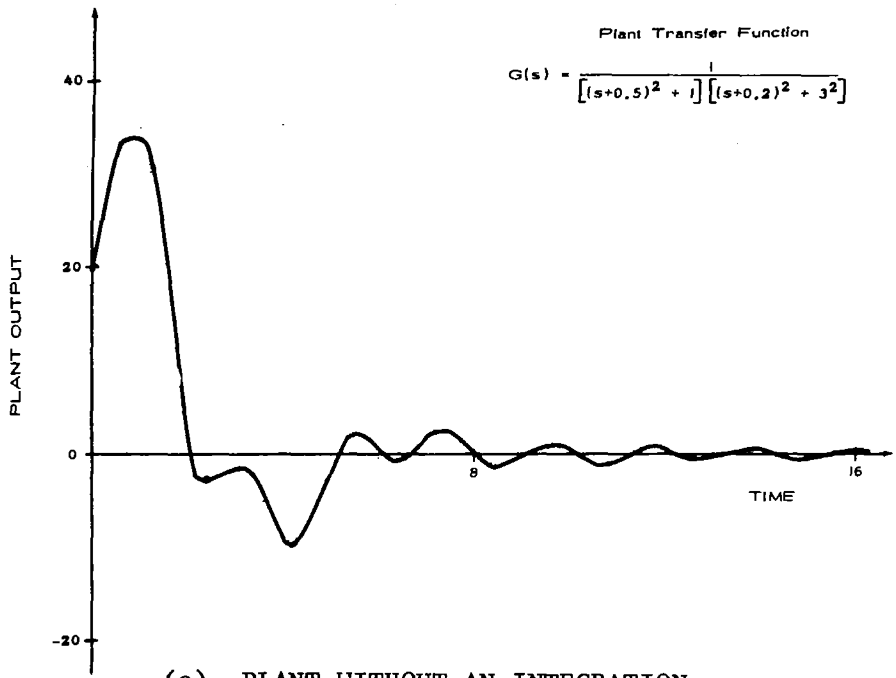


(a) PLANT WITH AN INTEGRATION



(b) OSCILLATORY PLANT

FIGURE 2-5 FREE RESPONSES OF THREE TYPES OF PLANTS



(c) PLANT WITHOUT AN INTEGRATION

FIGURE 2-5 FREE RESPONSES OF THREE TYPES OF PLANTS (Cont'd.)

STABILITY BOUNDARY RESULTS

As has been noted, the decision interval (T) and the weighting factor (h) are the two design parameters of our control system. Previous DACS research has shown the close correspondence between analytical T-h Stability boundaries established by Liapunov's second method, and those obtained by experimental studies. Since this relationship was previously established, the analytical T-h stability boundaries provided a very convenient method of determining the stability boundaries for the system controlling each plant of the representative set. Also, of importance is the fact that the stability boundaries could be established by this method for any desired type of prediction; i.e. Exact, Mixed, Taylor, and Interpolation Prediction.

Exact Prediction - Pole Configurations.-Exact Prediction T-h stability boundaries were obtained for about sixty plants of second through ninth order. The system stability boundaries for control of all the second order plants consisted of the entire T-h plane examined. Figure 2-6 shows the comparison between the common third order system stability boundary and the common fourth order system stability boundary. The common stability boundary of a particular order system is that set of points in the T-h plane common to all plants of the representative set for that order. The stable region is indicated in Figure 2-6 where the common boundary for third order systems is open to the right for large T-h values. This should be interpreted to mean that if a boundary to the right exists, it is at larger T-h values than those shown on the graph. The fourth order boundary is closed on all sides except the top where, if the graph were extended to include larger values of T, the boundary would probably close at the top. The stable region in the stability graphs that follow may be interpreted in a similar manner where, in general, the stable region lies to the right of the boundary as it is traversed in a clockwise direction.

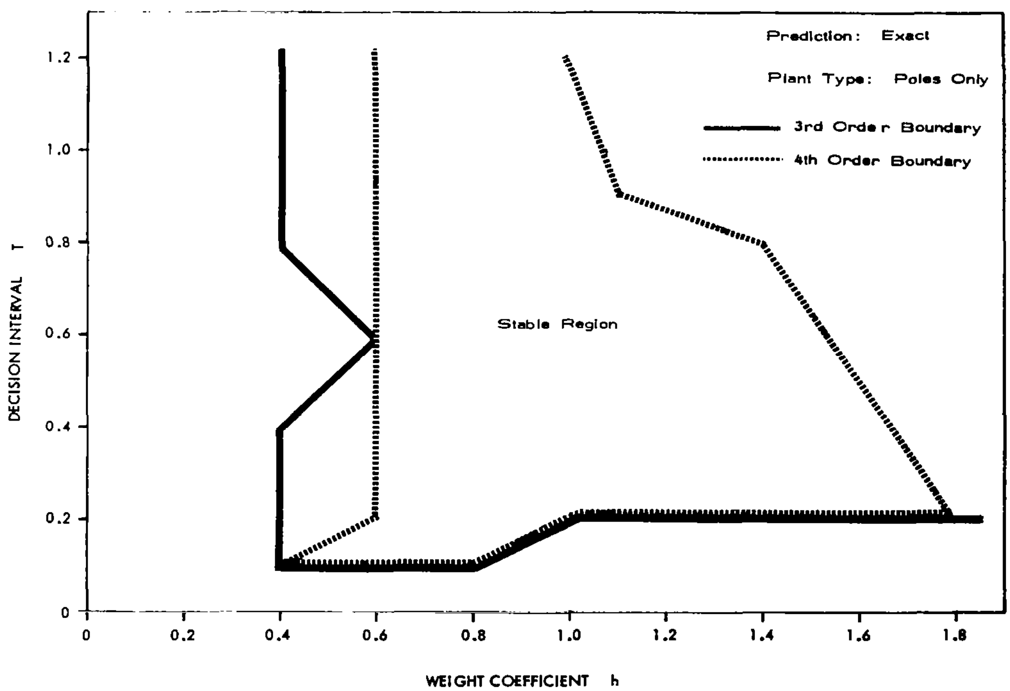


FIGURE 2-6 COMMON EXACT STABILITY BOUNDARIES OF 3rd AND 4th ORDER SYSTEMS

The obvious effect of higher plant order is the common stable operation region of the system becomes smaller. This fact is also illustrated by Figures 2-7 and 2-8, which present the common fifth and sixth order system stability boundaries. The sixth order system region of common stability is quite small, and that of even higher order systems consists of only stable neighborhoods of isolated T-h points. A fact of equal importance is that for all plants of each order through six there exists a common stable region, and that for all the plants of order through six there exists a small common region of stable performance. This is particularly significant since many of the plants are poorly damped to the point of being on the difficult side, even for more conventional linear compensation techniques with full knowledge of the plant transfer function.

Exact Prediction - Pole-Zero Configurations.-Exact Prediction T-h stability boundaries were obtained for about forty plants of second through ninth order. As in the previous poles only case, all the second order system stability boundaries consisted of the entire T-h plane examined. Figures 2-9, 2-10, 2-11, and 2-12 present the common stable boundaries for third, fourth, fifth, sixth, and seventh order systems. Figure 2-9 shows the effect of higher plant order is to decrease the common region of stable performance as was the case with pole configuration transfer functions. This same point is illustrated by the other figures for plants through seventh order. The seventh order system common boundary is still a sizable region, but for higher order systems the region decreased to a few isolated T-h points. The important facts again are that all plants of each order thorugh seven have a common stability boundary, and that all the plants of order through seven have a small common region of stable performance.

Also, in general the plants with zeros have a larger system stability boundary. This is easily seen by comparing the common boundaries for any like order, and is brought out by the existence of a larger common boundary for pole-zero configuration plants of all orders. This result is very easily demonstrated by Figures 2-13, 2-14, and 2-15 for a third, fourth, and fifth order system respectively. Figure 2-13 presents the stability boundary for a third order pole configuration plant, and that for a pole-zero configuration plant having the same set of poles. It may be observed from this figure that the pole-zero configuration plant has the larger stability boundary. This same fact is evident for the higher order systems presented in the other figures.

Mixed Prediction.-Mixed Prediction was introduced and partially investigated on pole configuration plants of low order in previous DACS research (reference 3). These early studies indicated that this type of prediction worked with some success on the limited number of plants examined. However, due to the limited number of plants considered, no general conclusions could be made for this type of prediction.

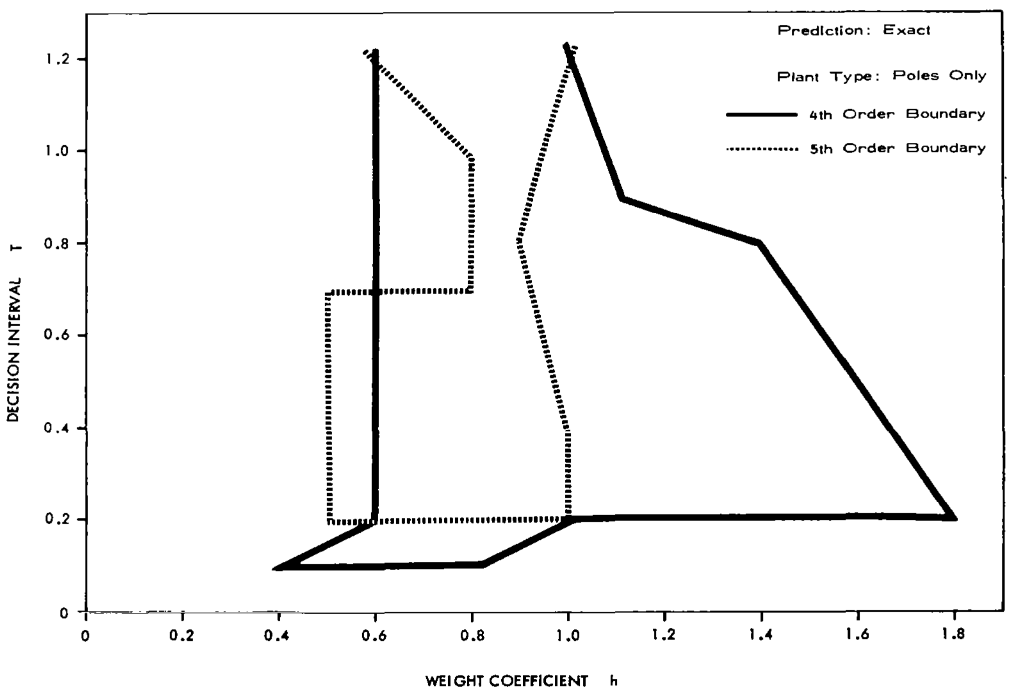


FIGURE 2-7 COMMON EXACT STABILITY BOUNDARIES OF 4th AND 5th ORDER SYSTEMS

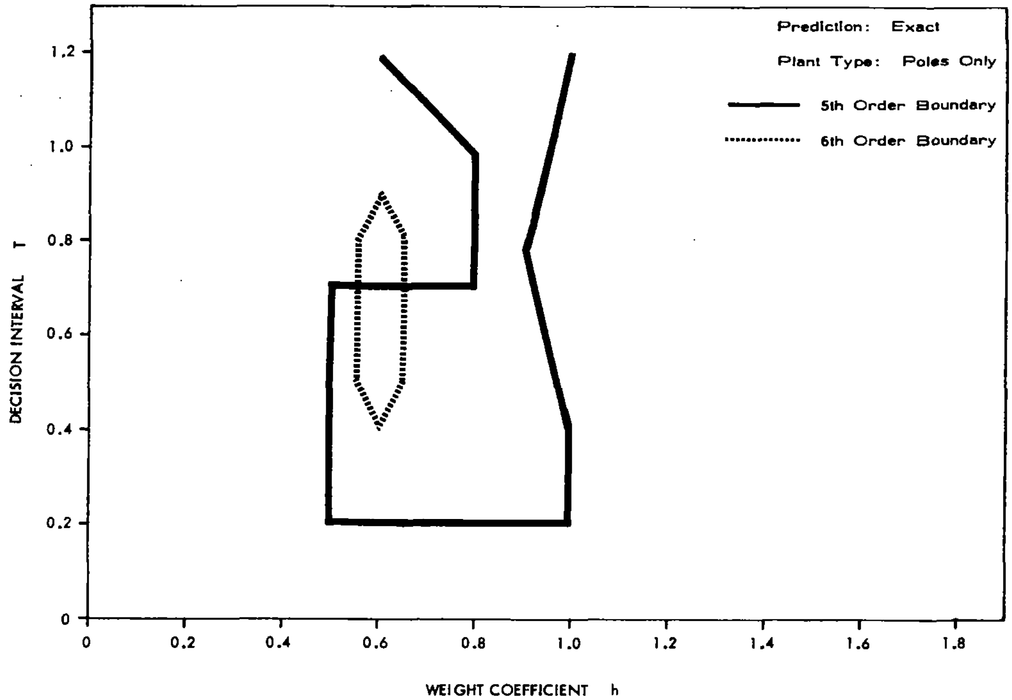


FIGURE 2-8 COMMON EXACT STABILITY BOUNDARIES OF 5th AND 6th ORDER SYSTEMS

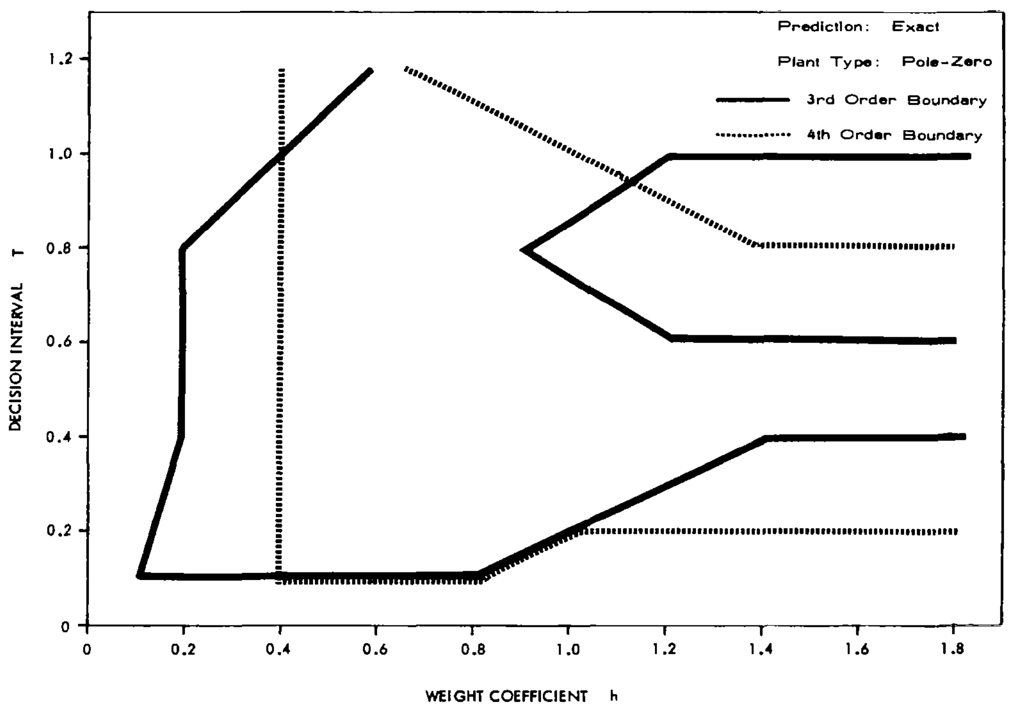


FIGURE 2-9 COMMON EXACT STABILITY BOUNDARIES OF 3rd AND 4th ORDER SYSTEMS

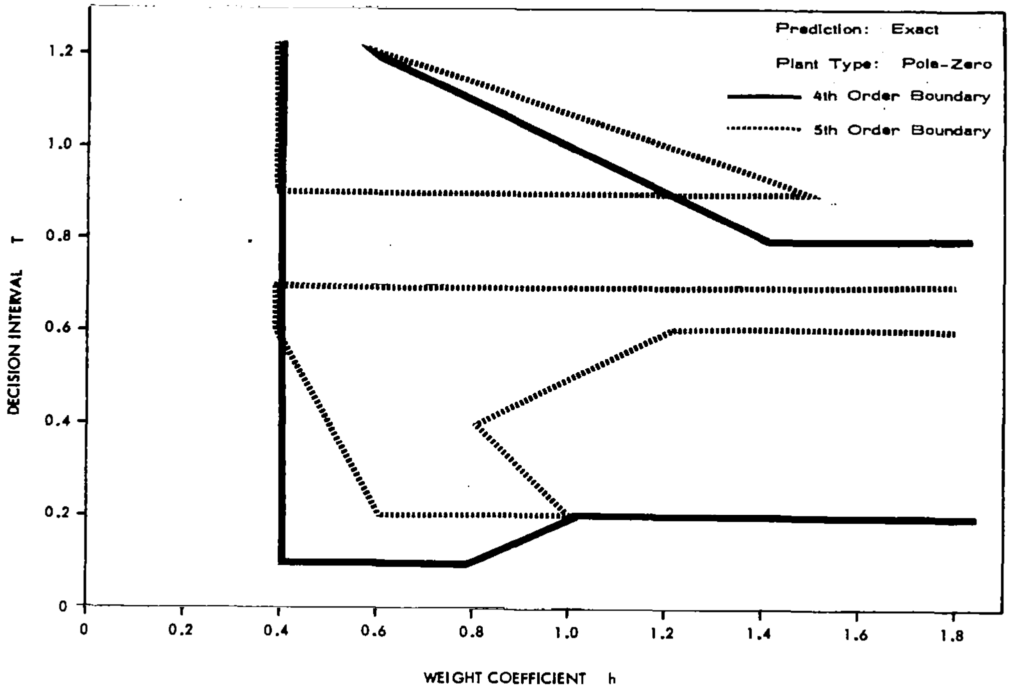


FIGURE 2-10 COMMON EXACT STABILITY BOUNDARIES
OF 4th AND 5th ORDER SYSTEMS

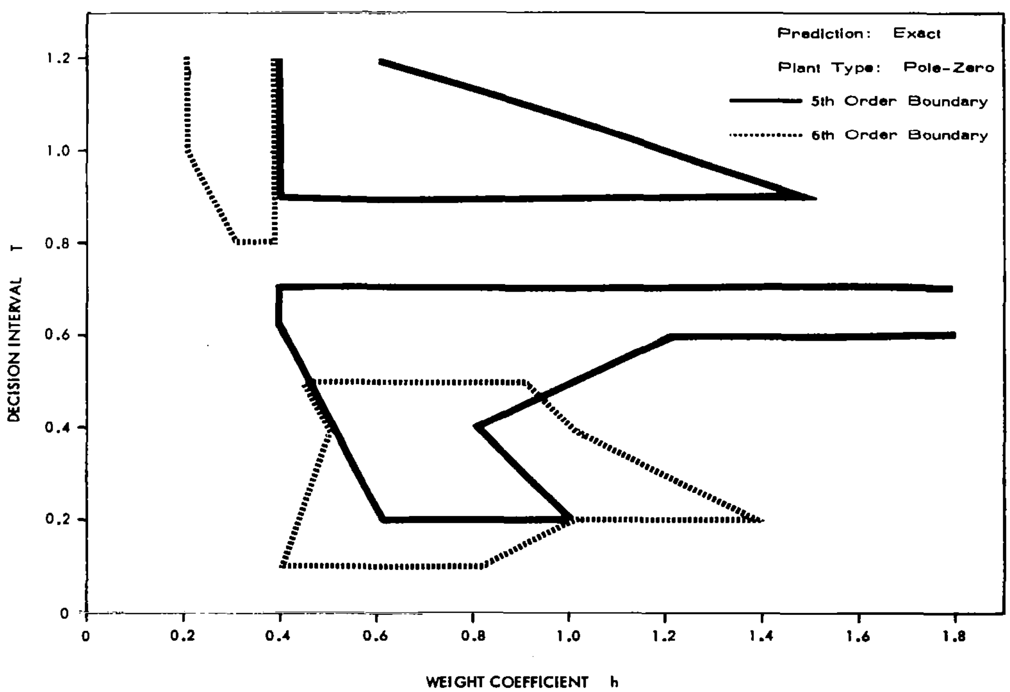


FIGURE 2-11 COMMON EXACT STABILITY BOUNDARIES
OF 5th AND 6th ORDER SYSTEMS

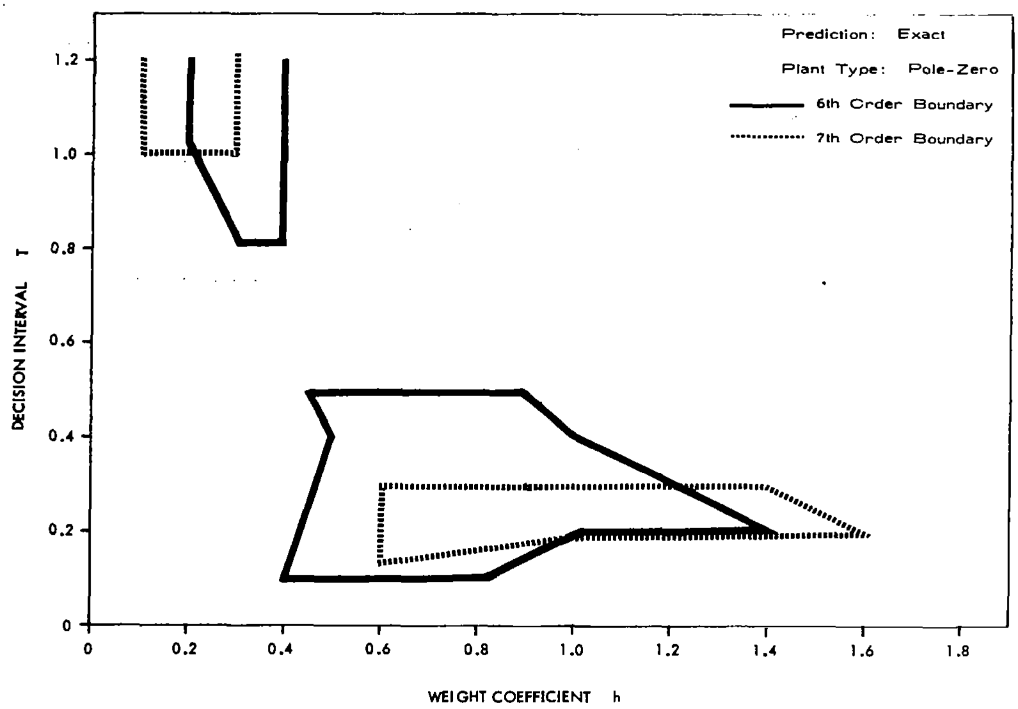


FIGURE 2-12 COMMON EXACT STABILITY BOUNDARIES OF 6th AND 7th ORDER SYSTEMS

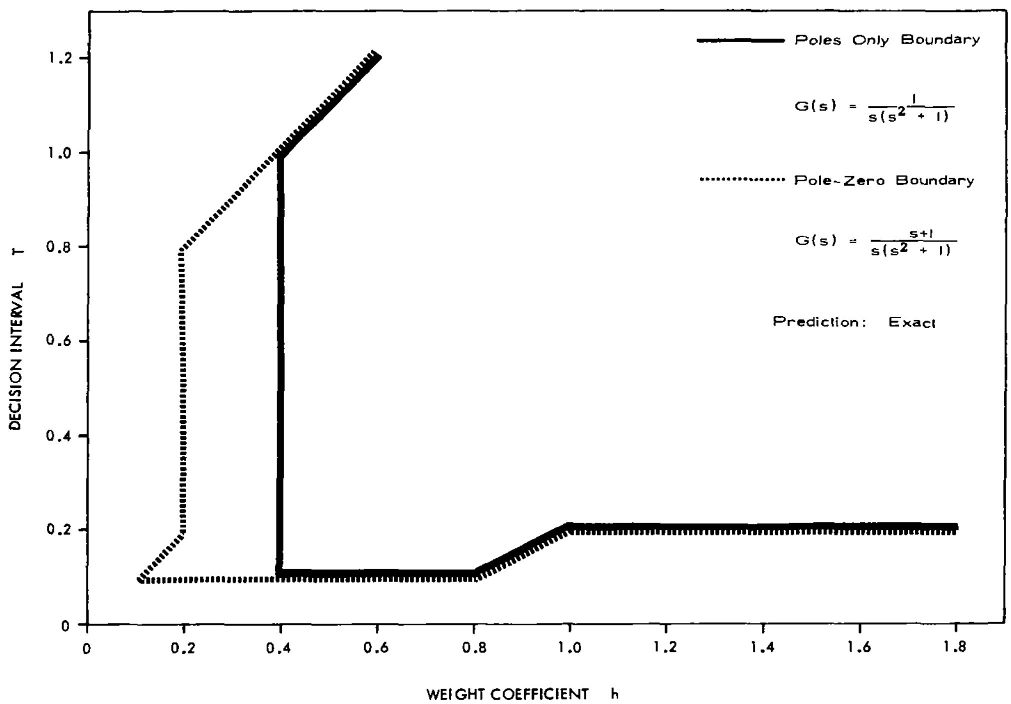


FIGURE 2-13 STABILITY BOUNDARIES OF TWO 3rd ORDER PLANT CONFIGURATIONS

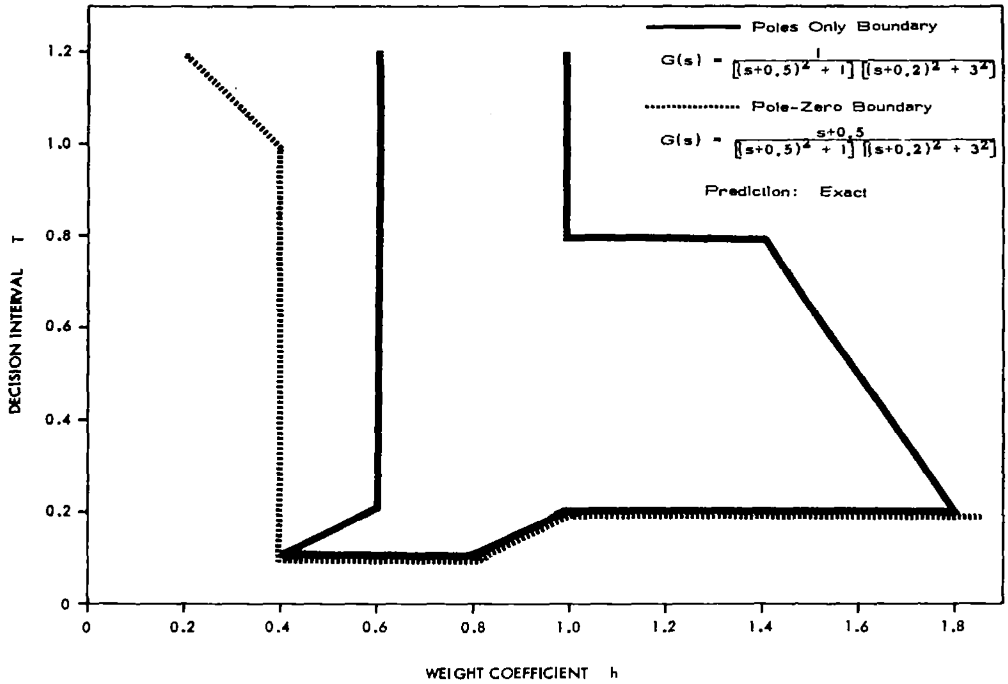


FIGURE 2-14 STABILITY BOUNDARIES OF TWO 4th ORDER PLANT CONFIGURATIONS

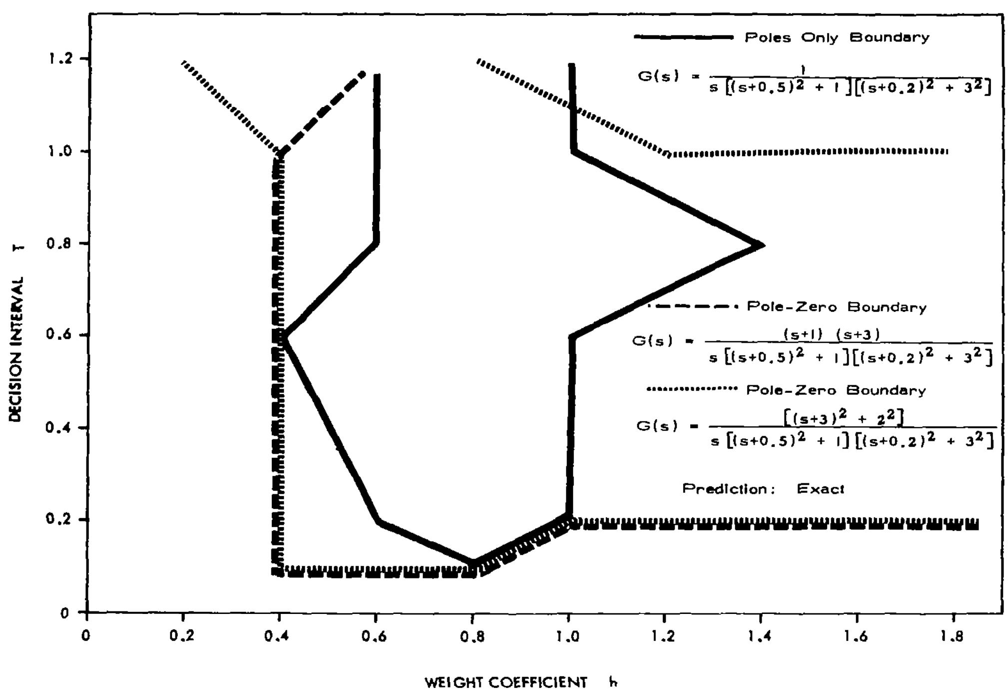


FIGURE 2-15 STABILITY BOUNDARIES OF THREE 5th ORDER PLANT CONFIGURATIONS

The present experimental study was conducted on a subset of fifty pole and pole-zero configuration plants of order thorugh eight. Figure 2-16 and 2-17 present the Exact and Mixed Prediction system stability boundaries for a second order pole and pole-zero configuration plant respectively. It may be observed in both figures that the Mixed Prediction resulted in smaller stability boundaries than those for Exact Prediction. This same result is illustrated by Figures 2-18 and 2-19 for third order systems and by Figures 2-20 and 2-21 for fourth order systems. These results are typical for all systems studied through fourth order. However, the systems of order greater than four in general proved to have very poor stability boundaries. The only common stability boundary existed for the second order systems.

In summary, the Mixed Prediction study resulted in adequate stability boundaries for all second and third order systems, and fair to poor stability boundaries for fourth order systems. The higher order systems resulted in very poor stability regions, and in many cases, no stable region existed.

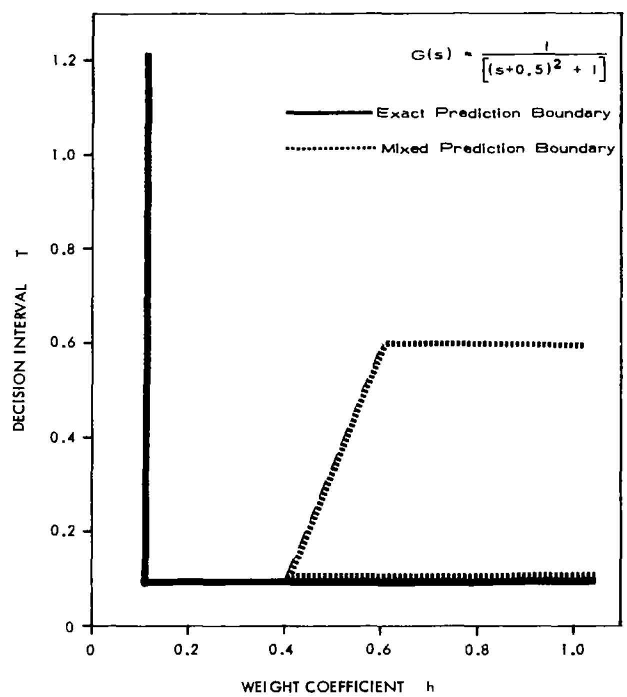


FIGURE 2-16 TWO TYPES OF STABILITY BOUNDARIES OF A 2nd ORDER SYSTEM

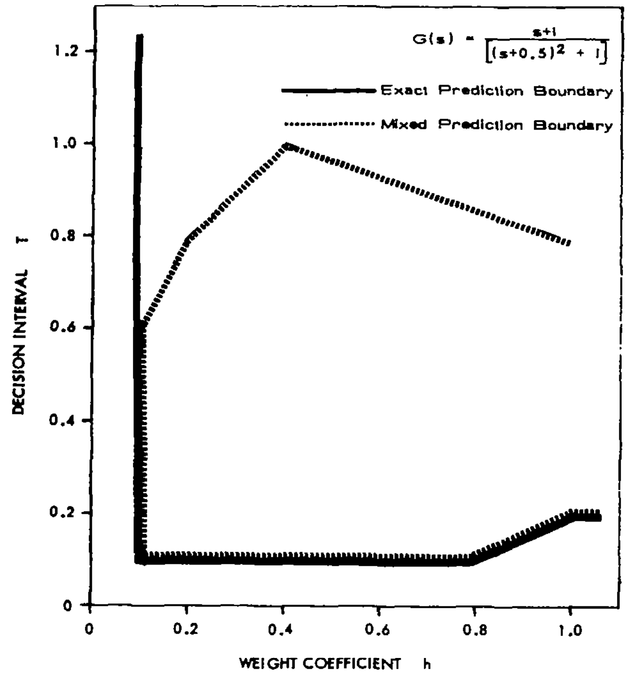


FIGURE 2-17 TWO TYPES OF STABILITY BOUNDARIES OF A 2nd ORDER SYSTEM

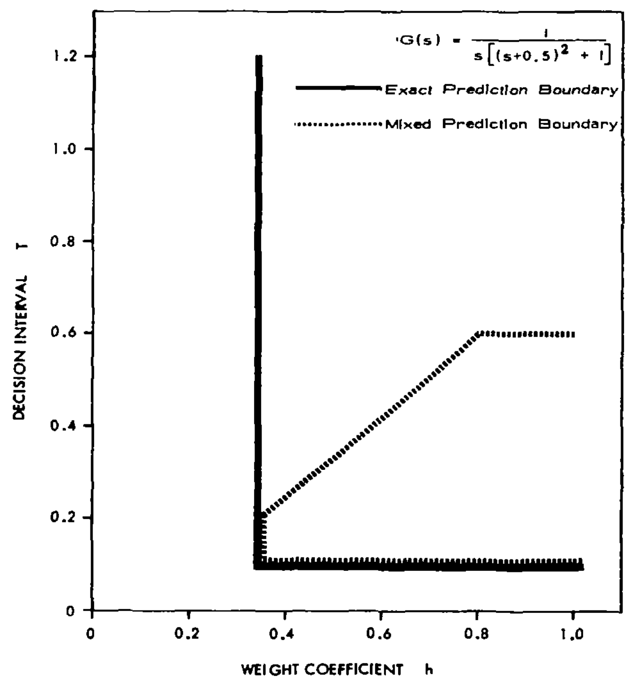


FIGURE 2-18 TWO TYPES OF STABILITY BOUNDARIES OF A 3rd ORDER SYSTEM

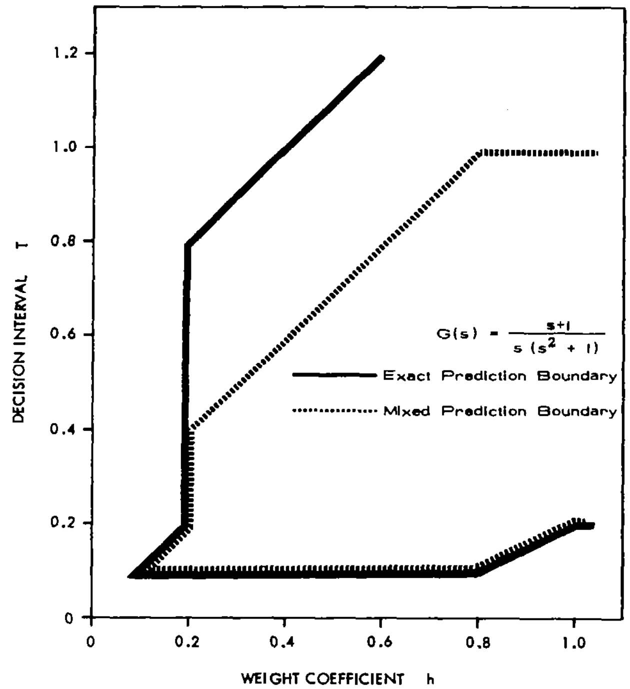


FIGURE 2-19 TWO TYPES OF STABILITY BOUNDARIES OF A 3rd ORDER SYSTEM

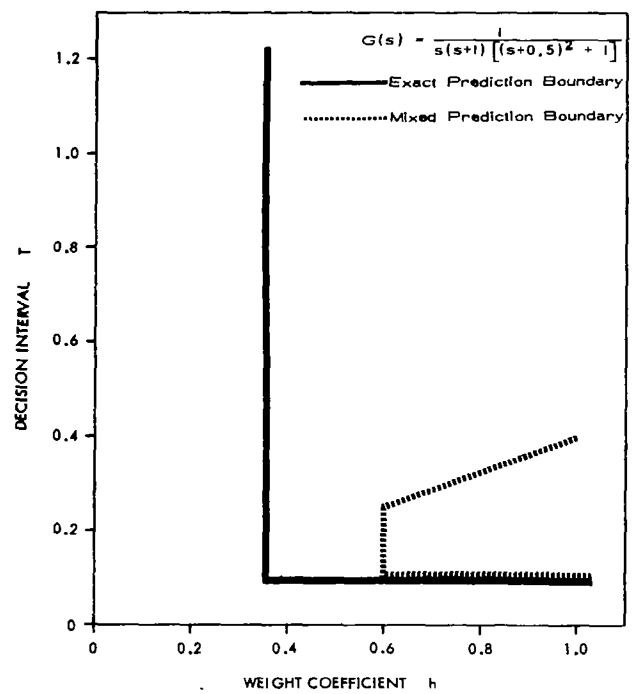


FIGURE 2-20 TWO TYPES OF STABILITY BOUNDARIES OF A 4th ORDER SYSTEM

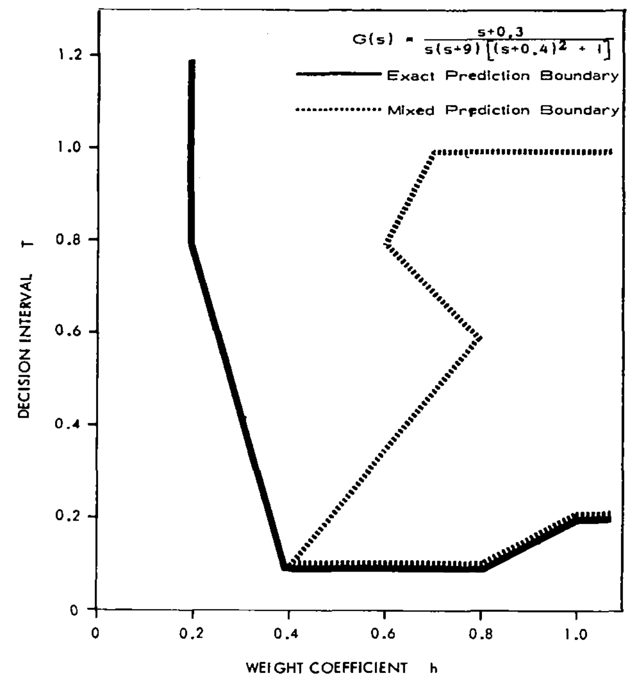


FIGURE 2-21 TWO TYPES OF STABILITY BOUNDARIES OF A 4th ORDER SYSTEM

Taylor Prediction.-Taylor Prediction was also introduced in previous DACS research. Several experimental studies of some depth were conducted on plants through fifth order using this type of prediction. These early experimental studies indicated that Taylor Prediction possessed stable boundaries of adequate size, and that the control system response compared quite well to that obtained from Exact Prediction. However, no general conclusions could be made from this early work due to the limited number of plants used, and the regulator response being the only desired output state examined in any depth.

The present experimental study was conducted on about sixty plants of second through ninth order. These plants were all of pole configuration since no counterpart of Taylor Prediction has been developed for plants of pole-zero configuration. The system stability boundaries for control of all second order plants consisted of the entire T-h plane examined. Figure 2-22 through 2-26 present the common stability boundaries for all third through eighth order systems. Two stability boundaries are represented on each figure to allow easy comparison as plant order increases from third to fourth, fourth to fifth, ... seventh to eighth. For example, Figure 2-22 shows the common stability boundary of the third order plants and that of the fourth order plants. As in the other types of prediction studied, the lower order system stability boundary is better than or equal to that of the higher order system. This same result is evident in the Figures 2-23 through 2-26. Rather sizable common stability regions existed for all plants of each order through eight. Also, a common region of stability exists for all the plants of order through six. These results are of considerable significance when it realized that only the plant order was utilized in the Taylor Prediction method.

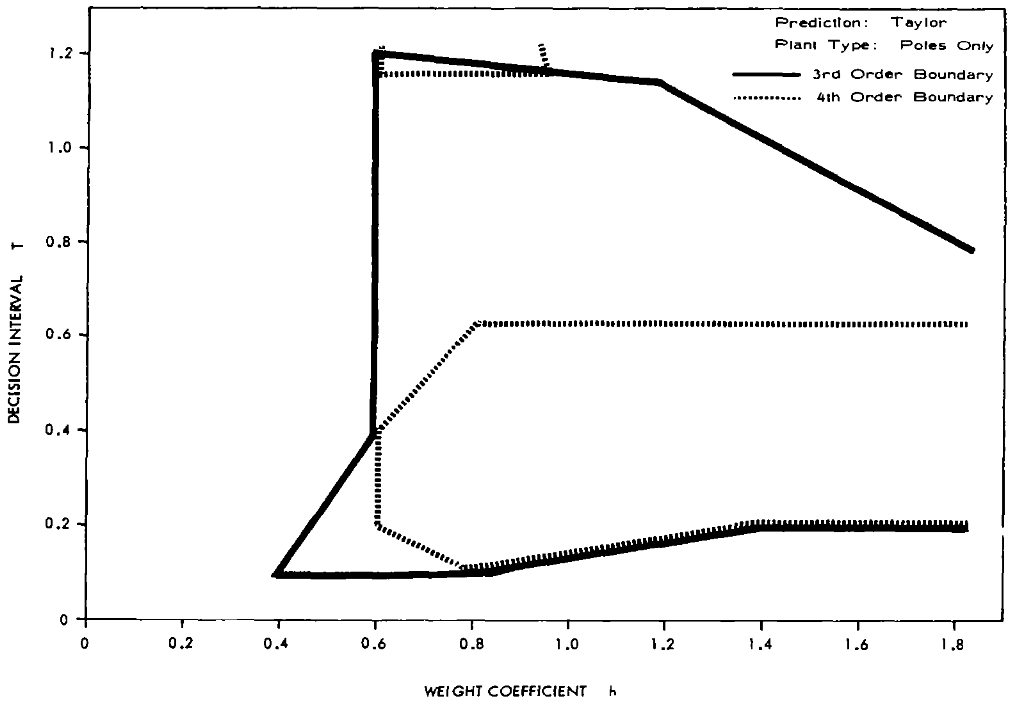


FIGURE 2-22 COMMON TAYLOR STABILITY BOUNDARIES OF 3rd AND 4th ORDER SYSTEMS

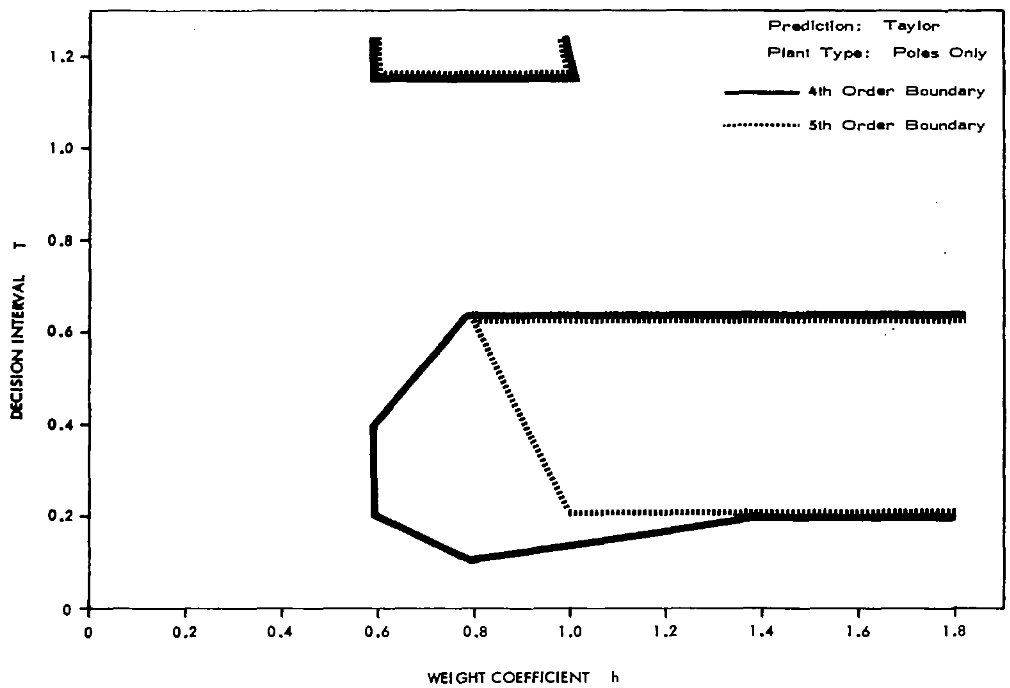


FIGURE 2-23 COMMON TAYLOR STABILITY BOUNDARIES OF 4TH AND 5th ORDER SYSTEMS

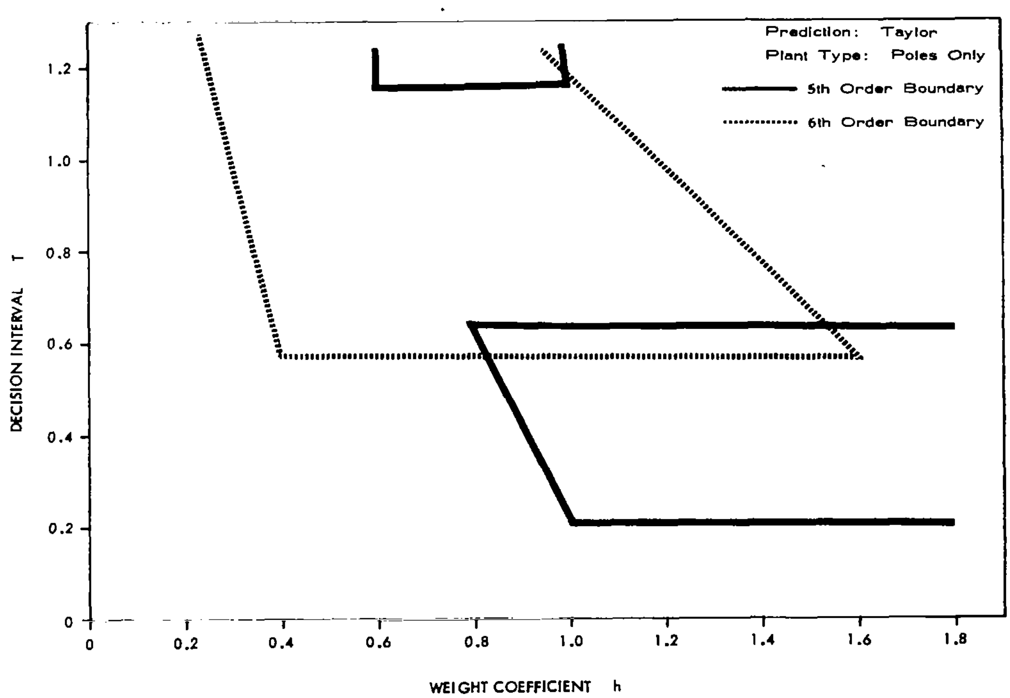


FIGURE 2-24 COMMON TAYLOR STABILITY BOUNDARIES OF 5th AND 6th ORDER SYSTEMS

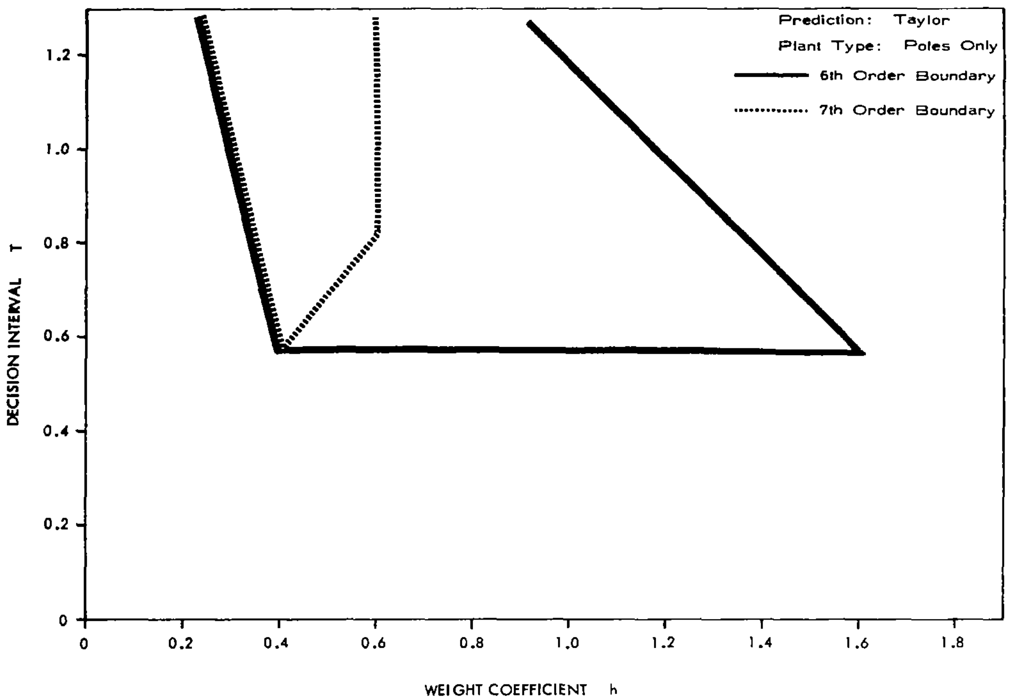


FIGURE 2-25 COMMON TAYLOR STABILITY BOUNDARIES OF 6th AND 7th ORDER SYSTEMS

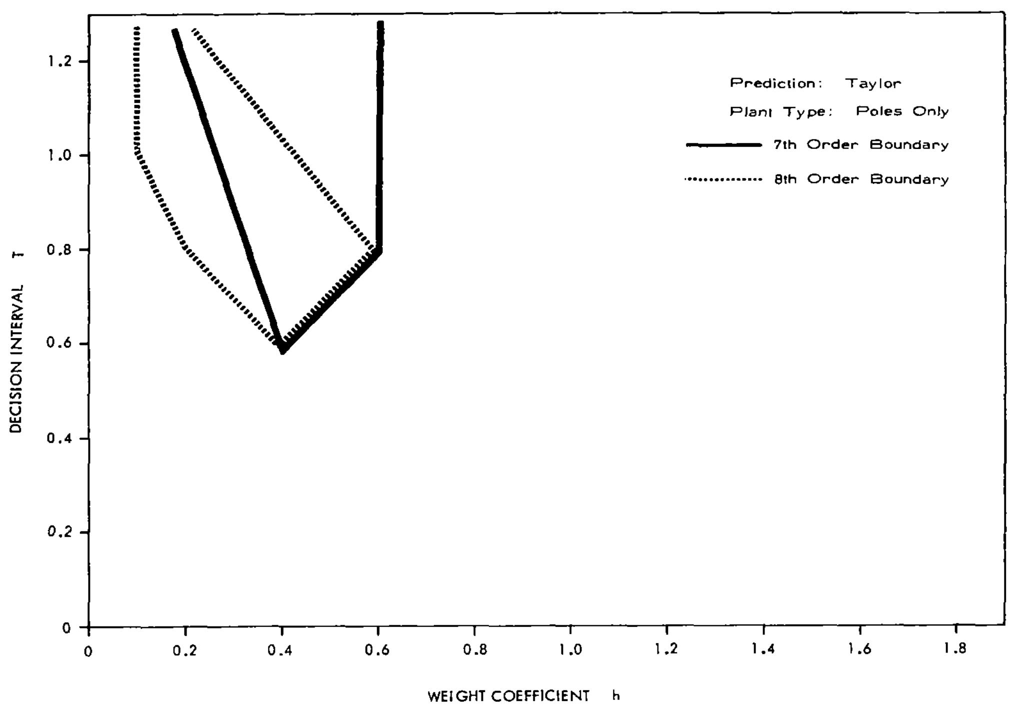


FIGURE 2-26 COMMON TAYLOR STABILITY BOUNDARIES OF 7th AND 8th ORDER SYSTEMS

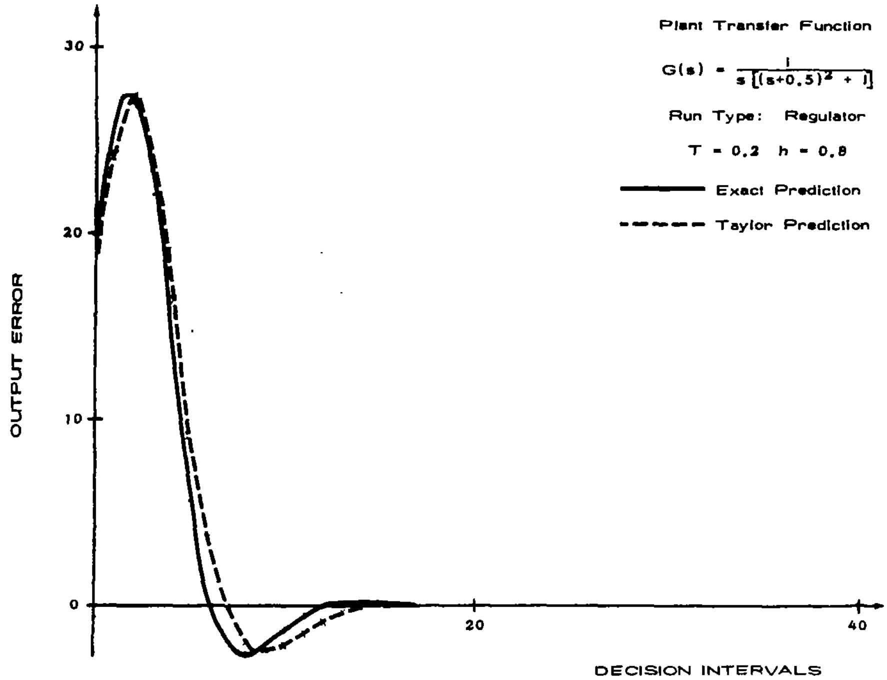
The comparison between Exact and Taylor stability boundaries is subject to qualification. The Taylor stability boundaries for fifth and higher order systems consists largely of marginally stable T-h points. Marginally stable points are those which have at least one eigenvalue between 0.9 and 1.0. The past DACS research has shown that the system response is sluggish at such T-h points, and sometimes unacceptable due to the slow system performance. However, such regions are of importance as an ideal starting place to commence experimentation for establishing desirable T and h values in a learning process.

TAYLOR PREDICTION CONTROL TO ACTUAL SYSTEM ORDER

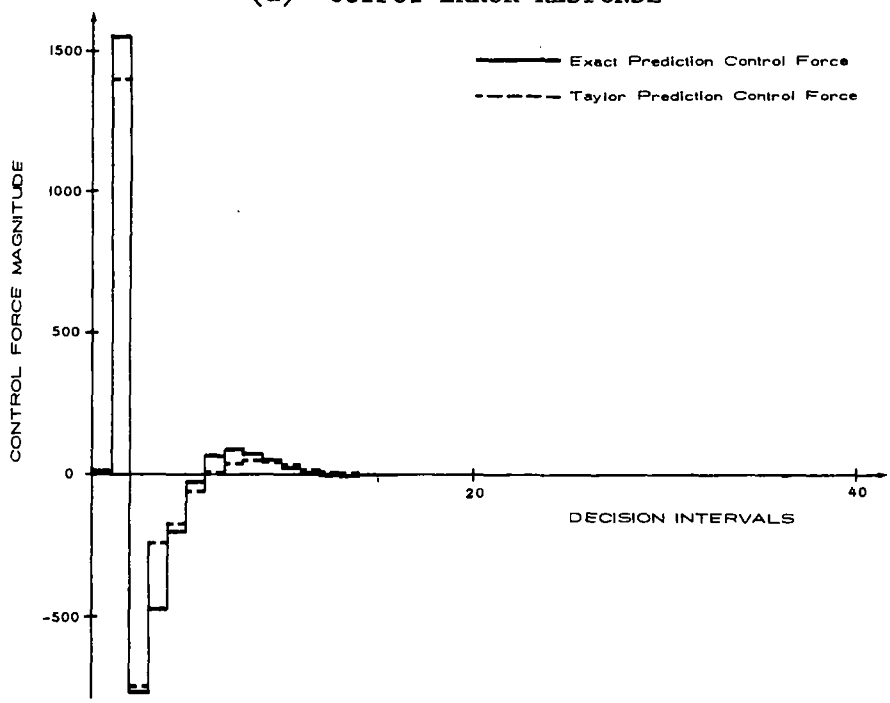
Exact Prediction control studies were conducted to provide a basis for evaluating the Taylor Prediction control performance. Studies were conducted to determine the control performance of our system for various desired output states. Approximately three hundred control simulations were conducted on sixty plants of second through ninth order.

The method of start-up consisted of one arbitrarily selected control force being applied to the plant. The only purpose of this control force was to start the simulation. The value of this control force was usually very small, and was of no significance when compared to the initial state of the plant. The following results summarize the numerous control simulations conducted to determine the effect of using Taylor Prediction.

Regulator Results.-An initial value of -20 units was used for each of the state vector components giving each component an initial error of +20 units. Figures 2-27, 2-28, and 2-29 illustrate typical Exact and Taylor Prediction control responses for stable T-h points for third order systems. The control force sequence has been included in Figures 2-27 and 2-29 to illustrate the type of control action effected by the control policy using Exact and Taylor Prediction. On low order systems both Exact and Taylor prediction resulted in approximately the same control performance. However, in some cases, Taylor Prediction control was somewhat more sluggish, as is seen by Figure 2-29.



(a) OUTPUT ERROR RESPONSE



(b) CONTROL FORCE SEQUENCE

FIGURE 2-27 ERROR RESPONSE AND CONTROL FORCE SEQUENCE OF A 3rd ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

Figures 2-28 and 2-29 also show the Exact and Taylor control performance for low order unstable plants. These results are typical of those obtained for the unstable plants of fourth and lower order.

Typical control performance results for Exact and Taylor Prediction at stable T-h points for fourth and higher order systems is presented by Figures 2-30 through 2-37. It is easily observed that in many instances Taylor Prediction produces a less oscillatory response, and that in the majority of cases it is as good or better than Exact Prediction performance.

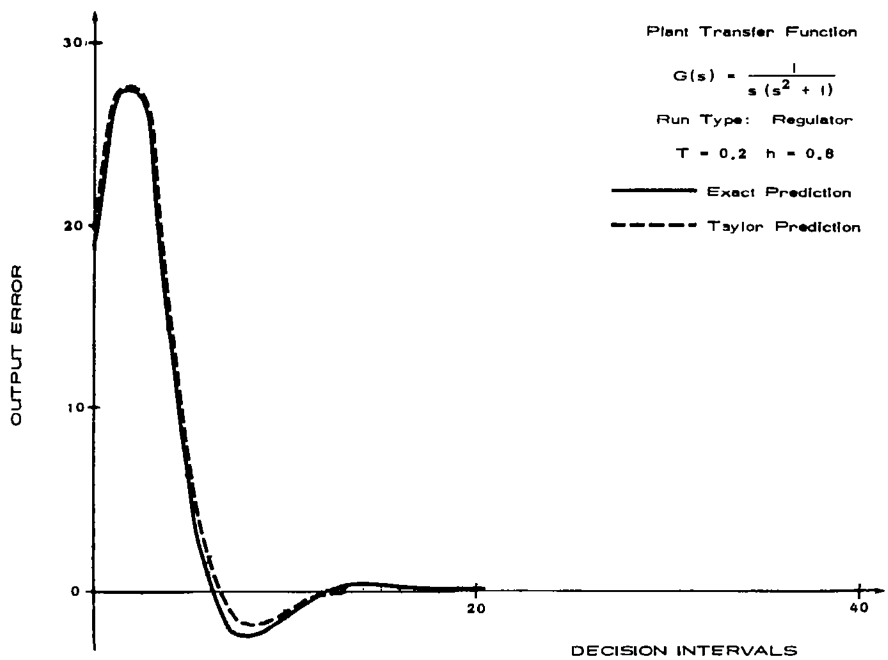
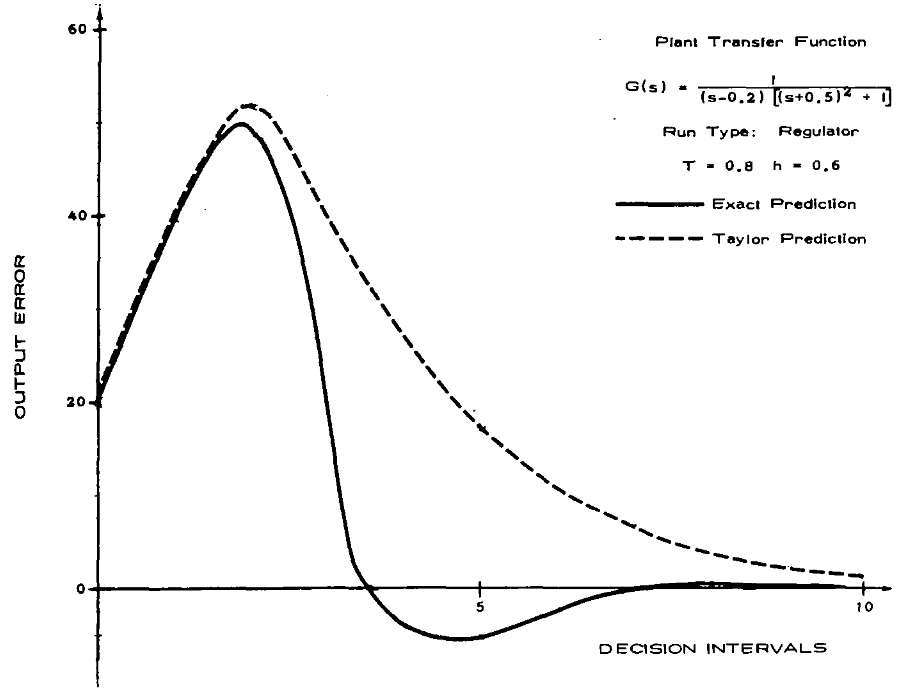
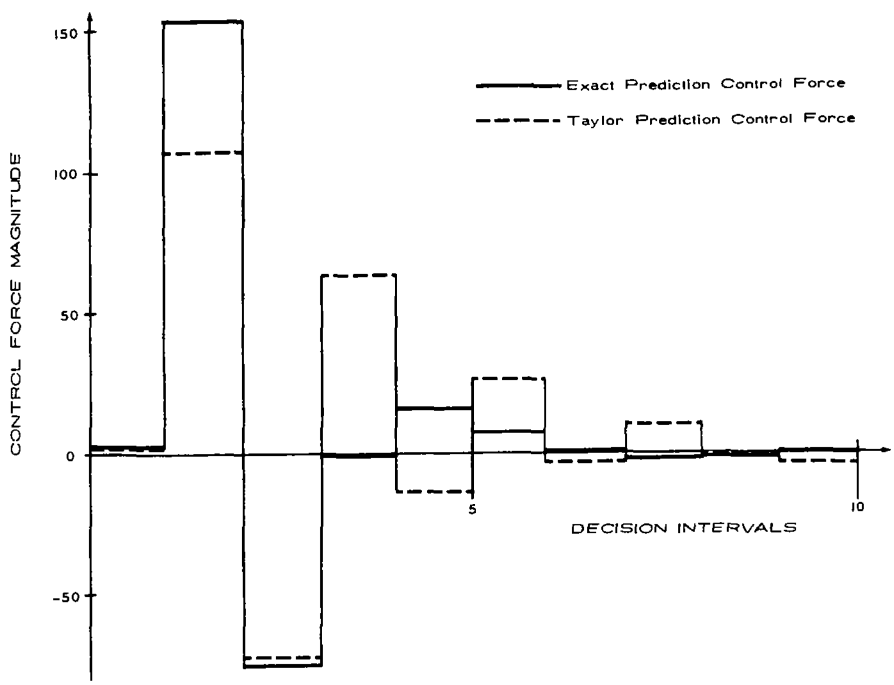


FIGURE 2-28 ERROR RESPONSE OF A 3rd ORDER SYSTEM - EXACT AND TAYLOR PREDICTION



(a) OUTPUT ERROR RESPONSE



(b) CONTROL FORCE SEQUENCE

FIGURE 2-29 ERROR RESPONSE AND CONTROL FORCE SEQUENCE OF A 3rd ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

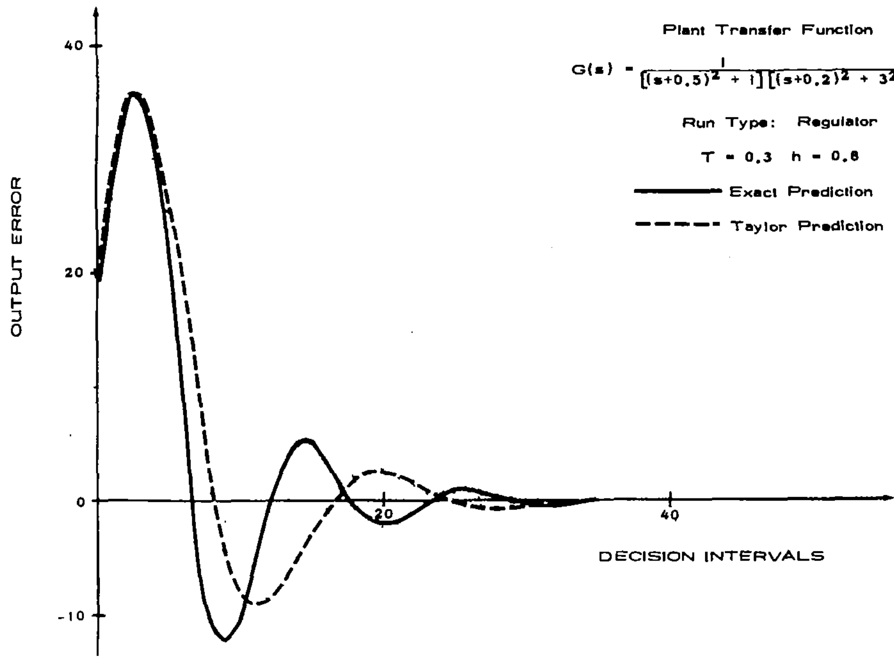


FIGURE 2-30 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

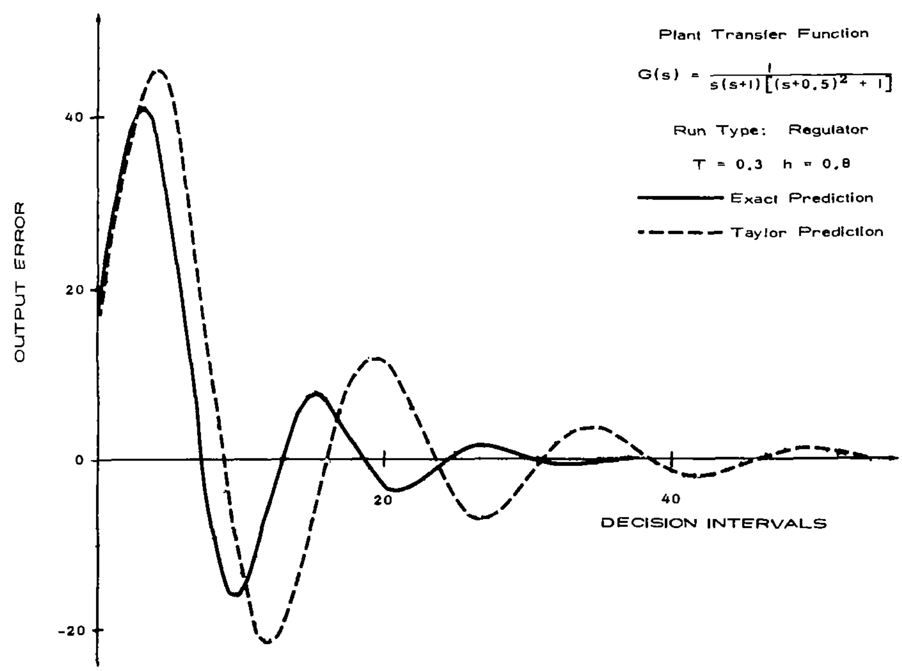


FIGURE 2-31 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

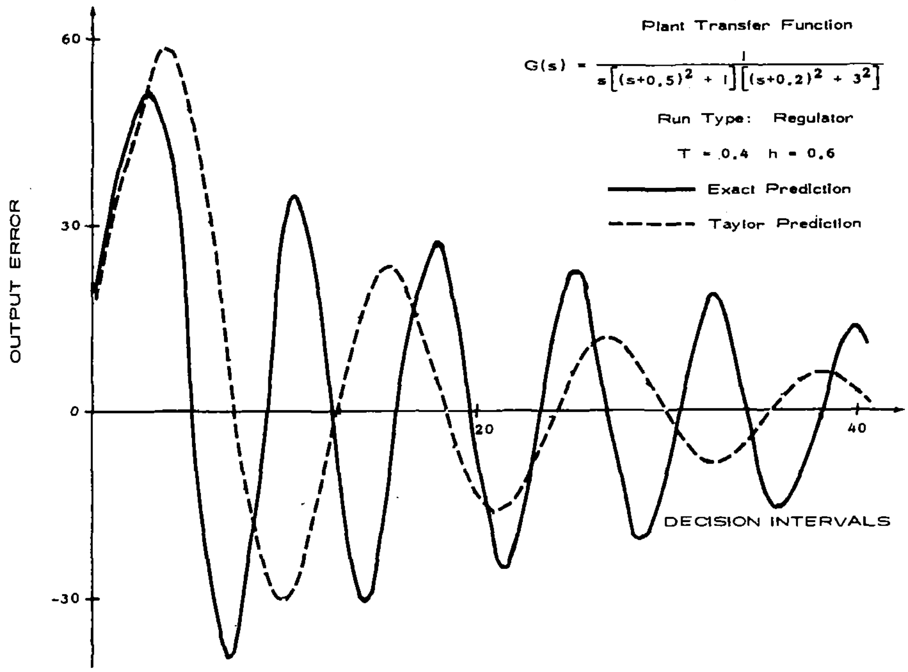


FIGURE 2-32 ERROR RESPONSE OF A 5th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

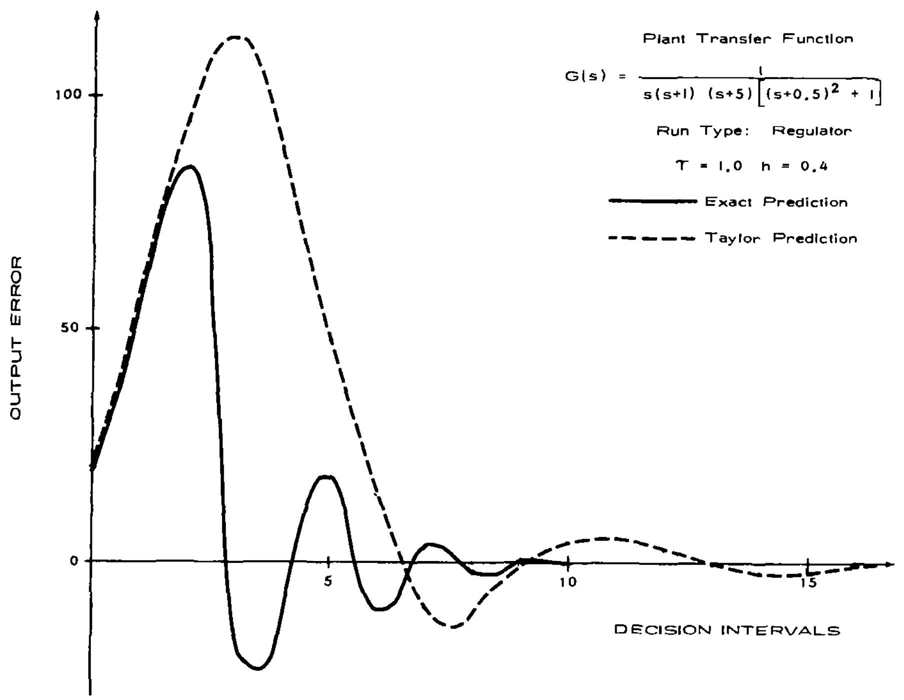


FIGURE 2-33 ERROR RESPONSE OF A 5th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

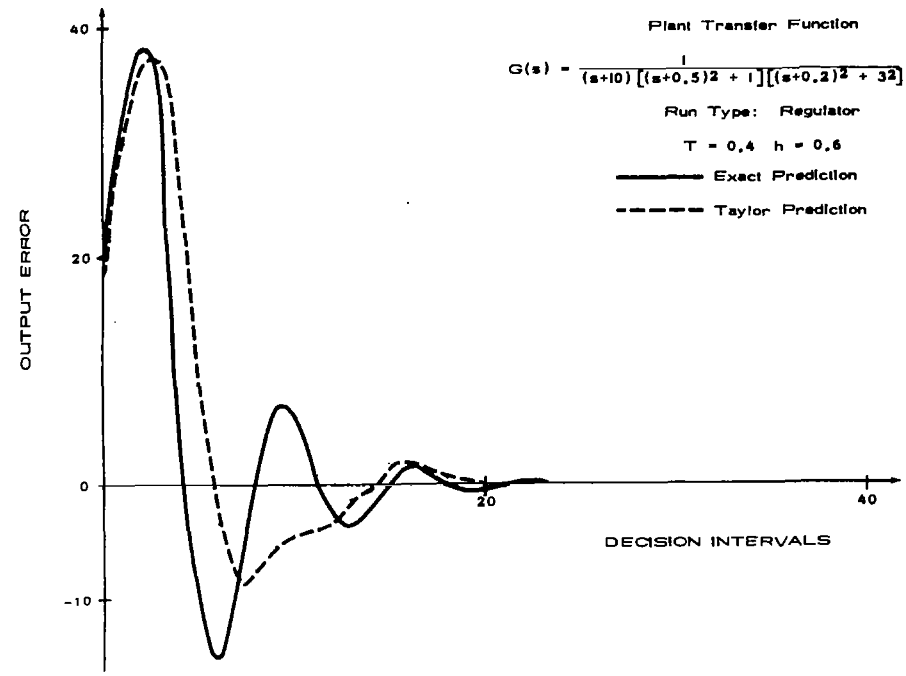


FIGURE 2-34 ERROR RESPONSE OF A 5th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

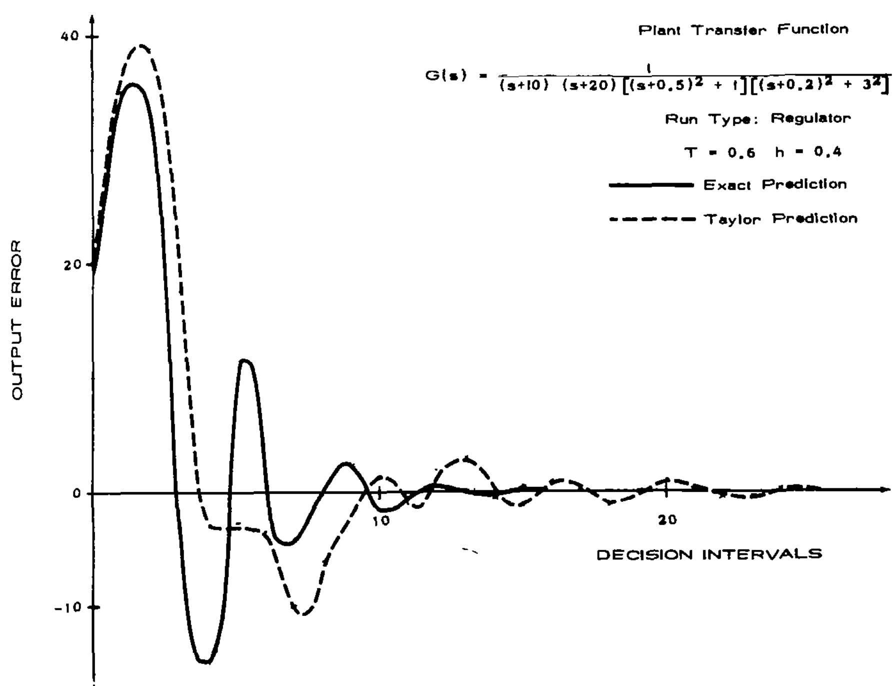


FIGURE 2-35 ERROR RESPONSE OF A 6th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

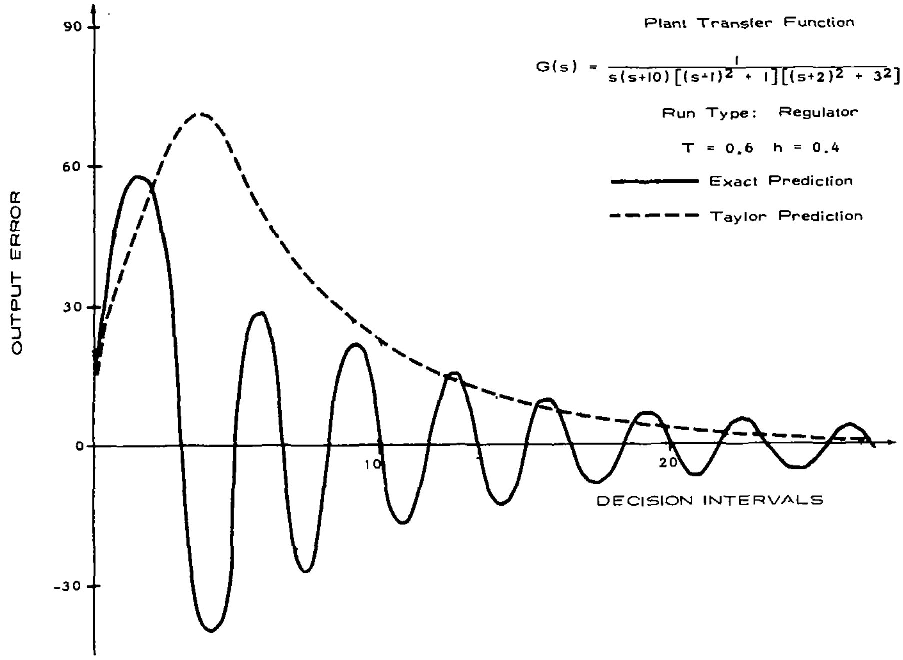


FIGURE 2-36 ERROR RESPONSE OF A 6th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

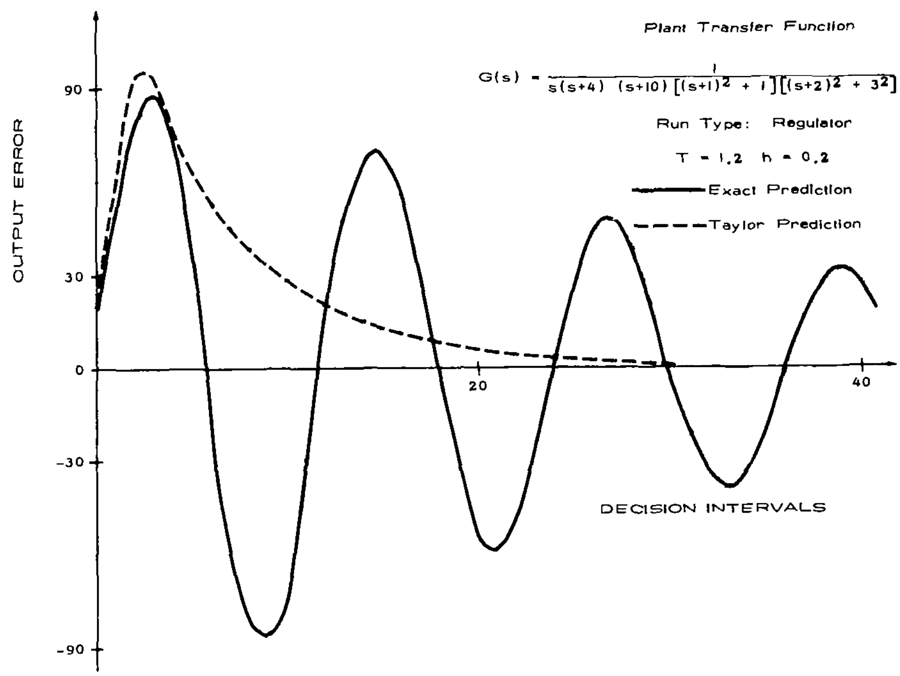


FIGURE 2-37 ERROR RESPONSE OF A 7th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

Trajectory Results.-Exact and Taylor Prediction control performance was examined for step and ramp desired output states. This was done to determine the capability of the controlled system to follow a desired state other than zero. In all of the runs shown, the initial value of each of the state variable components is zero. Illustrative results of this experimentation are presented for two fourth order plants in Figures 2-38 through 2-41. Exact Prediction provided good control performance for both the step and ramp desired states. However, Taylor Prediction performance was dependent on plant configuration. Control of plants containing a pole at the origin (an integration) resulted in good performance for steps, and a steady state error for ramps. The control of plants without a pole at the origin resulted in a steady state error for steps, and an increasing error for ramps. Figures 2-38, 2-39, 2-40, and 2-41 present these results, and are typical of the Exact and Taylor Prediction control performance for systems through ninth order. The Taylor Prediction control results are discussed in paragraph 2.3, and the analytical reasons for such performance are presented in Appendix E.

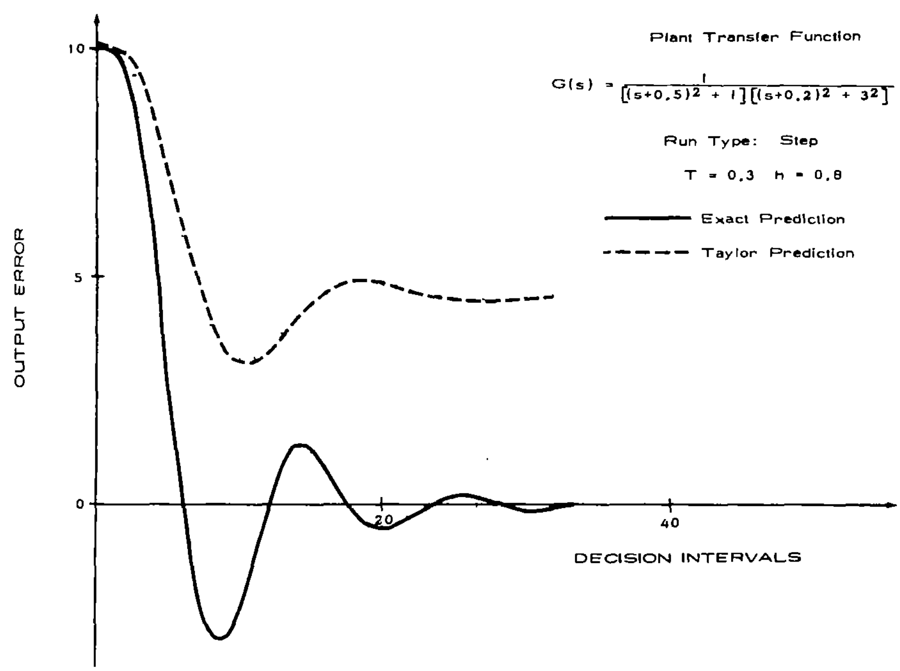


FIGURE 2-38 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

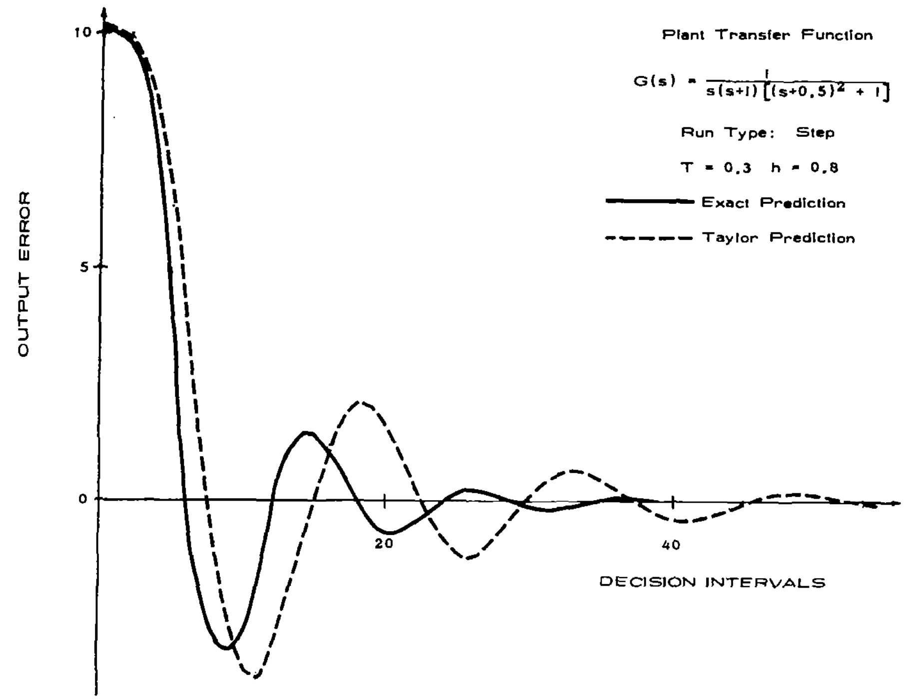


FIGURE 2-39 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

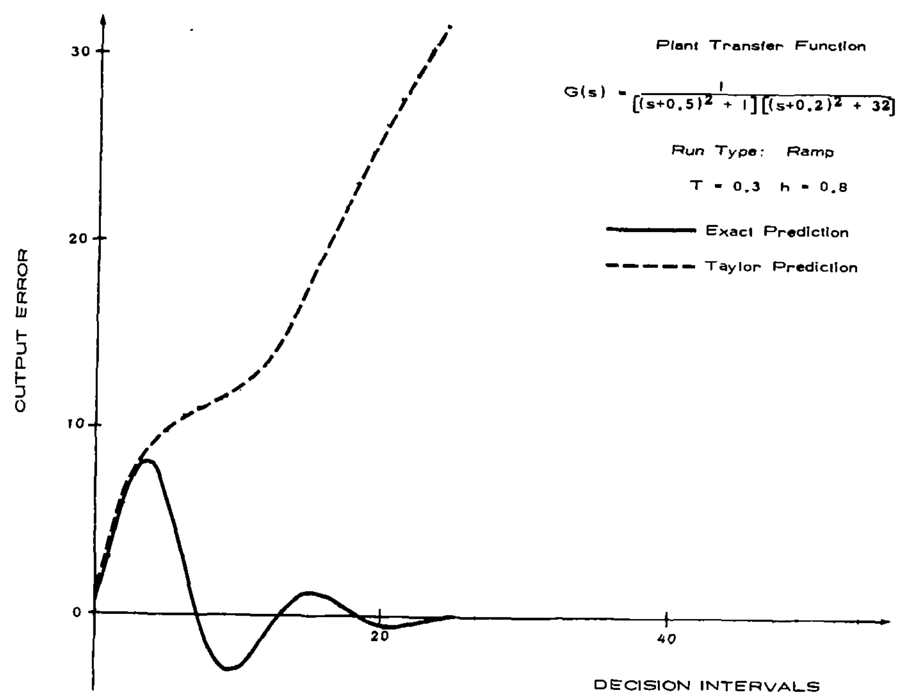


FIGURE 2-40 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

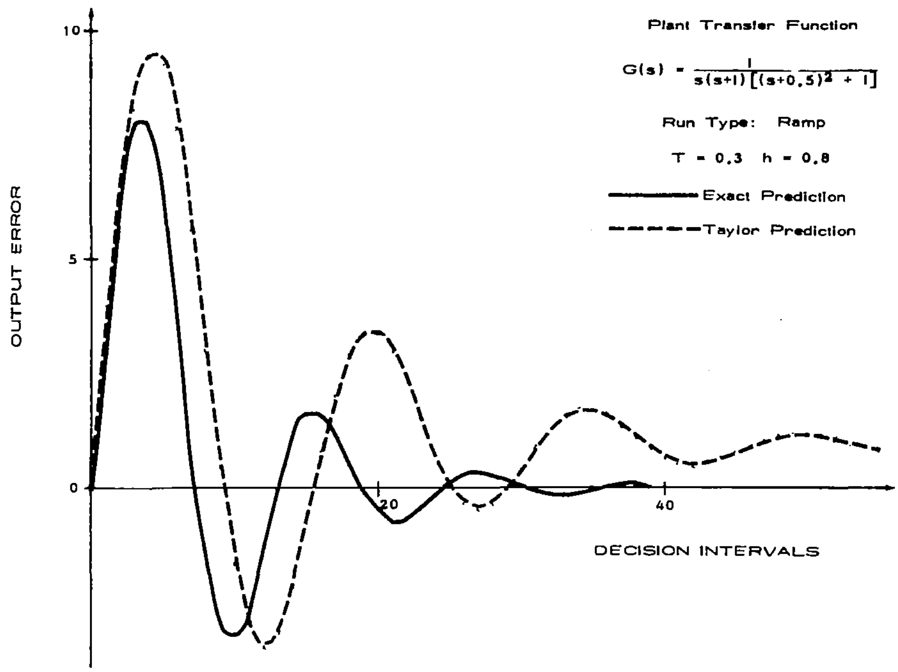


FIGURE 2-41 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND TAYLOR PREDICTION

TAYLOR PREDICTION CONTROL TO LOWER THAN ACTUAL SYSTEM ORDER

Up to this point the system order has been assumed to be known. In the following experimental study both the system and its order are unknown in the sense that the assumed order is less than the true order. This area is of interest since it in fact uses no knowledge of the plant to be controlled.

Stability Boundaries.-System stability was investigated for about thirty plants of third through eighth order. Typical stability boundaries are presented in Figures 2-42, 2-43, and 2-44. These results indicate the general result that the region of stability decreases as the assumed system order becomes smaller. Also of interest is the result that no common stability boundaries exist for any order system assumed as a second order system. However, of more importance is the result that small common boundaries exist for fourth through eighth order systems alternately assumed to be: one less than actual order or third order. It must be noted that these common regions of stability consisted mostly of marginally stable points.

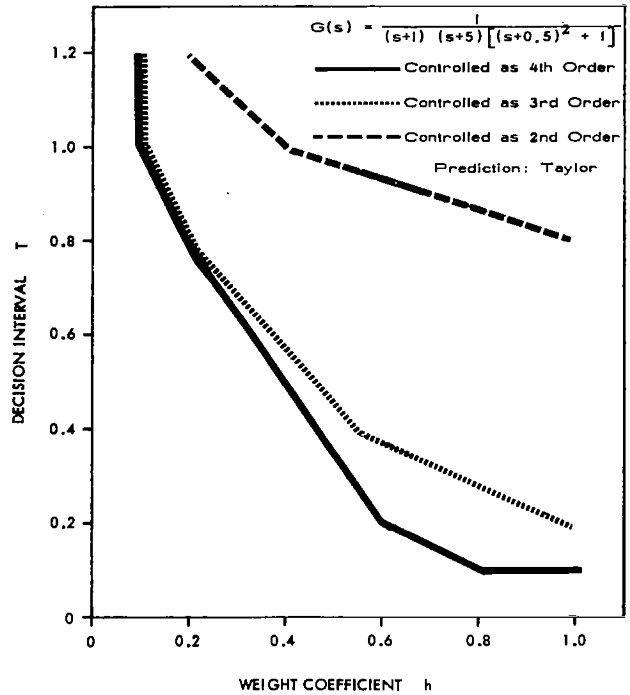


FIGURE 2-42 TAYLOR LOWER THAN ACTUAL ORDER CONTROL STABILITY BOUNDARIES

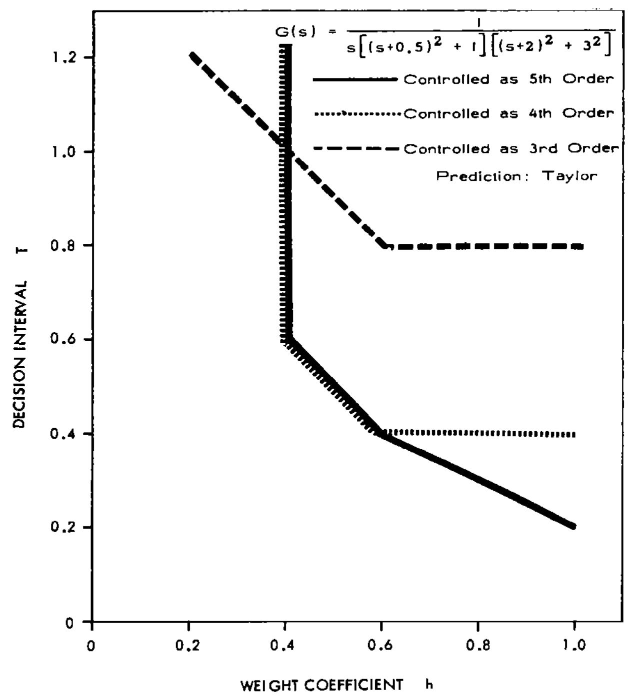


FIGURE 2-43 TAYLOR LOWER THAN ACTUAL ORDER CONTROL STABILITY BOUNDARIES

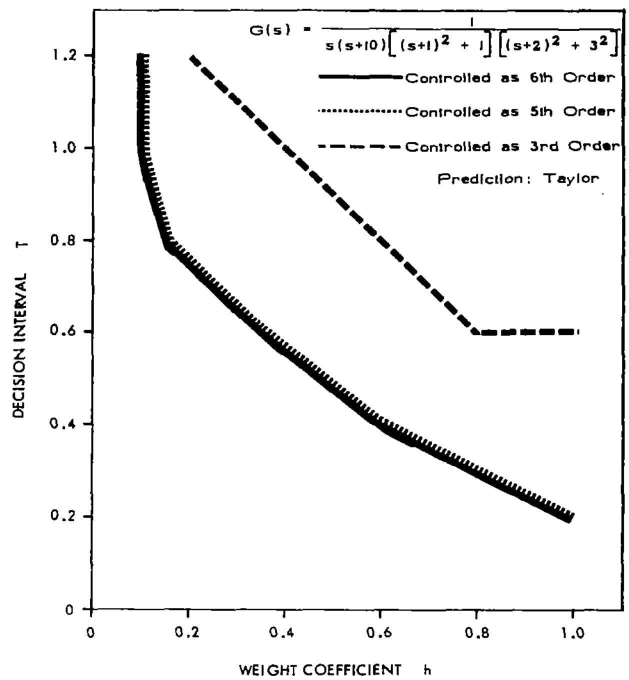


FIGURE 2-44 TAYLOR LOWER THAN ACTUAL ORDER CONTROL STABILITY BOUNDARIES

Regulator Results.-Due to the trajectory results for Taylor Prediction actual order control, only the regulator problem was considered for lower than actual order control. A limited number of such control experiments were made on fourth through eighth order plants. Figures 2-45, 2-46, and 2-47 present experimental results for fourth, fifth, and sixth order plants controlled as lower than actual order at stable T-h points. A typical set of responses is shown in Figure 2-46 where a fifth order plant is controlled with respectively a fifth, fourth, and third order control law. As the assumed system order decreases, the controlled response becomes more sluggish and in many cases, as seen by Figure 2-47, results in completely unacceptable performance. Such sluggish response was usually encountered in the higher order systems controlled as lower order. In most cases, the T-h point was a marginally stable point since the boundary consisted largely of such points.

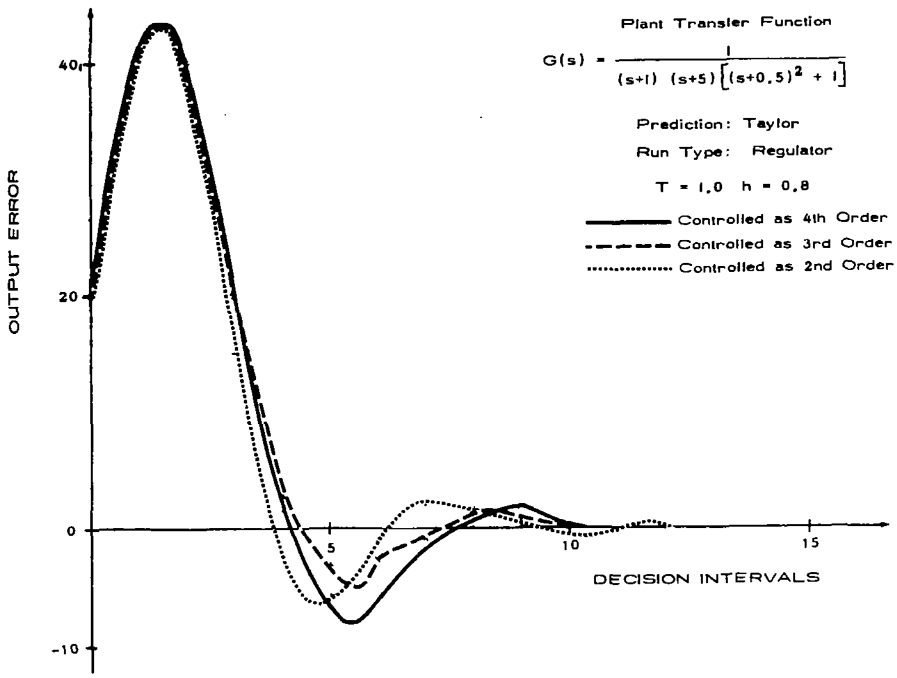


FIGURE 2-45 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 4th ORDER SYSTEM - TAYLOR PREDICTION

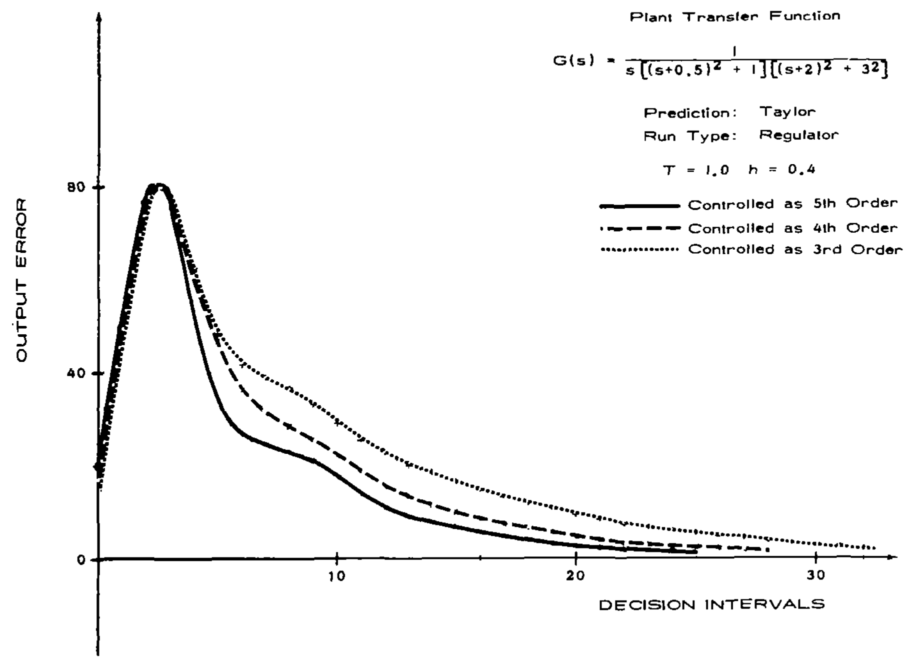


FIGURE 2-46 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 5th ORDER SYSTEM - TAYLOR PREDICTION

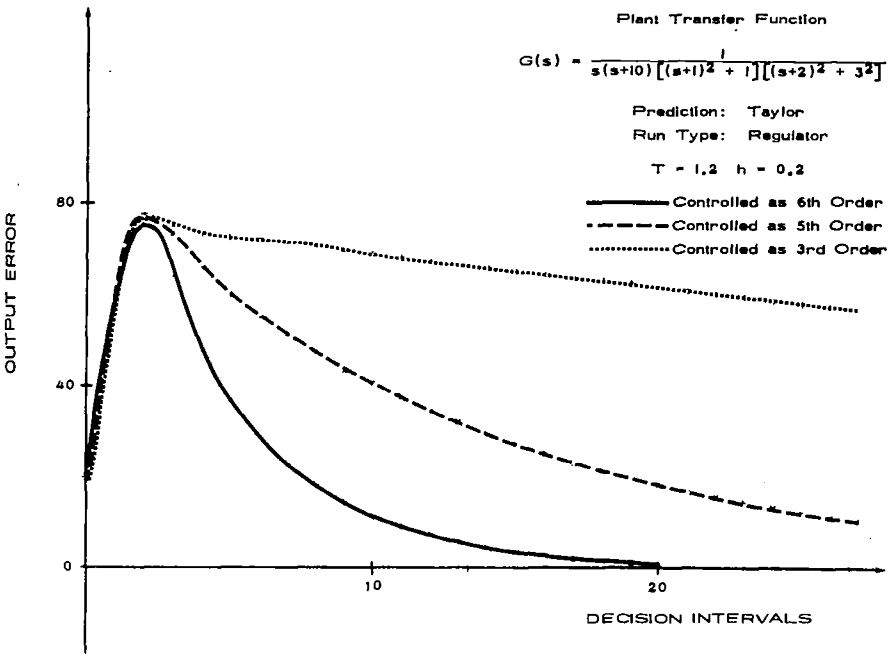


FIGURE 2-47 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 6th ORDER SYSTEM - TAYLOR PREDICTION

INTERPOLATION PREDICTION CONTROL TO ACTUAL ORDER

The last area of experimental investigation was Interpolation Prediction. The previous experimental procedure used for Mixed and Taylor Prediction started with a stability investigation. However, theoretically the Interpolation Prediction is equivalent to the Exact Prediction provided no singularity or ill conditioning is encountered in the matrix of basis vectors and all of the state variables are measurable. Thus, the Exact stability boundaries were considered to be applicable to Interpolation Prediction as well as Exact Predition. The Exact stability boundaries presented and discussed earlier are, therefore, of practical interest with regard to Interpolation Prediction.

The Interpolation Prediction method is not self-starting, i.e., it requires the accumulation of the results of a set of initial control actions. The experimental start-up procedure used Exact or Taylor Prediction for control over N + 2 (N being the assumed system order) decision intervals. After these intervals, the Interpolation Predicition method was utilized for system control. In the initial set of simulation experiments the interpolation matrices were determined once at the start of each run. The start-up period was considered to be part of the run and in all cases shown Exact Prediction was used during the start-up phase. The start-up portion of the run is indicated on each figure. Figure 2-48 has the start-up portion labled and the remaining figures have the start-up portion indicated by arrows. Since the purpose of the present study was to demonstrate the equality between Interpolation and Exact Prediction, the Exact or Taylor start-up method is justified as an experimental procedure to accumulate the matrix of basis vectors. The above and later experimental start-up procedures proved convenient tools to investigate causes, effects, and possible solutions for singularity and ill conditioning of the matrix of basis vectors (Interpolation Matrix).

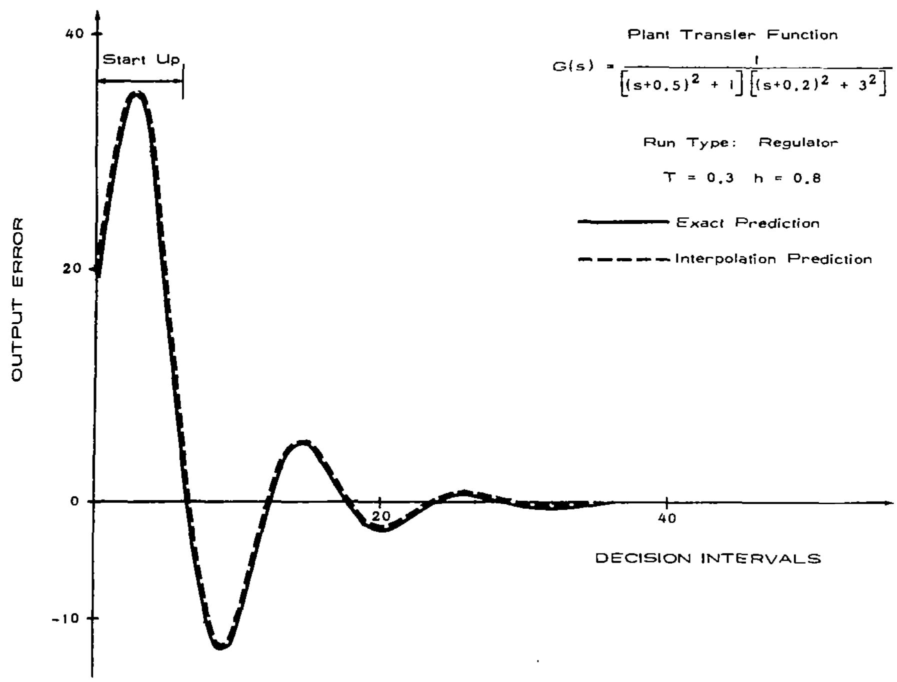


FIGURE 2-48 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND INTERPOLATION PREDICTION

The singularity problem is of particular concern in the linear plant control case. This problem is a direct result of the linear control policy equation, and as such must be avoided by some remedy. One simple remedy is to use a non-control policy force, which when included in the matrix of basis vectors prevents the matrix from being singular. In the case where the matrix is assembled once, and used for the entire run (i.e. no updating) the arbitrary non-control policy force was the first (in some cases the first and second) control force applied to the plant. In later studies where the matrix of basis vectors was updated during the run the one non-control policy force was computed by multiplying the calculated control force by an arbitrary constant (usually 1.1 or 1.5). This altered value was then used as the applied control force.

The problem of the matrix of basis vectors becoming ill conditioned is directly associated with the Interpolation Prediction method, and not with the control policy equation of any plant class. The ill conditioning may occur in either linear or non-linear plant control situations, and in the linear case even if a non-control policy force is included in the matrix of basis vectors.

Over three hundred and fifty Interpolation Prediction control simulations were conducted on pole and pole-zero configuration plants of seventh and lower order. These studies were performed for zero, step, and ramp desired output states.

Regulator Results.-The following results summarize the numerous experiments performed to establish the equality of Exact and Interpolation Prediction control performance. In all cases the initial condition of each state vector component was -20 units giving each an initial error of +20 units.

Figures 2-48 and 2-49 represent the typical correspondence between Exact and Interpolation control performance for fifth and lower order systems. None of the fifth and lower order systems examined resulted in a singular or ill conditioned matrix of basis vectors. This observation is of considerable interest since only one arbitrary control force not calculated by the control policy was included in the matrix of basis vectors.

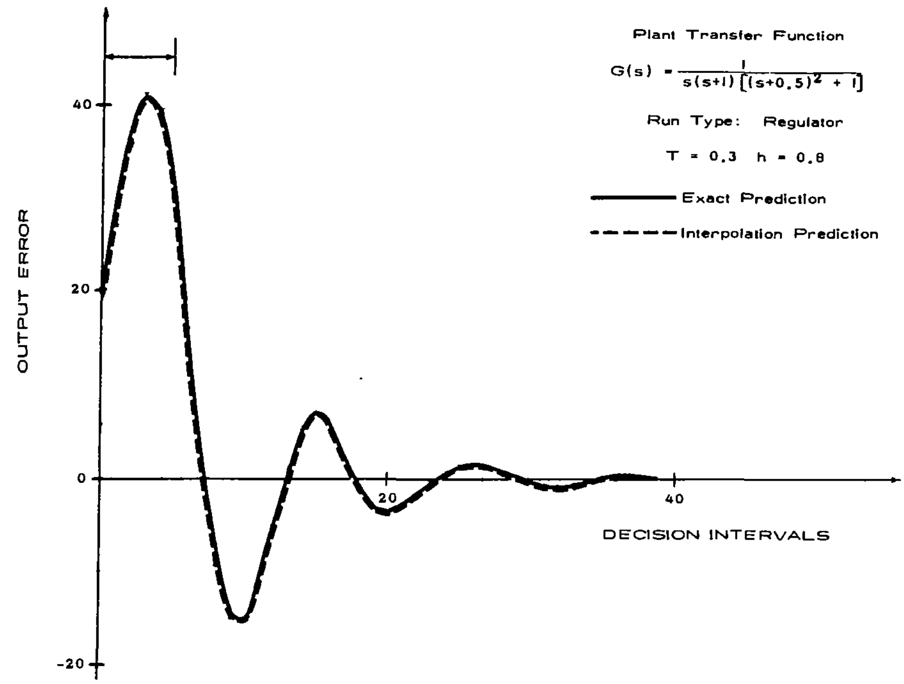


FIGURE 2-49 ERROR RESPONSE OF A 4th ORDER SYSTEM - EXACT AND INTERPOLATION PREDICTION

Typical Interpolation Prediction control system performance for pole and pole-zero configuration plants of fifth and lower order are presented in Figures 2-50 through 2-58. The pole-zero plant configuration control performance is of special interest, since it experimentally demonstrates the control capability for such plant configurations.

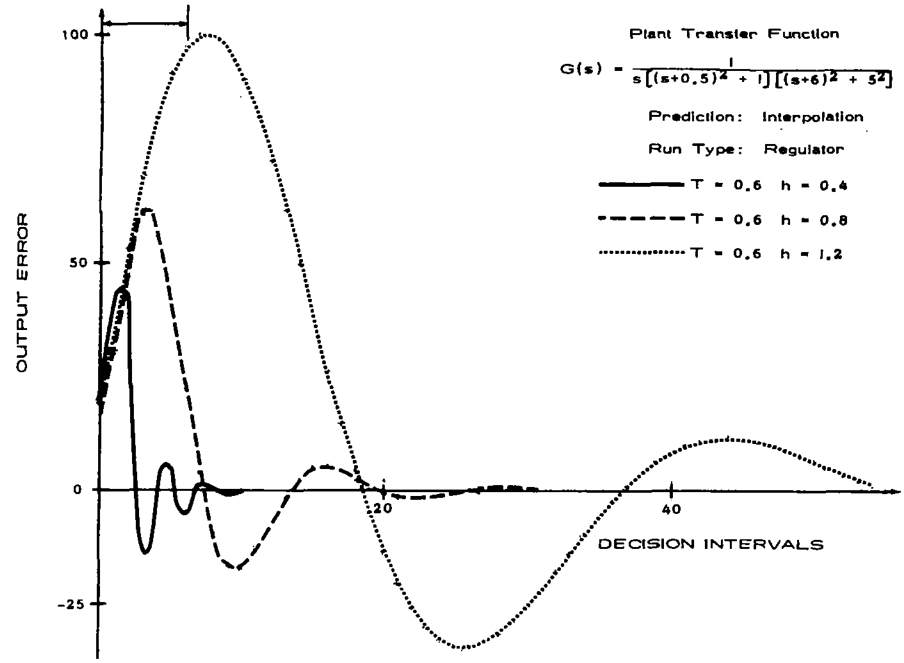


FIGURE 2-50 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

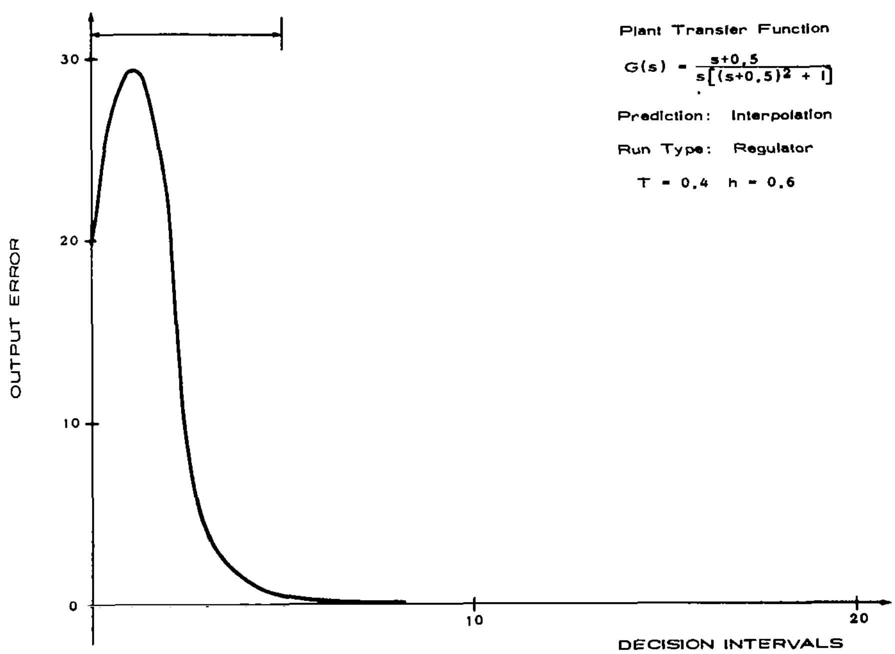


FIGURE 2-51 ERROR RESPONSE OF A 3rd ORDER SYSTEM - INTERPOLATION PREDICTION

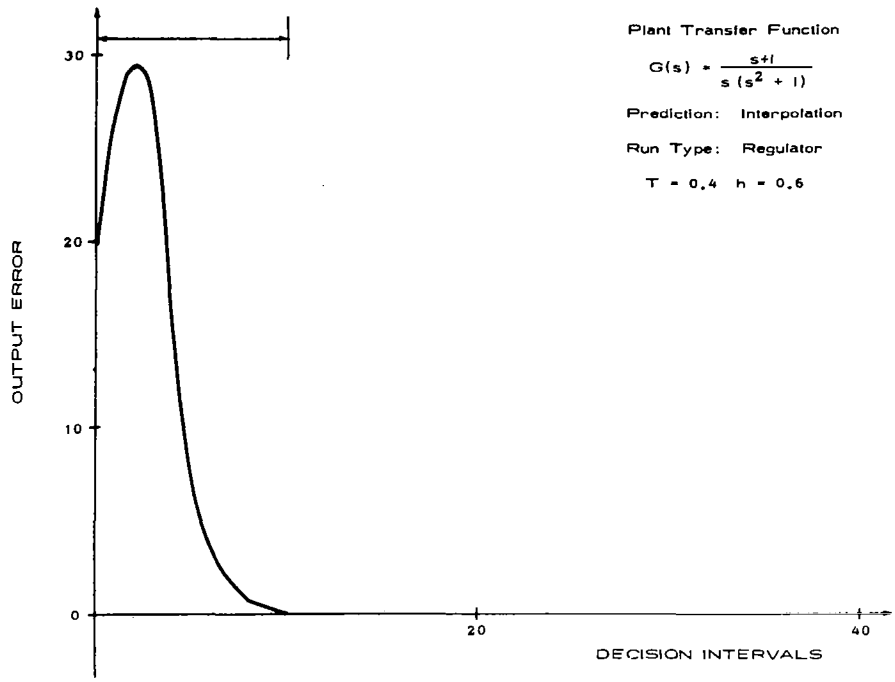


FIGURE 2-52 ERROR RESPONSE OF A 3rd ORDER SYSTEM - INTERPOLATION PREDICTION

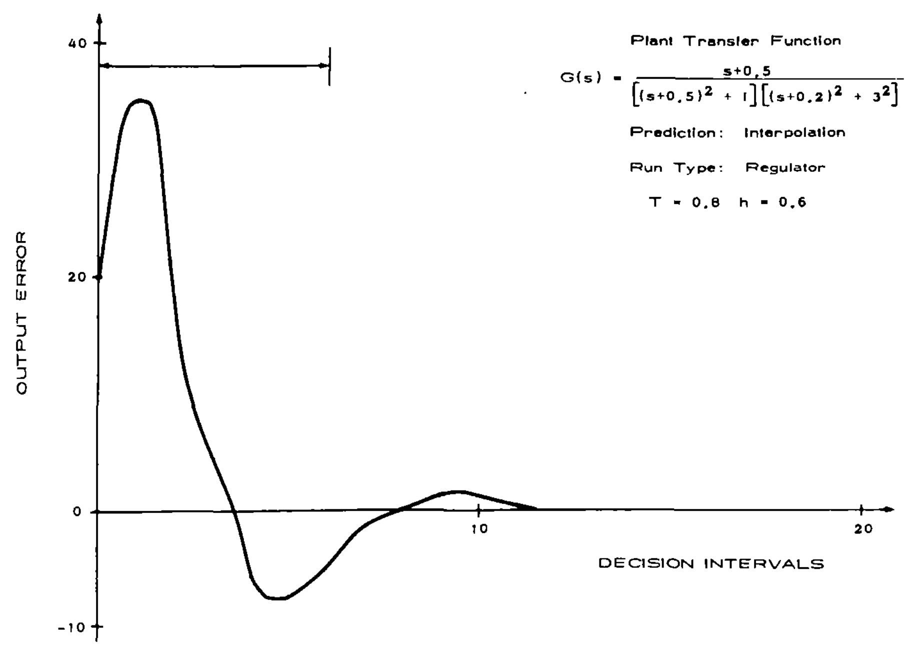


FIGURE 2-53 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

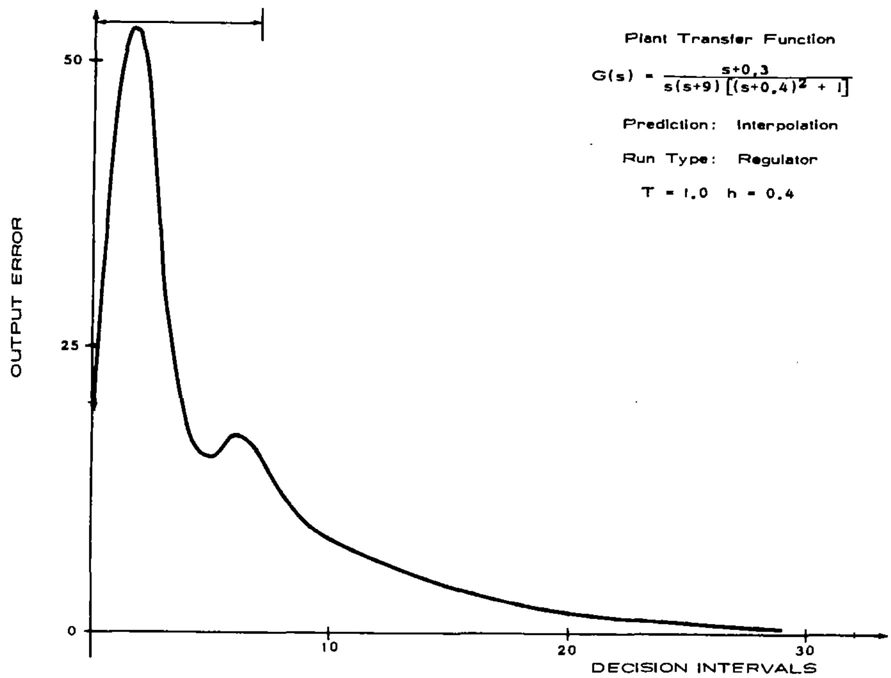


FIGURE 2-54 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

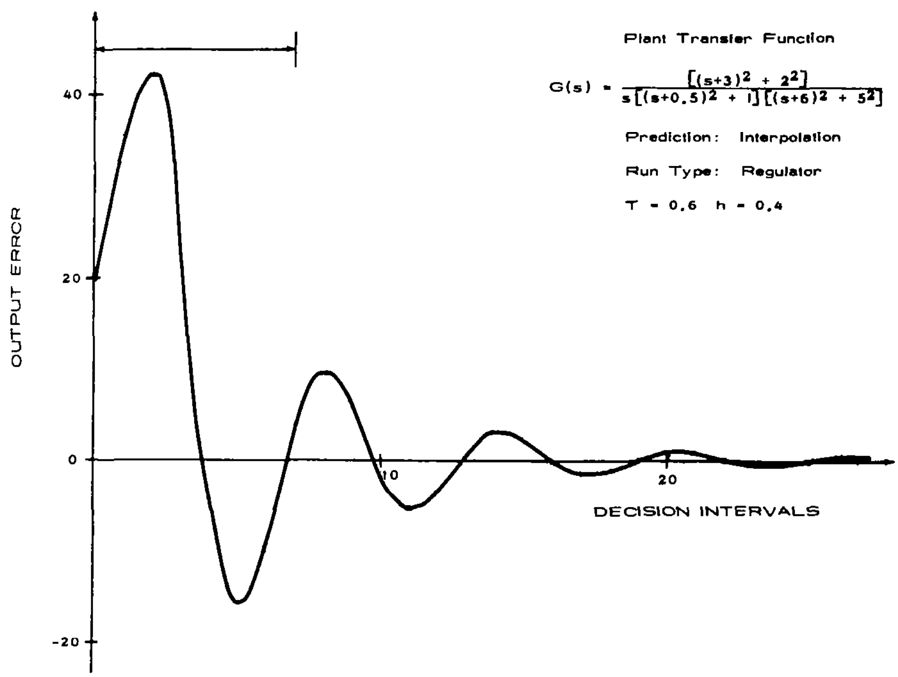


FIGURE 2-55 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

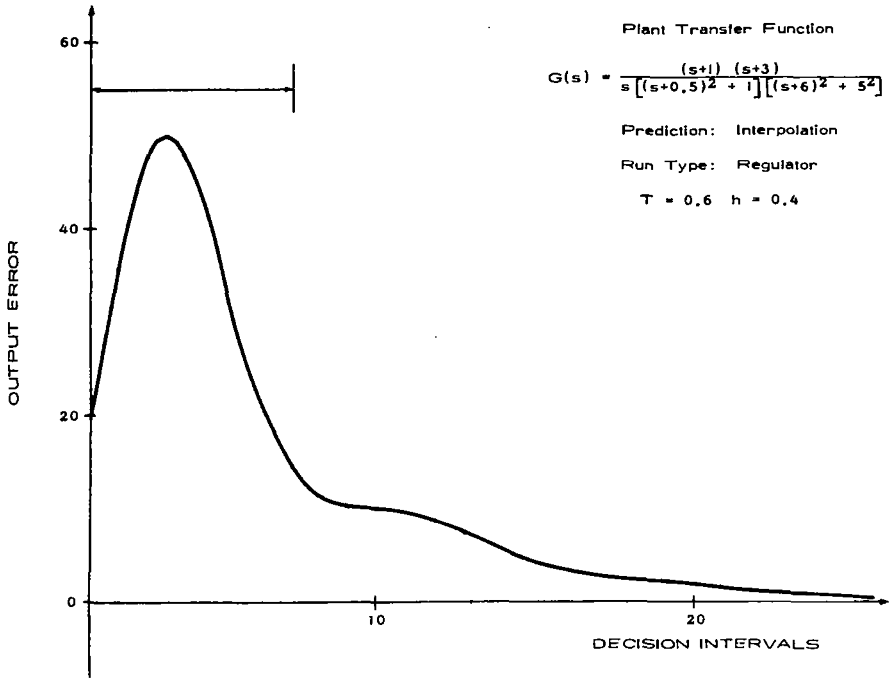


FIGURE 2-56 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

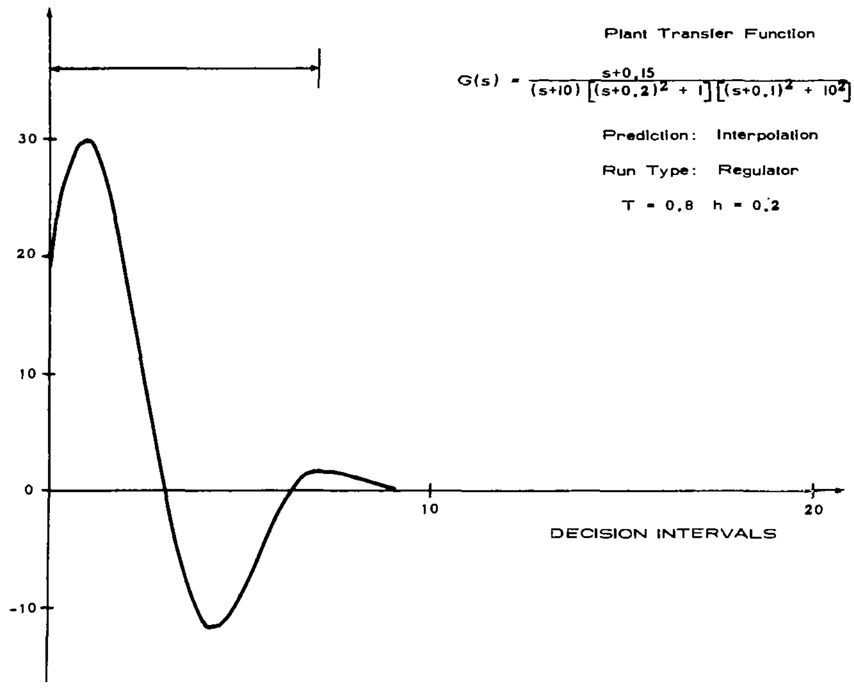


FIGURE 2-57 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

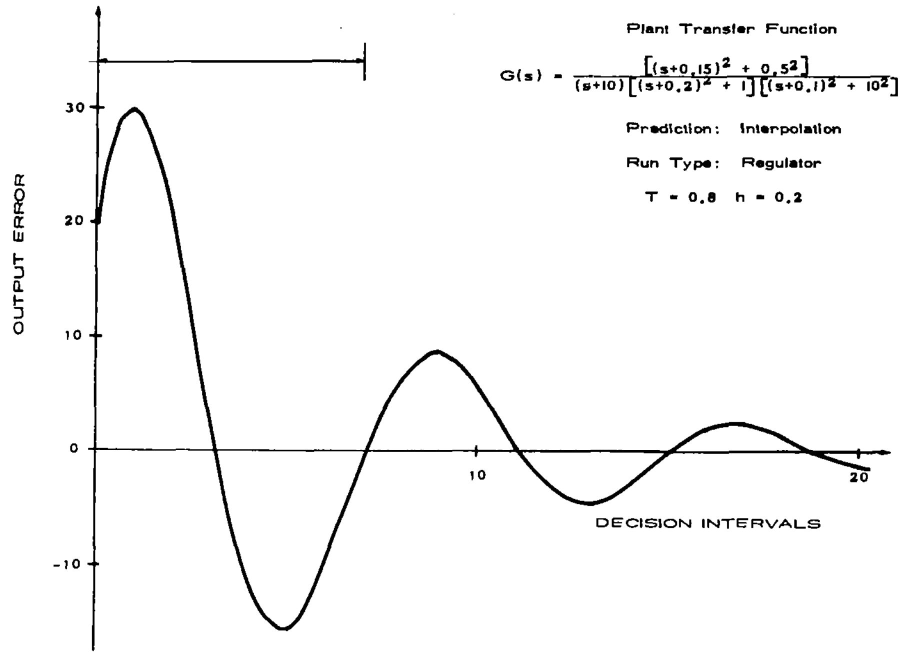


FIGURE 2-58 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

The experimental results for sixth and seventh order systems indicate that in about eighty percent of the cases one arbitrary control force did not prevent matrix singularity or ill conditioning. However, of interest is the result that only about half of the cases encountered very serious control performance deterioration. The problem of matrix singularity was eliminated in all cases by including two arbitrary control forces in the Interpolation Matrix. These arbitrarily selected control forces were of small magnitude, and did not seriously affect the control performance. Figure 2-59 illustrates the control improvement when two non-control policy forces are included in the Interpolation Matrix. However, ill conditioning still occurred in a small number of cases.

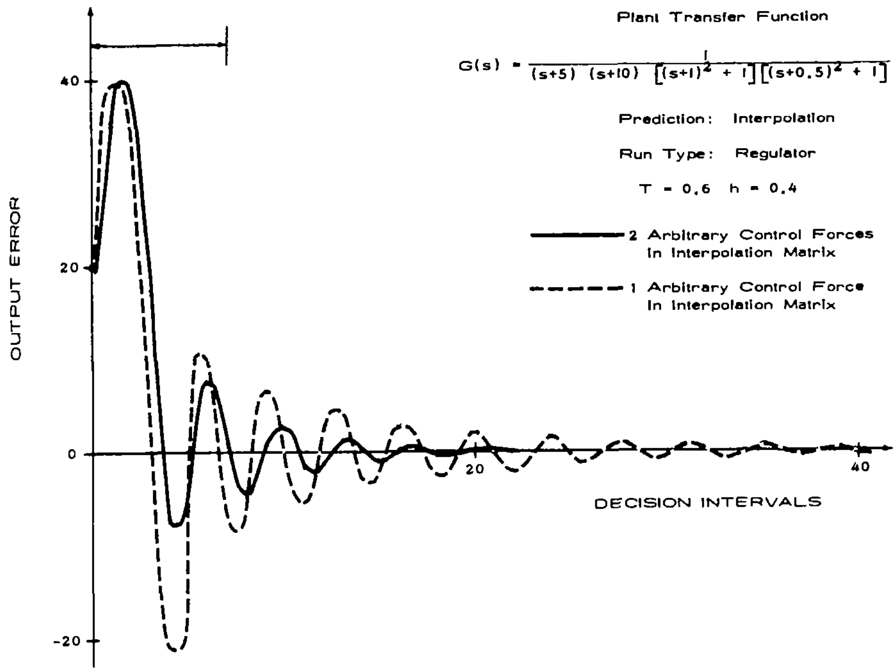


FIGURE 2-59 ERROR RESPONSE OF A 6th ORDER SYSTEM - INTERPOLATION PREDICTION

<u>Trajectory Results.</u>-Exact and Interpolation Prediction control performance was examined for step and ramp desired states for plants through seventh order. An initial value of -20 units was used for each of the state vector components. This fact is evident in the initial values of the output error at time zero in Figure 2-60 (step of +10 units) and 2-61 (ramp starting at zero). The trajectory investigation results were identical to those discussed in the previous presentation of regulator results.

Interpolation Prediction provided good control performance for both step and ramp desired output states on all the fifth and lower order pole configuration systems. Typical results for the pole configuration plants are presented in Figures 2-60 and 2-61. It may be noted in both figures that the error went to zero with approximately the same controlled response.

The control of pole-zero configuration plants was found to be dependent on the plant transfer function. If the plant contained an integration good performance was obtained for steps, and a constant error for ramps. These results are illustrated in Figures 2-62 and 2-63. The plants without an integration exhibited a constant error for steps, and an increasing error for ramps. Figures 2-64 and 2-65 show typical control results for such plants. The failure of adequate control for pole-zero configuration plants is discussed in paragraph 2.3, and analytical reasons for such performance are presented in Appendix E.

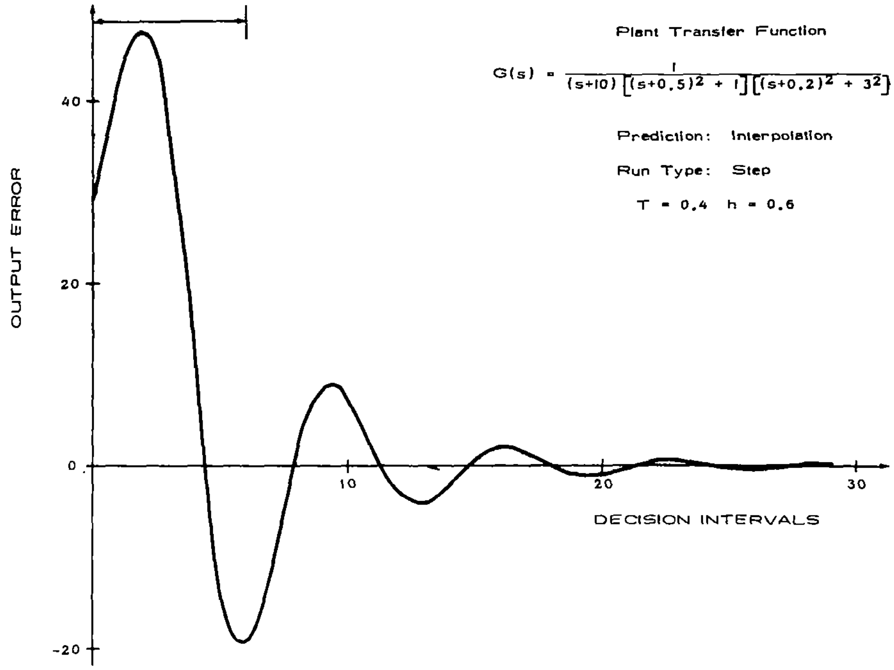


FIGURE 2-60 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

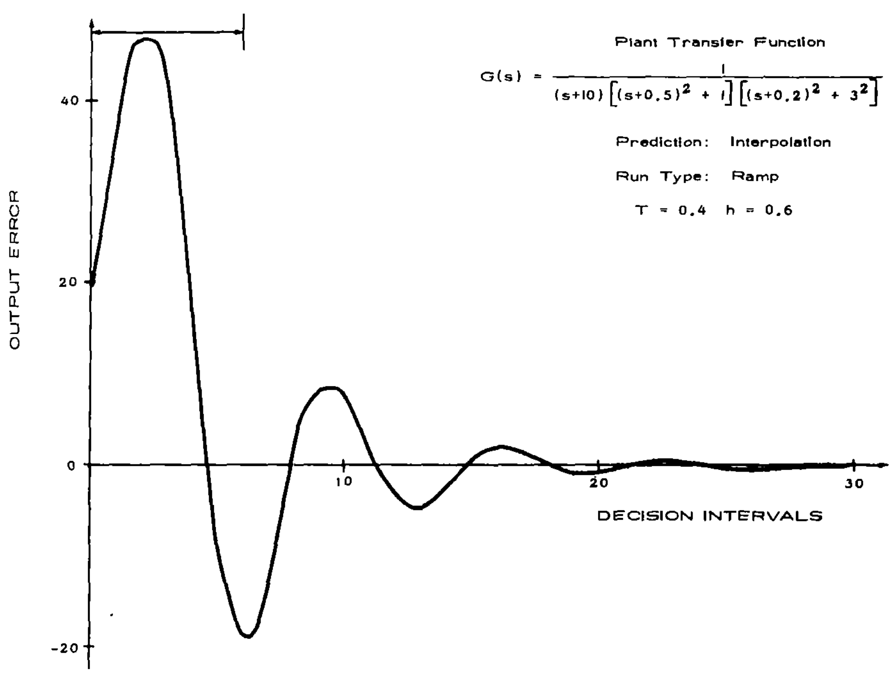


FIGURE 2-61 ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

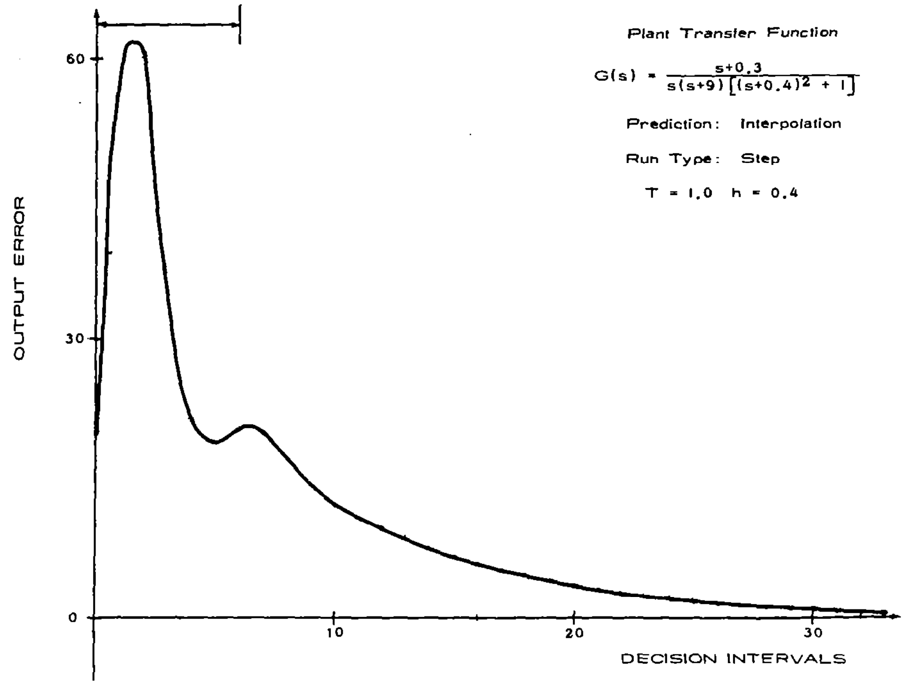


FIGURE 2-62 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

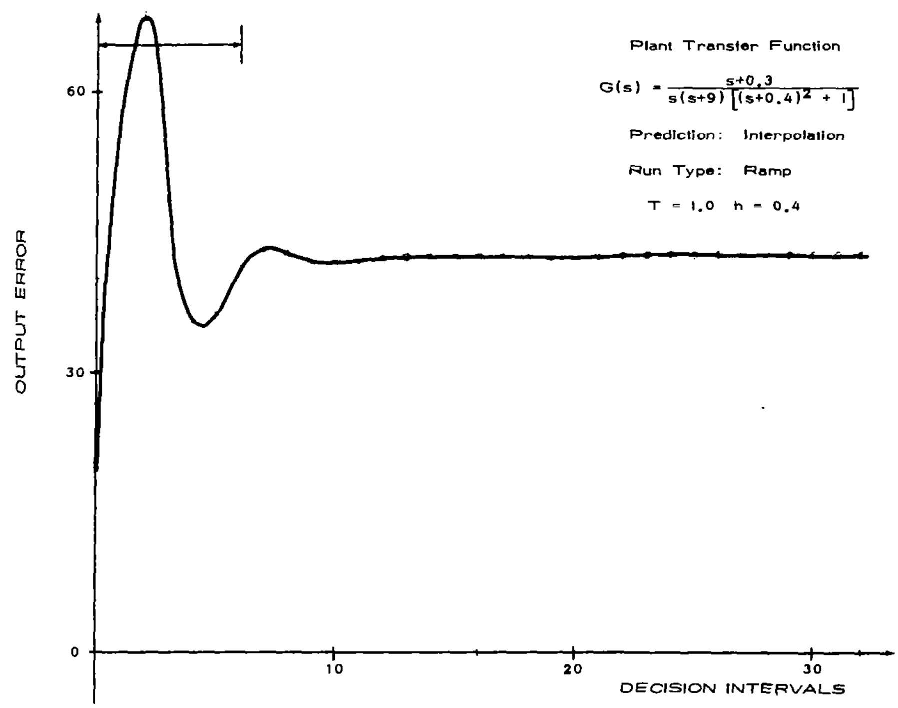


FIGURE 2-63 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

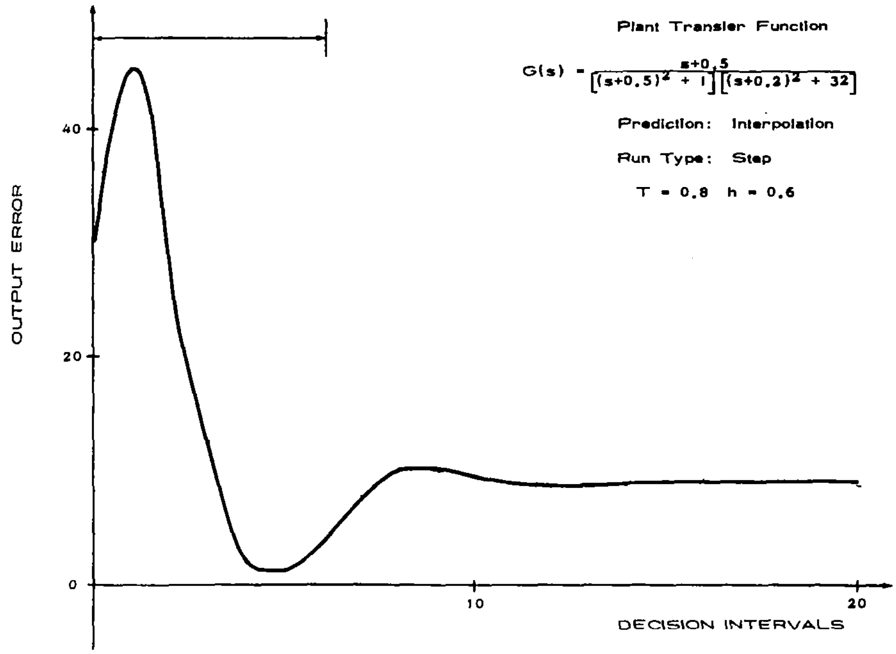


FIGURE 2-64 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

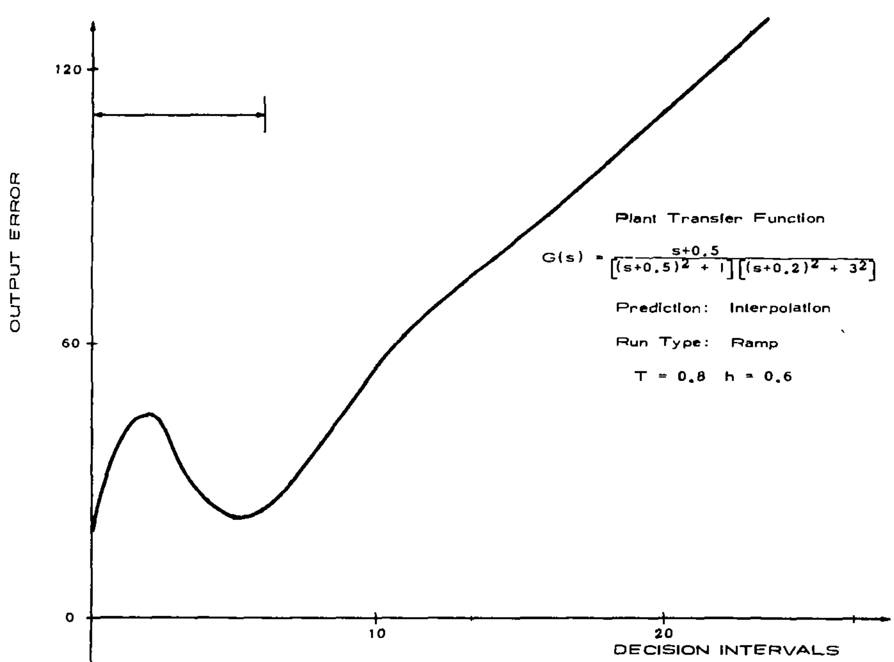


FIGURE 2-65 ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

INTERPOLATION PREDICTION CONTROL TO LOWER THAN ACTUAL SYSTEM ORDER

The present experimental study was undertaken to determine the control effectiveness for Interpolation Prediction carried out in less coordinates than the true system order. Since a method of establishing stability boundaries for such prediction does not exist at the present time, the lower than actual order Exact Prediction boundaries were used as a guide to possible stable T-h operation points. Approximately one hundred control simulations were made for systems of third through sixth order. In about ten per cent of these cases, unstable system control was observed for the T-h values taken from lower than actual order Exact Prediction boundaries.

Example results are presented in Figures 2-66 through 2-69 for third and fourth order systems controlled as second and third order respectively. When lower than actual order control is employed the start-up time becomes less as is indicated by the arrows in the figures. This results from the fact that the interpolation matrices are smaller and less data is required. In most cases the Interpolation Predicition with actual system order provided the best control performance. Interpolation Prediction with lower than actual system order is usually less precise and more sluggish. This observation is also true for the fifth and sixth order systems controlled as lower than actual order. Figures 2-70, 2-71, and 2-72 illustrate typical experimental results for the higher order systems.

WEIGHTING FACTOR PARAMETER STUDY RESULTS

The two parameters of our control system are the length of the decision interval (T), and the weighting factor (h). These parameters were discussed in paragraph 2.1. This study was concerned with establishing experimentally the effect of the weighting factor on control system performance. The experimental procedure consisted of obtaining the control system performance for three h values with a constant T value of 0.6 seconds. In most cases, the T-h combination provided stable control points. Both pole and pole-zero configuration plants through sixth order were used in this parameter study. Approximately fifty Interpolation Prediction control simulations were conducted on a total of eight systems.

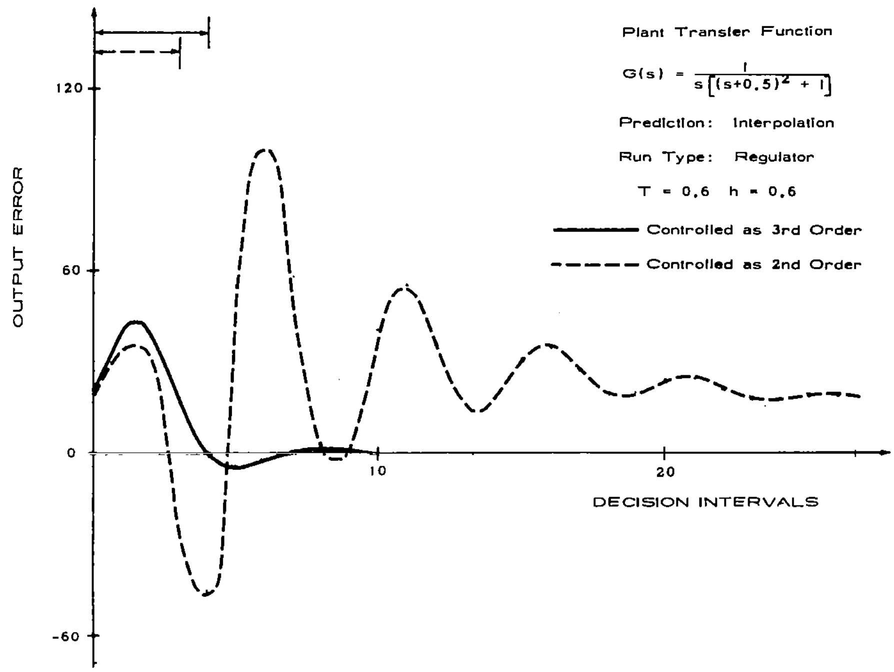


FIGURE 2-66 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 3rd ORDER SYSTEM - INTERPOLATION PREDICTION

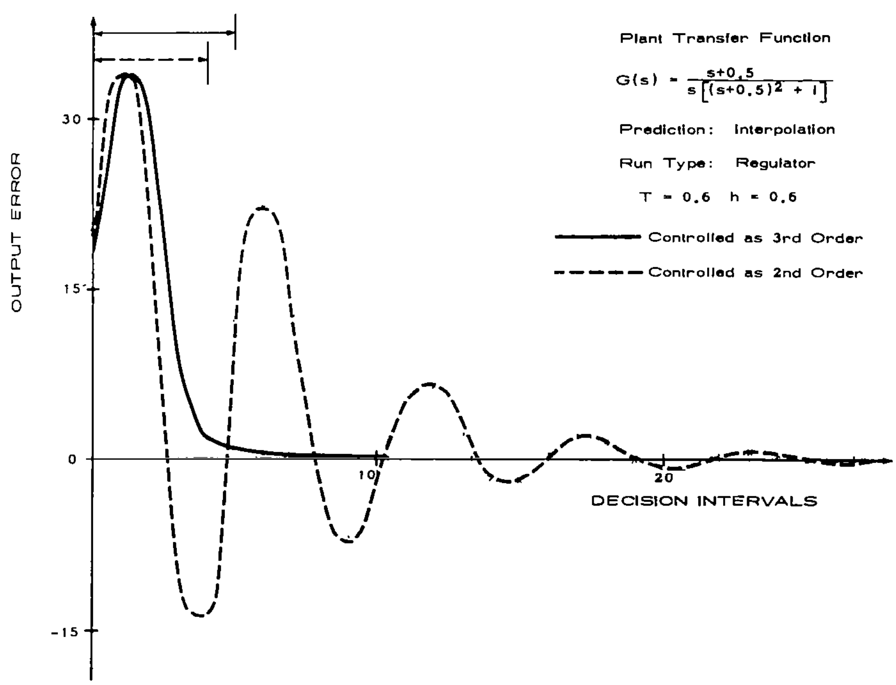


FIGURE 2-67 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 3rd ORDER SYSTEM - INTERPOLATION PREDICTION

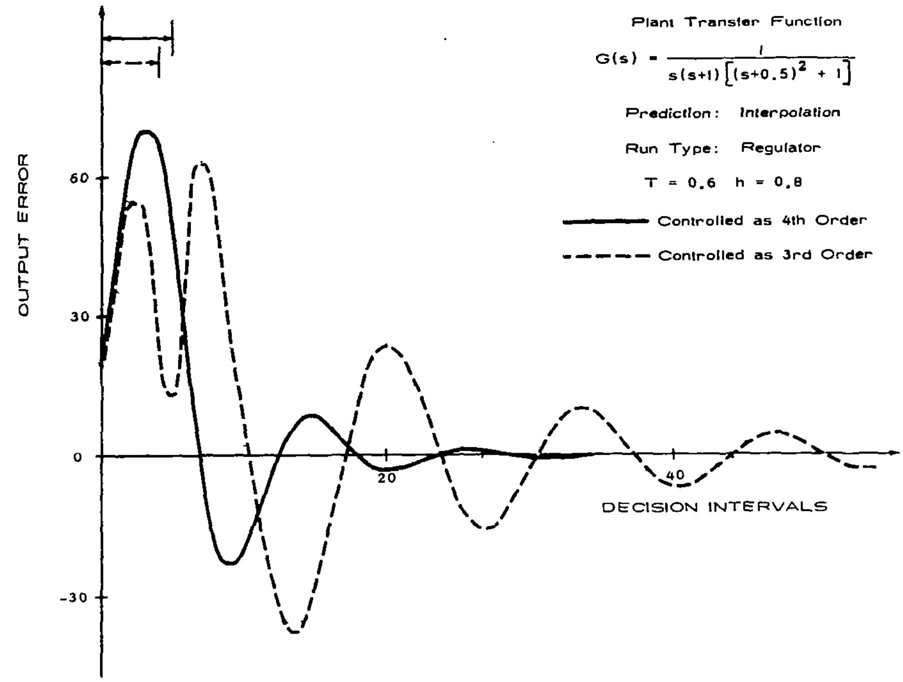


FIGURE 2-68 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

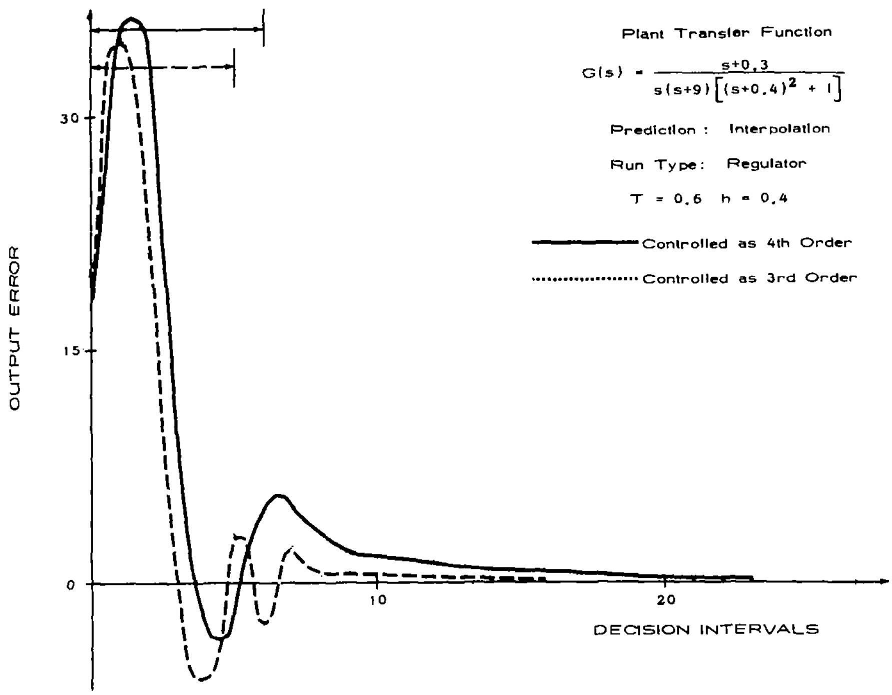


FIGURE 2-69 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 4th ORDER SYSTEM - INTERPOLATION PREDICTION

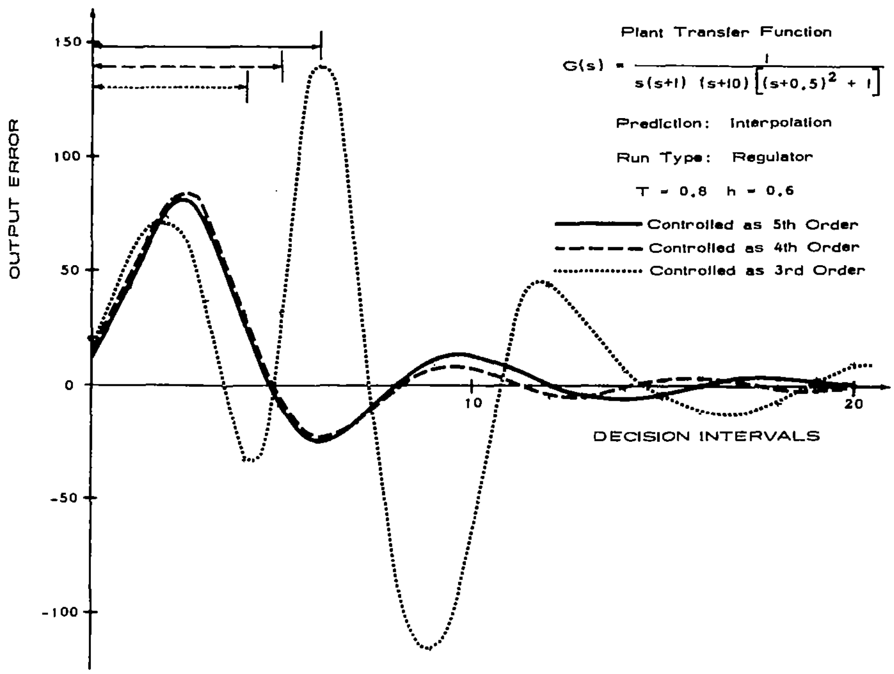


FIGURE 2-70 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

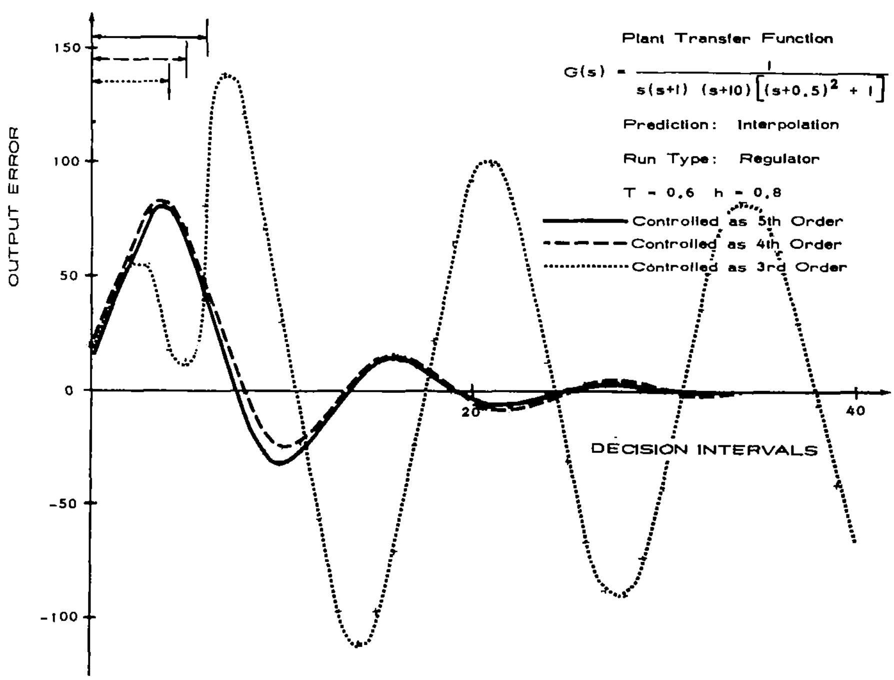


FIGURE 2-71 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 5th ORDER SYSTEM - INTERPOLATION PREDICTION

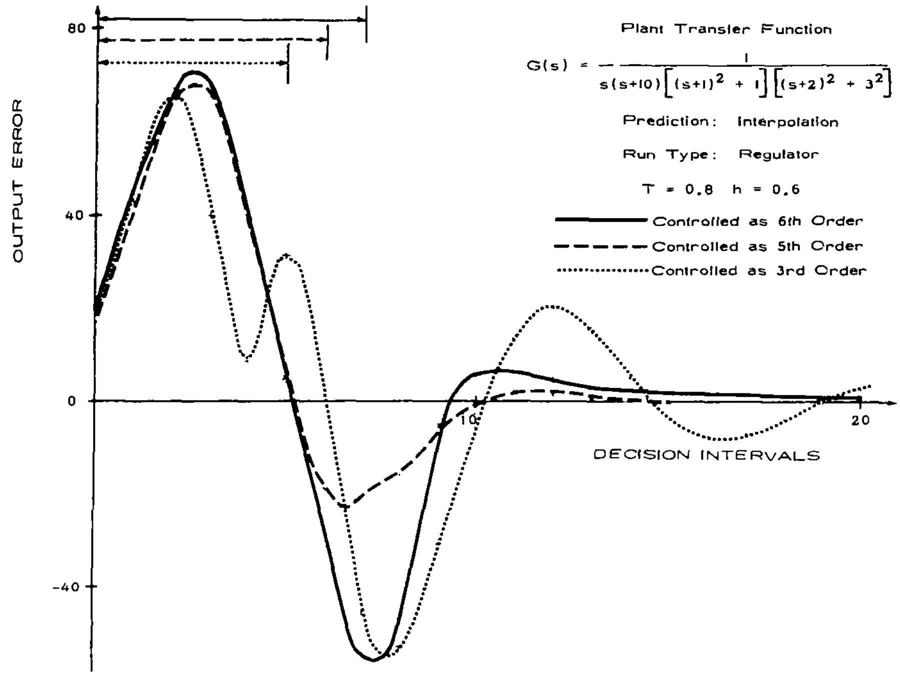


FIGURE 2-72 LOWER THAN ACTUAL ORDER CONTROL ERROR RESPONSE OF A 6th ORDER SYSTEM - INTERPOLATION PREDICTION

Similar results were obtained for all systems examined with both zero and step desired output states. In all cases each state vector components' initial condition was -20 units, and Exact Prediction was utilized to obtain the matrix of basis vectors during the start-up simulation phase. Typical results of the pole configuration plants are presented in Figure 2-73 for h values of 0.4, 0.8, and 1.2. It is easily observed that the greater weighting factors resulted in more sluggish control. This same observation may be seen in Figure 2-74 in control of a pole-zero configuration plant. Also, it may be noted that in all cases the low value of h provided the fastest and usually most oscillatory response.

CONTROL FORCE SATURATION STUDY RESULTS

In the previously described experimental studies the control force was unlimited in magnitude. However, in practical control situations, a limit would be imposed on the applied control force. Past research into this area was of a limited nature, and considered only a limited number of low order pole configuration plants.

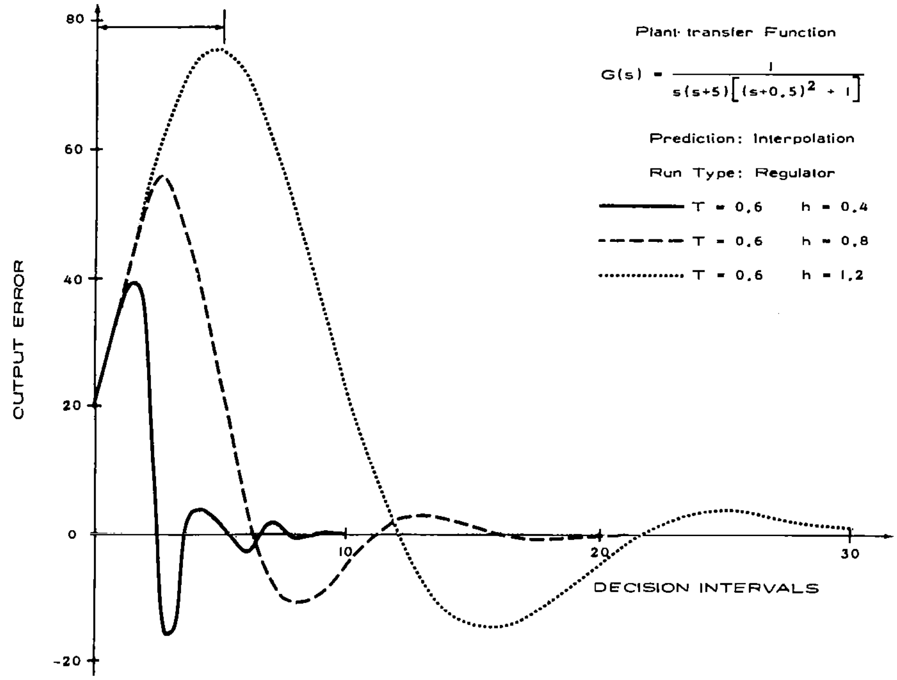


FIGURE 2-73 ERROR RESPONSE OF A 4th ORDER SYSTEM FOR THREE VALUES OF h

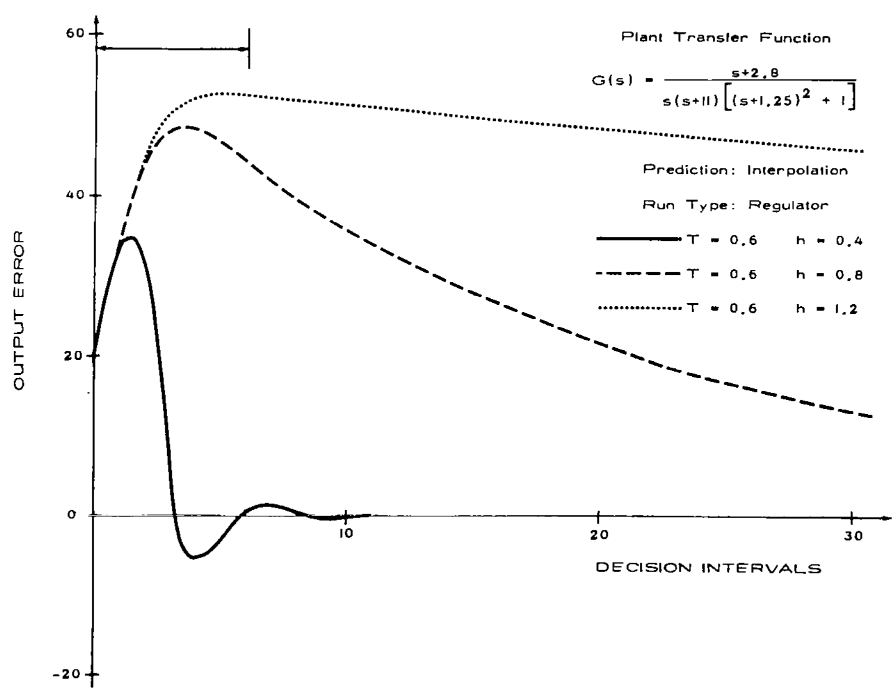


FIGURE 2-74 ERROR RESPONSE OF A 4th ORDER SYSTEM FOR THREE VALUES OF h

The present study was conducted on both pole and pole-zero configuration systems of third through sixth order. Also, this study considered three different stable points (T = 0.6 for h = 0.4, 0.8, 1.2) for each system. It should be noted that experimentation was conducted for both zero and step desired output states. As in previous experimentation, each state vector component initial condition was -20 units, and Exact Prediction was used to obtain the matrix of basis vectors during start-up. Approximately one hundred control force saturation simulations were completed for seven different systems.

The control limiting study consisted of determining the system response with unlimited control, and 50%, 25%, and 10% control force saturation. The experimental procedure was to limit the available control force magnitude to a percentage of the maximum control force requested in the unlimited run of the particular system being studied. Typical results for the T = 0.6 and h = 0.4 combination are presented in Figures 2-75 and 2-76 for two different fourth order systems. The results at this low h value indicate that control force saturation can cause serious deterioration in control performance. It should be noted that no objectionable deterioration is seen in the system response until the control force limit is reached about 70% or more of the time. Figures 2-77 and 2-78 show typical results for the T = 0.6 and h = 0.8 combination for the same two systems. Also, for this h value no series deterioration is seen in the system response until the limit is reached about 70% of the time. The h = 0.8 system response was unacceptable due to sluggishness, whereas the lower h value case was unacceptable due to the resultant limit cycle. The experimental results for the T = 0.6 and h = 1.2 combination were approximately like those obtained for the h = 0.8 value. The system response was unacceptable due to sluggishness once the control force limit was reached about 70% of the time.

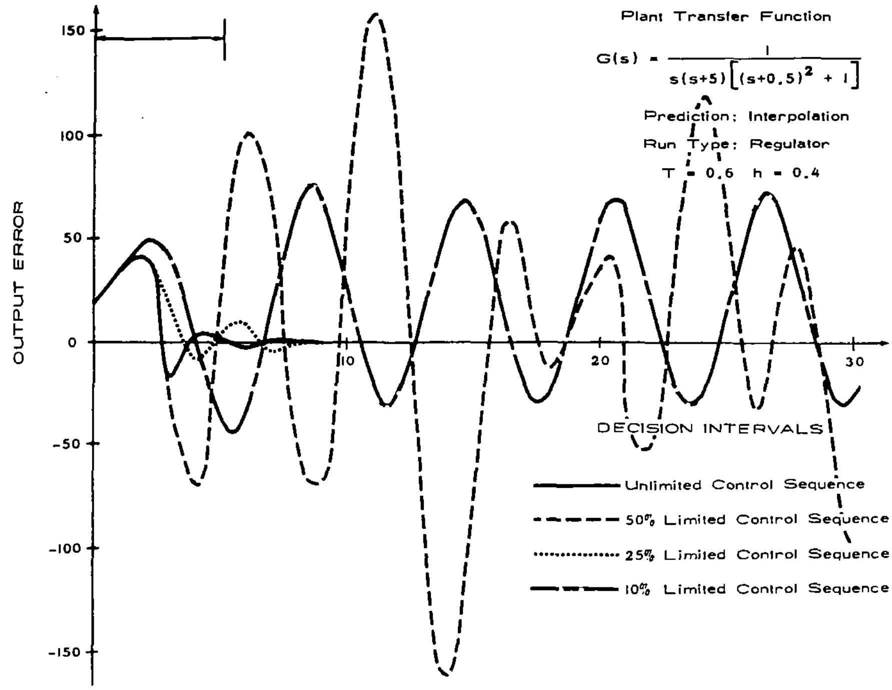
INTERPOLATION PREDICTION CONTROL - WITH UPDATING

In the previously described Interpolation Prediction results, the current sensitivity and current response were determined only once, and then used for the duration of the control simulation. This procedure was justified since the systems under study were stationary, and the only purpose of the previous studies was to demonstrate control system performance when Interpolation Prediction provided good current sensitivity and current response matrices. In light of the assumption that knowledge of the plant with respect to such factors as time variation is not available, the effects of updating the current sensitivity and current response are of practical interest. This study considered the problems associated with Interpolation Prediction with updating, and their resultant effects on the control system performance.

The critical problem of avoiding a singular matrix of basis vectors (see Appendix F) for zero or step desired output states required that at least one non-control policy force be present in this matrix. The control force perturbation required to eliminate the singularity was investigated on a limited basis. One of two alternate multiplying factors, 1.5 or 1.1, was used to alter the calculated control policy force. Therefore, the one non-control policy force contained in the matrix of basis vectors consisted of either 1.5 or 1.1 times the control force actually calculated by the control policy. The effect of such an applied force on control system performance is of considerable importance.

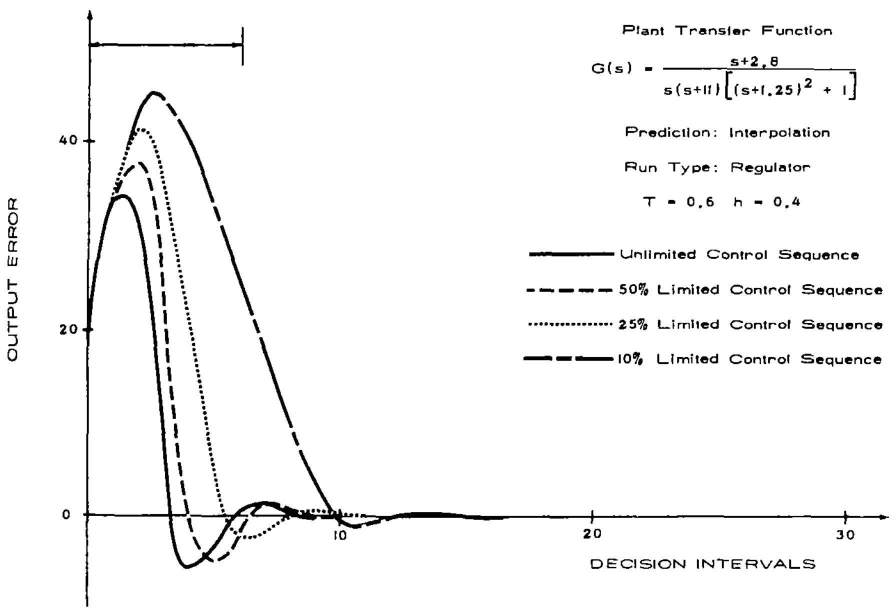
Another problem associated with updating is how and when new data should be included in the matrix of basis vectors. The experimentation in this area consisted of shifting the most current data into the matrix of basis vectors every k intervals. Values of $k = 1, 2,$ or 10 were used for this investigation. The related problem of how often to recalculate the current sensitivity and current response also was investigated in a very limited degree. The new current sensitivity and current response matrices were obtained every fifth interval in the majority of the cases.

The following experiments were conducted using the Interpolation Prediction which is not a self-starting method. Therefore, a start-up procedure was developed to allow Interpolation Prediction control to start at any desired initial system state. This procedure required an arbitrary set (equal to one greater than the dimension of the basis vector) of control forces to be applied to the system from some desired state at time t_o. The procedure



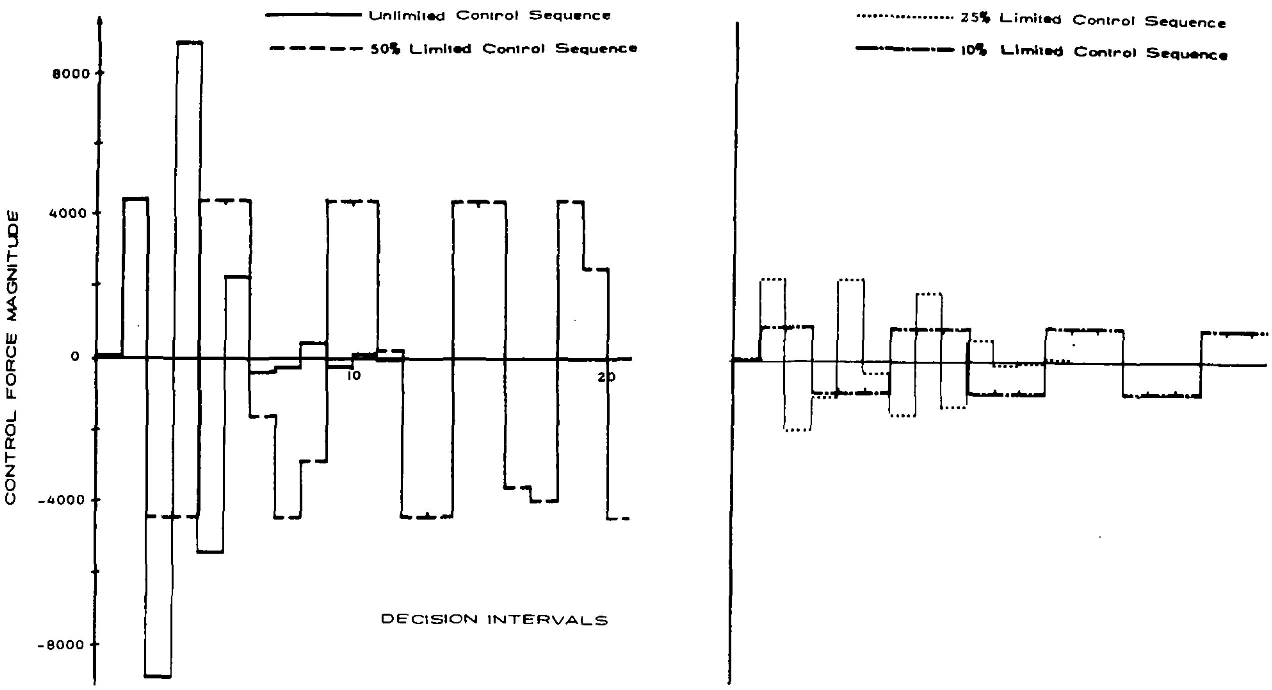
(a) OUTPUT ERROR RESPONSE

FIGURE 2-75 CONTROL FORCE SATURATION EFFECTS



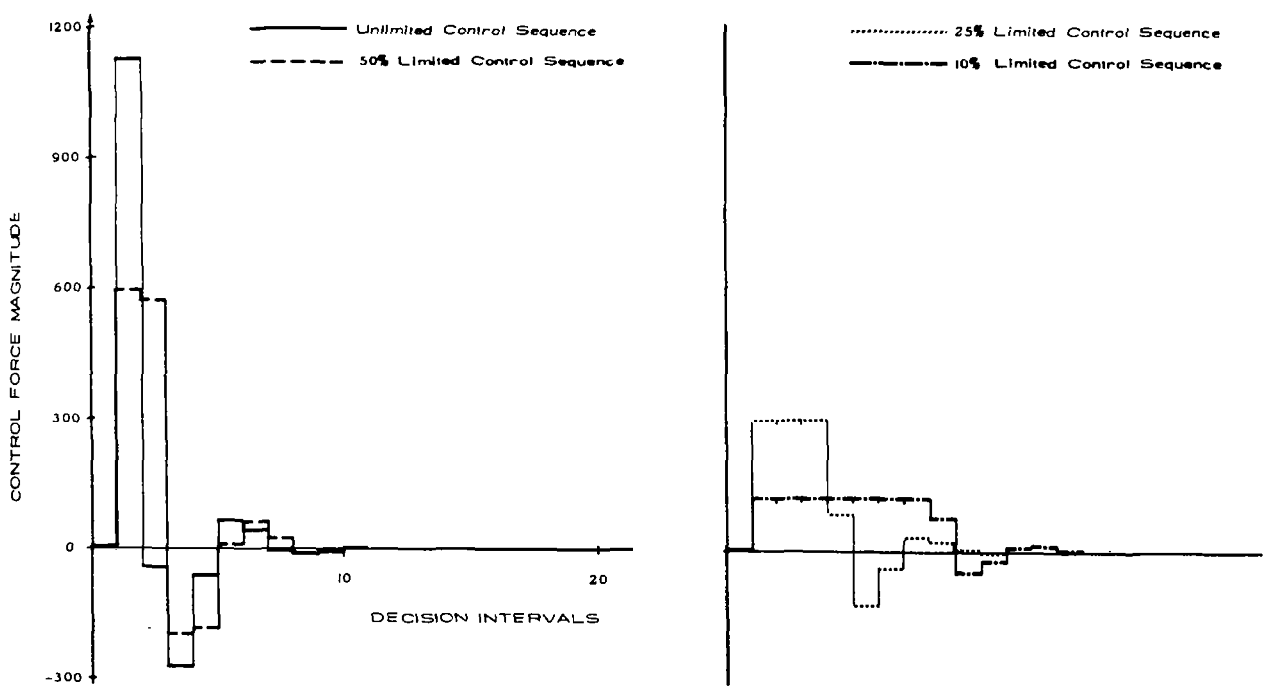
(a) OUTPUT ERROR RESPONSE

FIGURE 2-76 CONTROL FORCE SATURATION EFFECTS



(b) CONTROL FORCE SEQUENCE

FIGURE 2-75 CONTROL FORCE SATURATION EFFECTS



(b) CONTROL FORCE SEQUENCE

FIGURE 2-76 CONTROL FORCE SATURATION EFFECTS

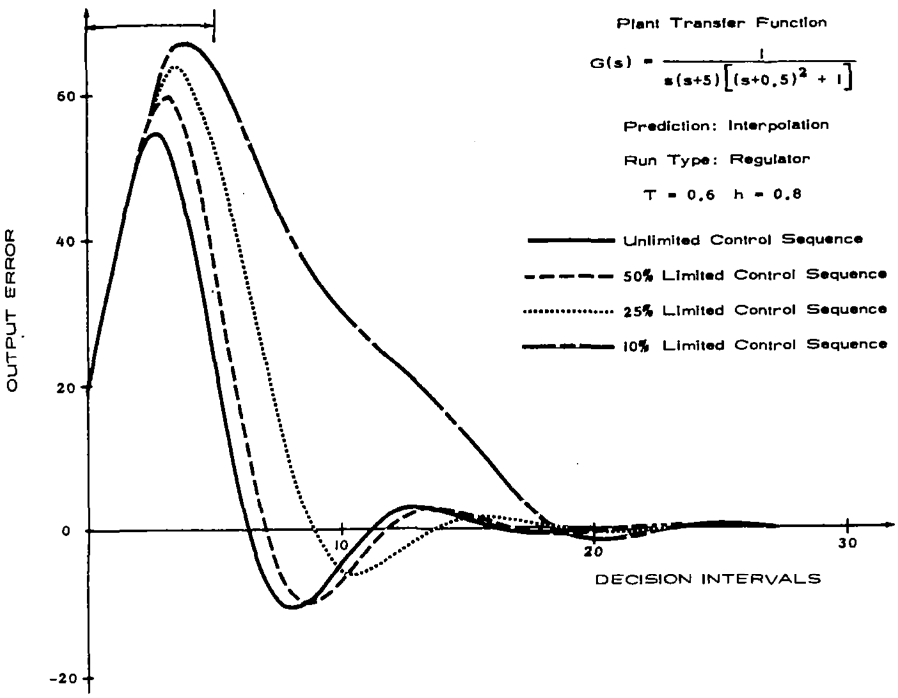


FIGURE 2-77 CONTROL FORCE SATURATION EFFECTS

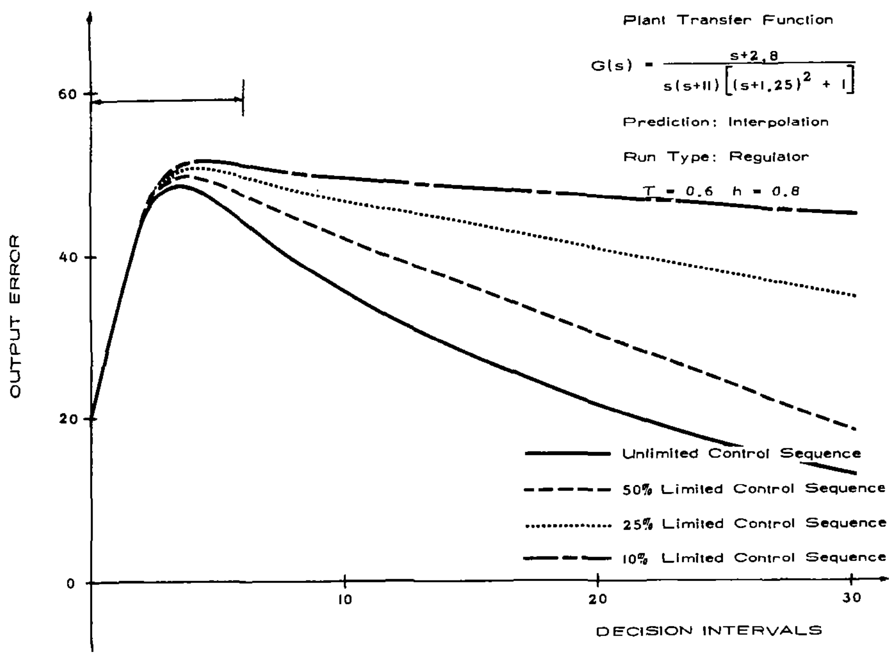


FIGURE 2-78 CONTROL FORCE SATURATION EFFECTS

develops the matrix of basis vectors from the system response resulting from the arbitrary control forces being applied backwards from the desired state at time t_o. Once the matrix of basis vectors is developed, the current sensitivity and current response matrices are calculated, and used to control the system from the initial state at time t_o. This start-up method is of no practical importance, and is used only as an experimental procedure.

The start-up simulation phase is not included in the runs. At the start of each of the runs shown in the following figures the current sensitivity and current response matrices have been predetermined by the Interpolation method, and so the control is effected by the Interpolation method during the entire run. Also, the figures are reproductions of the graphs plotted by the computer program with the exception that the axes have been relabeled, and the data points have been connected using straight line approximation.

Experimental Results.-The following results summarize the many experiments performed to investigate the above problems associated with updating. About sixty control simulations were conducted on twelve pole configuration plants of second through fifth order. The first two sets of figures are presented to illustrate the effects of the 1.5 and 1.1 multiplying factors. Figures 2-79 through 2-81 show that the matrix of basis vectors indeed did not go singular, but that the 1.5 factor did cause undesirable control for the step and ramp desired output states. Figures 2-82 through 2-84 present the results obtained using the 1.1 multiplying factor. The results illustrated by these figures indicate that the 1.1 factor was sufficient to keep the matrix of basis vectors from becoming singular, and was a small enough perturbation so as not to degrade the control performance. It should be noted that the above results were typical of those for fourth and lower order sytems. However in some cases, the 1.1 multiplication factor was not a large enough perturbation to prevent ill conditioning of the matrix of basis vectors. This fact was usually true for the fifth order systems. Typical results for fifth order systems utilizing the 1.5 factor are presented in Figures 2-85 and 2-86. All of the results thus far presented shifted new data into the matrix of basis vectors every other interval, and recalculated the current sensitivity and current response every fifth interval.

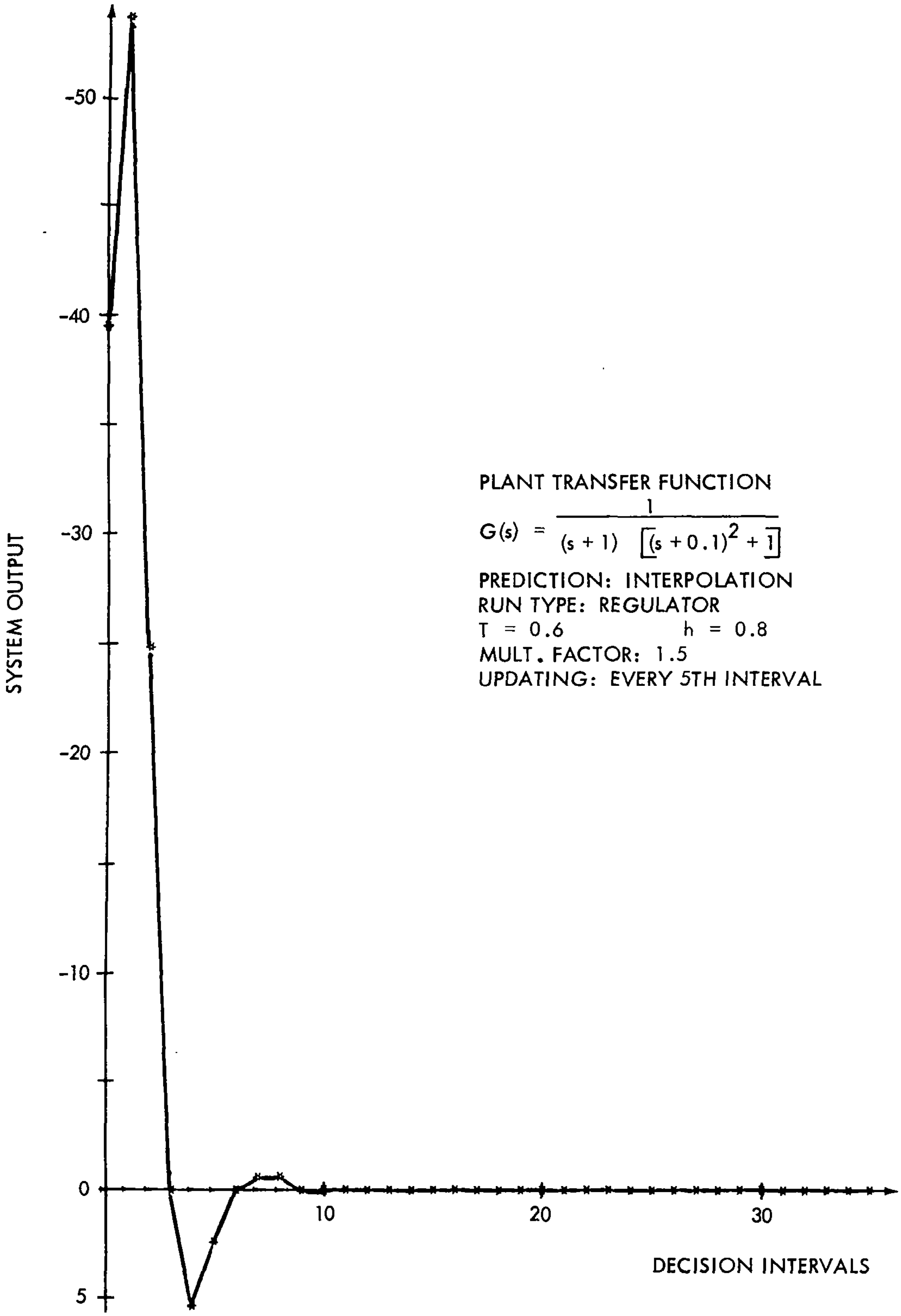


FIGURE 2-79 REGULATOR RUN USING INTERPOLATION PREDICTION WITH UPDATING

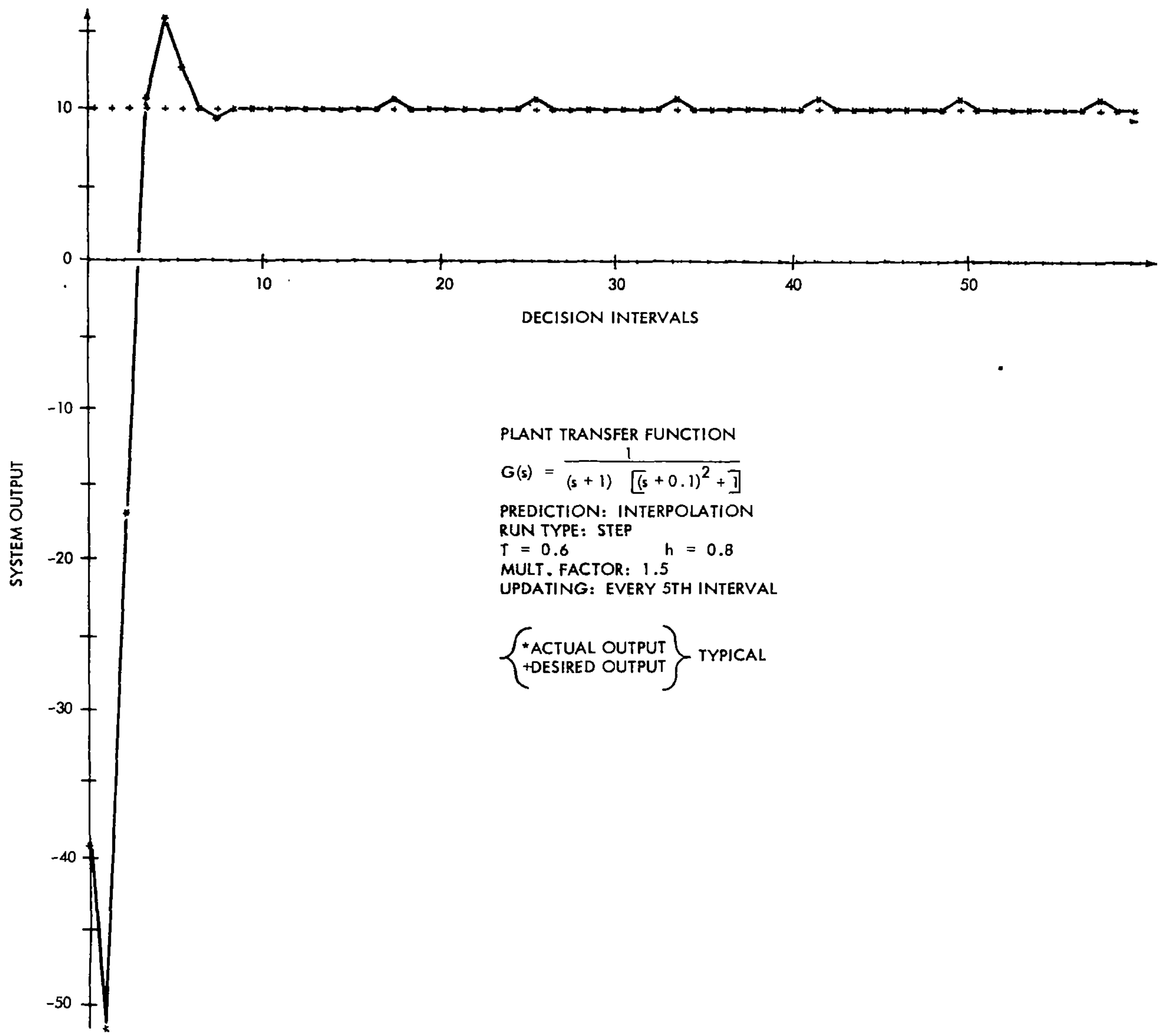


FIGURE 2-80 STEP RUN USING INTERPOLATION PREDICTION WITH UPDATING

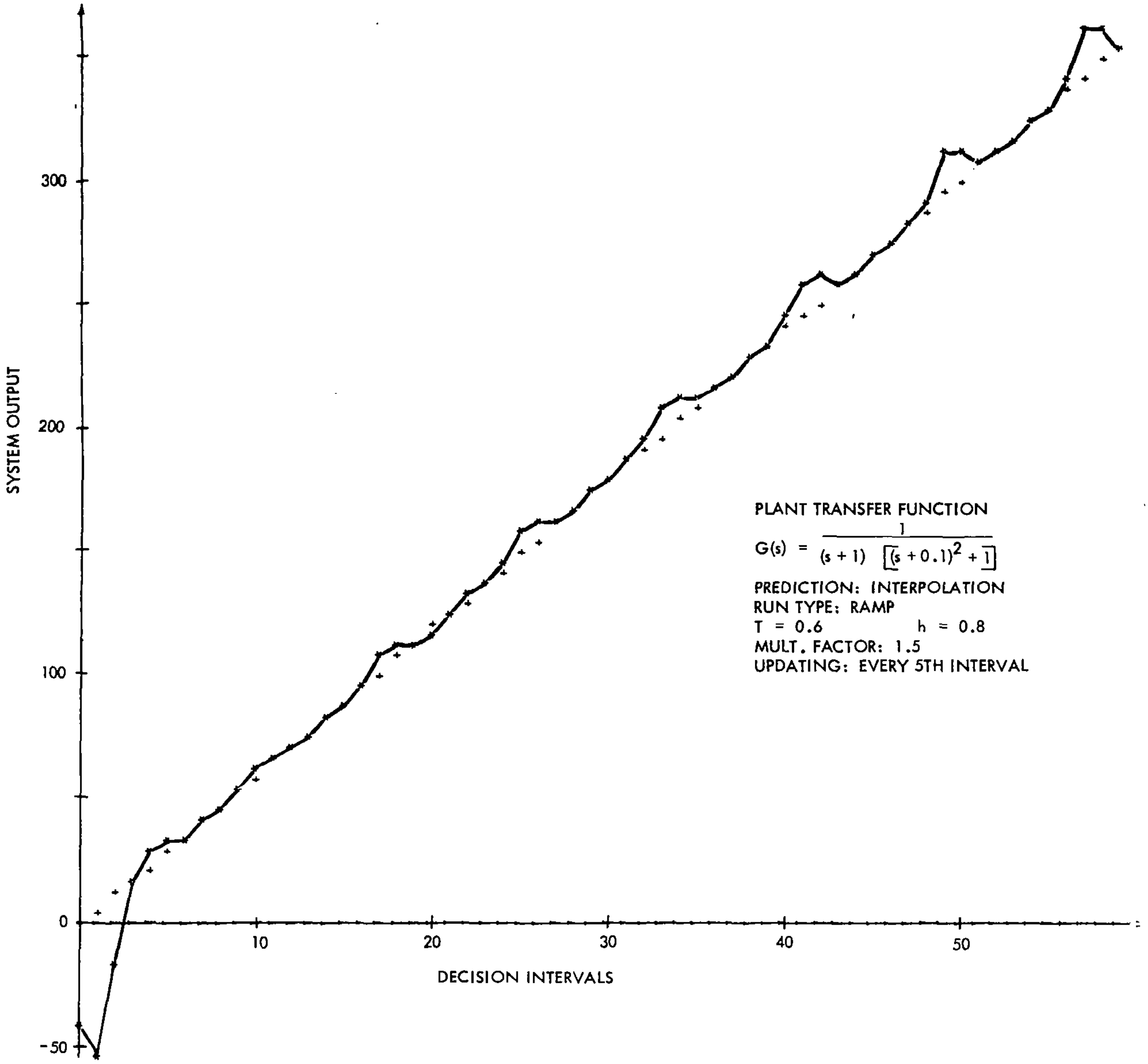


FIGURE 2-81 RAMP RUN USING INTERPOLATION PREDICTION WITH UPDATING

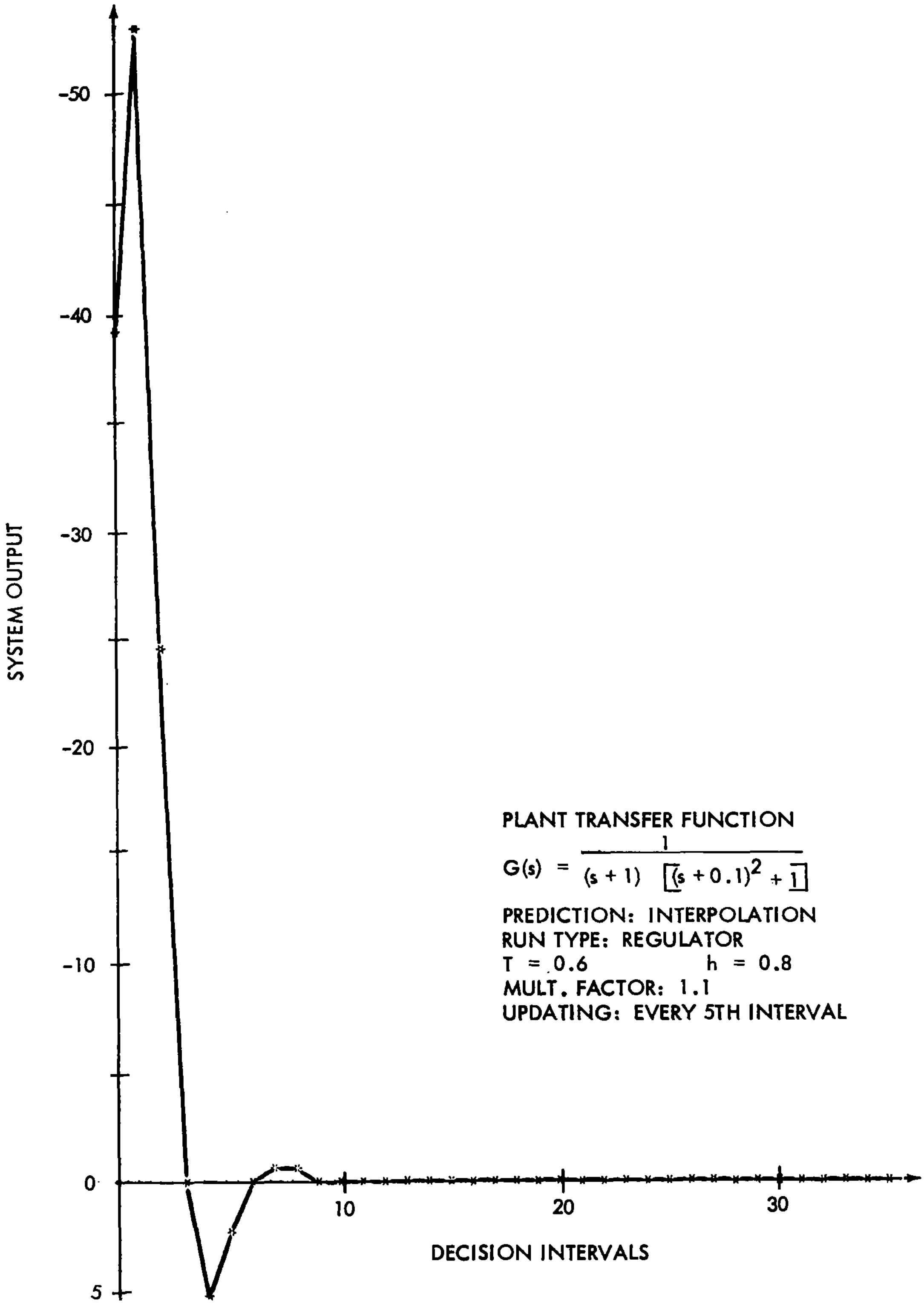


FIGURE 2-82 REGULATOR RUN USING INTERPOLATION PREDICTION WITH UPDATING

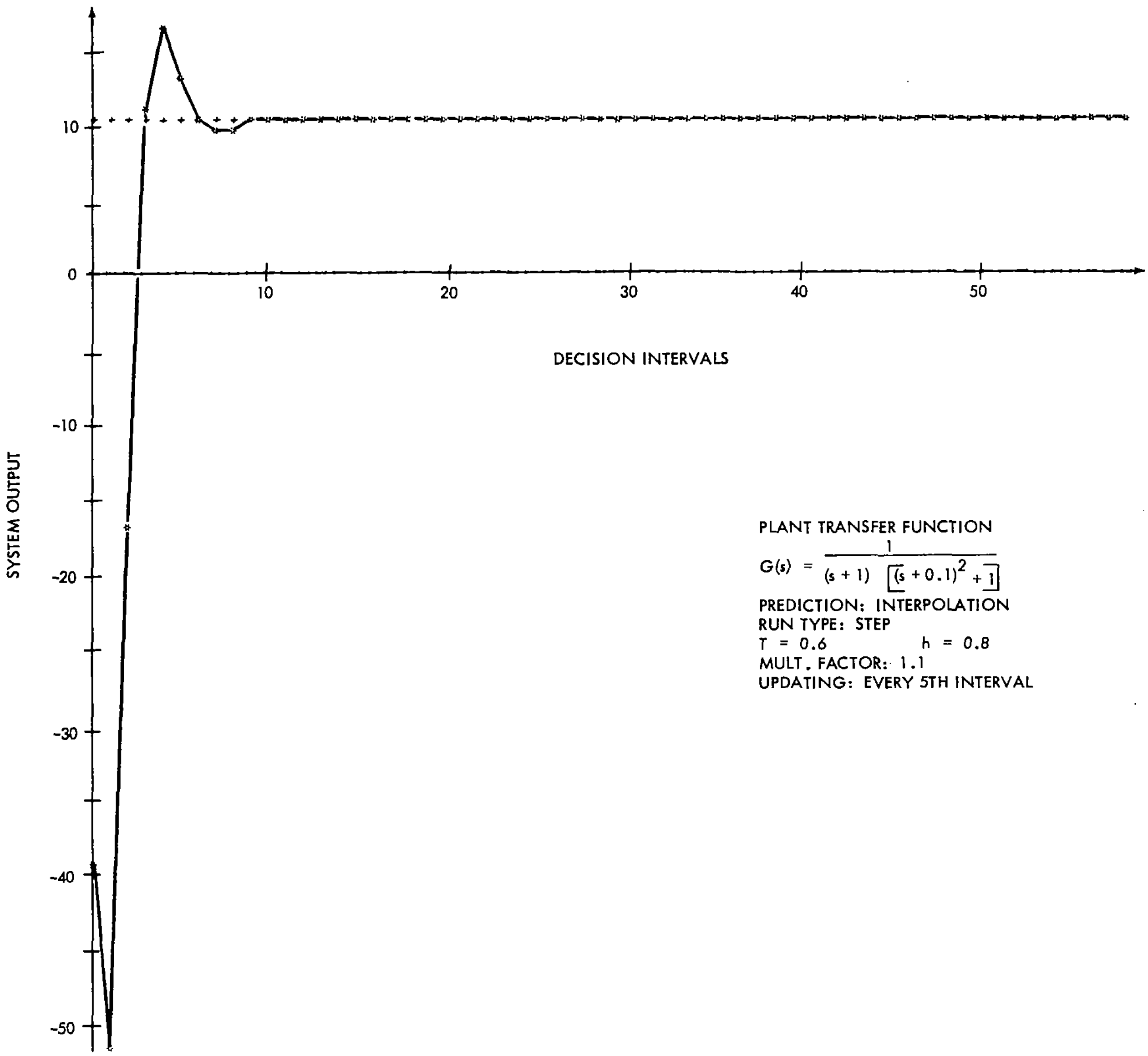


FIGURE 2-83 STEP RUN USING INTERPOLATION PREDICTION WITH UPDATING

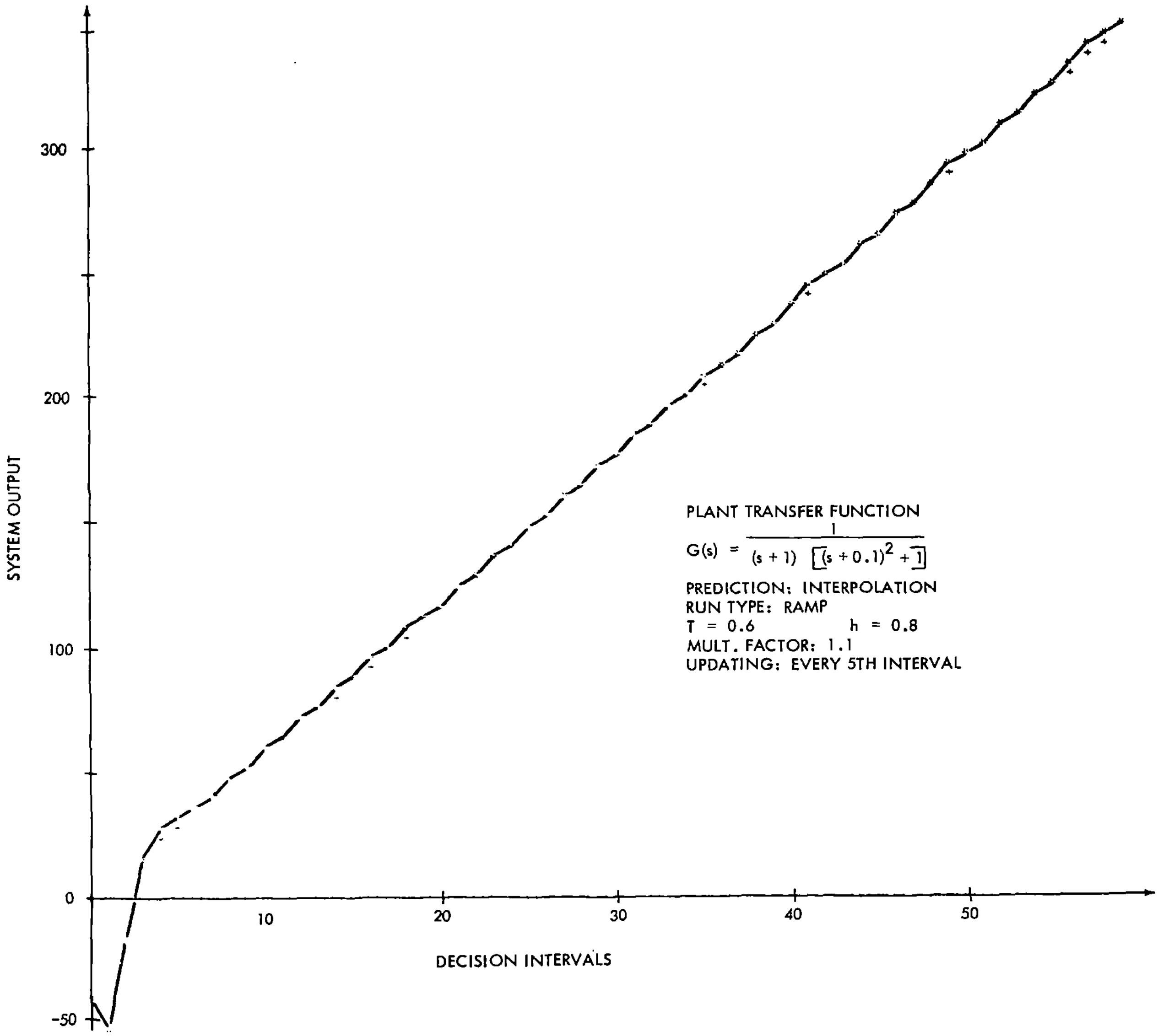


FIGURE 2-84 RAMP RUN USING INTERPOLATION PREDICTION WITH UPDATING

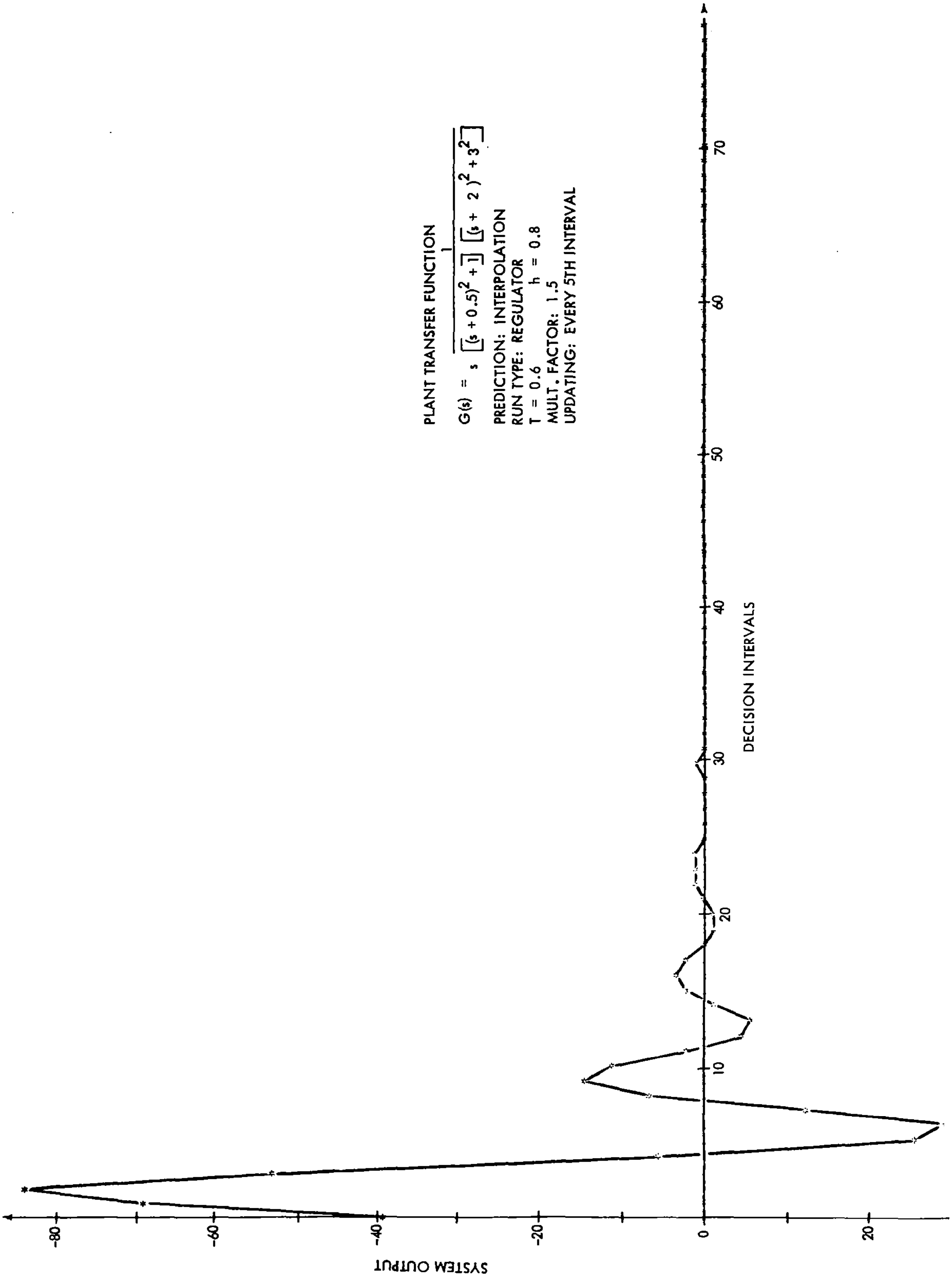


FIGURE 2-85 REGULATOR RUN USING INTERPOLATION PREDICTION WITH UPDATING

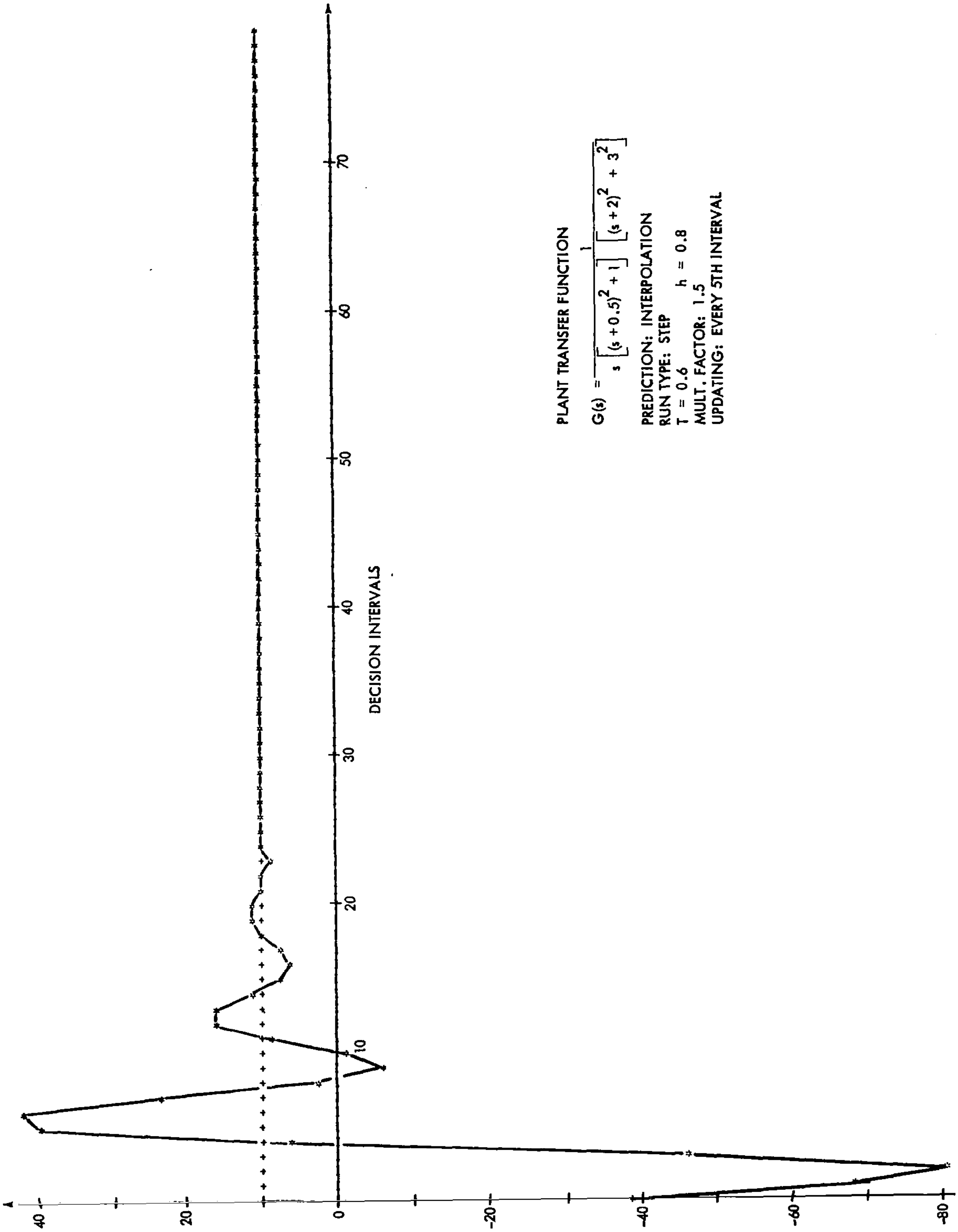


FIGURE 2-86 STEP RUN USING INTERPOLATION PREDICTION WITH UPDATING

The next set of figures are presented to experimentally demonstrate that no non-control policy force need be applied for cases where the desired output state is some trajectory. Figures 2-87 and 2-88 show the control performance for a ramp desired state with new data shifted into the matrix every interval and every other interval respectively. In both cases, the current sensitivity and current response were recalculated every tenth interval. No significant difference in control performance was encountered for the various k values. No matrix singularity was observed in any of the trajectory runs for fifth or lower order systems. However, matrix ill conditioning was noted in several cases. Figure 2-89 shows the typical matrix ill conditioning effect on the control system performance. The matrix of base vectors became ill conditioned at 26 decision intervals into the run. The resultant current sensitivity and current response matrices were used for the control calculations during the next ten intervals. Since these matrices were in error, poor control performance is seen during these ten intervals. At the end of these ten intervals new current sensitivity and current response matrices were computed, and these were used to control over the next ten intervals. These new matrices were once again a good approximation to the true plant matrices, and so good control resulted during the following portion of the run.

The last set of figures demonstrate the control performance for two pole configuration third order systems for a trajectory desired output state. The trajectory is described by the equation:

$$r(t) = 30t - 1.5t^2 + 0.01666t^3$$

In both cases no non-control policy forces were used, new data was shifted into the matrix of basis vectors every interval, and new current sensitivity and current response matrices were calculated every fifth interval. The output state and its first derivative are presented along with actual desired state in Figures 2-90, 2-91, 2-92, and 2-93 for both systems. These are typical control results for the second and third order pole configuration systems used in this brief experimental study.

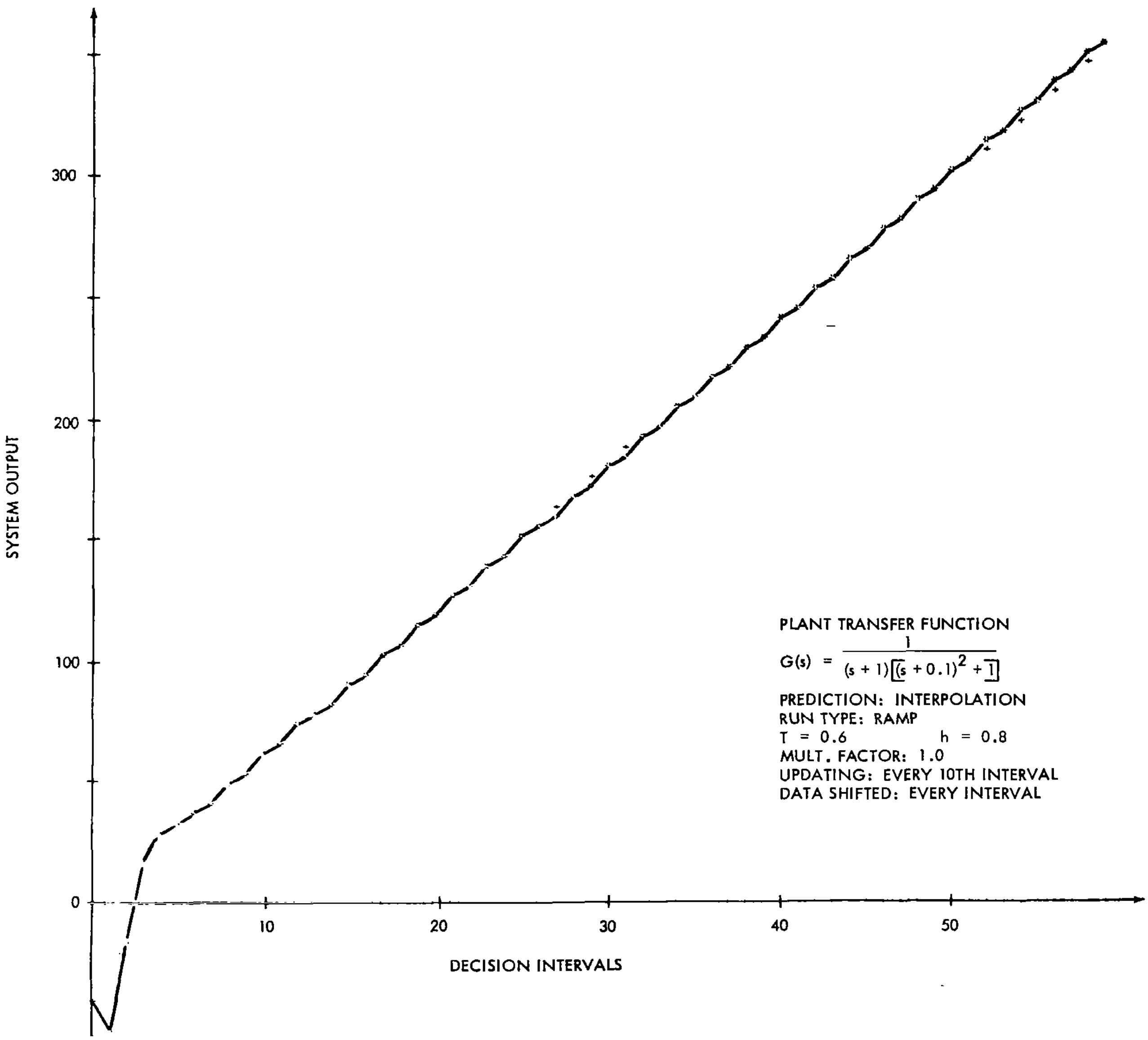


FIGURE 2-87 RAMP RUN USING INTERPOLATION PREDICTION WITH UPDATING

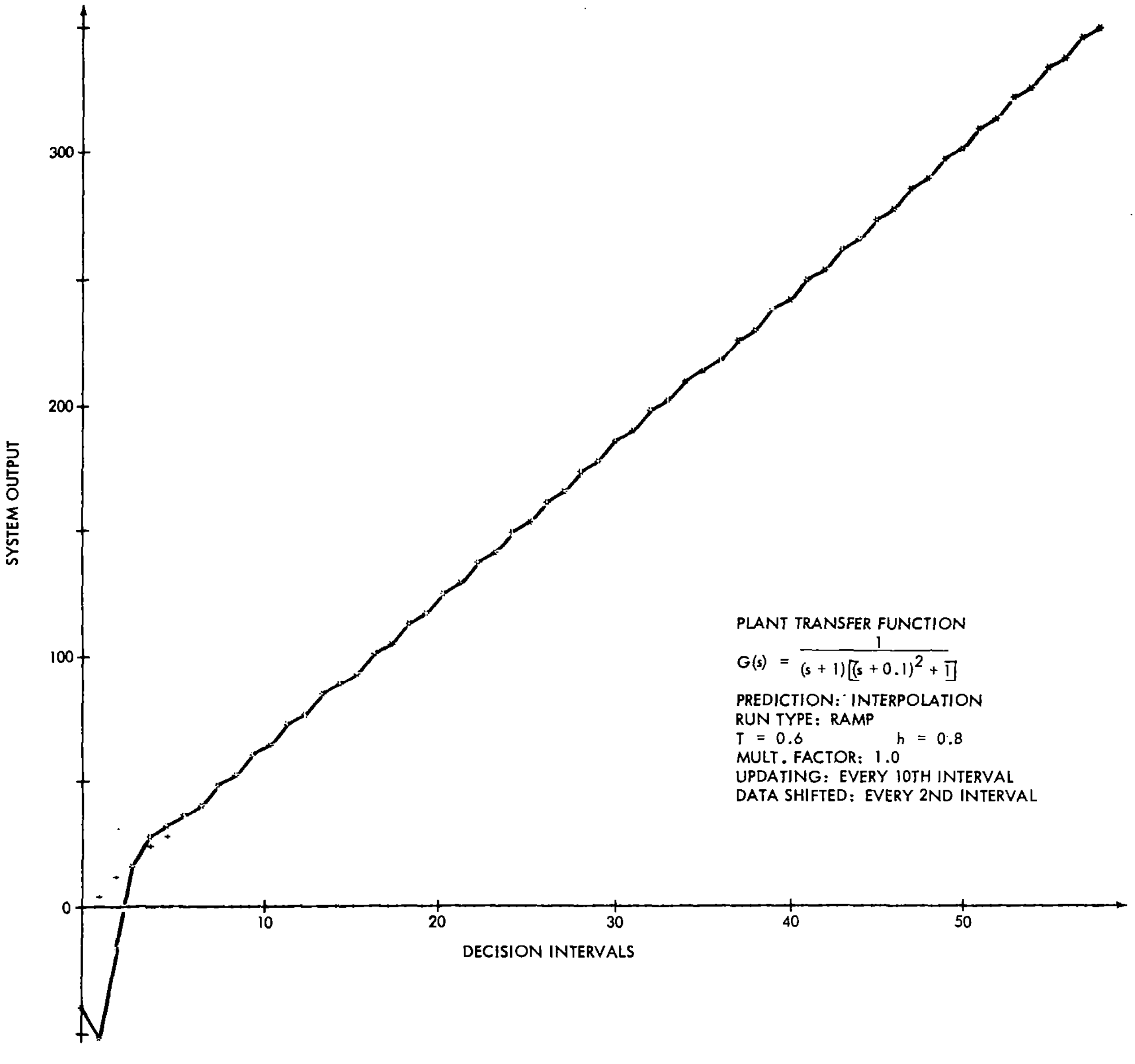


FIGURE 2-88 RAMP RUN USING INTERPOLATION PREDICTION WITH UPDATING

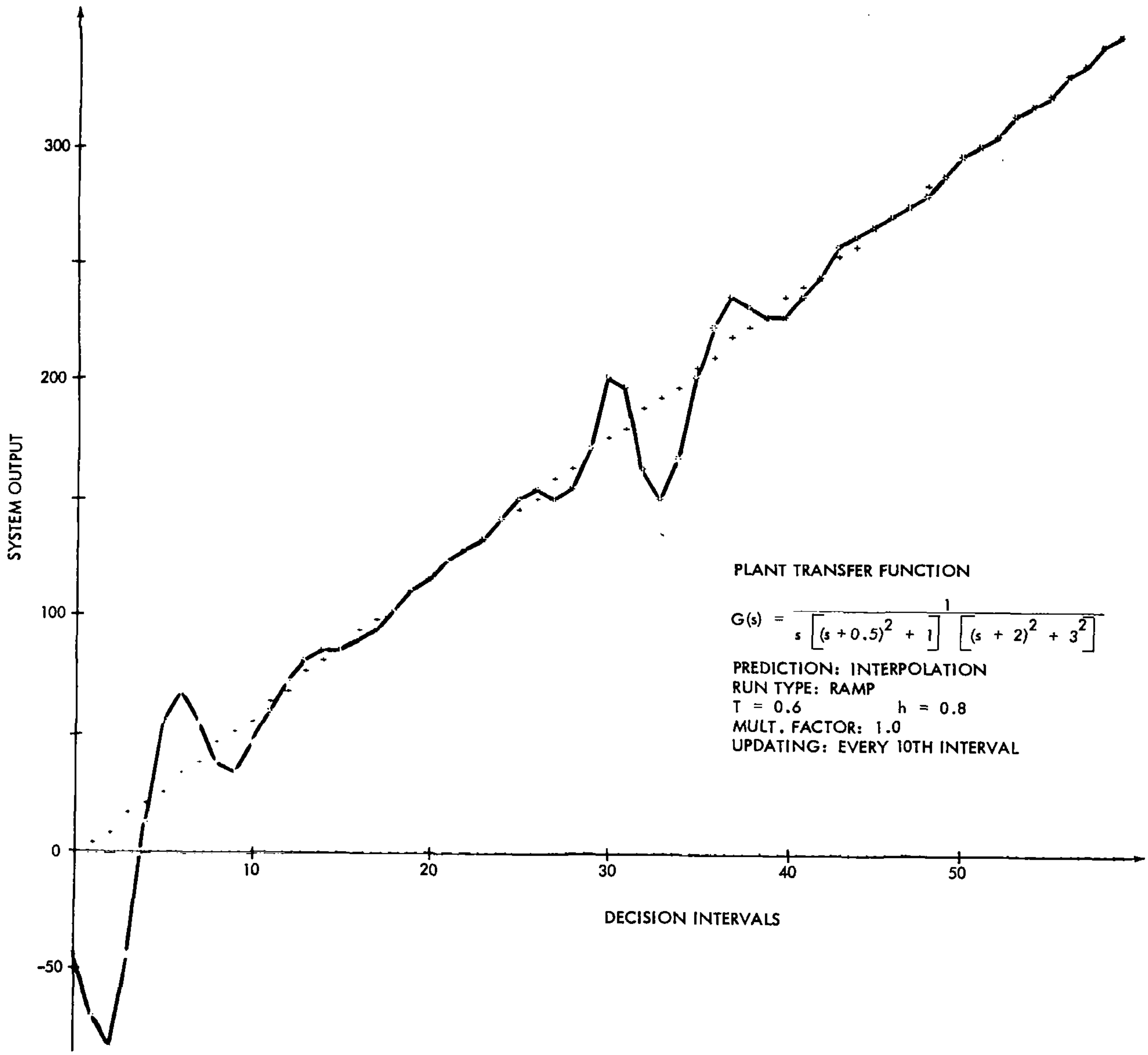


FIGURE 2-89 RAMP RUN USING INTERPOLATION PREDICTION WITH UPDATING

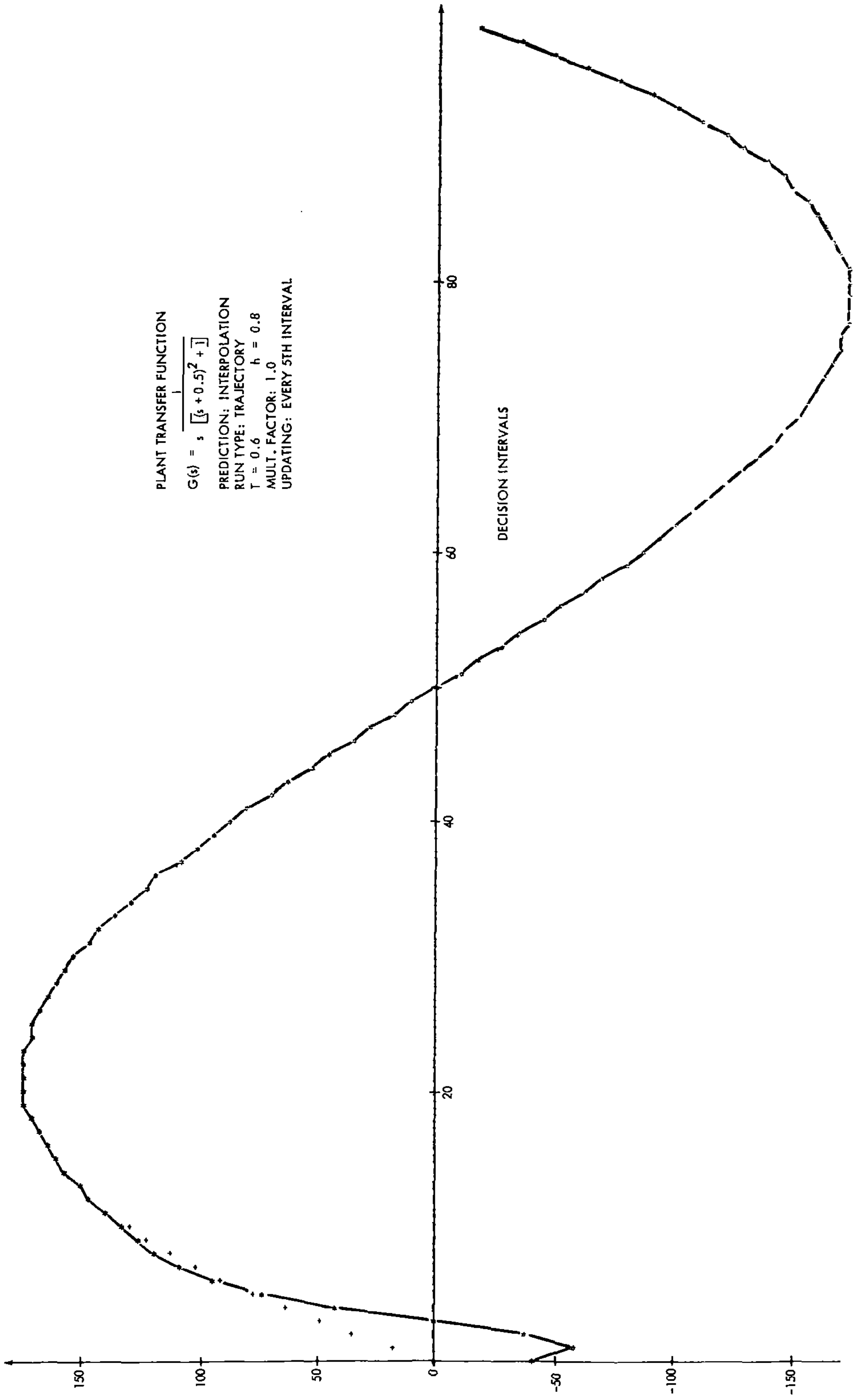


FIGURE 2-90 3rd ORDER PLANT OUTPUT RESPONSE TO A POLYNOMIAL TRAJECTORY

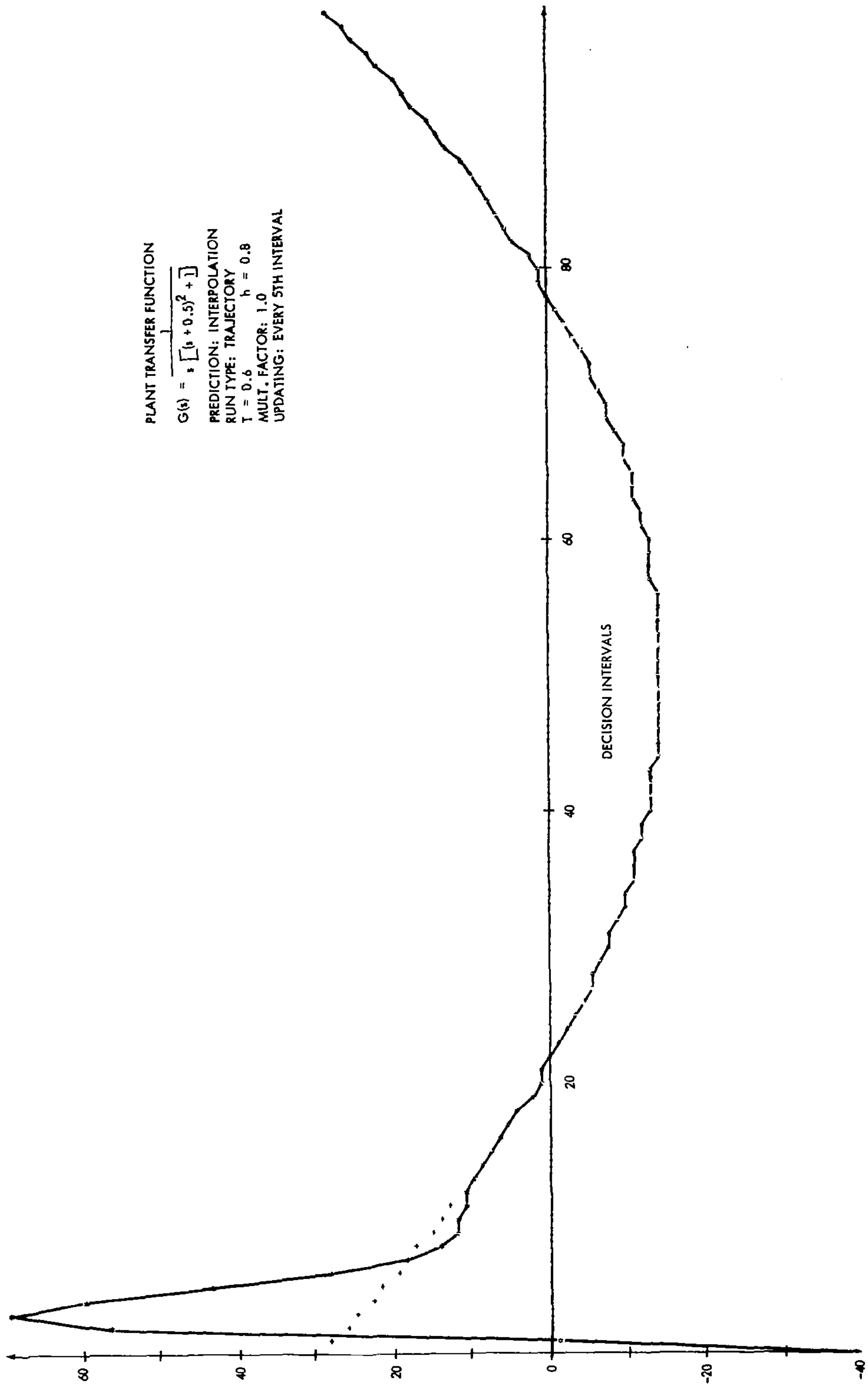


FIGURE 2-91 3rd ORDER PLANT VELOCITY RESPONSE TO A POLYNOMIAL TRAJECTORY

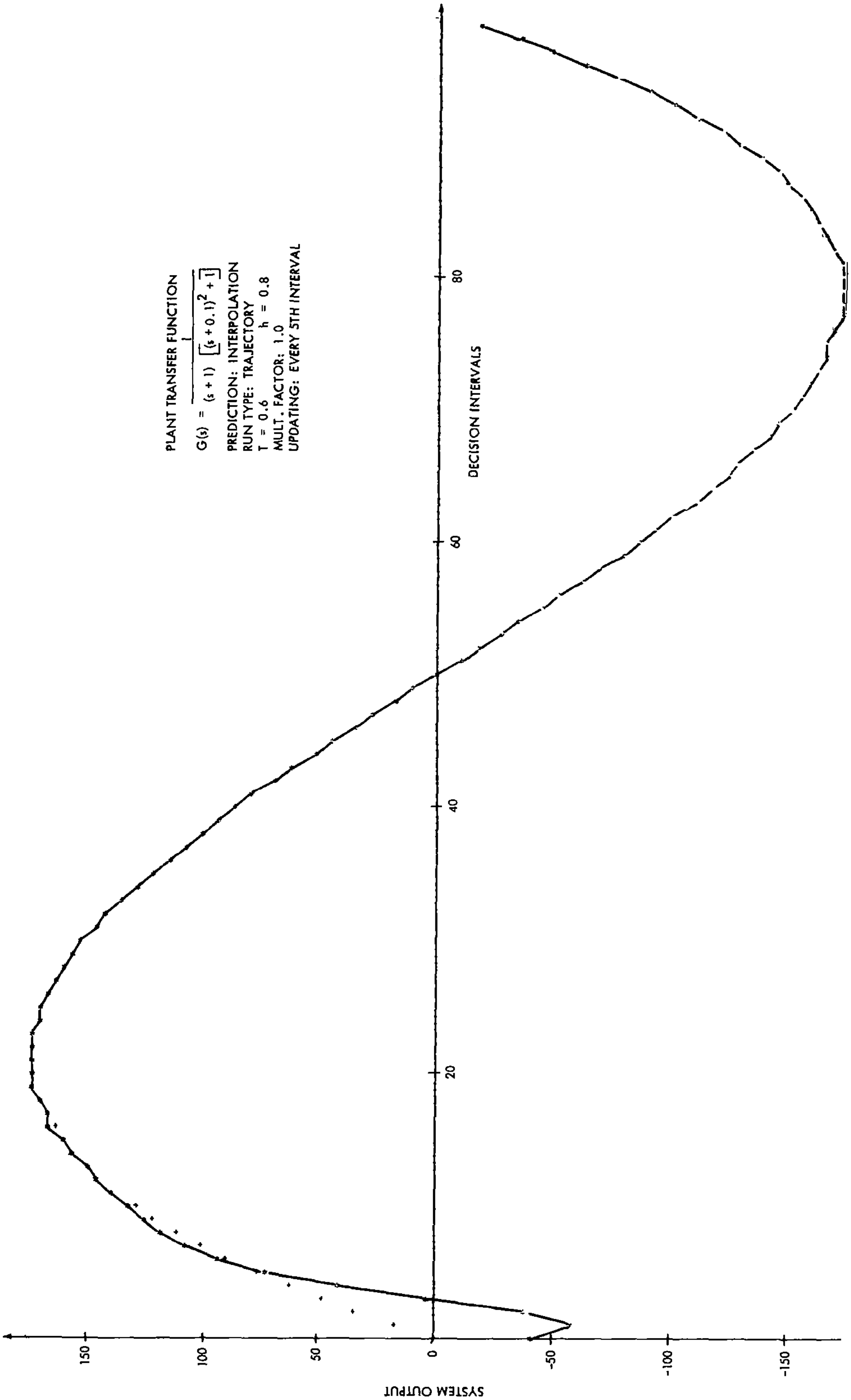


FIGURE 2-92 3rd ORDER PLANT OUTPUT RESPONSE TO A POLYNOMIAL TRAJECTORY

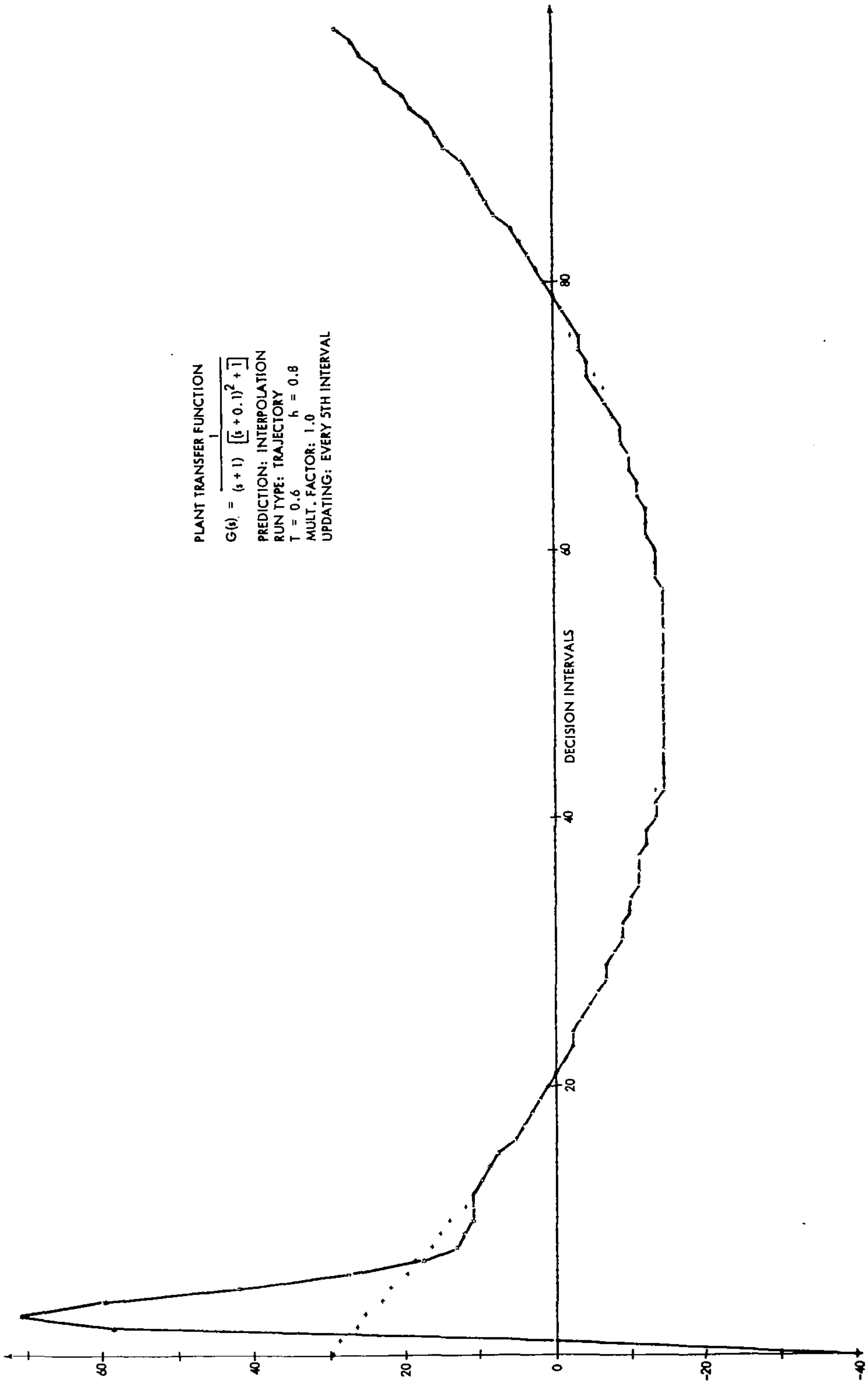


FIGURE 2-93 3rd ORDER PLANT VELOCITY RESPONSE TO A POLYNOMIAL TRAJECTORY

2.3 SUMMARY OF ANALYTICAL AND EXPERIMENTAL STUDIES

ANALYTICAL STUDIES

The mathematical tools for the study of the control method were developed in paragraph 2.1. These included a mathematical description of the type of plants considered, a state transition equation relating the values of the state variable of the plant at the sampling times, and the control policy equation by which the control input to the plant is calculated. Involved in implementing the control policy is the prediction of future states of the output of the plant without assuming any specific knowledge of the plant. Various techniques were introduced as possible predicition methods of which Exact Prediction (exact knowledge of the plant dynamical equations) serves as a reference against which to measure the effectiveness of other approximate types of prediction.

A Liapunov function is used to provide an analytical basis from which to establish stable regions of control operation in terms of the two control policy parameters:

T, the decision or sampling interval length in seconds

h, an arbitrary weighting factor which is used to emphasize or de-emphasize the higher order state variables.

EXPERIMENTAL STUDIES

A representative set of experimental data is presented in paragraph 2.2. It must necessarily be of a summary nature due to the immense amount of data compiled. Over 260 T-h stability planes were plotted for approximately 150 plants of orders ranging from second through ninth, and over 1500 control simulation runs were made to test the actual control characteristics of the control policy employing several types of response prediction.

The types of prediction studied consisted of a form termed "Taylor Prediction" in which a truncated Taylor series is used as an approximation of the free and forced response modes of the plant; a form termed "Mixed Prediction" in which the truncated Taylor series is again used as the approximation of the free response modes and an averaging technique is used to obtain an approximation of the forced response mode, and a form termed "Interpolation Prediction" in which an interpolation technique is used to select a functional which approximates the actual plant input-output functional without attempting to identify the plant.

Taylor Prediction.-Taylor Prediction produced stability boundaries which were almost without exception as large as those obtained using Exact Prediction. The Taylor boundaries for the high order plants were actually larger than the exact boundaries although a good portion of the larger Taylor boundary consisted of T-h points which would give relatively sluggish performance. Taylor Prediction is pretty well restricted to those plants whose corresponding transfer functions contain only poles as there is no clear cut extension to cover the case where the transfer function is allowed to contain zeroes. The lower than actual order Taylor boundaries were smaller than the actual order boundaries, but this might well be expected.

The control simulation runs showed that when the actual order of the plant was assumed using Taylor Prediction, the control action for regulator runs compared very favorably with that using Exact Prediction and was in some cases actually better in that the response was less oscillatory. Lower than actual order control gave relatively sluggish response in most cases and, in particular, when a relatively high order plant (i.e. sixth) was controlled as a third order, for example, the response would have to be termed prohibitively sluggish although it was asymtotically stable. The control action for trajectory runs depended upon the form of the plant and the type of trajectory. Steady state errors exist for both a step and ramp input if the plant transfer function does not contain a pole at the origin, and for a ramp input if the transfer function contains one pole at the origin. In contrast, Exact Prediction produces no such steady state errors regardless of the plant configuration. An analysis of the reason for this characteristic of Taylor Prediction is presented in Appendix E. In simple terms, the difficulty is due to the fact that Taylor Prediction yields approximations of the free response matrix and the forced response vector which are understandably not very accurate, and trajectory 'tracking' requires relatively good estimates.

Mixed Prediction.-Mixed Prediction produced stability boundaries which were not considered to be very large as compared with the other prediction methods. Reasonable boundaries were observed if the plant order was, at most, fourth, but for higher order plants T-h planes were either void of stable regions or consisted at most of isolated stable points. For this reason, Mixed Prediction experimental studies were terminated after the stability study phase except for a few runs to establish the validity of the analytically established T-h boundaries.

Interpolation Prediction.-Actual order Interpolation Prediction yielded estimates of the free and forced response modes of the plants which were sufficiently accurate to allow use of the stability boundaries obtained using Exact Prediction as a guide for the control simulation runs. In all cases when the plant transfer function contained poles only, exact and interpolation control action simulation runs yielded identical performance for regulator, step and ramp desired output states. The lower than actual order interpolation control action simulation runs gave results very similar to those obtained using Taylor Prediction except that the region of stable performance appeared to be larger. Controlling plants, whose transfer functions contain zeroes as well as poles, yielded actual order results very similar to those obtained using Taylor Prediction for plants with poles only, in that steady state errors which depended on the plant pole configuration existed for step and ramp desired output states. An analysis of the reason for this characteristic is given in Appendix E, where it is shown that the errors are not due to any shortcoming of the interpolation procedure. Instead they are due to the control policy equation used for plants with poles and zeroes. It is quite possible that a simple remedy exists and would be an interesting area for further investigation.

The Weighting Parameters.-The weighting matrix introduced in paragraph 2.1 has the effect of weighting the state vector of the plant so as to control the importance of the higher order state vector components. The experimentatic presented in paragraph 2.2 illustrated that for a given value of T, the decisic interval length in seconds, small values of h, the arbitrary weighting factor, tend to make the control action oscillatory and the response very fast. Large values of h make the response more sluggish as the control policy seeks to control more precisely the higher order state variable components. As the value of T becomes smaller, the response for a given value of h is more oscillatory, resulting in the area of the T-h stability plane near the origin being unstable for virtually all plants.

Control Force Saturation.-Although imposing a limit on the available control force is strictly not in the realm of linear studies, it is recognized that all practical situations have limitations of this nature. For this reason, an experimental study of the effect of control force limitation on the linear control policy was performed. The effect of control force limiting or saturation is to prevent the system response from following the optimum trajectory as defined by the linear control policy in the state space. For relatively large h values, the response was made more sluggish by control force saturation in cases where the linear control policy requested control forces exceeding the limit a majority of the time. For small values of h, the response was made more oscillatory in many cases. In some cases, a limit cycle occurred with the control policy requesting control forces alternately greater than the positive and negative limits.

Updating the Interpolation Predicition.-The Interpolation procedure is very compatible with one of the basic premises of the control method under study, i.e., control of a plant in terms of estimates of its current response and current sensitivity. The interpolation estimates of the plant response may be calculated in terms of the most applicable sets of measured basis vectors. Updating implies revising the interpolation estimates of the plant response in terms of, usually, the most recent measured data. Theoretically, the estimates need not be updated when a system is linear and stationary, but a study in this area provides a convenient starting point.

The central problem in the Interpolation procedure is to avoid using a matrix of basis vectors which is singular. When the desired state of a system is the origin of its state space, any linear control policy which determines the control forces in terms of a linear combination of the state variables will result in a singular matrix (see Appendix F). The experimental studies conducted on linear stationary systems showed that this problem may be circumvented by including one or two non-control policy forces in the matrix. This took the form of multiplying the calculated control forces by a fixed constant which may for fourth and lower order systems be as small as 1.1, which alters the calculated control force by only 10%. Higher order systems which result in larger matrices may require larger multipliers to prevent the matrix from being 'ill conditioned'. Another preliminary conclusion is that when the system is performing well, it is best not to update. This is exemplified by the case where the input is a constant position and the initial transients have died out. In this case, all of the derivatives will become infinitesimally small. Ill conditioning then results from the fact that several of the rows of the matrix of basis vectors will be close to zero and therefore very close to being linearly dependent.

REFERENCES

1. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," Journal of the Society of Industrial and Applied Mathematics, Ser. A: On Control, Vol. 1, No. 2, pp. 152-192, 1963.

2. Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identification," Proceedings of the JACC, 1964, pp. 366-375.

3. "Digital Adaptive Control System Research, Summary Report," Emerson Electric Report No. 1544-5, March 31, 1964.

4. Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpolation and Lagrange Expansion for Functionals," Proceeding of the JACC, 1965, pp. 169-177.

5. Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Missouri, 1965.

6 Blakelock, J. H., <u>Automatic Control of Aircraft and Missiles</u>, John Wiley and Sons, Inc., 1965, Chapter 2.

7 Serex, H. M. and Holt, J. N., "Comparison of High-Gain-Linear and Self Adaptive Flight Control Systems with Nonlinearities Introduced," Master of Science Thesis, Air Force Institute of Technology, Thesis GGC/EE/64-11.

SECTION 3

CONTROL OF LINEAR TIME-VARYING PLANTS

3.1 DERIVATION OF PLANT AND SYSTEM EQUATIONS

A study of the control of linear, time-varying plants is a second step in the investigation of the general feasibility of the control method. Although the principle of superposition still applies, it, along with many other of the general linear analytic tools, is not quite as all-powerful as one is usually led to believe by the study of linear stationary systems.

Before considering the analysis of time-varying plants, it is best to digress for a moment and consider just what is meant by a time-varying plant. In developing a mathematical description of a physical quantity, usually a group of elements are isolated and termed the plant, and any outside events which affect the elements of the plant are considered to be inputs or disturbances (reference 1). In the case of a single input plant, the primary input signal to the plant is identified and the analysis proceeds to consider the effects of this input on the plant. Those quantities which are insensitive to the input signal are termed plant parameters, and those which are affected by the input are termed plant variables.

One of the basic assumptions in this single input, plant parameter, plant variable method of analysis is that the plant is autonomous with respect to other outside events. In many cases this is approximately true and the autonomous assumption yields meaningful results. In other cases, however, the effect of outside events on the plant cannot be ignored, and it becomes difficult to decide which elements of the plant shall be considered parameters and which shall be considered plant variables. To preserve the single input concept, if the effect of the secondary inputs may be considered independently of the primary signal, and affect only those quantities which are parameters as far as the primary input is concerned, then the parameter variations due to the secondary inputs may be described as functions of time and the plant termed "time-varying". Many physical examples of this situation exist. A high performance aircraft moving through a wide range of Mach numbers at various altitudes is one. The change in Mach number and altitude reflect in the variation of certain parameters of the plant over wide ranges which cannot be ignored. For the purposes of analysis, these parameters may be considered time variable parameters as far as the primary input is concerned.

METHODS OF ANALYSIS

The physical plant is assumed to be describable by a linear differential equation with coefficients which are continuous functions of time.

$$L(p,t)\ c(t) = M(p,t)\ m(t) \tag{3-1}$$

The plant is also assumed to possess a single input, m(t), and a single output, c(t). L(p,t) and M(p,t) are linear differential operators of orders n and m respectively, where the derivative coefficients are functions of time. Although equation 3-1 suffices to describe the type of plant to be considered, it is advantageous to write the mathematical description in state-space notation:

$$\dot{\underline{x}}(t) = \underline{H}(t)\ \underline{x}(t) + \underline{G}(t)\ \underline{u}(t) \tag{3-2}$$

$$\underline{y}(t) = \underline{D}(t)\ \underline{x}(t) \tag{3-3}$$

where $\underline{x}$, $\underline{u}$, and $\underline{y}$ are n, p, and q vectors respectively, and the matrices $\underline{H}(t)$, $\underline{G}(t)$, and $\underline{D}(t)$ are continuous functions of time. Equations 3-2 and 3-3 are termed the dynamical equations of the plant (reference 2). Equation 3-1 and the equation pair 3-2 and 3-3 become equivalent if $\underline{u}(t)$ and $\underline{y}(t)$ are identified as:

$$\underline{u}'(t) = \lfloor m(t) \quad \dot{m}(t) \quad \ldots\ldots \quad \overset{m}{m}(t) \rfloor \tag{3-4}$$

$$\underline{y}'(t) = \lfloor c(t) \quad \dot{c}(t) \quad \ldots\ldots \quad \overset{n-1}{c}(t) \rfloor \tag{3-5}$$

in which case p = m+1 and q = n. The quantity $\underline{x}(t)$ is identified as the state variable of the plant. As was the case for linear stationary plants, the state variable is not unique; however, any choice, $\underline{x}(t)$, is relatable to any other choice, $\underline{z}(t)$, by a linear transformation of the form:

$$\underline{z}(t) \;=\; \underline{E}(t)\;\underline{x}(t) \tag{3-6}$$

subject to the condition:

$$\left\|\underline{E}(t)\right\| \leq c_1 \qquad \text{and} \qquad \left\|\underline{E}^{-1}(t)\right\| \leq c_2 \tag{3-7}$$

where c_1 and c_2 are fixed constants, and $\|\cdot\|$ is the Euclidean norm (reference 2).

As was the case in the linear stationary plant study, it is necessary to restrict the matrix $\underline{D}(t)$ to be the identity matrix $\underline{I}$. The plant dynamic equation pair 3-2 and 3-3 therefore reduces to the single equation:

$$\dot{\underline{x}}(t) = \underline{H}(t)\;\underline{x}(t) + \underline{G}(t)\;\underline{u}(t) \tag{3-8}$$

where:

$$\underline{x}'(t) \equiv \underline{y}'(t) = \lfloor\, c(t) \quad \check{c}(t) \quad \ldots\ldots \overset{n-1}{c}(t) \,\rfloor \tag{3-9}$$

and the forms of $\underline{H}(t)$ and $\underline{G}(t)$ are:

$$\underline{H}(t) = \begin{bmatrix} & \ddots \; 0 \; \ddots & \\ & \ddots \; 1 \; \ddots & \\ -A_0(t) & -A_1(t) \;\ldots\ldots\ldots & -A_{n-1}(t) \end{bmatrix} \tag{3-10}$$

$$\underline{G}(t) = \begin{bmatrix} & & \\ \text{———} & 0 & \text{———} \\ & & \\ B_0(t) & B_1(t) \;\ldots\ldots\ldots & B_m(t) \end{bmatrix} \tag{3-11}$$

where $A_i(t)$ and $B_i(t)$ are the time-varying coefficients of the i^{th} derivatives of the left and right hand sides, respectively, of the plant differential equation 3-1. $\underline{H}(t)$ is a square matrix of order n, and $\underline{G}(t)$ is a rectangular matrix with n rows and m+1 columns.

THE STATE EQUATION

The general solution of the plant dynamical equation 3-8 is given by:

$$\underline{x}(t) = \underline{\varphi}(t,t_o, \underline{x}_o) = \underline{F}(t,t_o)\, \underline{x}(0) + \int_{t_o}^{t} F(t,\tau)\, \underline{G}(\tau)\, \underline{u}(\tau)\, d\tau \qquad (3\text{-}12)$$

where $\underline{F}(t,t_o)$ is the transition matrix of the free differential equation, and $\underline{x}(0)$ is the value of the state variable at $t = t_o$. Equation 3-12 is valid for any $t \geqslant t_o$.

The General State Equation in Terms of $t = kT^\circ$ Initial Conditions.-Since the control action is effected by an on-line digital computer, the specific control functions (plant inputs) considered are those which are piecewise constant over decision or sampling intervals of equal lengths (T seconds). Thus, as was the case for linear stationary systems, the plant input, m(t), is constant over the intervals $kT \leq t < (k+1)T$. Equation 3-12 may conveniently be written in the discrete form of equation 3-13, where T° is the ficticious time defined in Appendix A and discussed in Section 2 on linear stationary systems:

$$\underline{x}((k+1)T^\circ) = \underline{F}((k+1)T,kT)\, \underline{x}(kT^\circ) + \underline{\lambda}((k+1)T,kT)\, u_k \qquad (3\text{-}13)$$

Equation 3-13 is not as useful as was its linear stationary counterpart, as $\underline{F}(t,t_o)$ and $\underline{\lambda}(t,t_o)$ are no longer constant matrices for constant T. If the plant is relatively slowly time-varying, $\underline{F}((k+1)T,kT)$ and $\underline{\lambda}((k+1)T,kT)$ will be relatively slowly time-varying. In the limit as the time variation is extremely slow, it may be possible to assume $\underline{F}((k+1)T,kT) \longrightarrow \underline{F}$ and $\underline{\lambda}((k+1)T,kT) \longrightarrow \underline{\lambda}$ over the time interval (t_a,t_b) during which control is desired. Equation 3-13 will be referred to as the general discrete state equation for time-varying systems.

The General State Equation in Terms of Measureable States.-As was the case for linear stationary plants, the state equation in terms of the states at kT° is of limited usefulness when the control policy must be implemented on the basis of physically existent states. For this reason, a state equation of the form of equation 2-17 of paragraph 2.1 is written:

$$\underline{x}((k+1)T^-) = \underline{F}((k+1)T,kT)\ \underline{x}(kT^-) + \underline{b}_1((k+1)T,kT)\ u_k + \underline{b}_2((k+1)T,kT)\ u_{k-1} \tag{3-14}$$

It should be noted that equation 3-14 differs from equation 2-17 in that the free and forced response matrices are time-varying. If the differential equation which describes the plant does not contain derivatives of the input, m(t), $\underline{b}_2((k+1)T,kT)$ will be zero and the forced response of the plant is describable in terms of the single forced response vector $\underline{b}_1((k+1)T,kT)$.

ESTIMATION AND PREDICTION

The response of the plant may be broken down into two parts, the first being what may be termed the free response which would occur in the absence of any control input. The second part may be termed the forced response, or that part of the response which is due to the control input. The state equation may be written in terms of measurable states which exhibit the two responses:

$$\underline{x}((k+1)T^-) = \underline{x}_k((k+1)T^-) + \underline{b}_1((k+1)T,kT)\ u_k + \underline{b}_2((k+1)T,kT)\ u_{k-1} \tag{3-15}$$

where $\underline{x}_k((k+1)T^-)$ is the free response of the plant and the second two terms in equation 3-15 comprise the forced part of the response. The general problem of prediction is concerned with the estimation of the free response $\underline{x}_k((k+1)T^-)$ as compared with the desired state. An appropriate control force, u_k, is calculated by the control policy to better align the actual and desired states on the basis of the estimate of the state $\underline{x}((k+1)T^-)$ if no control force were applied. For a more complete discussion of the prediction problem, refer to paragraph 2.1 on linear stationary plants.

Interpolation Prediction.-For a general development of the basic interpolation equations, refer to Appendix B, as only that part which is applicable to the linear time-varying case is considered here. In terms of the interpolation development (references 3 and 4), the estimate of the state at $t = (k+1)T^-$ is given by the equation:

$$\underline{\tilde{x}}((k+1)T^-) = {}_D\underline{X}\,\underline{\Phi}^{-1}\,\underline{\phi}(u_k,\underline{\eta}_k) \tag{3-16}$$

where $\underline{\tilde{x}}$ is the estimate of $\underline{x}$ at $t = (k+1)T^-$

A linear term in t may be added to the set of base functionals to account for the time variation of the plant. If the plant is relatively slowly time-varying, the approximation may be sufficiently accurate for a relatively long period of time. As the interpolation estimate becomes insufficiently accurate, a new estimate may be made by an updating procedure which uses more current data for the set of measured base functionals included in $\underline{\phi}$.

The base vector including a linear term in t will be:

$$\underline{\phi}'(u_i,\underline{\eta}_i) = \lfloor \underline{x}'(iT) \quad u_i \quad u_{i-1} \quad T \rfloor \tag{3-17}$$

where T is the length of the sampling interval in seconds. The matrix of vector base functions, $\underline{\Phi}$, consists of an appropriate set of $\underline{\phi}_i$'s which need not be consecutive. Since the base vector $\underline{\phi}_i$ contains one more term (T) than the linear stationary case, one more interval of data is required and the interpolation matrices will be increased by one in order. The matrix $\underline{B}$ is now partitioned into four submatrices:

$$\underline{B} = {}_D\underline{X}\;\underline{\Phi}^{-1} = \left[\begin{array}{c|c|c|c} \underline{\theta}_1 & \underline{\varphi}_1 & \underline{\varphi}_2 & \underline{\varphi}_3 \end{array}\right] \tag{3-18}$$

where, if the assumed order of the matrix is p, $\underline{\theta}_1$ is a p^{th} order square matrix, and $\underline{\varphi}_1$, $\underline{\varphi}_2$, and $\underline{\varphi}_3$ are p vectors. The interpolation procedure yields as an estimate of $\underline{x}((k+1)T^-)$:

$$\underline{\tilde{x}}((k+1)T^-) = \underline{\theta}_1\ \underline{x}(kT^-) + \underline{\varphi}_1\ u_k + \underline{\varphi}_2\ u_{k-1} + \underline{\varphi}_3\ T \tag{3-19}$$

where $\underline{\theta}_1$, $\underline{\varphi}_1$, $\underline{\varphi}_2$, and $\underline{\varphi}_3$ are constant matrices.

The correspondence between the interpolation estimate of $\underline{x}((k+1)T^-)$ and the exact value as given by equation 3-15 is not as direct as was the case for linear stationary plants. The time-varying nature of the actual free and forced response matrices is accounted for in a single term $\underline{\varphi}_3$ T in the interpolation estimate.

THE CONTROL POLICY

As was the case for linear stationary systems, the control policy for linear time-varying systems will be to minimize the quadratic form (reference 5):

$$\underset{u_k}{\text{Min}} \left[Q_k\right] = \underset{u_k}{\text{Min}} \left[\underline{e}'((k+1)T)\ \underline{K}\ \underline{e}((k+1)T)\right] \tag{3-20}$$

The error state vector, $\underline{e}(t)$, is as it was defined in paragraph 2.1:

$$\underline{e}(t) = \underline{r}(t) - \underline{x}(t) \tag{3-21}$$

In order to provide continuity, the control policy equation is written in terms of the fictitious time kT°. In Appendix A it is shown that the interpolation equation 3-19 may be written in terms of kT° in the form:

$$\underline{\tilde{x}}((k+1)T^\circ) = \underline{\theta}_1\ \underline{x}(kT^\circ) + \underline{\varphi}_e\ u_k + \underline{\varphi}_3\ T \tag{3-22}$$

where $\underline{\varphi}_e$ is given by:

$$\underline{\varphi}_e = \underline{\varphi}_1 + \underline{\theta}_1^{-1}\ \underline{\varphi}_2 \tag{3-23}$$

Substituting the interpolation estimate of $\underline{x}((k+1)T^\circ)$, equation 3-22, into the expression for Q_k according to the definition equation 3-21 yields an expression for Q_k in terms of u_k:

$$Q_k = \left[\underline{r}'((k+1)T^\circ) - \underline{x}'(kT^\circ)\,\underline{\theta}_1' - u_k\,\underline{\varphi}_e' - T\,\underline{\varphi}_3'\right]\underline{K}$$

$$\left[\underline{r}((k+1)T^\circ) - \underline{\theta}_1\,\underline{x}(kT^\circ) - \underline{\varphi}_e\,u_k - \underline{\varphi}_3\,T\right] \tag{3-24}$$

The assumption is made that $\underline{r}(t)$ is continuous at $t = (k+1)T^\circ$ and the T° notation has been included in $\underline{r}(t)$ for the sake of uniformity only.

Differentiating Q_k with respect to u_k yields:

$$\frac{dQ_k}{du_k} = -2\,\underline{\varphi}_e'\,\underline{K}\left[\underline{r}((k+1)T^\circ) - \underline{\theta}_1\,\underline{x}(kT^\circ) - \underline{\varphi}_3\,T\right] + 2\,\underline{\varphi}_e'\,\underline{K}\,\underline{\varphi}_e\,u_k \tag{3-25}$$

Setting equation 3-25 equal to zero will yield the value of u_k which will minimize the quadratic form Q_k. Equation 3-26 is the control policy equation in terms of the fictitious time kT°:

$$u_k = \frac{\underline{\varphi}_e'\,\underline{K}\left[\underline{r}((k+1)T^\circ) - \underline{\theta}_1\,\underline{x}(kT^\circ) - \underline{\varphi}_3\,T\right]}{\underline{\varphi}_e'\,\underline{K}\,\underline{\varphi}_e} \tag{3-26}$$

Assurance that equation 3-26 yields a minimum rather than a maximum is provided by differentiating 3-25 and realizing that $\underline{K}$ is defined to be positive definite:

$$\frac{d^2Q_k}{du_k^2} = 2\,\underline{\varphi}_e'\,\underline{K}\,\underline{\varphi}_e > 0 \tag{3-27}$$

In order to place the control policy equation in a form which contains only measurable states, use is made of a relationship derived in Appendix A:

$$\underline{x}(kT^\circ) = \underline{x}(kT^-) + \underline{\theta}_1^{-1}\,\underline{\varphi}_2\,u_{k-1} \tag{3-28}$$

substituting equation 3-28 into equation 3-26 yields as a practical control policy equation:

$$u_k = \frac{\underline{\varphi}_e' \; \underline{K} \left[\underline{r}((k+1)T) - \underline{\theta}_1 \, \underline{x}(kT^-) - \underline{\varphi}_2 \, u_{k-1} - \underline{\varphi}_3 \, T \right]}{\underline{\varphi}_e' \; \underline{K} \; \underline{\varphi}_e} \qquad (3\text{-}29)$$

SOME COMMENTS ON STABILITY

As is pointed out by Gibson (reference 1), the subject of stability of time-varying systems is one about which the definitive word has probably not been said yet. Many of the basic theorems which apply to the second method of Liapunov are valid for time varying systems; however, because of the difficulties involved with forming useable Liapunov functions, engineering applications are rare.

There is a strong tendency to discuss time-varying systems in terms of poles and zeroes which move about in the complex plane. This concept is not without its pitfalls, as there is a tendency to connect the transient response of the system with the ephemeral locations of the 'poles' when no such connection exists. Describing stability of the plant in terms of the movement of the 'poles' back and forth between the left and right half planes can be misleading. Gibson quotes an example where the 'poles' of a time-varying plant are restricted to the left half plane, but the actual response can be unstable under certain conditions.

3.2 EXPERIMENTAL STUDIES

The function of this section is to present the experimental control system performance for low order linear time-varying systems. The Interpolation Prediction method was used throughout this experimentation, since it displayed the most rewarding results on the linear nontime-varying systems.

The objectives of this experimental program were:

> To determine the control performance of the control system using the Interpolation Prediction method for a selected set of linear time-varying plants.

To investigate areas of interest such as updating, and inclusion of the decision interval time (T) in the basis vector.

These primary objectives are considered along with some discussion of the experimental procedures in the following paragraphs.

LINEAR TIME-VARYING PLANTS

In order to meaningfully establish the feasibility of the control system for time-varying plants, it was necessary to select a limited, but representative set of such plants. This set consisted of second and third order pole configuration plants. Also, only linear and sine time variations were considered in this experimentation. The range and speed of variation of a single time varied parameter were studied for each plant configuration. The free (uncontrolled) response of two types of time-varying plants studied is shown in Figures 3-1 and 3-2.

EXPERIMENTAL RESULTS

This experimental study consisted of approximately seventy-five control simulations on the selected time-varying plants. Areas of interest such as how often new data should be included into the matrix of basis vectors, the effect of using one non-control policy force on the control system performance, and how often the current sensitivity and current reponse should be recalculated were investigated in the course of the following experimentation. Rather severe parameter time variations with respect to range and speed of variation were considered in order to more firmly establish meaningful conclusions. All the following experimentation was conducted with -40 units as an initial condition on each component of the state vector.

The same experimental start-up procedure utilized and described for the linear stationary case with updating was used in these studies. This procedure is of no real importance, since it only provided an artificial method for starting the control simulations at any initial state.

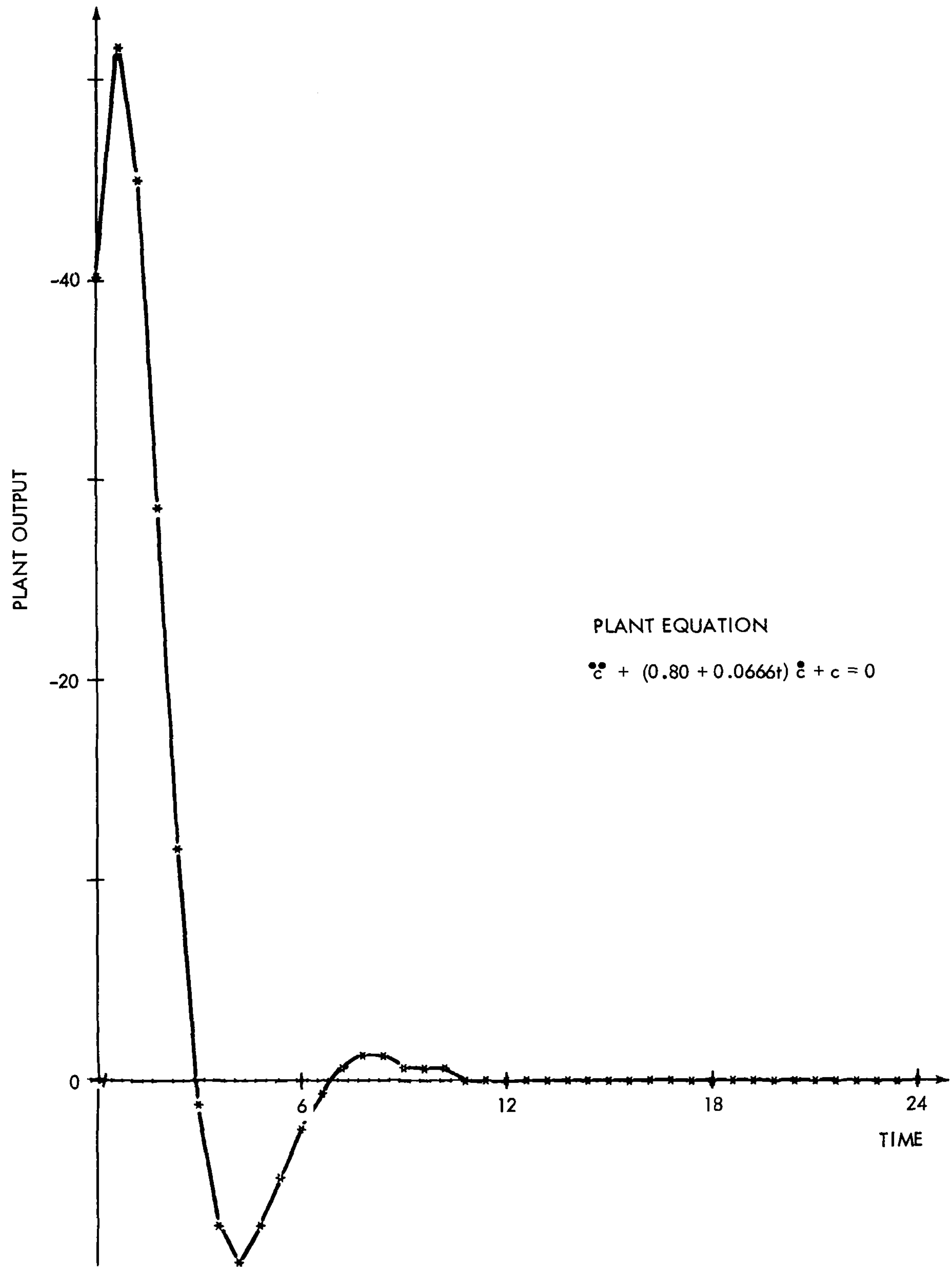


FIGURE 3-1 FREE RESPONSE OF TYPICAL 2nd ORDER TIME-VARYING PLANT

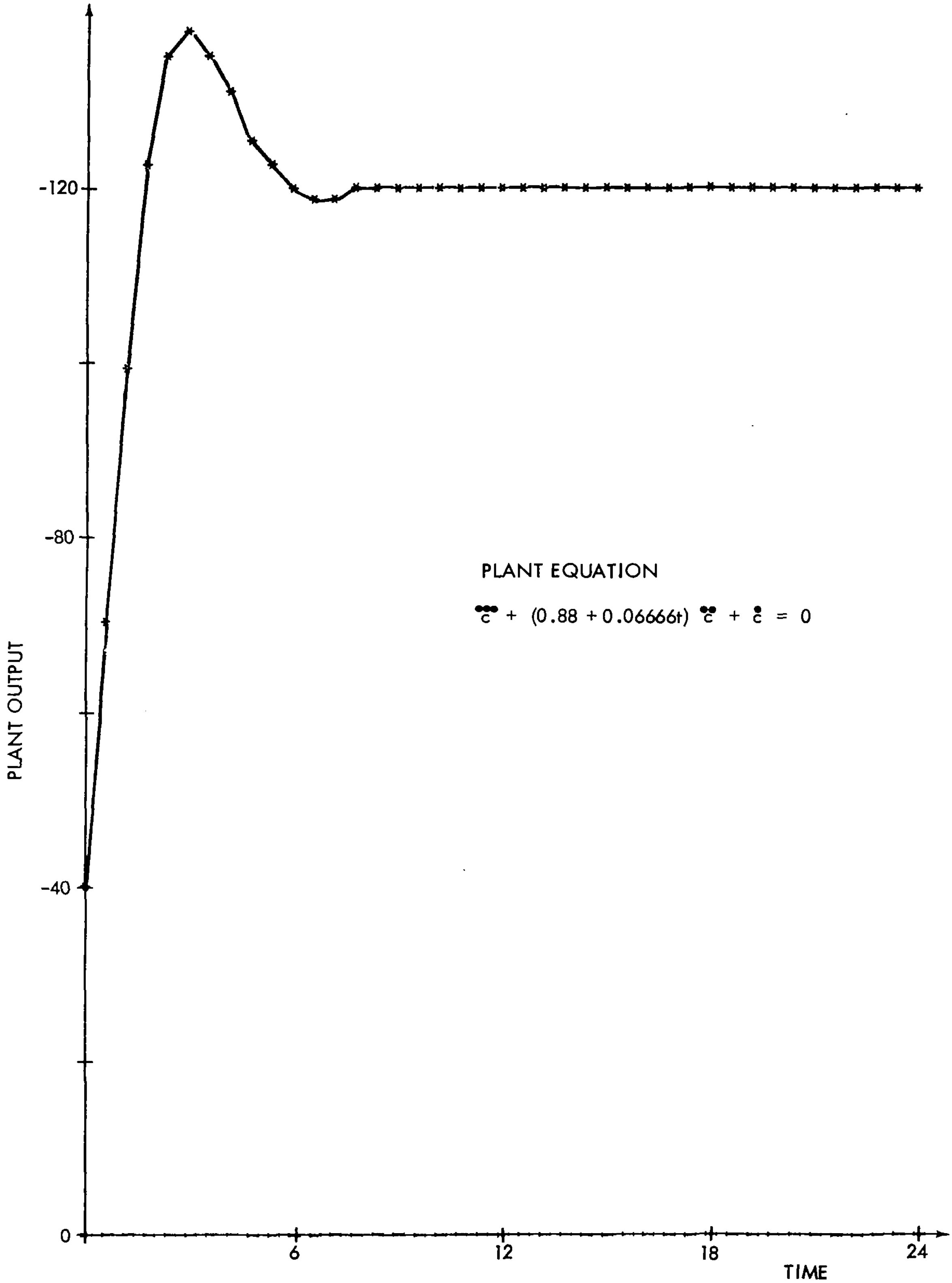


FIGURE 3-2 FREE RESPONSE OF TYPICAL 3rd ORDER TIME-VARYING PLANT

Second Order System Results.-Typical results of the second order time-varying system control experiments are given in Figures 3-3 through 3-14 for a step desired output state. In all of these cases the plant damping was a linear function of time. Two ranges of the time-variation are presented, approximately increasing the damping from 0.40 to 0.80 and decreasing it from 0.80 to 0.40 in twenty decision intervals. The equations describing these two systems are respectively:

$$\ddot{c} + (0.80 + 0.06666t)\,\dot{c} + c = m(t) \tag{3-30}$$

$$\ddot{c} + (1.60 - 0.06666t)\,\dot{c} + c = m(t) \tag{3-31}$$

The plant natural frequency in both cases is one. Control of the second order plants as linear nontime-varying plants is presented in Figures 3-3 and 3-4 whereas Figures 3-5 and 3-6 show the control performance for the same plants when the decision interval time (T) was included in the basis vectors. For both cases a constant error exists in the resultant output state. Figures 3-7 through 3-10 show the control results for these plants without (T) in the basis vector, but with updating the interpolation estimates of the system response matrices every fifth and second interval. In general, stable and adequate control was observed. Figures 3-11 through 3-14 show the result of including (T) in the basis vector, and recalculating the current sensitivity and current response matrices every fifth and second intervals. It appears that in most cases the best control is provided in the case where (T) is included in the basis vector and the system matrices are updated every second interval.

The effect of the inclusion of (T) in the basis vector is an area which will require more study to draw any concrete conclusions. It might be expected that inclusion of T may be of more significant help for more slowly time-varying plants where updating would not be necessary quite so often.

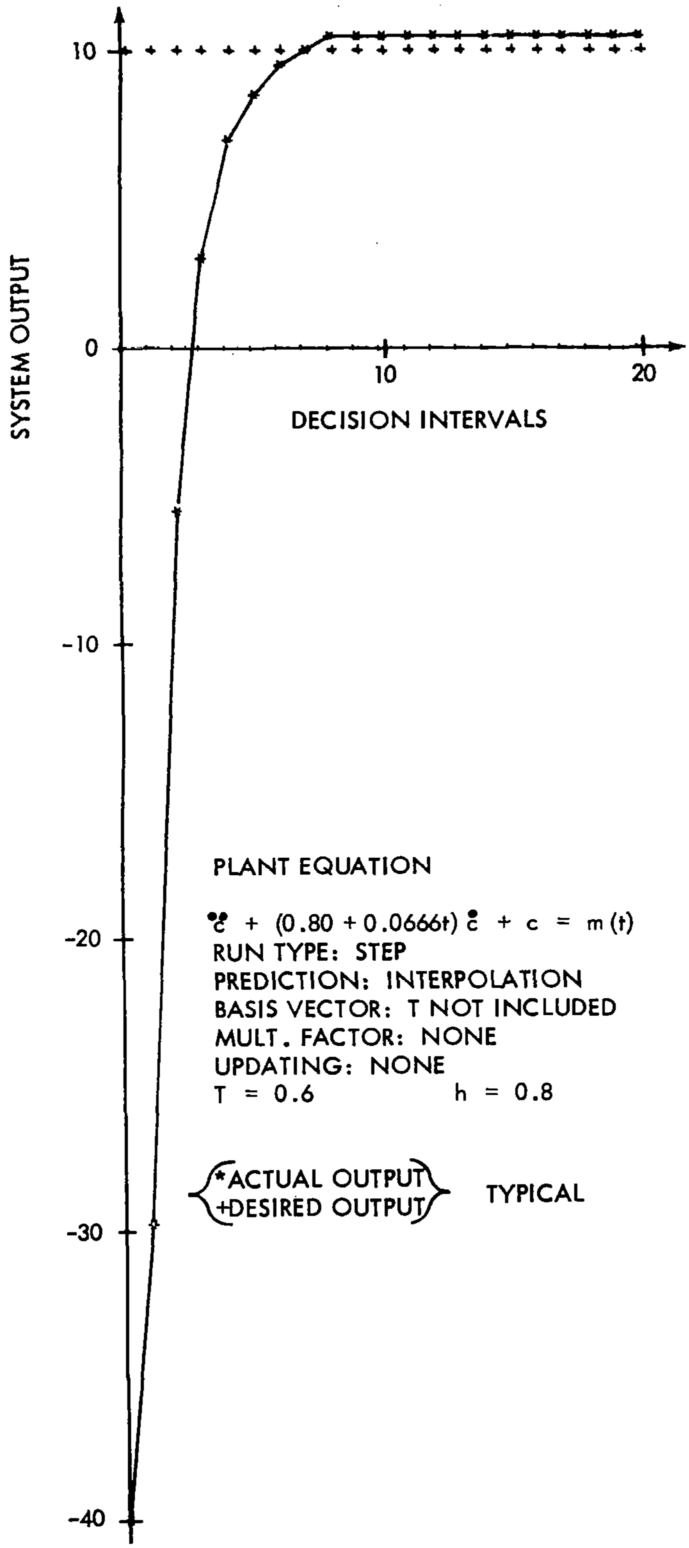


FIGURE 3-3 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

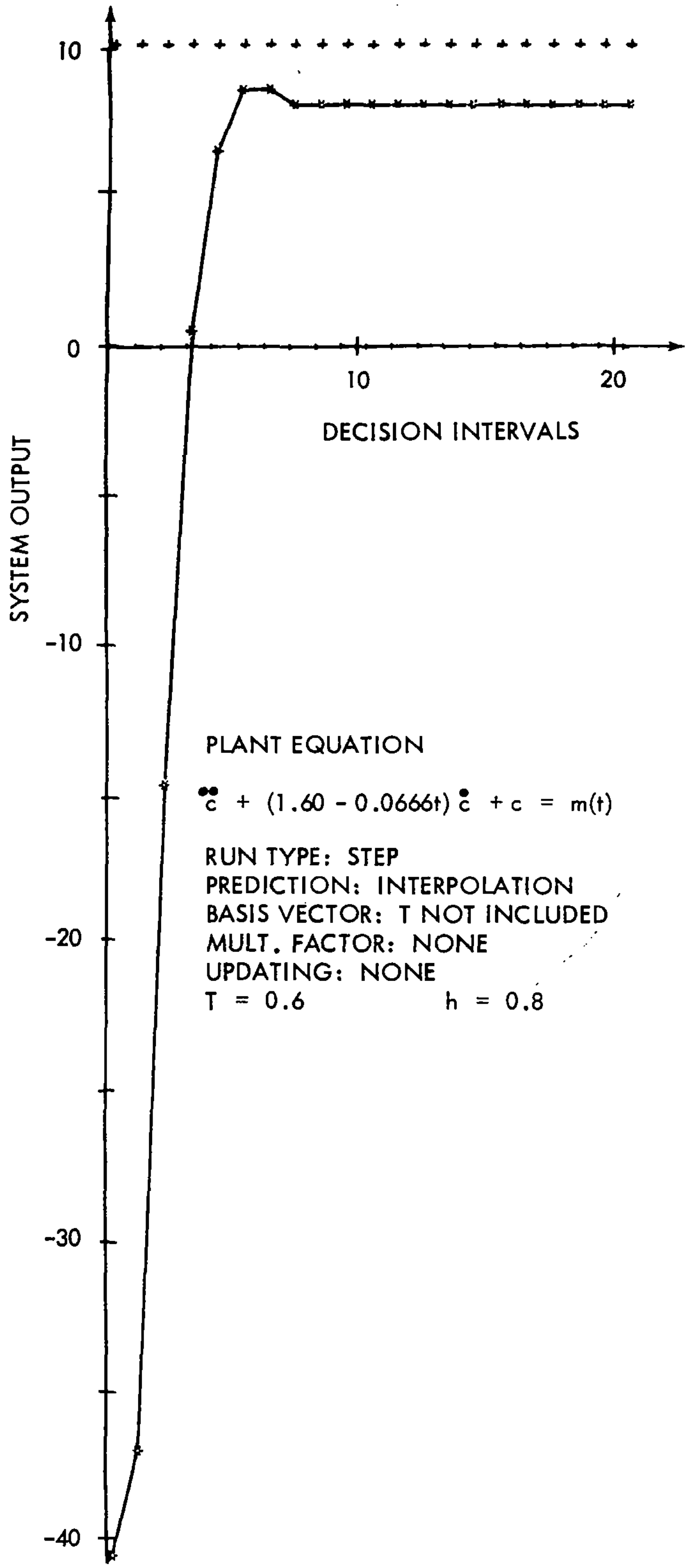


FIGURE 3-4 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

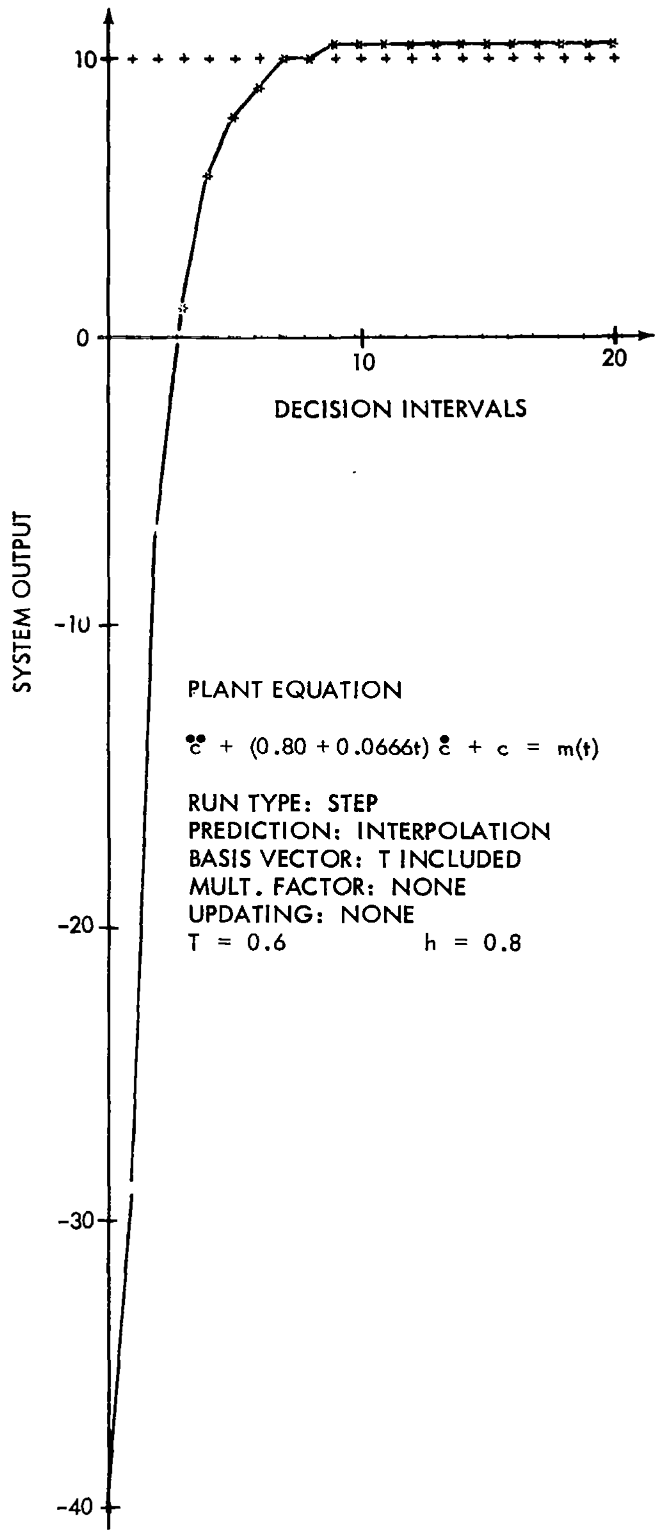


FIGURE 3-5 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

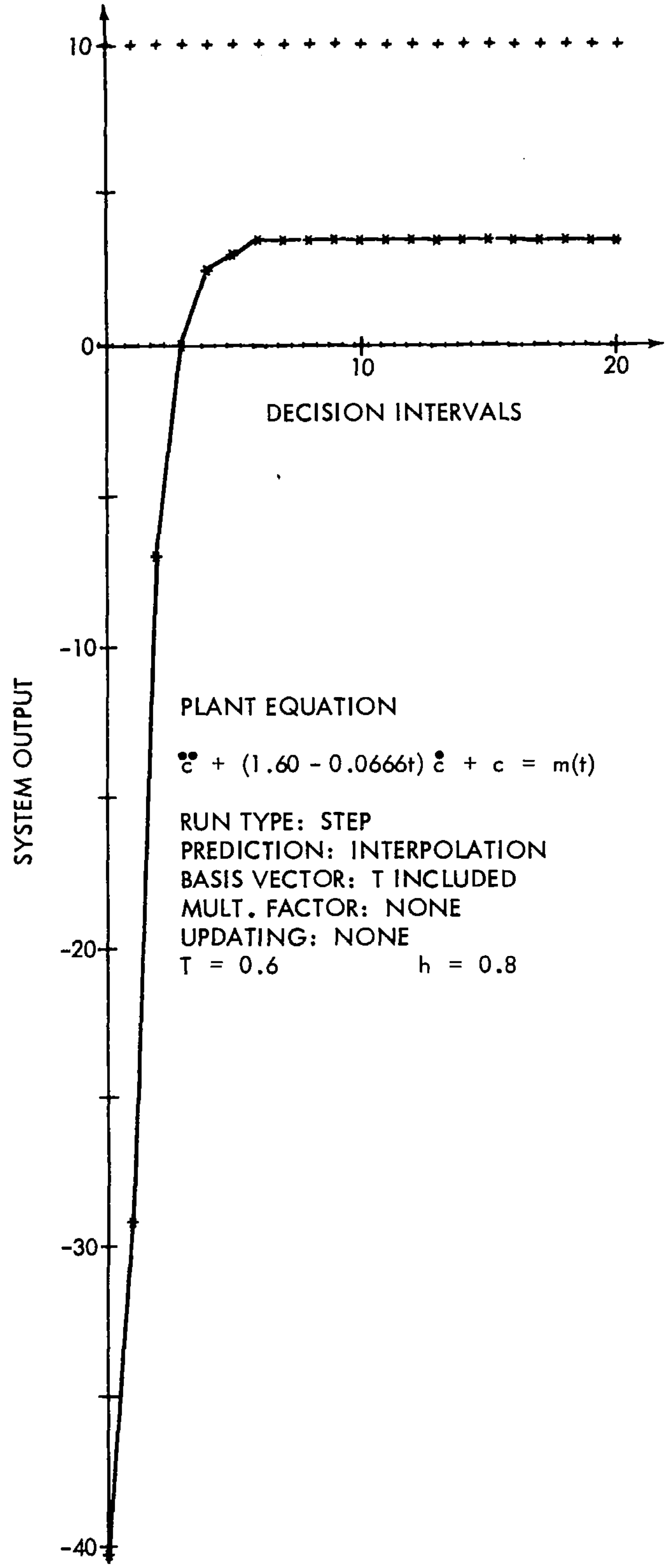


FIGURE 3-6 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

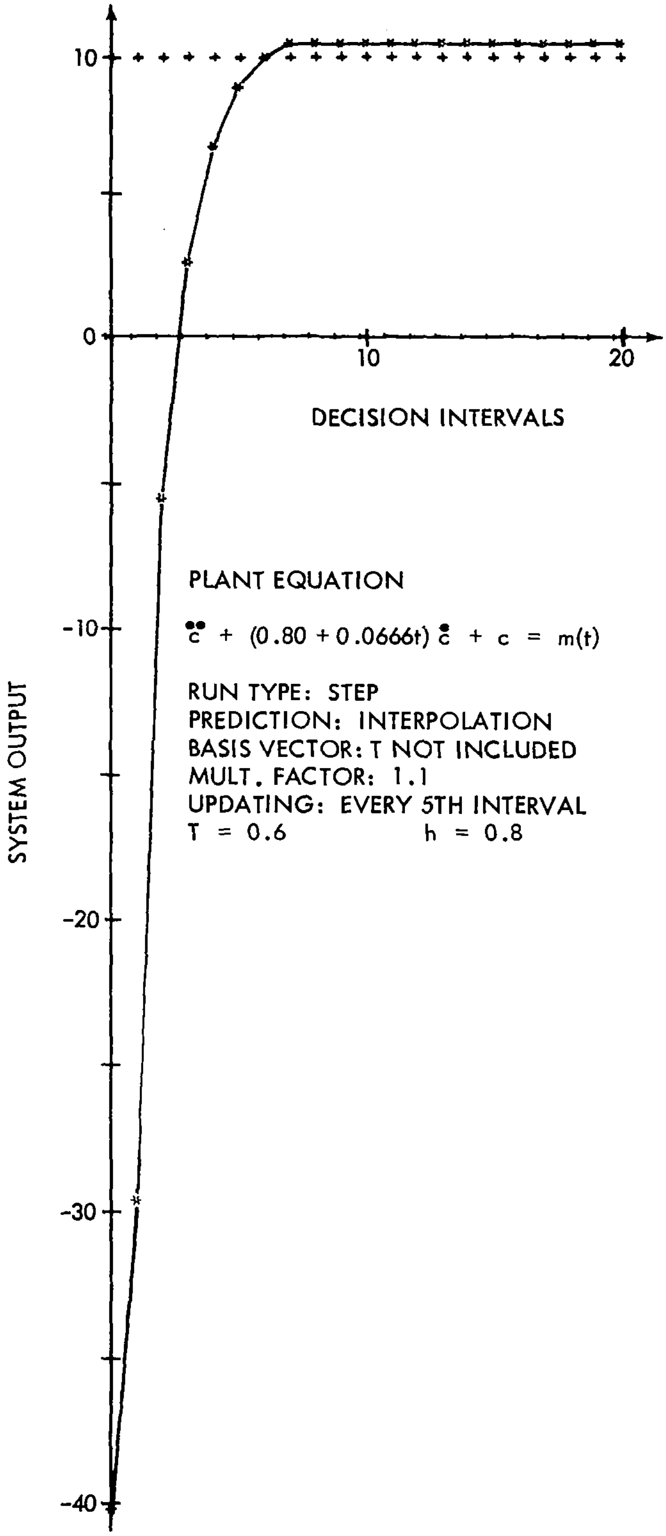


FIGURE 3-7 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

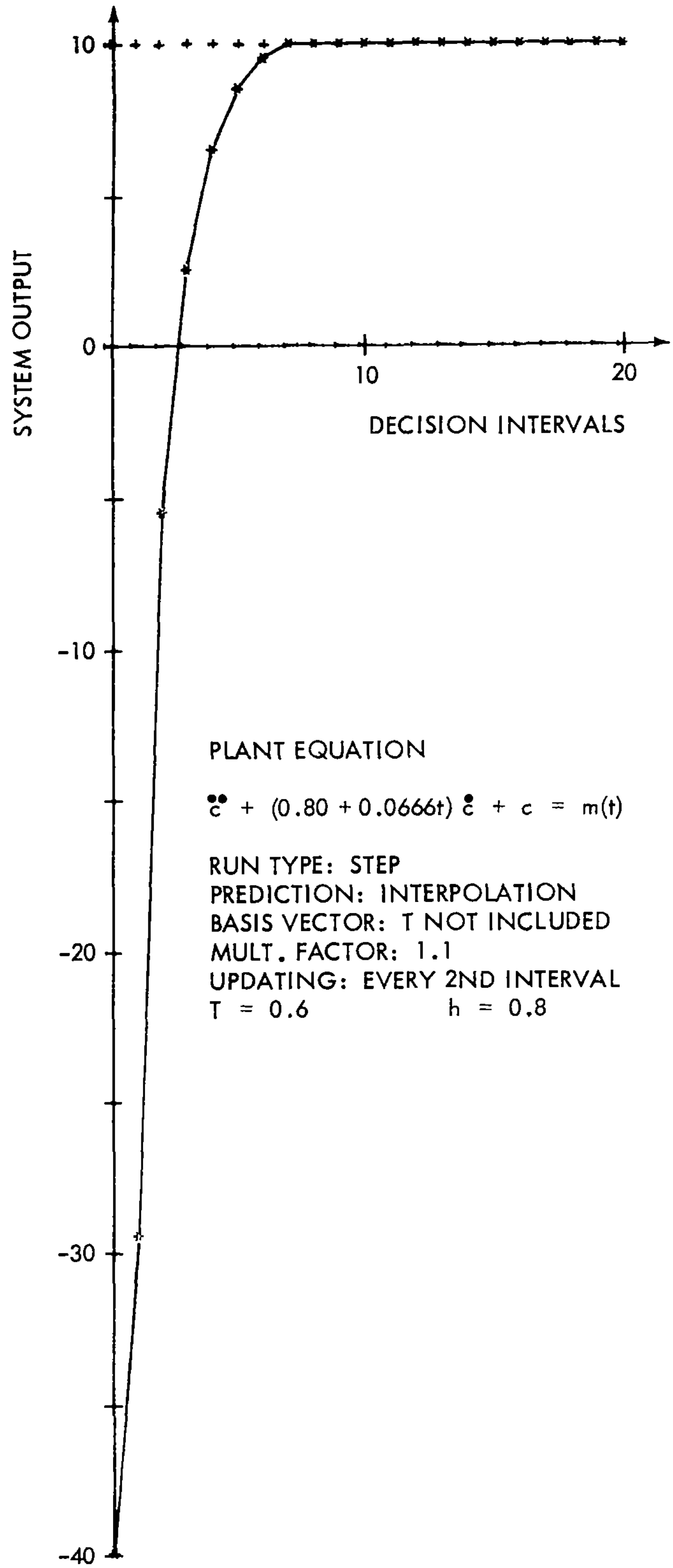


FIGURE 3-8 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

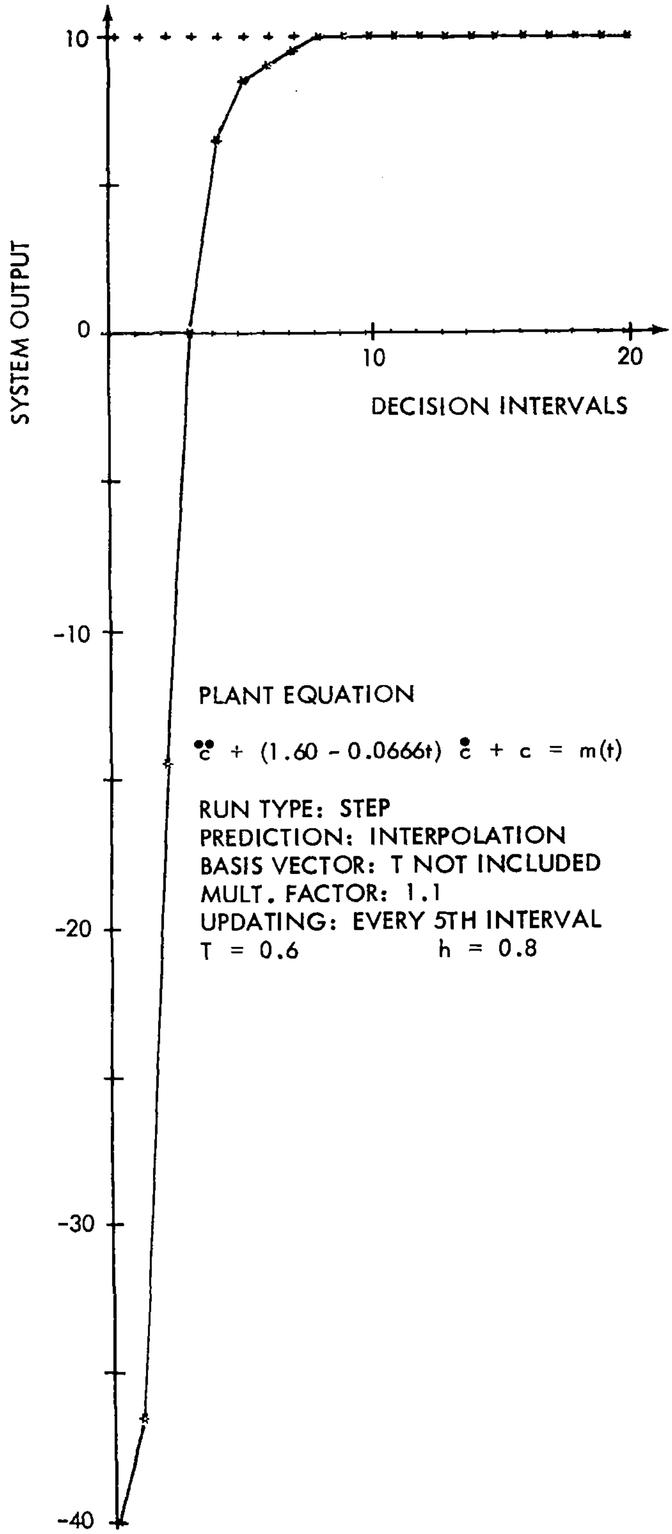


FIGURE 3-9 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

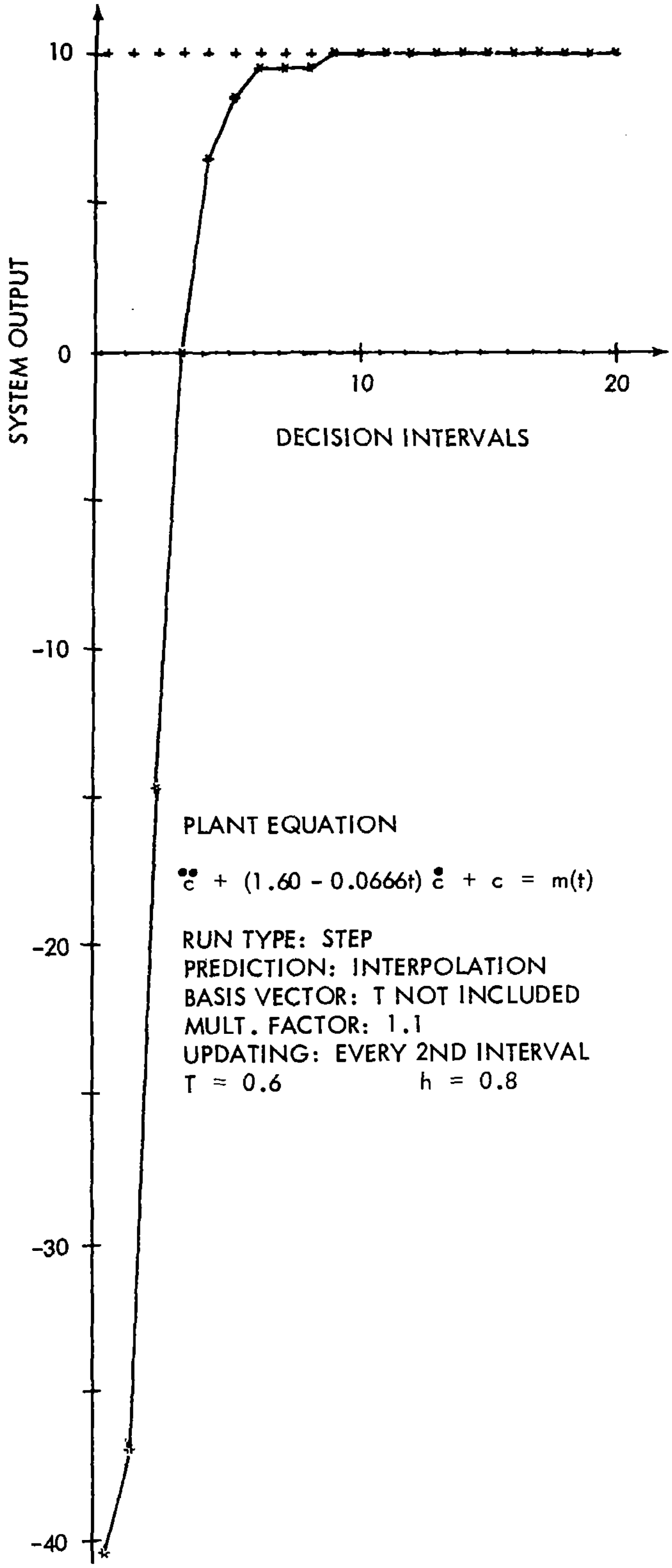


FIGURE 3-10 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

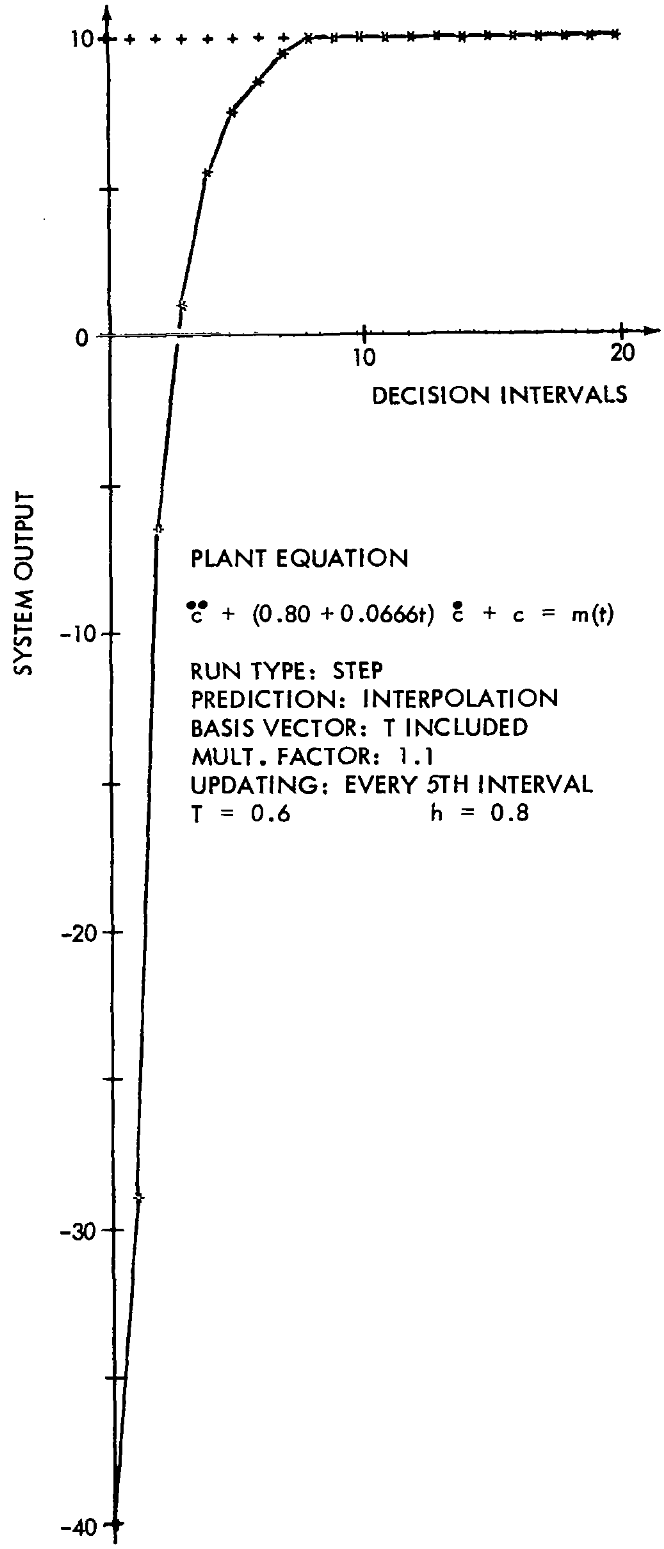


FIGURE 3-11 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

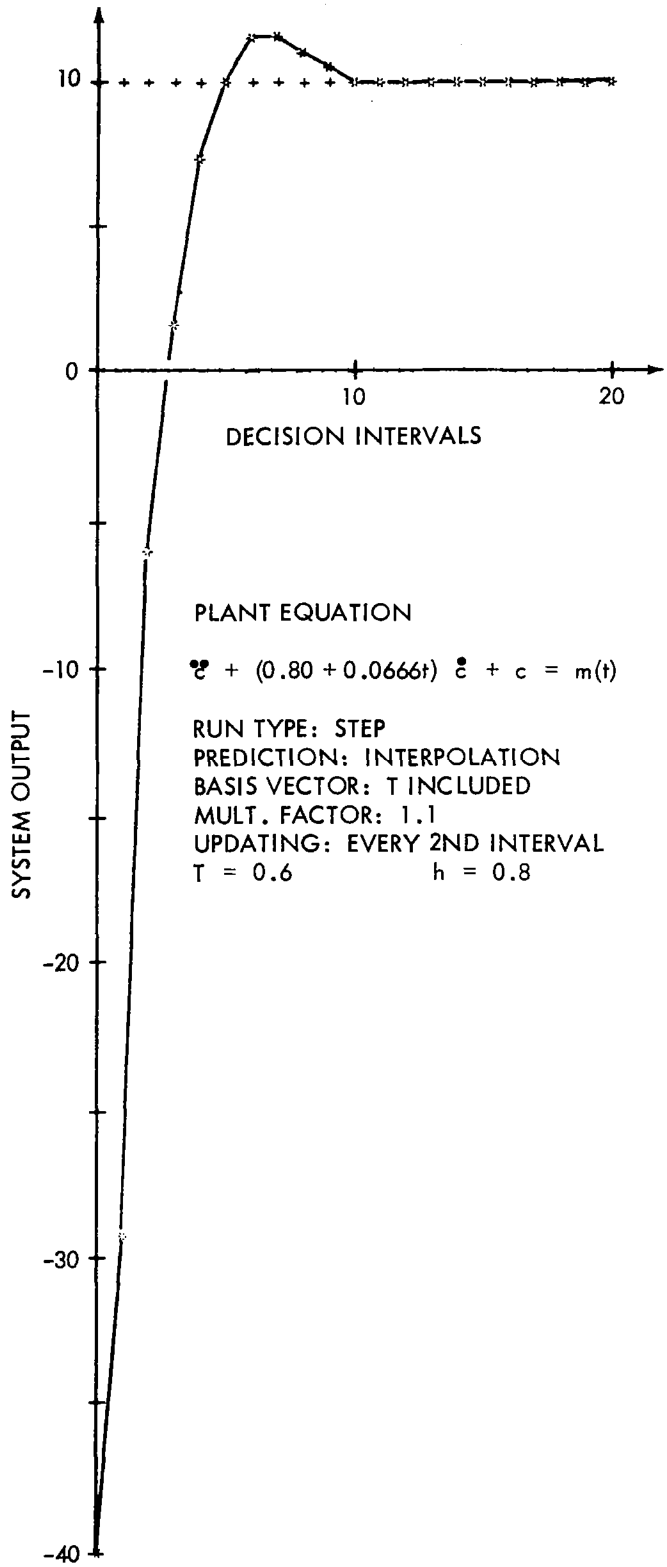


FIGURE 3-12 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

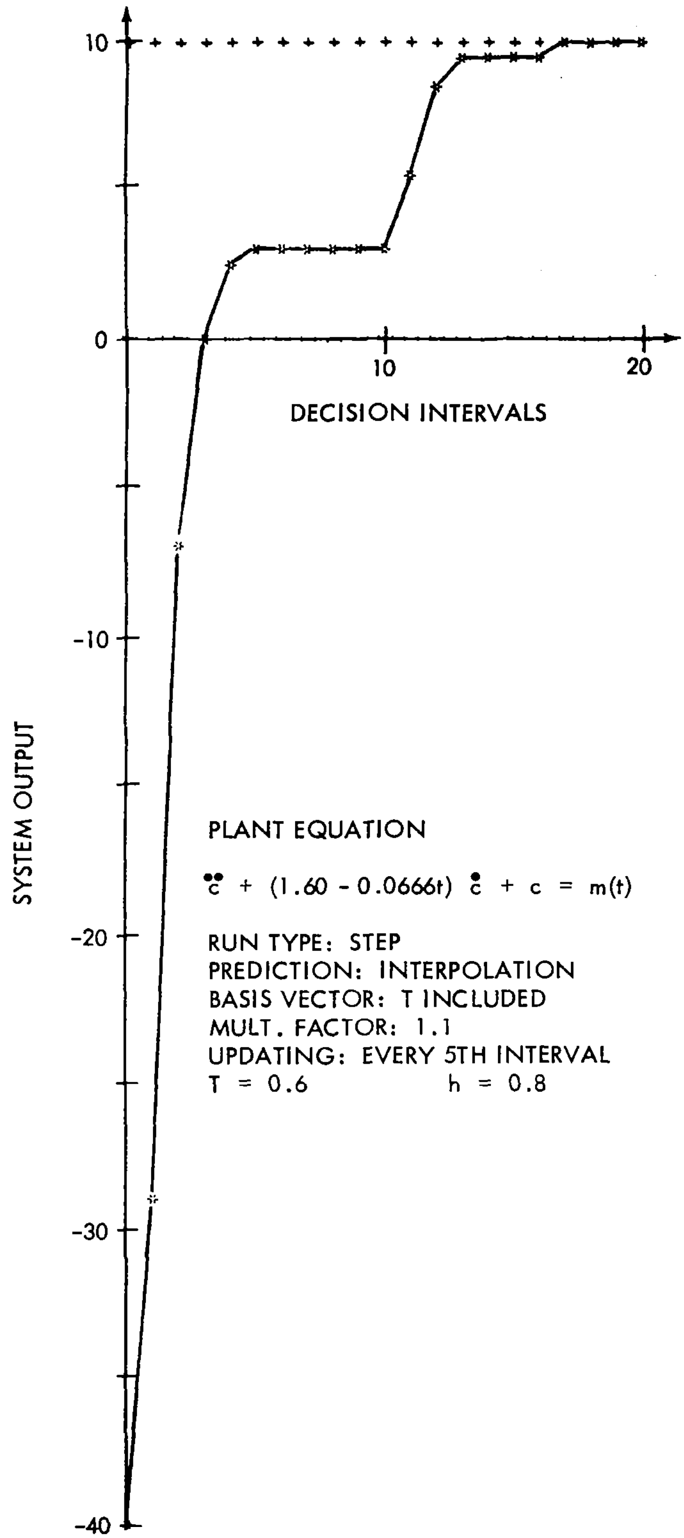


FIGURE 3-13 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

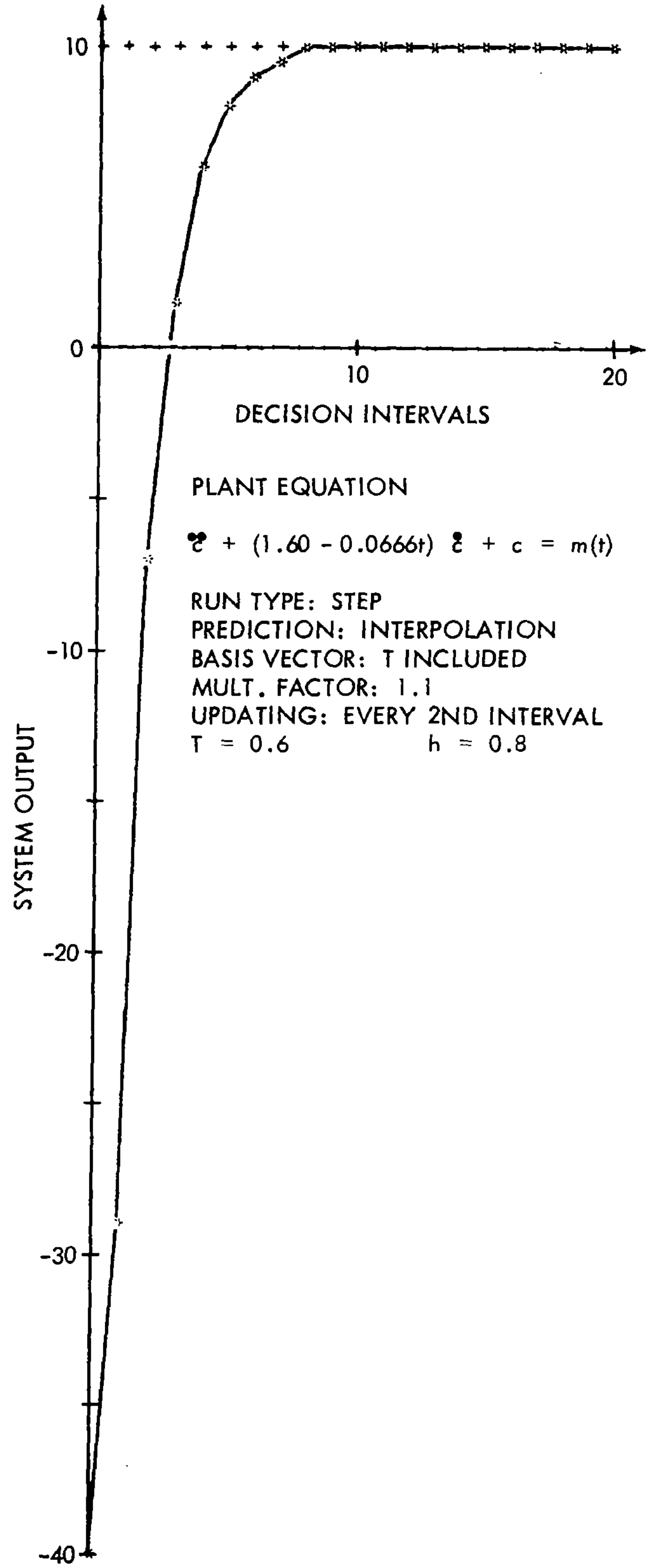


FIGURE 3-14 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

A few cases, such as exhibited by Figures 3-7 and 3-13, were encountered due to insufficient frequency of updating the system matrices. Figure 3-7 shows the control action resulted in a slight constant offset error; this may also be due to the 1.1 control force multiplying factor. The lack of updating with sufficient frequency is particularly evident in Figure 3-13 where the response arrived at several intermediate plateaus. When an update occurred the control action was such as to reduce the error in successive steps until the output arrived at the desired state. The plateau effect is removed by updating every second interval as is shown in Figure 3-14.

The results obtained should be expected due to the wide range and speed of the time-varying parameter. In all of the updating experimentation new data was shifted into the matrix of basis vectors every interval, and a multiplying factor of 1.1 was utilized to keep one non-control policy force present in this matrix.

Figures 3-15 through 3-18 show typical results for plants with slower time variation of damping for both step and ramp desired output states. The corresponding system equations are:

$$\ddot{c} + (0.72 + 0.0333t)\,\dot{c} + c = m(t) \qquad (3\text{-}32)$$

$$\ddot{c} + (1.68 - 0.0333t)\,\dot{c} + c = m(t) \qquad (3\text{-}33)$$

Figures 3-15 and 3-16 give the control results for the first plant, where the damping varied from about 0.36 to 0.76 in forty decision intervals. The following figures, 3-17 and 3-18, show the control response for the second plant, where the damping varied from about 0.84 to 0.44 in forty intervals. In these four runs, the decision interval time (T) was included in the basis vector, new data was incorporated in the matrix of basis vectors every interval, and the current sensitivity and current response was updated every fourth interval. Also, the 1.1 multiplying factor was used for the step desired output state runs of Figures 3-15 and

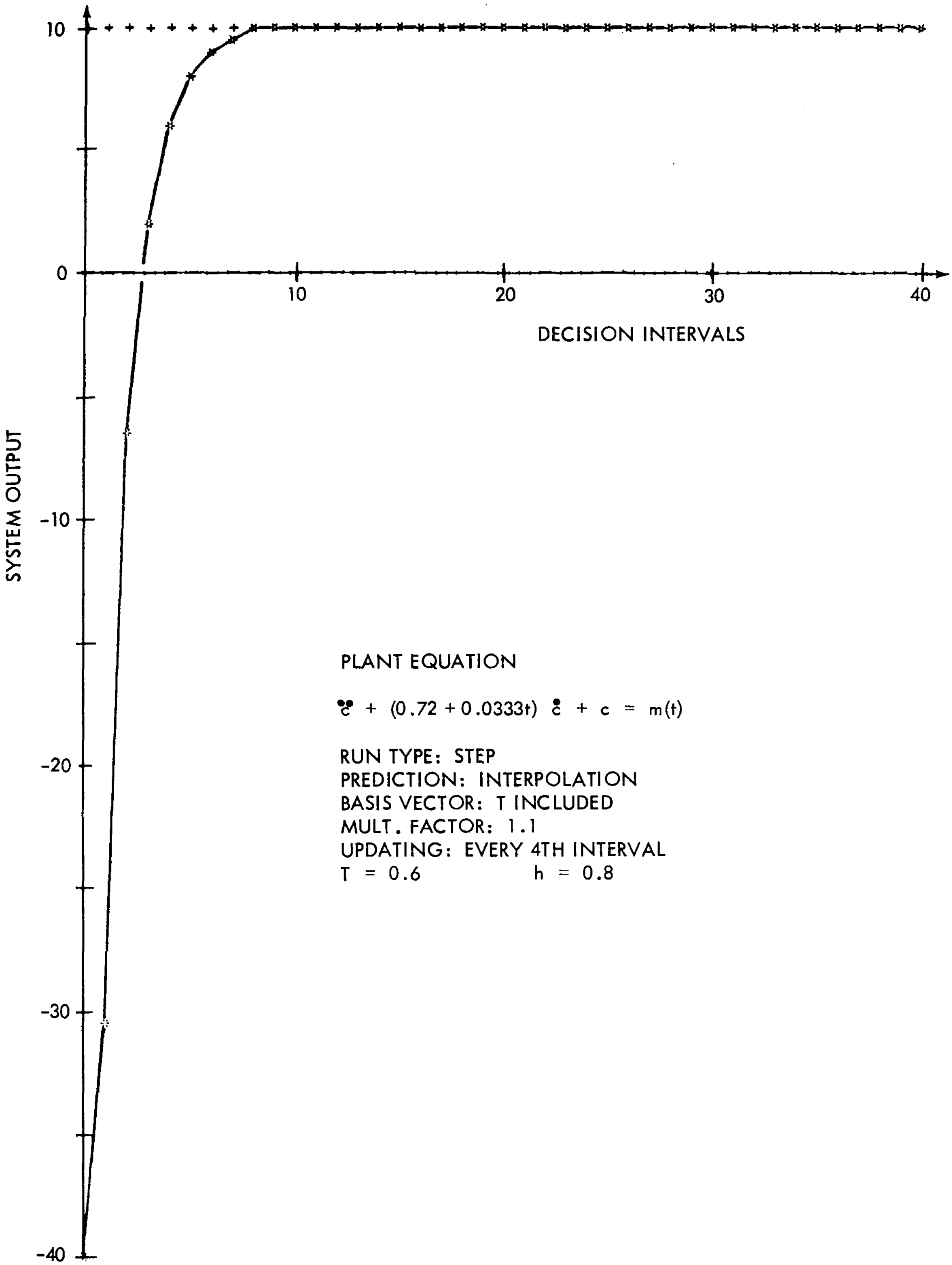


FIGURE 3-15 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

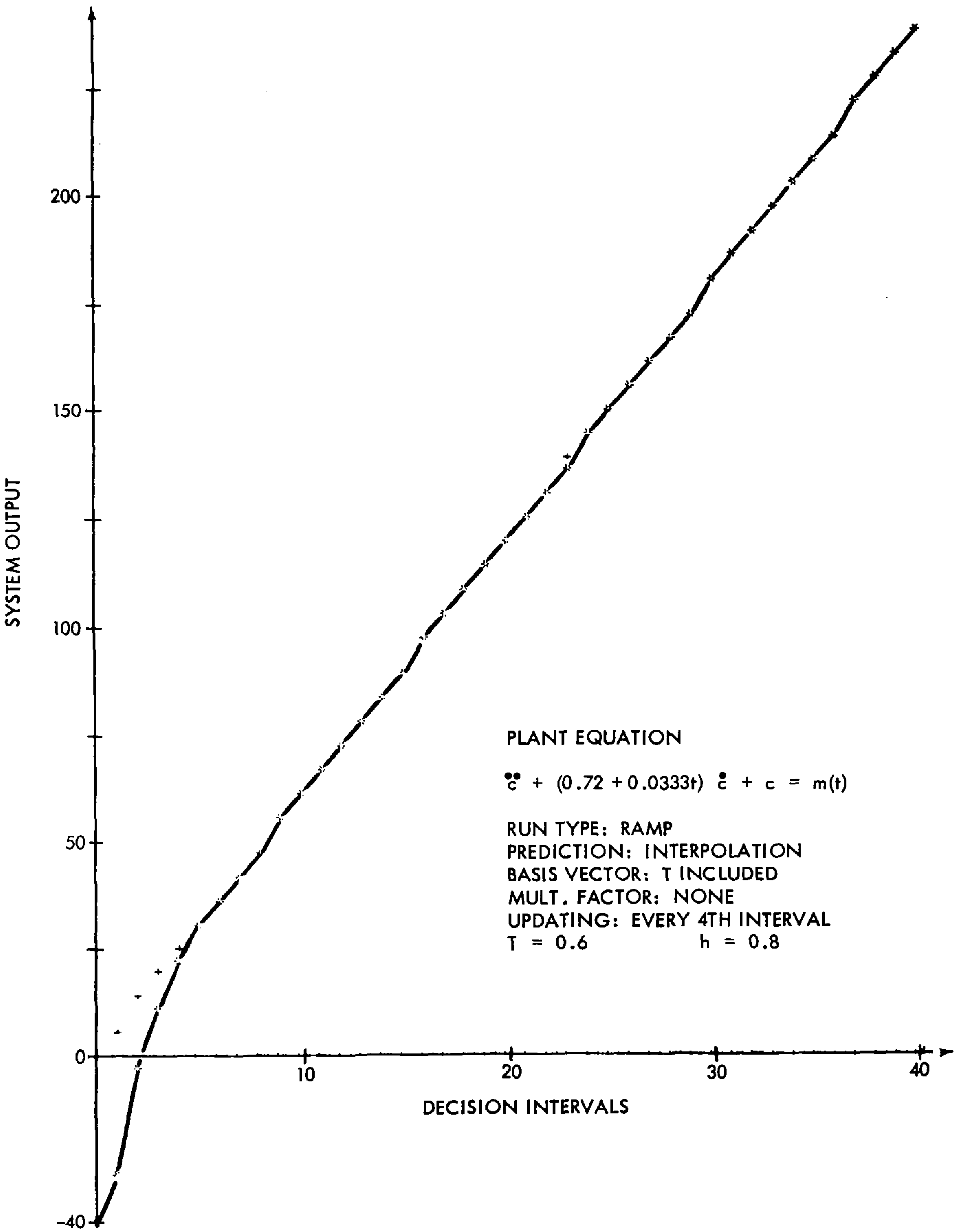


FIGURE 3-16 RAMP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

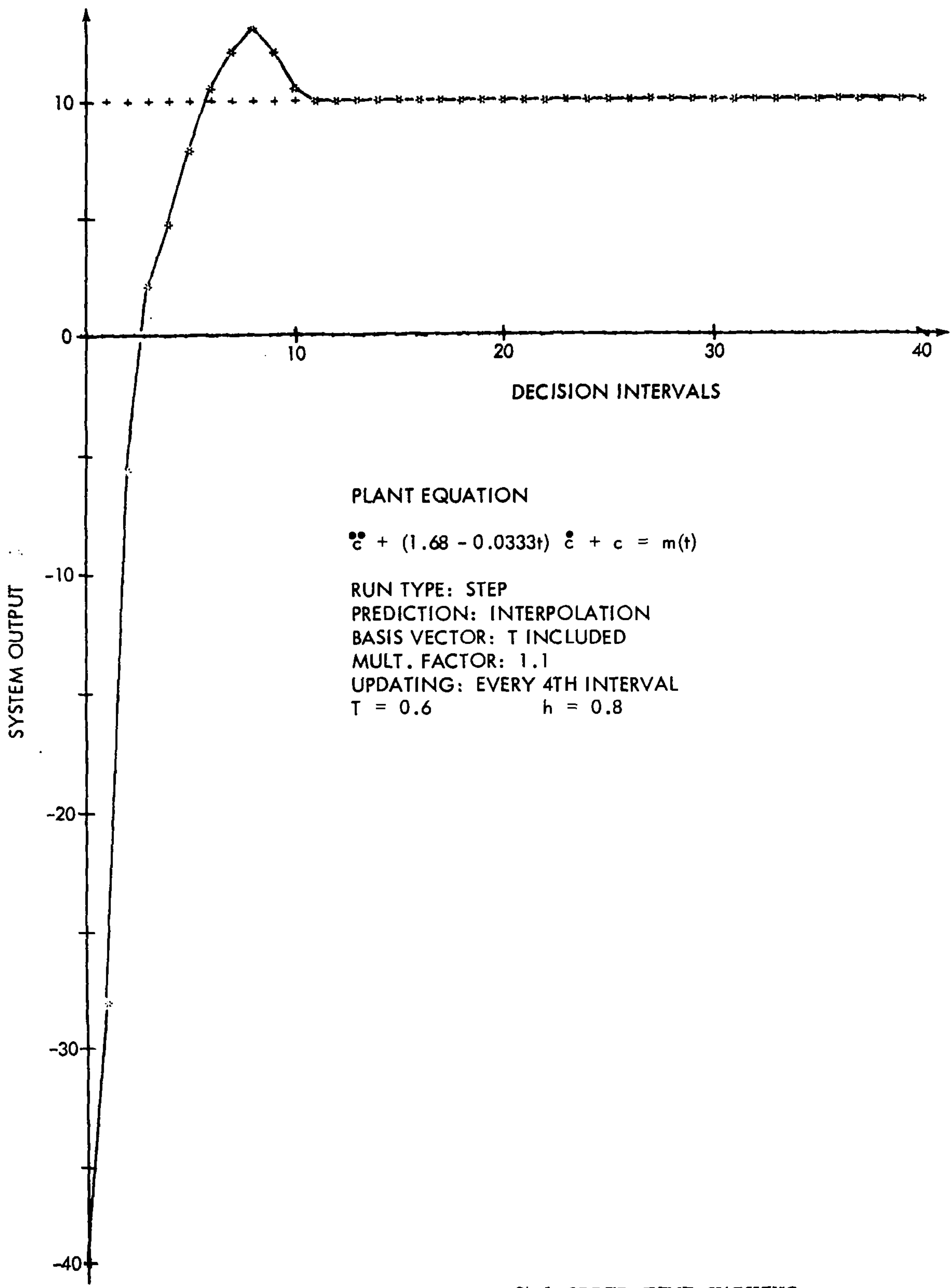


FIGURE 3-17 STEP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

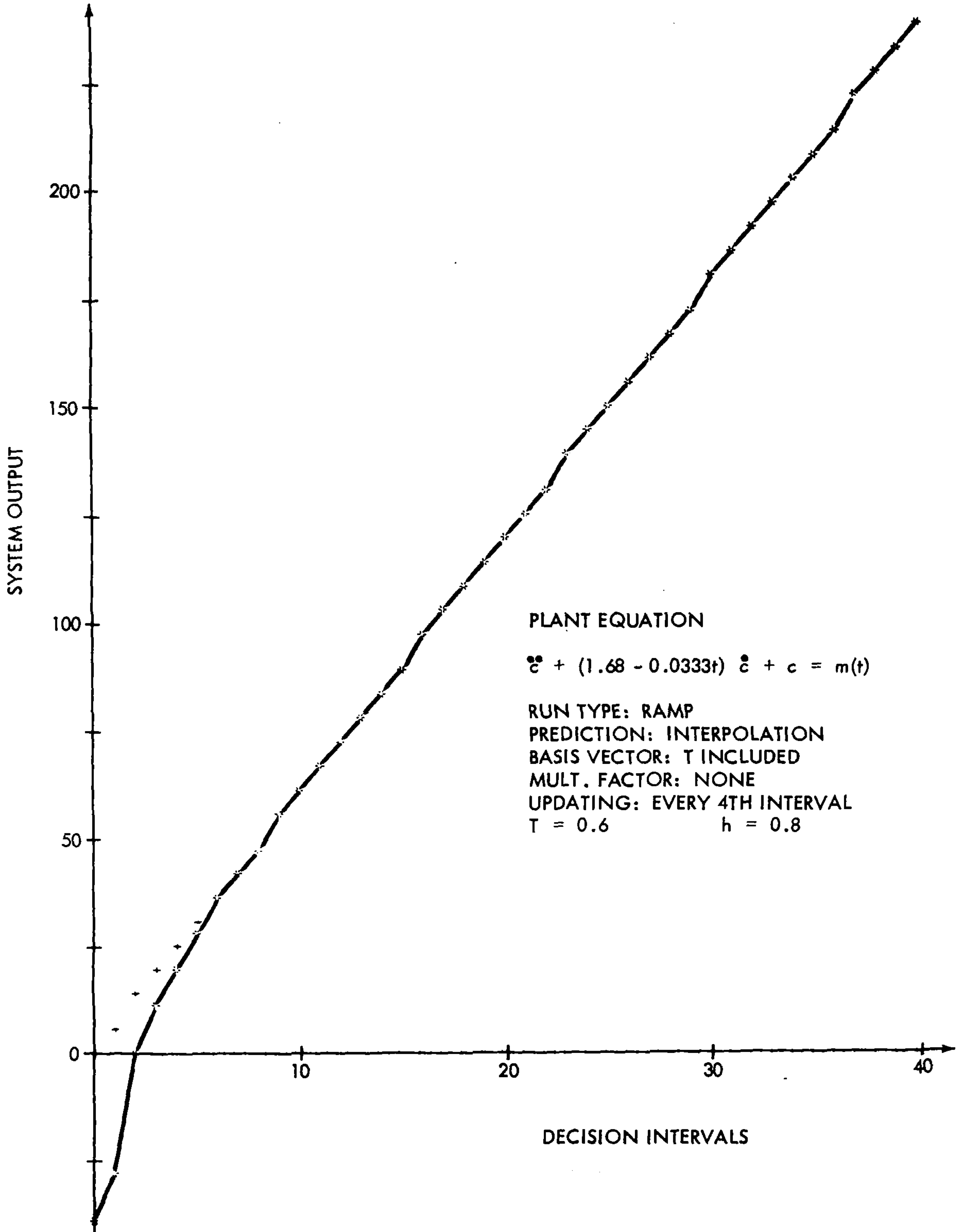


FIGURE 3-18 RAMP RUN FOR 2nd ORDER TIME-VARYING SYSTEM WITH UPDATING

3-17. These results indicate that the slower the parameter time variation, the less the necessity for frequent recalculation of the current sensitivity and current response matrices.

Third Order System Results.-The first set of experiments were conducted on a plant with an integration, a natural frequency of one, and a damping with linear time dependence. The system equations are:

$$\dddot{c} + (0.88 + 0.06666t)\ddot{c} + \dot{c} = m(t) \tag{3-34}$$

$$\dddot{c} + (1.52 - 0.06666t)\ddot{c} + \dot{c} = m(t) \tag{3-35}$$

Figures 3-19 through 3-22 illustrate the control performance for the case where the damping decreases from about 0.76 to 0.16 in thirty intervals. The first two Figures, 3-19 and 3-20, show that with (T) in the basis vector and no updating of the system matrices, a constant error results for both zero and step desired output states. The latter two figures along with Figures 3-23 and 3-24 demonstrate the control performance for decreasing (0.76 to 0.16) and increasing (0.44 to 1.04) damping respectively. These four figures illustrate that with the decision interval (T) included in the basis vector and updating every second interval, satisfactory control resulted for both zero and step desired output states.

A few control experiments were conducted on a plant with an integration, a damping of 0.3, and a natural frequency with linear time dependence. In the worse case examined the natural frequency varied from 2.44 to 9.4 in thirty decision intervals. Example control results are presented in Figures 3-25 and 3-26.

The third set of experiments were made on a plant with a natural frequency of one, a damping of 0.3, and a real pole with linear time dependance. The real root was varied from 1.9 to 9.4 in twenty-five and fifty-six intervals. The control results obtained for the case of the parameter variation over twenty-five intervals were very poor. However, good control

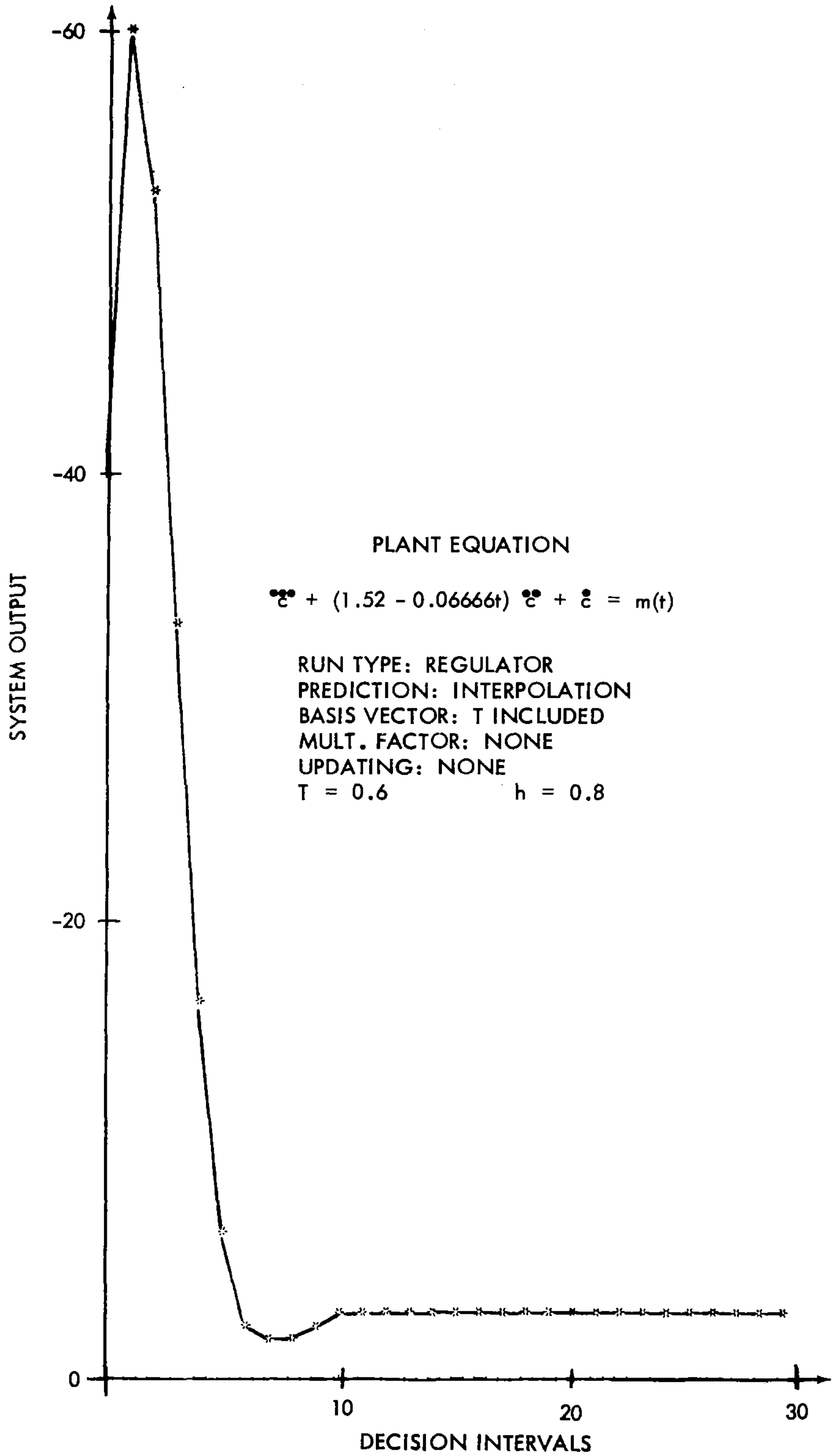


FIGURE 3-19 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

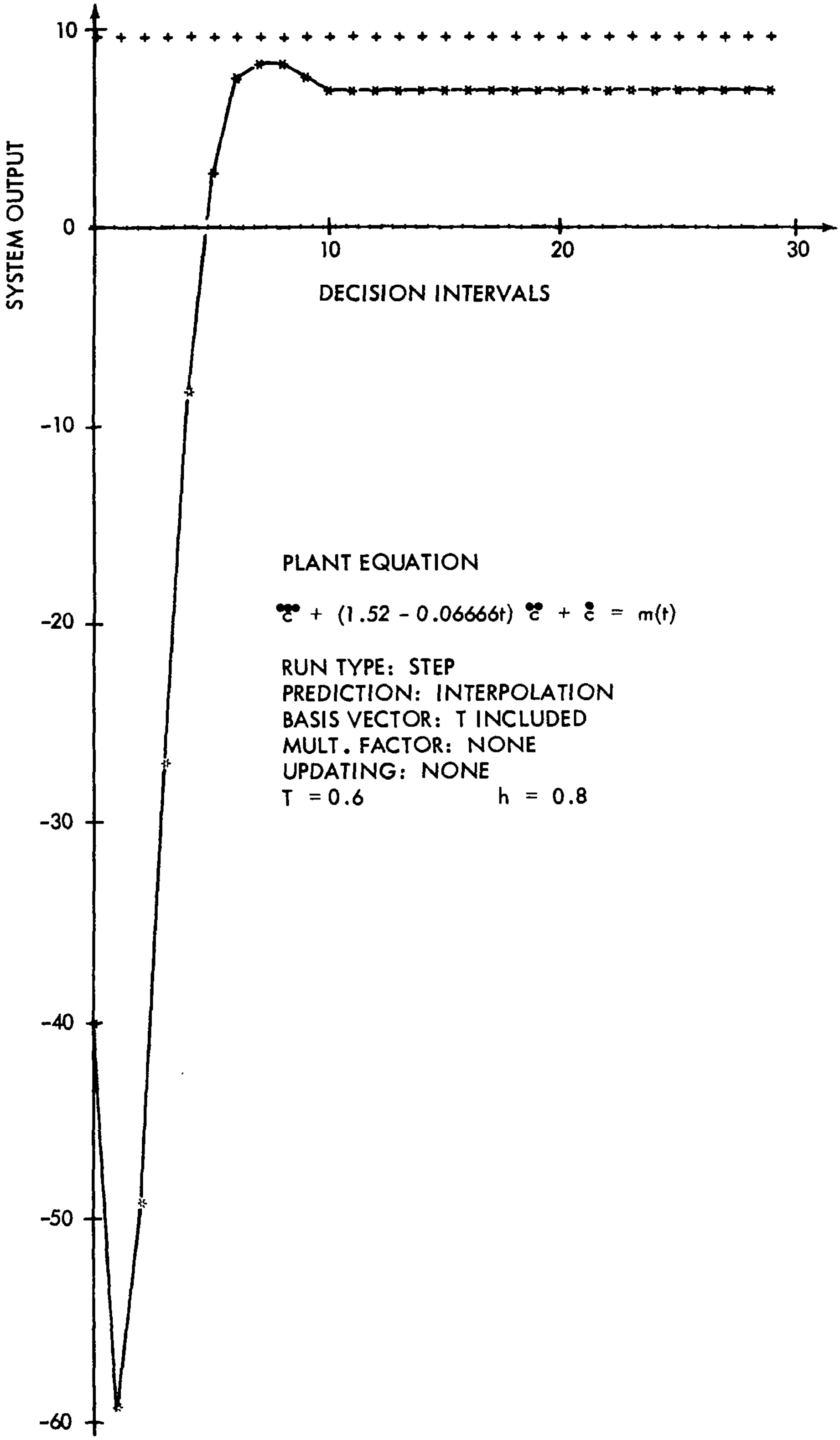


FIGURE 3-20 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH NO UPDATING

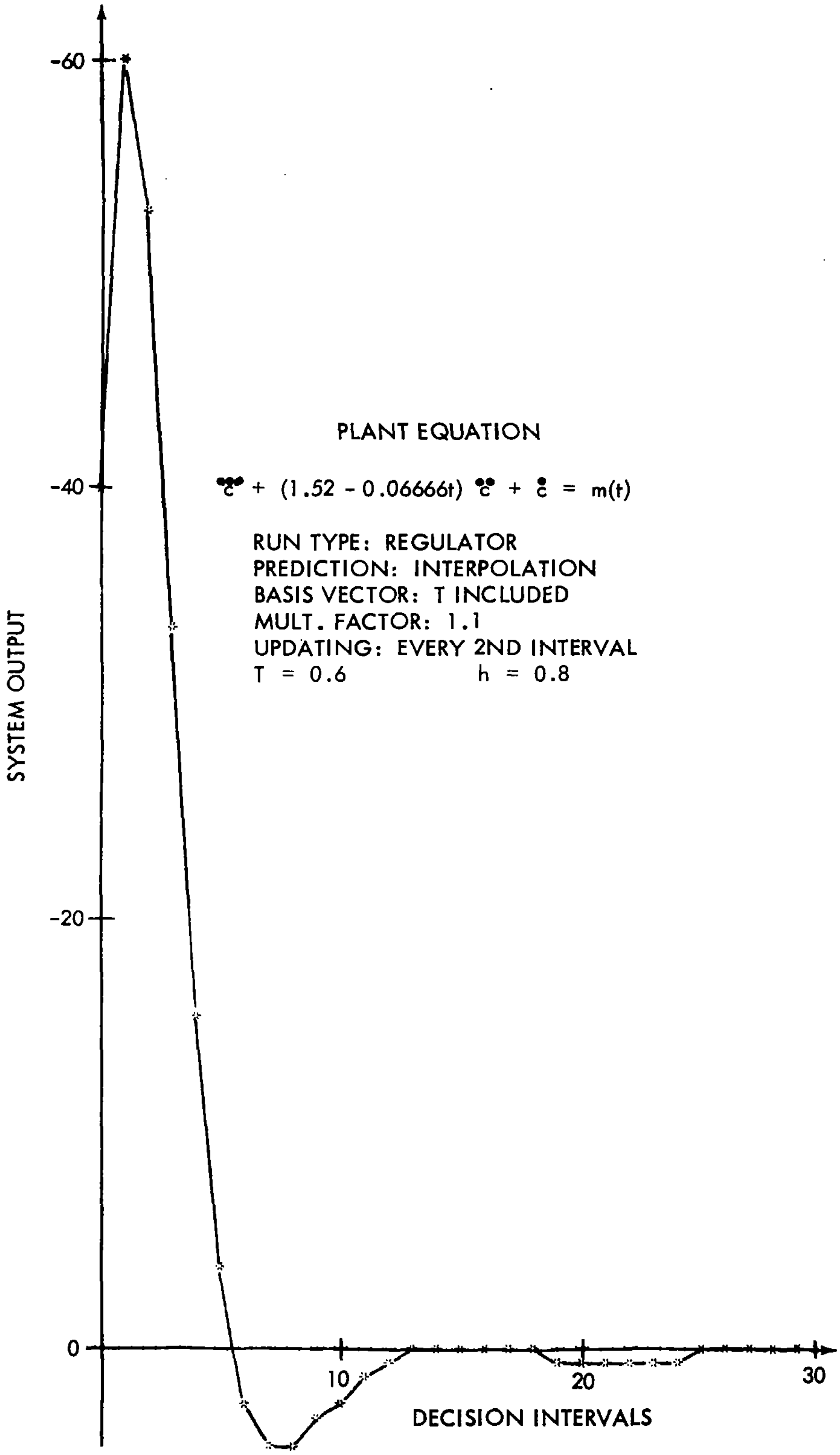


FIGURE 3-21 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

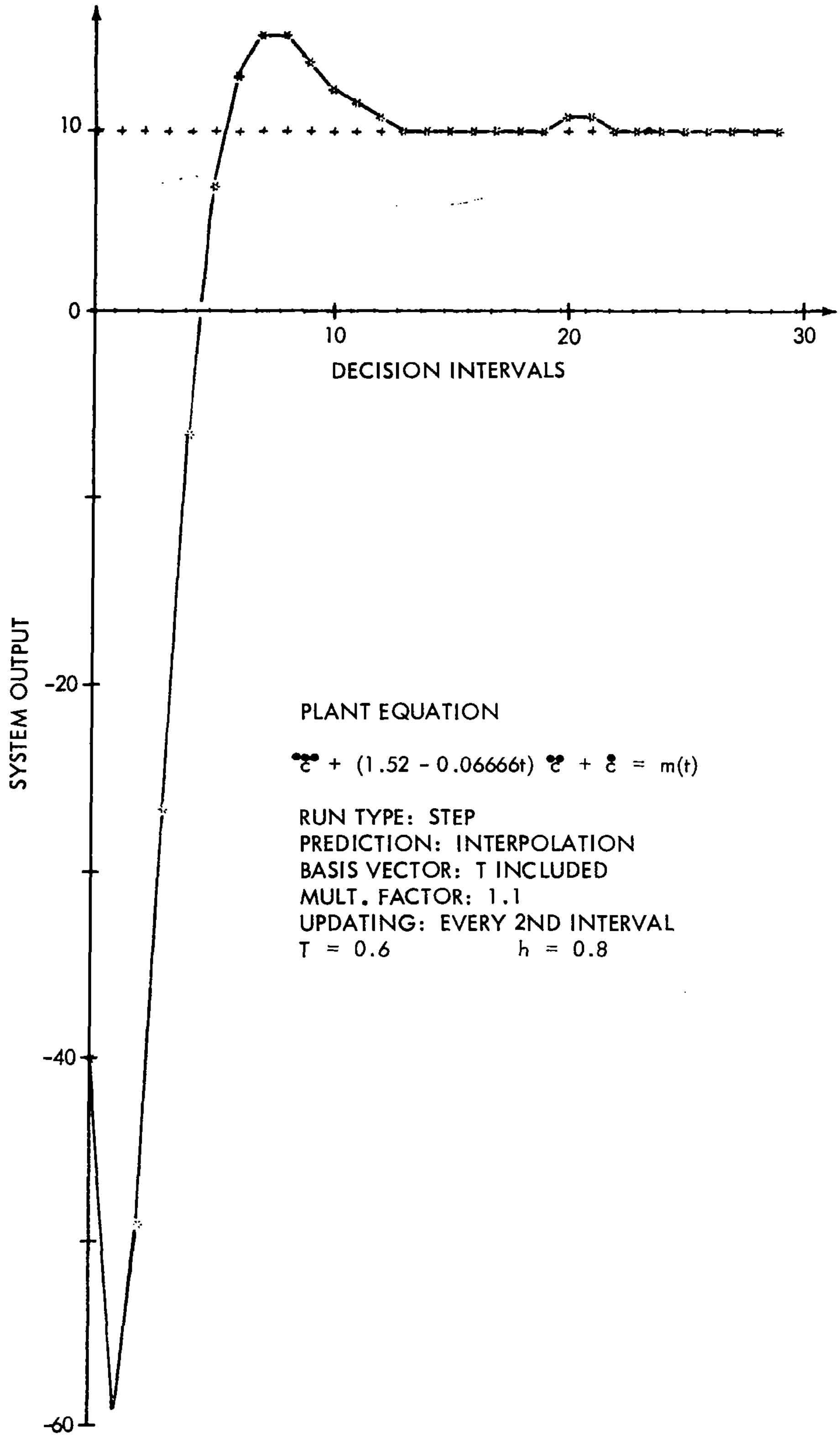


FIGURE 3-22 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

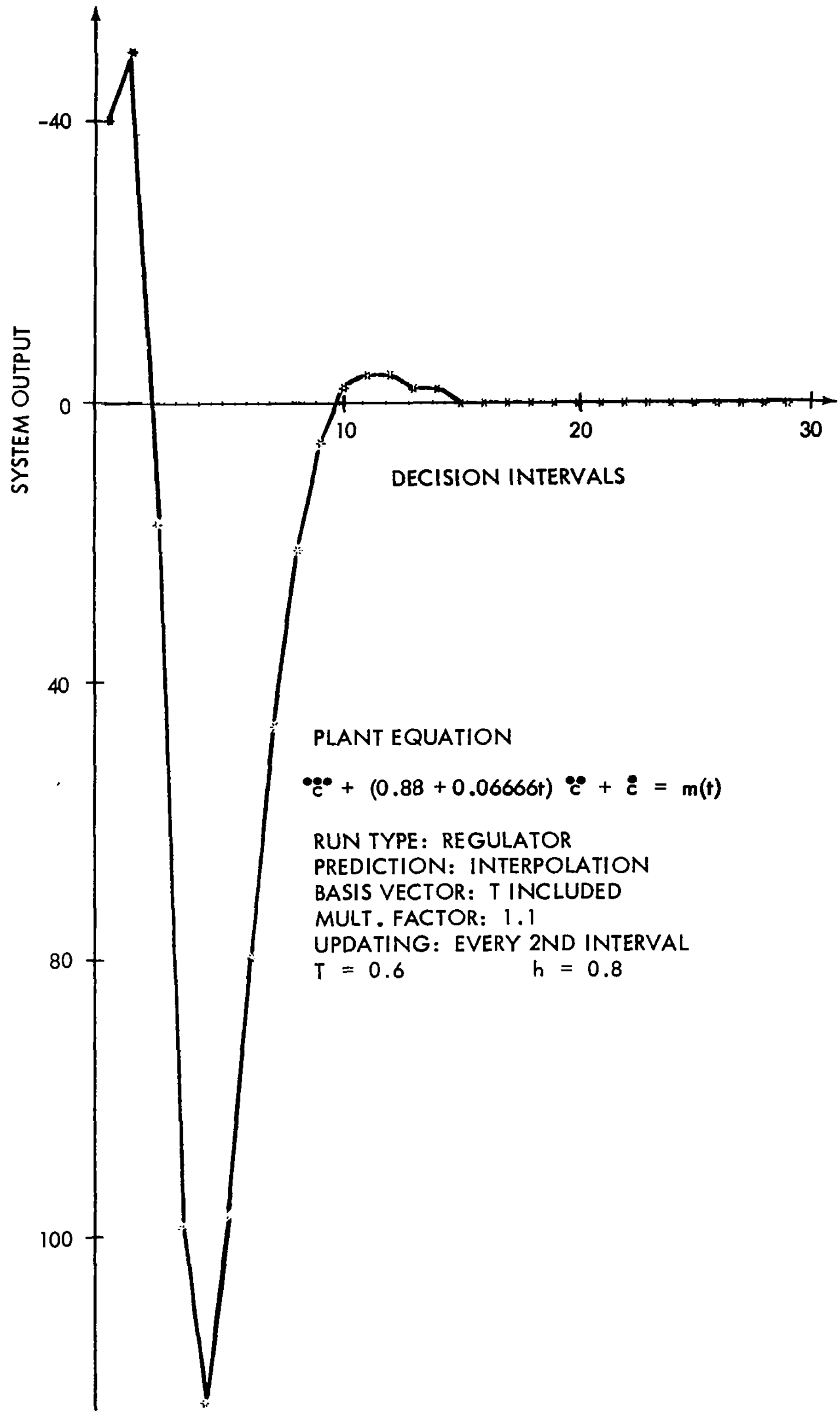


FIGURE 3-23 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

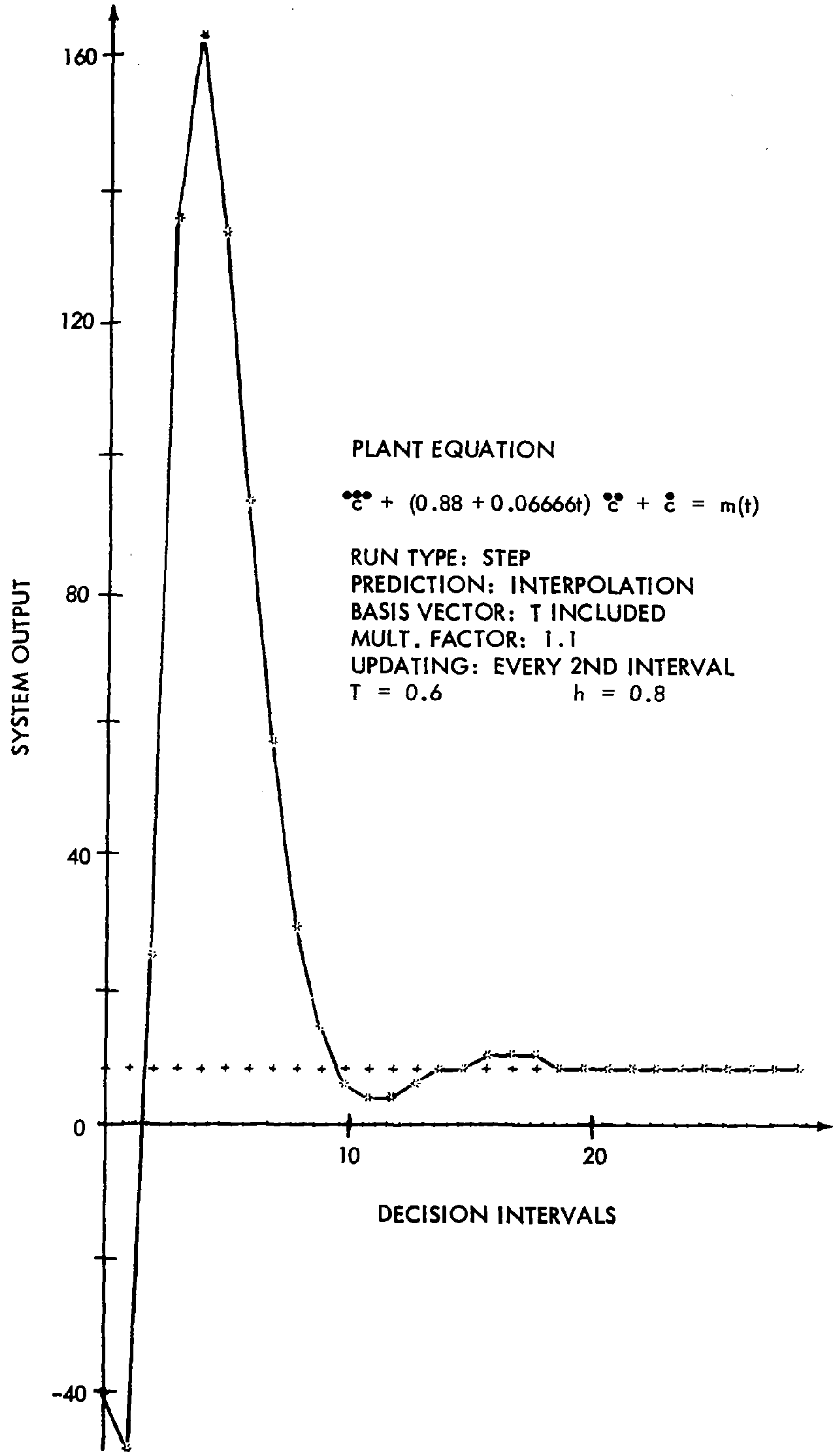


FIGURE 3-24 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

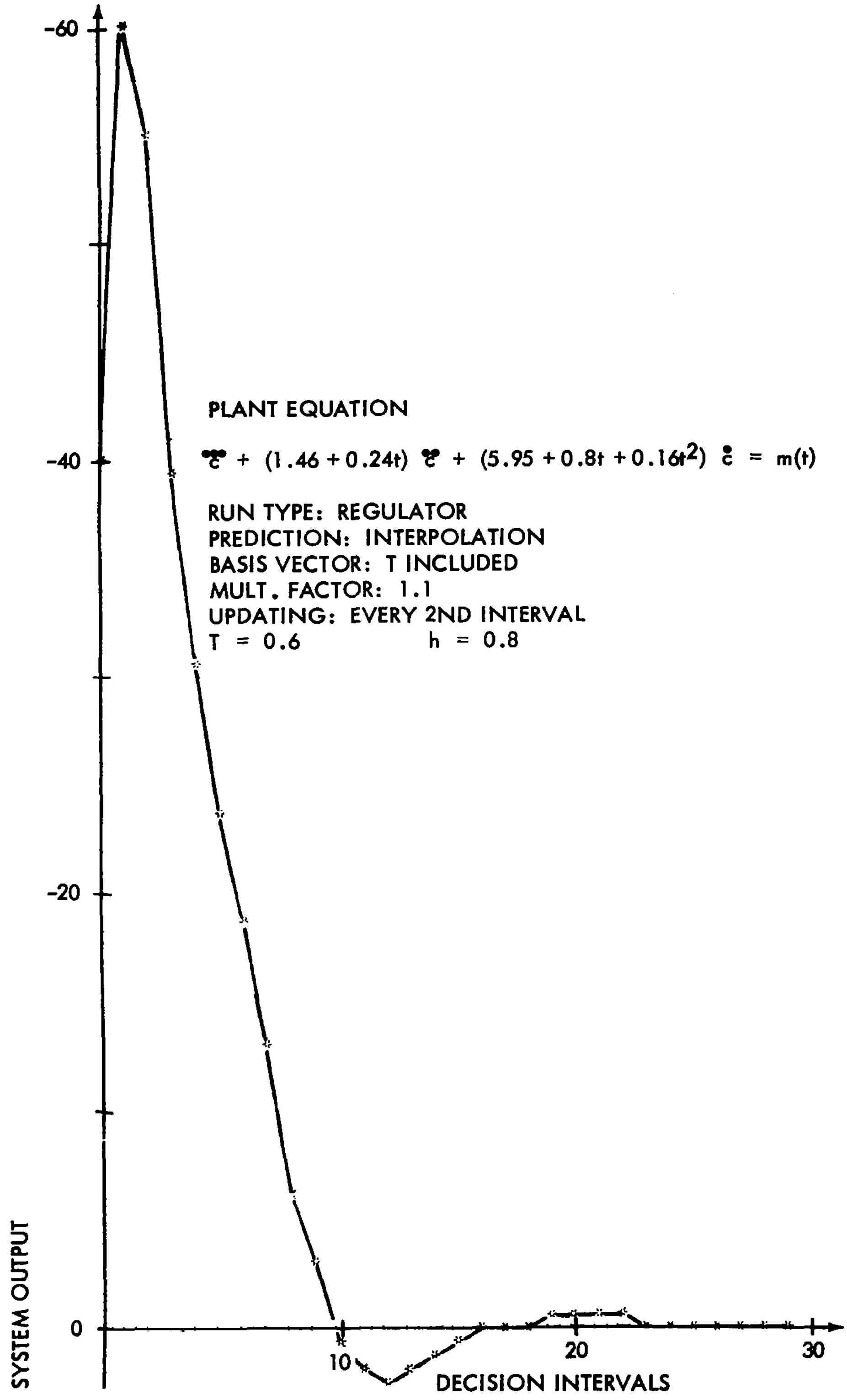


FIGURE 3-25 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

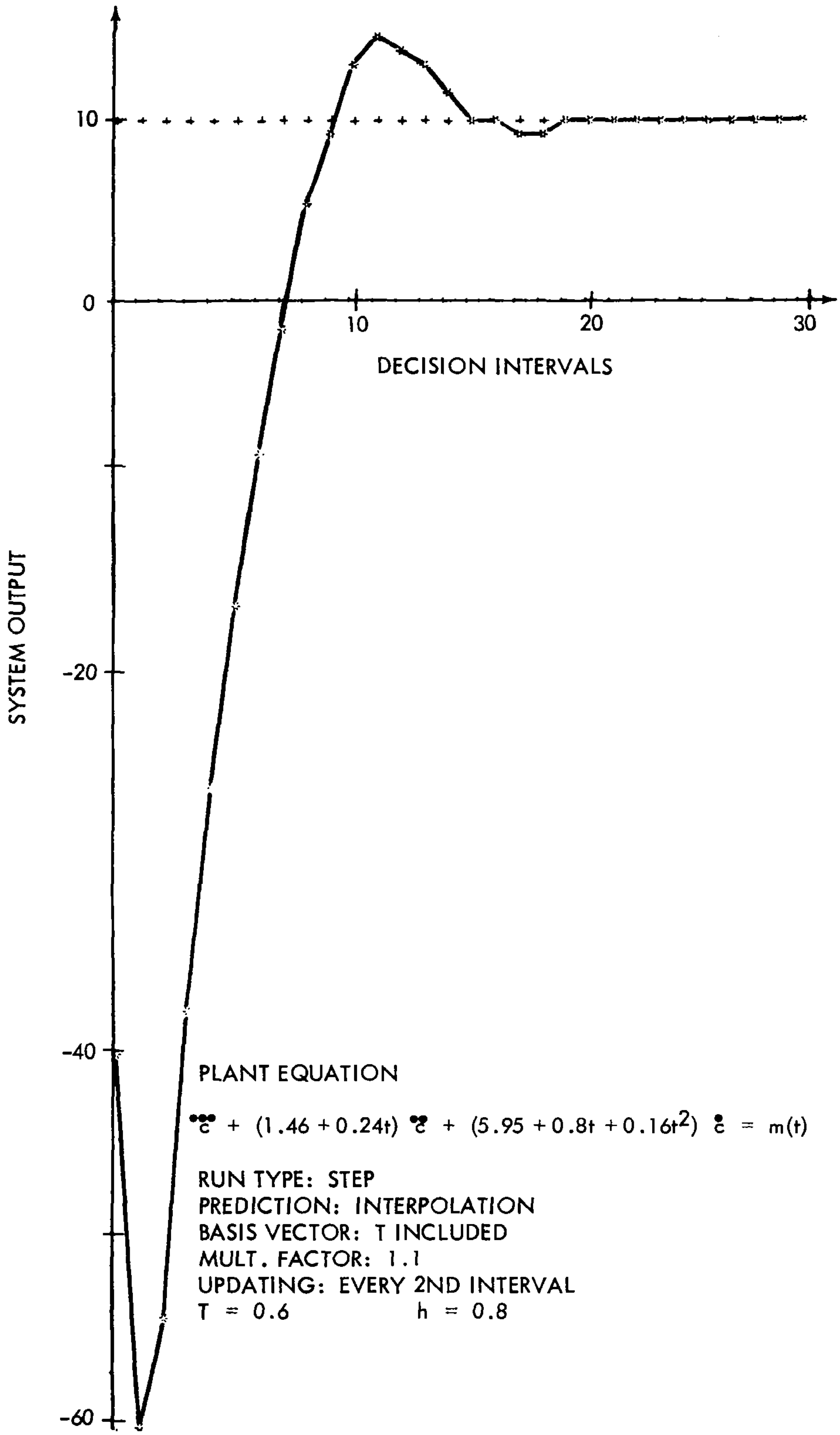


FIGURE 3-26 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

performance was observed for the case of the parameter variation over fifty-six intervals. Figures 3-27 and 3-28 present the latter experimental results. Figure 3-28 shows the controlled system response differs from the desired output state at about 20 decision intervals into the run. This event was due to a slight ill conditioning of the matrix of basis vectors, which resulted in poor current sensitivity and current response matrices being used for two decision intervals.

A limited number of control simulations for plants with the real root as a sine function of time were made in the last stages of this investigation. The majority of these experiments were done on a plant with a natural frequency of one and a damping of 0.3. The real root was varied from about 3.95 to 5.55 as a sine function of time with a frequency of 1/8, 1/4, and 1/2 the natural frequency of the plant. Figures 3-29 and 3-30 show typical results obtained for zero desired output state. In these results the real root varied at 1/8 and 1/4 the plant's natural frequency respectively. Figures 3-31 and 3-32 show the control performance under the same conditions, but for a step desired output state. The deterioration of control performance due to matrix ill conditioning becomes worse as the real root variation frequency is increased.

In all the third order systems presented, a 1.1 multiplying factor was used to keep one non-control policy force in the matrix of basis vectors, new data was included into this matrix every interval, and the current sensitivity and current response was updated every other interval.

3.3 SUMMARY AND EVALUATION OF EXPERIMENTAL RESULTS

The experimental results presented in the last section indicate that low order systems with rather severe time variation may be stably and adequately controlled by our control system. It should be noted that only two T-h combinations were investigated in the experimental studies. These were T = 0.6, h = 0.8, and T = 0.3, h = 0.8. Stable performance was observed, and similar control results were obtained for both T-h cases. The control performance was affected by several factors which are associated with the

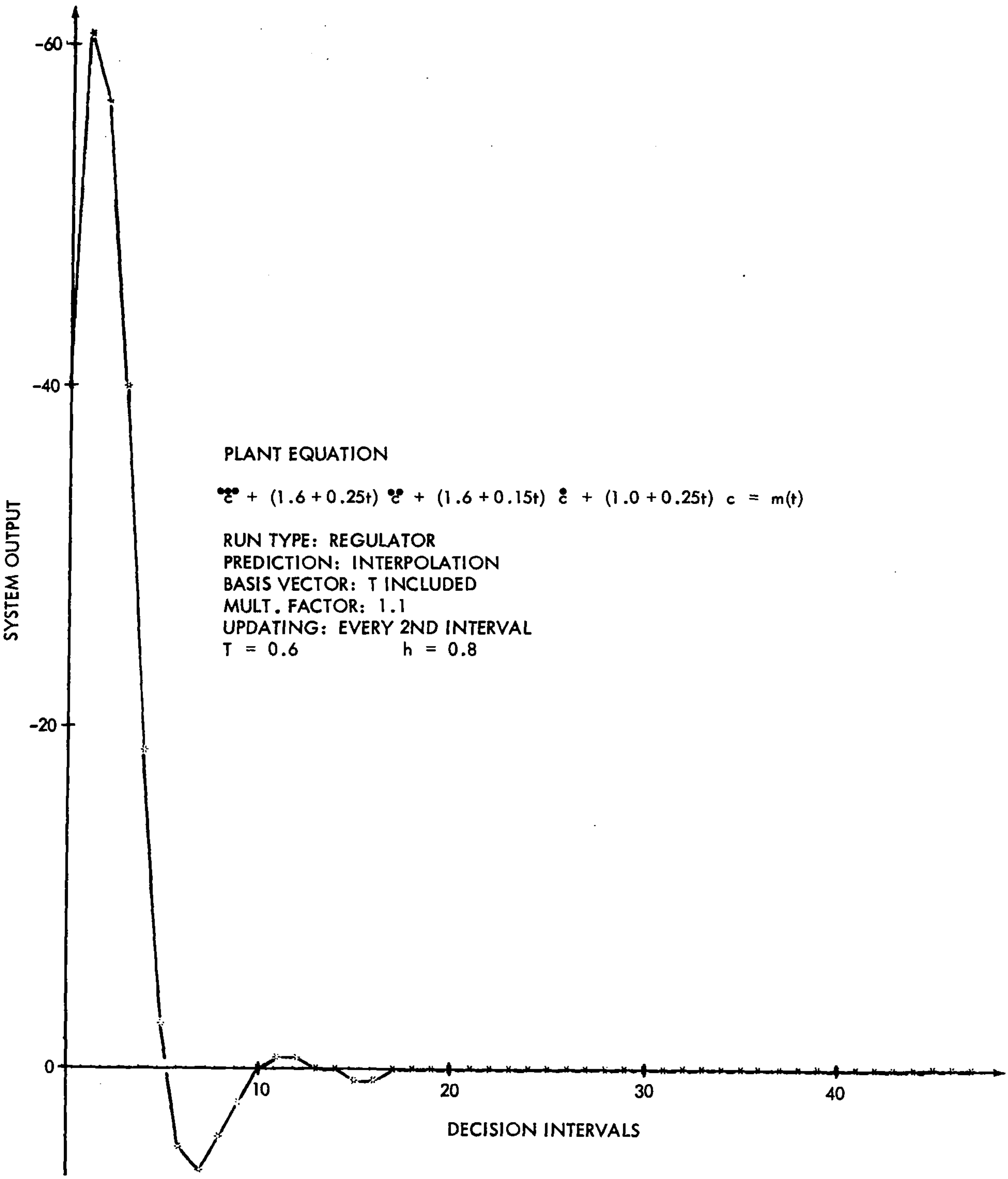


FIGURE 3-27 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

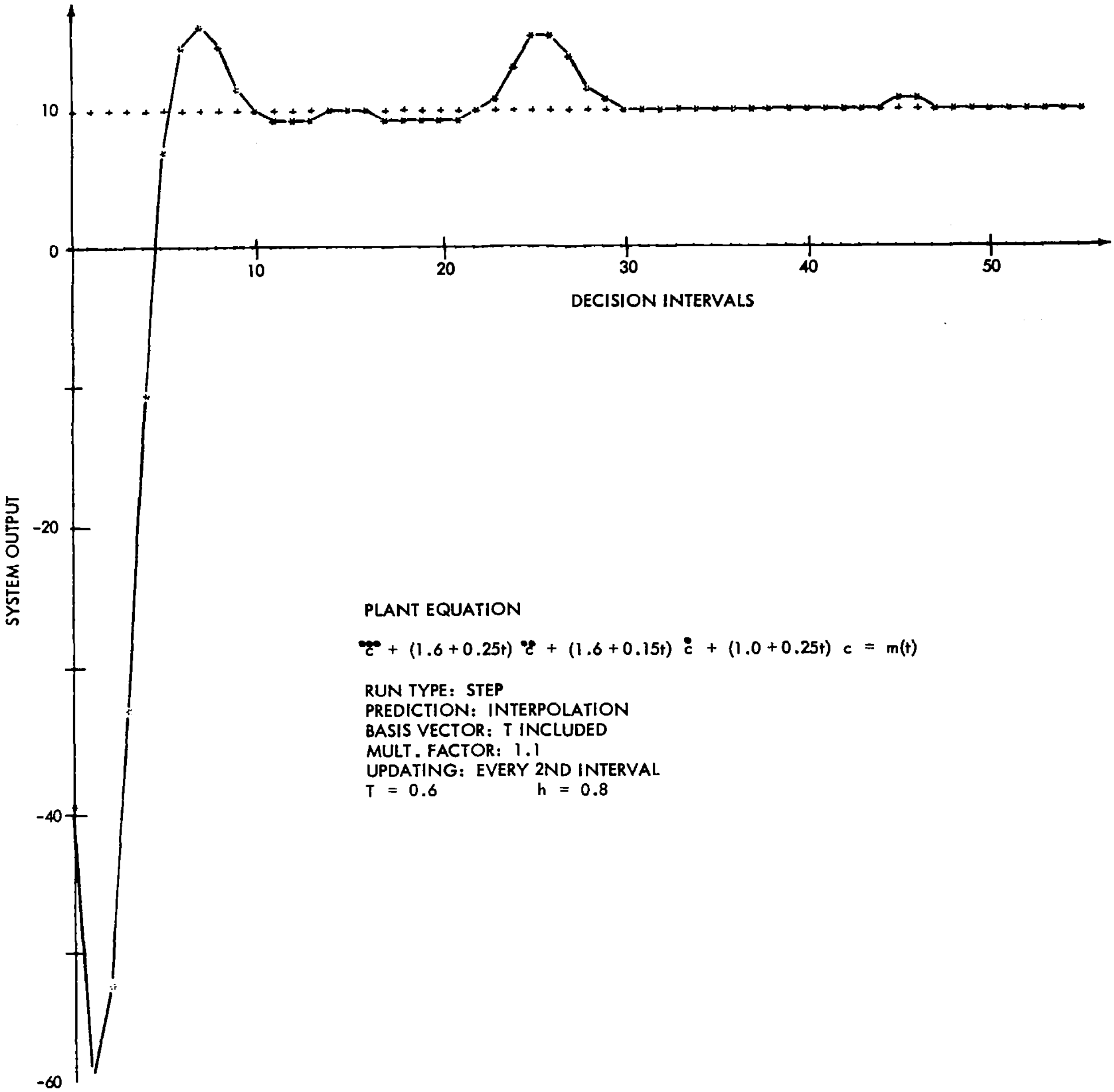


FIGURE 3-28 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

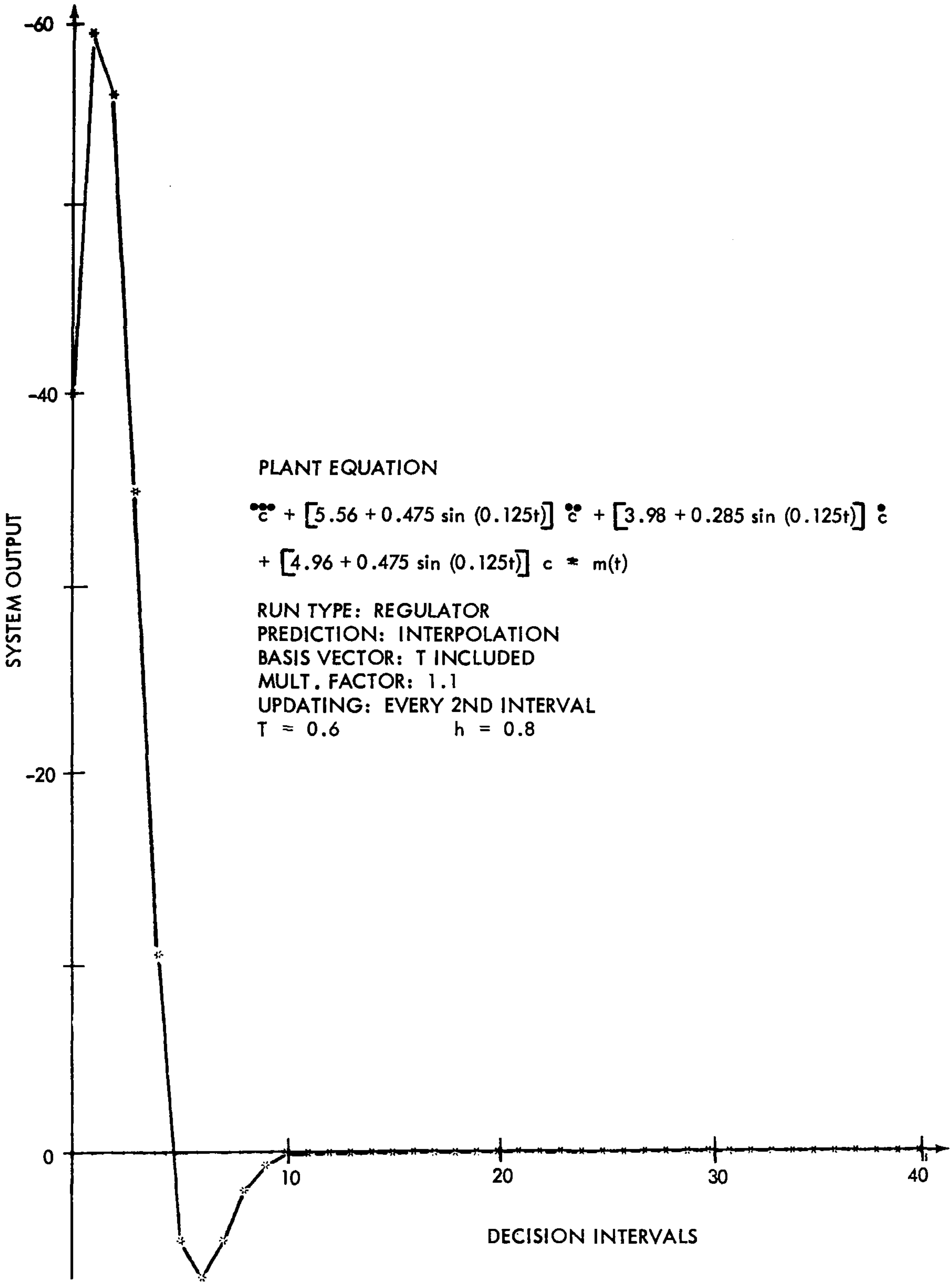


FIGURE 3-29 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

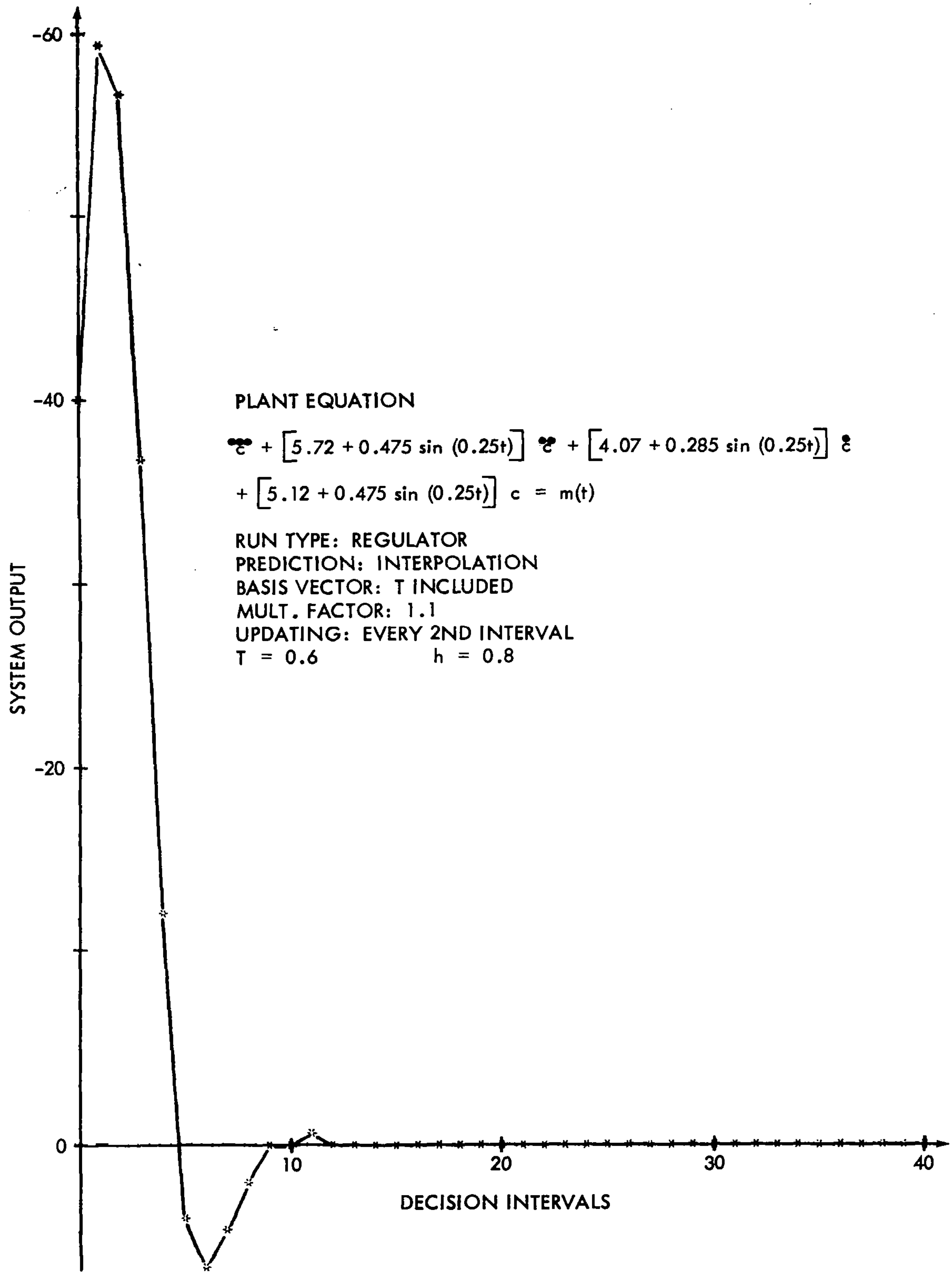


FIGURE 3-30 REGULATOR RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

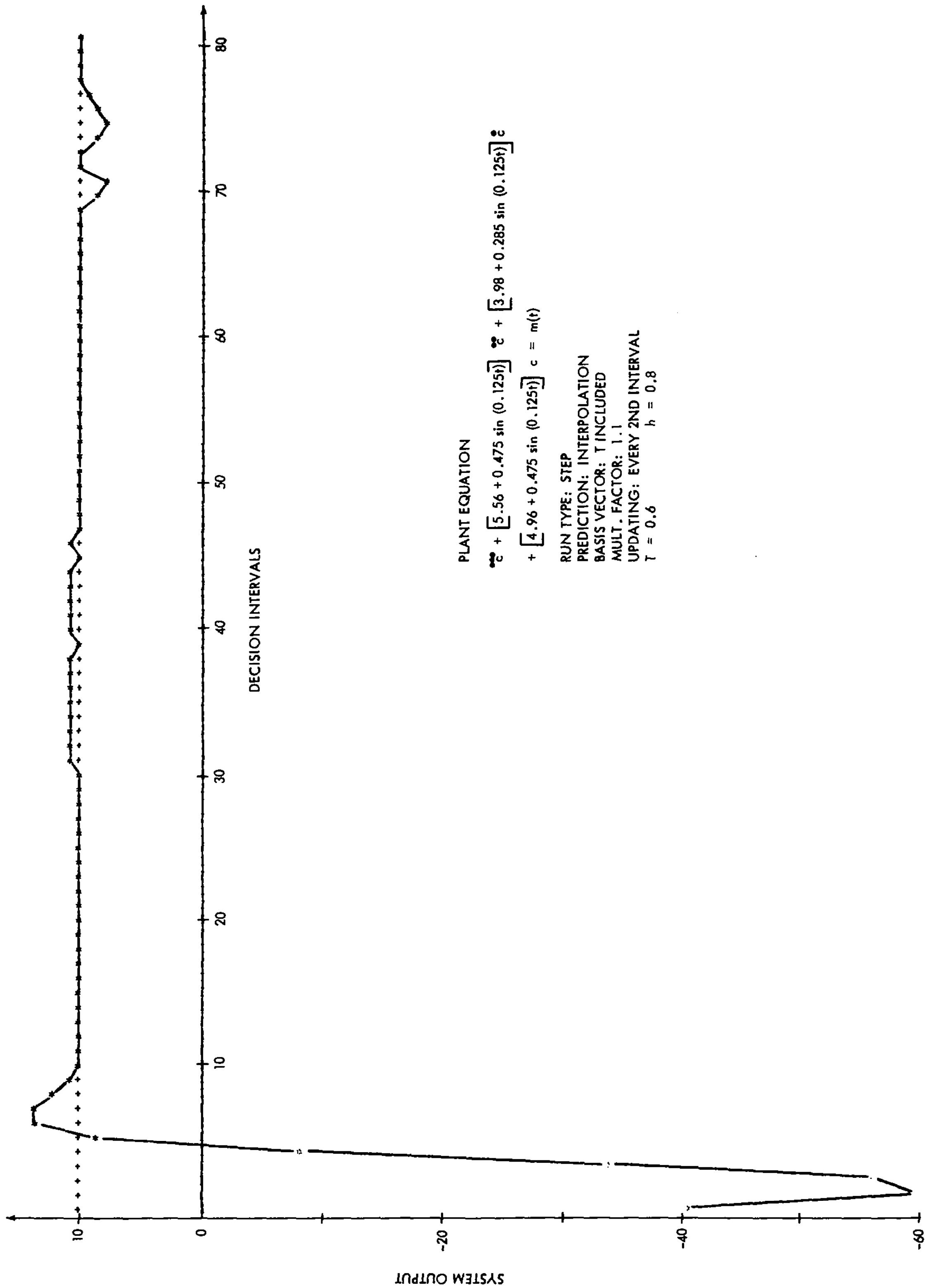


FIGURE 3-31 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

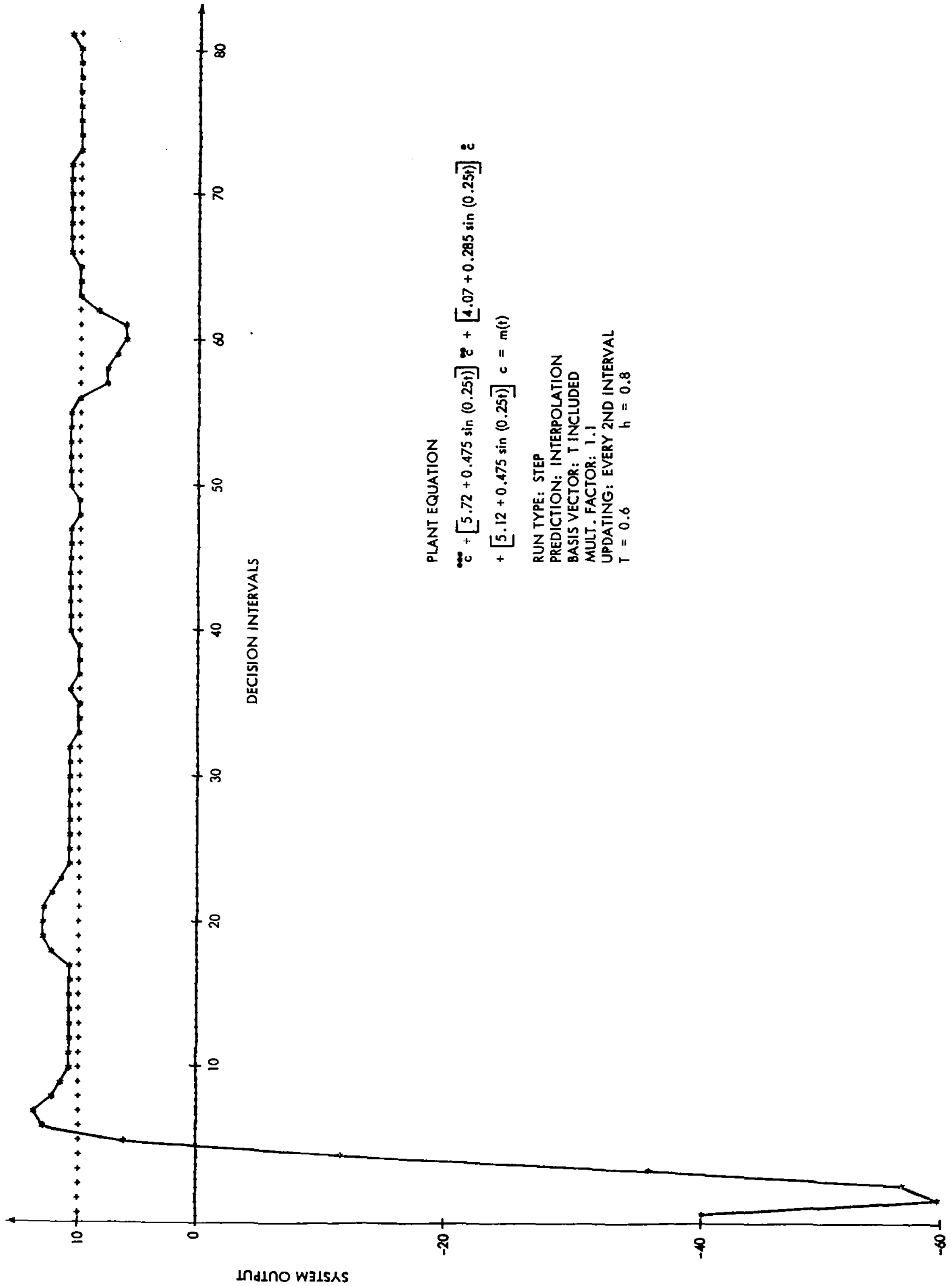


FIGURE 3-32 STEP RUN FOR 3rd ORDER TIME-VARYING SYSTEM WITH UPDATING

Interpolation Prediction method. These factors are considered in the following paragraphs.

Time Augmented Basis Vector.-The inclusion of the decision interval time (T) in the basis vector resulted in a stable controlled response, but with a steady state error for zero and step desired output states. This result is more than likely due to the very large and fast time variation studied in this investigation. The conclusion that the presence of (T) in the basis vector is not sufficient for plants with fast time variation seems to be well grounded. However, no conclusion may be made for slow time-varying plants of any order until investigations are conducted in this area.

Updating Current Sensitivity and Current Response Matrices.-A second way to account for the plant being time-varying is to include new data into the matrix of basis vectors each interval and to update the current sensitivity and current response matrices every n intervals. The basis vector did not include (T) for this study. The results lead to the conclusion that for fast time-varying systems good control is possible. However, for fast time-varying systems, the control performance is a strong function of the frequency of updating the current sensitivity and current response matrices. Also, the problem of inaccurate current sensitivity and/or current response matrices due to ill conditioning of the matrix of basis vectors becomes evident.

Time Augmented Basis Vector and Updating.-The last area of investigation combined the previous two methods utilized in control of time-varying plants. Based on the experimental results, the prime conclusion is that this combination resulted in control performance slightly better than the best previous method of only updating the system matrices. Since the systems investigated were rather fast time-varying, the inclusion of (T) in the basis vector does not outstandingly affect the control performance. Once again, the problem of matrix ill conditioning is noticeably present.

Ill Conditioning.-In the previously presented experimental results, the current sensitivity and current response matrices were updated every n intervals, where $n \geq 1$ was a program input. Such arbitrary updating resulted in the matrix of basis vectors becoming ill conditioned for one of two reasons. The ill conditioning occurring in the majority of the cases was due to two or more rows of the matrix of basis vectors being almost proportional to each other. This was noted many times when the actual and desired output states were approximately identical over a number of intervals. This condition resulted in several rows of the matrix of basis vectors being almost proportional. Matrix ill conditioning due to two or more columns being almost proportional was noted in a very few cases. Each time this event occurred the actual and desired output states were approximately identical over a number of intervals, and two approximately equal control forces were applied during this past set of intervals. In both of the above cases the fact that the desired output state was either zero or a step enhanced the possibility for such ill conditioning. The overpowering conclusion, with regard to avoiding matrix ill conditioning, is that if the actual and desired output states are identical to some degree of satisfaction no new updated current sensitivity and current response matrices should be obtained for system control. In noting this and previous conclusions concerning updating of the system matrices, it becomes quite evident that fast time-varying systems require frequent updating until the desired output state is realized, and then less frequent updating to avoid matrix ill conditioning. This implies the desirability for some monitoring procedure to decide when updating is desirable for continuing good system control.

REFERENCES

1. Gibson, J. E., Nonlinear Automatic Control, McGraw-Hill, Inc., New York, 1963, Chapters 5 and 10.

2. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," Journal of the Society of Industrial and Applied Mathematics, Ser. A: On Control, Vol. 1, No. 2, pp. 152-192, 1963.

3. Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpolation and Lagrange Expansion for Functionals," Proceedings of the JACC, 1965, pp. 169-177.

4. Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm Through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Mo., 1965.

5. Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identification," Proceedings of the JACC, 1964, pp. 366-375.

SECTION 4

CONTROL OF NON-LINEAR PLANTS

4.1 DERIVATION OF PLANT AND SYSTEM EQUATIONS

A study of the control of non-linear stationary and non-linear time-varying plants is the final step in the investigation of the general feasibility of the control method. This final step is a large one as the area of non-linear control systems is a highly segmented one, in that there are many isolated methods of analysis and synthesis which apply to limited classes of problems but no general method which applies to all. The word non-linear itself perhaps implies the difficulties involved in the field as it is a negative definition, relying on the definition of linear systems to define what is non-linear.

METHOD OF ANALYSIS

The dynamics of the physical plant are assumed to be describable by the following very general type of non-linear differential equation:

$$L(p,t)\ c(t) + F(t,\ c(t),\ \dot{c}(t),\ .\ .\ .,\ \overset{n-1}{c}(t)) = M(p,t)\ m(t) \tag{4-1}$$

where the plant is assumed to have a single input, m(t), and a single output, c(t). L(p,t) and M(p,t) are linear differential operators with variable coefficients and are of orders n and m respectively. The function, F, is a non-linear function of its arguments.

An input-output relationship for the plant is conveniently expressed in terms of a functional relationship:

$$c(t) = T\left[m(t)\right] \tag{4-2}$$

Volterra (reference 1) presents a proof that if the functional $T\left[m(t)\right]$ is continuous, it may be approximated to any desired degree of accuracy over finite time intervals by a finite series of the form:

$$c(t) = y(t) + \sum_{j=1}^{J} \int_{o}^{t} \cdot \cdot \int_{o}^{t} h_j \; (t, \tau_1, \; \ldots \; \tau_j) \; m(\tau_1) \; \ldots \; m(\tau_j) \; d\tau_1 \; \ldots \; d\tau_j \tag{4-3}$$

The essential restrictions are that the system produce continuous and bounded outputs for continuous and bounded inputs. If $T\left[m(t)\right]$ can be represented exactly by a converging infinite series ($J = \infty$) of the form of equation 4-3, it is called analytic (reference 1). Volterra and George (reference 2) show that equation 4-3 can be interpreted as a functional generalization of the Taylor series expansion for the analytic functional.

In equation 4-3, $y(t)$ represents the free response which would occur in the absence of any control input, $m(t)$. It is important to note that equation 4-3 does not imply superposition because the kernels, h_j, are not unique and depend upon $y(t)$. Analytical procedures have been outlined by George (reference 2), McFee (reference 3), and Flake (references 4 and 5) for determining the kernels.

The functional representation presented encompasses a broad class of plants. Continuous non-linearities and time-variations are permitted (reference 6). Discontinuous non-linearities, such as relays in the plant, are about the only features excluded. Discontinuous time-variations are permissible if their occurrences can be easily recognized, as in the staging of a missile.

The particular control method under study assumes the existence of a functional relationship of the form of equation 4-3 but makes no attempt to identify it. The control element instead senses the current response of the plant along with the current sensitivity to control action and extrapolates this into the near future. Two analytical developments are presented in the following sections which serve as a basis for such a type of control action.

PLANT FUNCTIONAL REPRESENTATION USING INTERPOLATION

By considering the functional to be analytic in the interval over which it is being approximated, it is convenient to expand the input-output relationship to a vector relationship in which the successive state variables possess a derivative relationship. The input state variable, $\underline{u}(t)$, and the output state variable, $\underline{x}(t)$, are defined by equations 4-4 and 4-5:

$$\underline{x}'(t) = \lfloor c(t) \; \dot{c}(t) \; \ldots \overset{n-1}{c}(t) \rfloor \tag{4-4}$$

$$\underline{u}'(t) = \lfloor m(t) \; \dot{m}(t) \; \ldots \; \overset{m}{m}(t) \rfloor \tag{4-5}$$

A convenient way of representing the state equation of the plant is:

$$\underline{x}((k+1)T) = \underline{x}_k((k+1)T) + \underline{A}_k((k+1)T)\, \underline{u}_k \tag{4-6}$$

where:

$$\underline{x}_k((k+1)T) = \underline{y}((k+1)T) + \sum_{i=0}^{k-1} \underline{A}_i((k+1)T)\, \underline{u}_i \tag{4-7}$$

Equation 4-7 defines the first term of the right hand side of equation 4-6 as the current response of the plant due to that response which would occur in the absence of any control inputs, $\underline{y}((k+1)T)$, and that due to all past control forces previous to u_k. Superposition is not implied as the $\underline{A}_i$'s will depend upon $\underline{y}$ and the previously applied $\underline{u}$'s. The second term on the right hand side of equation 4-6 is the current sensitivity of the plant to the control input $\underline{u}_k$. Again, superposition is not implied as $\underline{A}_k$ is not a unique constant of the plant, but is a function of past states, $\underline{y}(t)$, and the past control forces.

The Interpolation Equation.-Refer to Appendix B for the general development of the interpolation procedure. The interpolation estimate of the state at t = (k+1)T is given by (references 7 and 8):

$$\tilde{\underline{X}}\,((k+1)T) = {}_D\underline{X}\;\; \underline{\Phi}^{-1}\;\underline{\phi}\,(u_k, \underline{\eta}_k) \tag{4-8}$$

The vector of base functionals $\underline{\phi}$ is considered to be limited to quadratic terms. A linear term in t may be included to account for time variation of the plant parameters; however, for the sake of simplicity, it is omitted here. The basis vector may be written as:

$$\underline{\phi}'(u_i, \underline{\eta}_i) = \lfloor \underline{x}'(iT)\ \ \underline{x}'^2(iT)\ \ \underline{x_j x_k}'(iT)\ \ u_i\underline{x}'(iT)\ \ u_i\ \ u_i^2 \rfloor \tag{4-9}$$

where $\underline{x_j x_k}(iT)$ consists of all possible cross-products of the elements of $\underline{x}(iT)$. The matrix of vector basis functions, $\underline{\Phi}$, consists of an appropriate set of $\underline{\phi_i}$'s which need not be consecutive, but can be that set which best represents the current behavior of the plant. The dimension of each basis vector is given by:

$$d = 3p + \frac{p(p-1)}{2} + 2 \tag{4-10}$$

where p is the dimension of $\underline{x}(iT)$. The dimension of $\underline{\Phi}$ will therefore be d x d. The $\underline{B}$ matrix is factored into six submatrices:

$$\underline{B} = {}_D X\ \ \underline{\Phi}^{-1} = \left[\underline{\theta}_1 \,\middle|\, \underline{\theta}_2 \,\middle|\, \underline{\theta}_3 \,\middle|\, \underline{\theta}_4 \,\middle|\, \underline{\varphi}_1 \,\middle|\, \underline{\varphi}_2\right] \tag{4-11}$$

where for an assumed order p for x(iT), the order of $\underline{\theta}_1$, $\underline{\theta}_2$, and $\underline{\theta}_4$ is p x p, the order of $\underline{\theta}_3$ is p x $\frac{p(p-1)}{2}$, and $\underline{\varphi}_1$, and $\underline{\varphi}_2$ are p vectors.

The interpolation estimate of the first term on the right hand side of equation 4-6 is given by:

$$\underline{\tilde{x}}_k((k+1)T) = \left[\underline{\theta}_1\underline{x}(kT) + \underline{\theta}_2\underline{x}^2(kT) + \underline{\theta}_3\underline{x_i x_j}(kT)\right] \tag{4-12}$$

Similarly the estimate of the second term of the right hand side of equation 4-6 is given by:

$$\underline{\tilde{A}}_k((k+1)T) = \left[(\underline{\theta}_4\underline{x}(kT) + \underline{\varphi}_1) \,\middle|\, (\underline{\varphi}_2)\right] \tag{4-13}$$

and:

$$\underline{u}'_k = \lfloor u_k\ \ u_k^2 \rfloor \tag{4-14}$$

$\underline{\tilde{X}}_k((k+1)T)$ is a p vector, the order of $\underline{\tilde{A}}_k((k+1)T)$ is p x 2, and $\underline{u}_k$ is a 2 x 1 vector.

Assembling the estimates of $\underline{\tilde{X}}_k((k+1)T)$ and $\underline{\tilde{A}}_k((k+1)T)$ yields as the interpolation estimate of $\underline{x}((k+1)T)$:

$$\underline{\tilde{x}}((k+1)T) = \underline{\tilde{X}}_k((k+1)T) + \underline{\tilde{A}}_k((k+1)T)\ \underline{u}_k \qquad (4\text{-}15)$$

where $\underline{\tilde{X}}_k$, $\underline{\tilde{A}}_k$ and $\underline{u}_k$ are defined by equations 4-12, 4-13, and 4-14 respectively.

THE NON-LINEAR CONTROL POLICY

As was the case for the linear studies, the non-linear control policy will be to minimize the quadratic form Q_k (reference 6):

$$\min_{u_k}\left[Q_k\right] = \min_{u_k}\left[\underline{e}'((k+1)T)\underline{K}\ \underline{e}((k+1)T)\right] \qquad (4\text{-}16)$$

where the error state vector $\underline{e}(t)$ is as it was defined for the linear studies:

$$\underline{e}(t) = \underline{r}(t) - \underline{x}(t) \qquad (4\text{-}17)$$

Regardless of whether the interpolation procedures or the Volterra representation of Appendix G is used, the estimate of the state at $\underline{x}((k+1)T)$ is of the form:

$$\underline{\tilde{x}}((k+1)T) = \underline{\tilde{X}}_k((k+1)T) + \underline{\tilde{A}}_k(k+1)T)\ \underline{u}_k \qquad (4\text{-}18)$$

Substituting equation 4-18 into equation 4-16 according to the definition equation 4-17 and dropping the time notation for the sake of compactness yields:

$$Q_k = \left[\underline{r}'_k - \underline{\tilde{X}}'_k - \underline{u}'_k\ \underline{\tilde{A}}'_k\right] \underline{K}\left[\underline{r}_k - \underline{\tilde{X}}_k - \underline{\tilde{A}}_k\underline{u}_k\right] \qquad (4\text{-}19)$$

Differentiating Q_k with respect to u_k yields:

$$\frac{dQ_k}{du_k} = -2 \frac{d\underline{u}_k'}{du_k} \tilde{\underline{A}}_k \underline{K} \left[\underline{r}_k - \tilde{\underline{x}}_k - \tilde{\underline{A}}_k \underline{u}_k \right] \tag{4-20}$$

where:

$$\frac{d\underline{u}_k'}{du_k} = \lfloor 1 \quad 2\,u_k \rfloor \tag{4-21}$$

It is convenient to modify the notation slightly at this point according to the definitions of equation 4-22:

$$\tilde{\underline{A}}_k((k+1)T) \equiv \left[{}_k\underline{a}_o \quad {}_k\underline{a}_{oo} \right] \tag{4-22}$$

$${}_k\underline{x}_\Delta \; ((k+1)T) \equiv \tilde{\underline{x}}_k - \underline{r}_k$$

Substituting the notation of equation 4-22 into equation 4-20 and setting the derivative of Q_k equal to zero yields:

$$\lfloor 1 \quad 2\,u_k \rfloor \begin{bmatrix} {}_k\underline{a}_o' \\ {}_k\underline{a}_{oo}' \end{bmatrix} \underline{K} \left\{ {}_k\underline{x}_\Delta + \left[{}_k\underline{a}_o \quad {}_k\underline{a}_{oo} \right] \underline{u}_k \right\} = 0 \tag{4-23}$$

Collecting the terms of equation 4-23 according to powers of u_k yields:

$$(2\,{}_k\underline{a}_{oo}' \, \underline{K} \, {}_k\underline{a}_{oo}) \, u_k^{\,3} + (2\,{}_k\underline{a}_{oo}' \, \underline{K} \, {}_k\underline{a}_o + {}_k\underline{a}_o' \, \underline{K} \, {}_k\underline{a}_{oo}) \, u_k^{\,2}$$

$$+ (2\,{}_k\underline{a}_{oo}' \, \underline{K} \, {}_k\underline{x}_\Delta + {}_k\underline{a}_o' \, \underline{K} \, {}_k\underline{a}_o) \, u_k + {}_k\underline{a}_o' \, \underline{K} \, {}_k\underline{x}_\Delta = 0 \tag{4-24}$$

Equation 4-24 is a cubic equation in terms of u_k which guarantees that at least one real root exists which will minimize the quadratic form Q_k. Zaborszky (reference 6) outlines a convenient method of extracting the proper root of equation 4-24 for the case where the coefficient of the linear term is predominant. Other techniques would be numerical methods such as Bairstows or Newton. The degree of non-linearity of the system will have a lot to do with what technique is appropriate.

4.2 COMPARISON OF THE TWO METHODS OF PLANT FUNCTIONAL APPROXIMATION

A method for plant functional representation using interpolation has been established in paragraph 4.1, and Appendix G describes a Volterra series representation of a plant. As a practical matter, based on considerations of time and funding, it was necessary to choose one of the two representations for simulatory study and resultant analysis. The interpolation method was chosen largely for two reasons:

> Its linear analog had been previously successfully tested in our work, and had known validity. In contrast, the Volterra series description had not been so tested.
>
> The interpolation method is conceptually straightforward, and we had obtained an intuitive "feel" for it. The Volterra representation was less "familiar", both in its analytic origin and in its association, with particular plants.

Our commitment to the interpolation method in this instance was primarily to insure immediate results. We believe the Volterra series approach should be investigated at some future time, specifically because it has not been experimentally tested.

The following summary records some primary criteria for comparison of the two methods, and their relative merits in each area. Since both methods can be generalized almost arbitrarily by inclusion of higher order terms, an arbitrary basis for comparison was chosen as follows:

> The representation of plants is arbitrarily truncated at the $R = 2$ kernel for the Volterra series representation, and at a basis containing quadratic terms for the interpolation representation.
>
> Time variance is included as a linear time expansion of the kernels in the Volterra representation, and as a linear augmentation of the basis vector by a time component in the interpolation representation.

Criterion 1 - Domain of Representable Plants.-The overall objective of the DACS concept is to encompass as wide a spectrum of plants as possible in a single control algorithm. Thus, a primary point of comparison might be the relative number of plants encompassed in the respective representations. No a priori preferability of representation method has been discovered by this criterion. Two problems render its analytic solution intractable:

> The class of non-linear plants is so broad that identification of a "typical" set is prevented as was done in the linear analysis.
>
> While some non-linear plants can be identified as exactly described by one or both of the methods of representation, most are not (the important class of piecewise defined non-linearities is not exactly representable by low order in either method). However, since the identification needs, and indeed can be, only sufficiently approximate, such plants can not be excluded by lacking the possibility of exact describability.

Criterion 2 - Precision of Input Data.-Any comparison of the formalism of the interpolation method (paragraph 4.1) with that of the Volterra series method (Appendix G) reveals an important difference. The interpolation method basis vector is heavily dominated by the output state variables with a relatively small and highly immediate dependence on the control force. In contrast, the Volterra series expansions are primarily in terms of sequences of past values of the control force with relatively small incidence of the free response state variables. The commanded value of the control force is computer derived and is known with some precision. On the other hand, direct measurement of the state variables is difficult and forseeably loses precision with increasing order of the variable.

These considerations suggest that the Volterra representation is preferable on the basis of input data precision. This conclusion must be regarded as highly tentative, however, for the following reasons:

> Because of the control policy, the control force sequence is not an independent variable.
>
> The Volterra representation depends on curve fitting techniques as functions of time. The implications of this process have not been experimentally investigated.

Criterion 3 - Incorporation of Time-Variability.-From the first description of the Volterra representation in reference 6, time variability was included in the formulation. However, it was recognized that the dimensionality of the set of equations requiring solution at each control step grows linearly with the degree of power series utilized in the time expansions. Since the compution requirement grows somewhat faster than the cube of this dimensionality, only linear time expansion appears practical in this method.

The interpolation method as originally conceived and described in reference 7 did not innately include time variability. However, it was subsequently incorporated in our linear nonstationary studies as a single linear component of the basis vector, and in provisions for periodically updating the state transition matrix. The computing penalty for time augmentation of the basis vector is slight, becoming relatively smaller as the dimensionality of the unaugmented basis increases. For large dimensionality the fractional increase in computation required for its introduction is of the order of 3/m, where m is the unaugmented dimensionality.

Updating by full inversion of the Interpolation Matrix is another matter, and if performed as infrequently as m^2 decision intervals, can unsurp the computing load. Several techniques for updating by perturbation of the prior inverse were suggested but not tested in these investigations. If they can be successfully applied, updating at as few as m decision intervals appears possible with a fractional increase of computer load.

The criterion of incorporating time-variability clearly favors the interpolation method by the above considerations.

Criterion 4 - Data Storage and Computation Requirements.-This comparison can become involved in many details as is evident from the preceding discussion of just one factor, time variability. Admitting some inexactitude in making this comparison (particularly in that the Volterra representation has never been computer programmed in any form), the following conclusions are general:

> For "equivalent" representation of stationary non-linear systems the Volterra representation requires the inversion of a smaller matrix of coefficients than the interpolation representation.
>
> The Volterra method requires a full matrix inversion at each decision interval. No recursive methods for alleviating this requirement have been identified.

The interpolation method for stationary plants in principle requires only one matrix inversion when start-up is complete. However, practically it may be desirable to periodically reinvert, if the region of sustained operation deviates markedly from the region of the original determination.

All dimensionalities in the control and computation process are set by the assumed dimensionality of the basis vector in the interpolation method. This in turn is largely a function of the state vector dimensionality.

In the Volterra representation the dimensionality of the matrix inversion is largely independent of the dimensionality of the state vector, but critically dependent on the number of stored data sections.

The Volterra representation is self updating in contrast to the interpolation method where an auxiliary decision to update is required. A corollary is that the computing rate requirement is constant in the Volterra method, but problem variant in the interpolation method. Another corollary is that the Volterra system is non-learning, while the decision to update can be generalized to selectively enter and discard data in the interpolation method.

While the preceding summary is highly conditioned by the difficulty of establishing a common baseline for the two methods, it generally indicates the preferability of the interpolation technique according to the criterion of data storage and computation requirements.

4.3 EXPERIMENTAL STUDIES

OBJECTIVES

This section presents the very limited experimental program and results obtained therefrom for non-linear stationary systems. Once again, as in the linear time-varying experimental investigations, the Interpolation Prediction method was used in the control policy.

The primary objective of this experimentation was to establish the possible feasibility of the linear control system policy for control of non-linear plants. This area of investigation is of considerable interest if viewed from the standpoint that no plant knowledge is available. That is, the plant may be assumed linear and of Nth order, but in reality may be non-linear of Mth order where $M \geq N$. This experimentation utilized the correct plant order, but assumed the plant was linear when in fact it was not linear. It is important to establish if system control may be accomplished under such an error. This area is of prime importance also in the light that the plant under control may be linear over some operational range, and then become non-linear over another range. It is desirable that the control policy provide acceptable control under such conditions.

NON-LINEAR PLANTS

In order to accomplish the above objective two types of non-linear plants were used in the following experimentation. The first type considered was a plant which is continuously non-linear, and the second type considered was a plant which is non-linear due to state variable constraints. The plants used to be representative of these types of non-linearities were respectively the Van der Pol Equation and a third order plant with velocity saturation. Investigation was restricted to these two low order systems due to the limited nature of this experimentation.

EXPERIMENTAL RESULTS

The experimentation conducted on these plants consisted of less than twenty-five control simulations. No attempt was made to investigate such areas as need for one non-control policy force in the matrix of basis vectors, how and when to shift new data into this matrix, and how often the current response and current sensitivity should be updated. Therefore, all the experimentation used a 1.1 multiplying factor to keep one non-control policy force present in the matrix of basis vectors, shifted in the new data every interval, and recalculated the current sensitivity and current response either every or every other interval. The method of start-up used throughout this experimentaton is identical to that described in paragraph 2.2 for the linear stationary case with updating. The free response for the plant represented by the Van der equation 4-25 for $\epsilon = 1.0$ is presented in Figure 4-1:

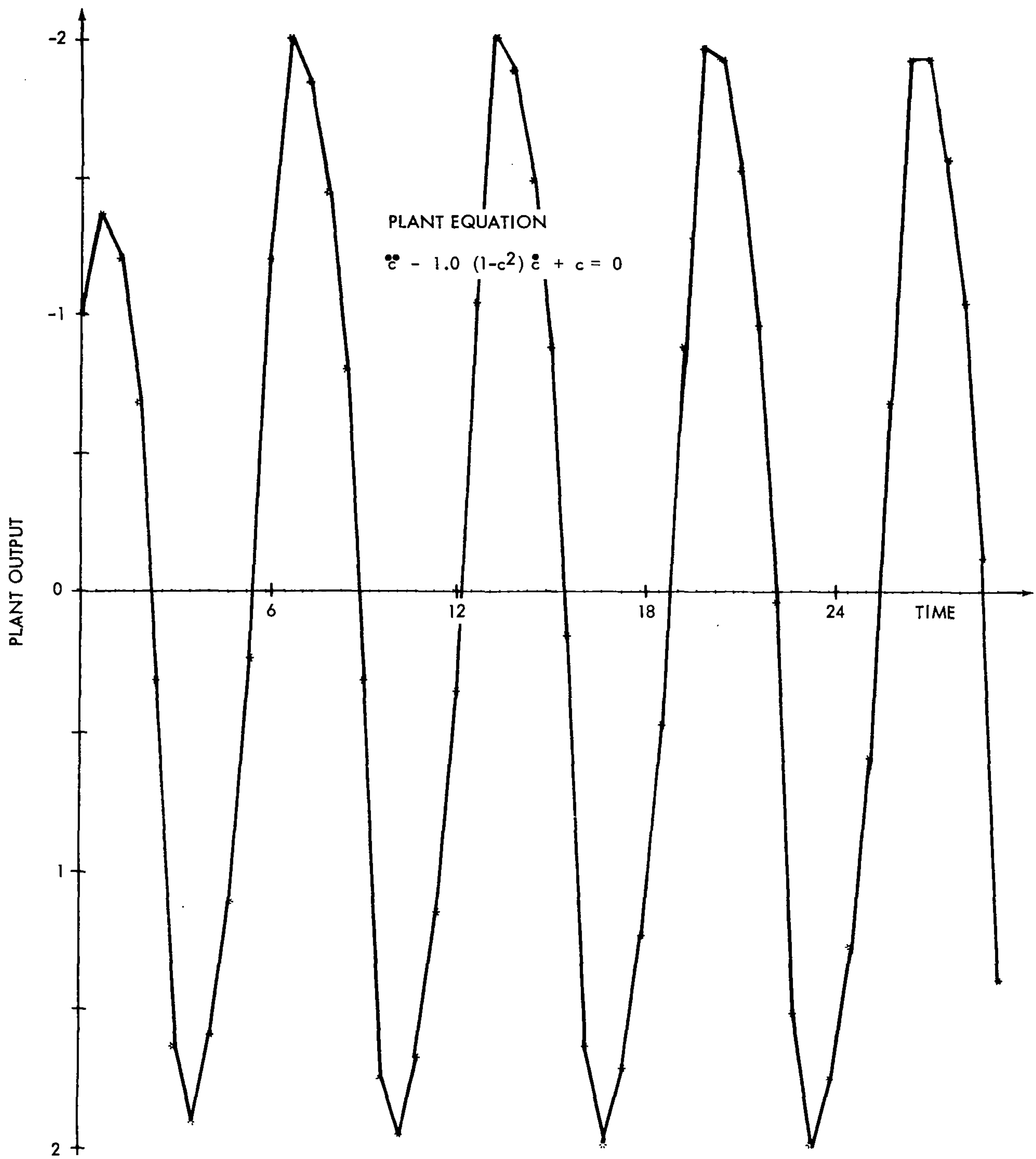


FIGURE 4-1 FREE RESPONSE FOR THE VAN DER POL PLANT

$$\ddot{c} - \epsilon(1 - c^2)\,\dot{c} + c = 0 \qquad (4\text{-}25)$$

Figures 4-2 and 4-3 show control performance of this plant for the cases where the current sensitivity and current response are recalculated every and every other interval respectively. It should be noted that in both cases the basis vector included the decision interval (T). Figures 4-4 and 4-5 illustrate the control performance for the same conditions with the decision interval (T) not included in the basis vector.

Example results obtained for the third order plant which is non-linear due to velocity saturation are given in Figures 4-6 and 4-7. In both of these cases, the current sensitivity and current response were recalculated every interval. The first figure corresponds to the case where the decision interval time (T) was included in the basis vector.

SUMMARY AND CONCLUSIONS

The results obtained from both non-linear plants indicate that adequate control may be realized with the linear control policy. However, of considerable interest is the fact that inclusion of (T) in the basis vector resulted in either very poor or no control of the non-linear plants, whereas the control performance for the cases without (T) in the basis vector resulted in very adequate control performance. The poor performance encountered for the cases with (T) in the basis vector was not due to matrix ill conditioning. Rather, it seems to be simply a result of inadequate description of the plant. The effect is that the vector associated with time variability does not go to zero as is the case for linear non-time varying systems, but varies each time it is recalculated due to the plant being non-linear. The important conclusion is that control of non-linear plants with the linear control policy is possible, but the inclusion of (T) in the basis vector is to be avoided if one is not sure the plant may be adequately considered linear. It must be noted that the conclusions reached in the above paragraphs are not considered general in any sense of the word. However, they are considered to be somewhat meaningful with respect to low order non-linear plants. In order to make general conclusions about the linear control system capabilities and/or limitations for non-linear plants, studies of more depth are required on higher order plants.

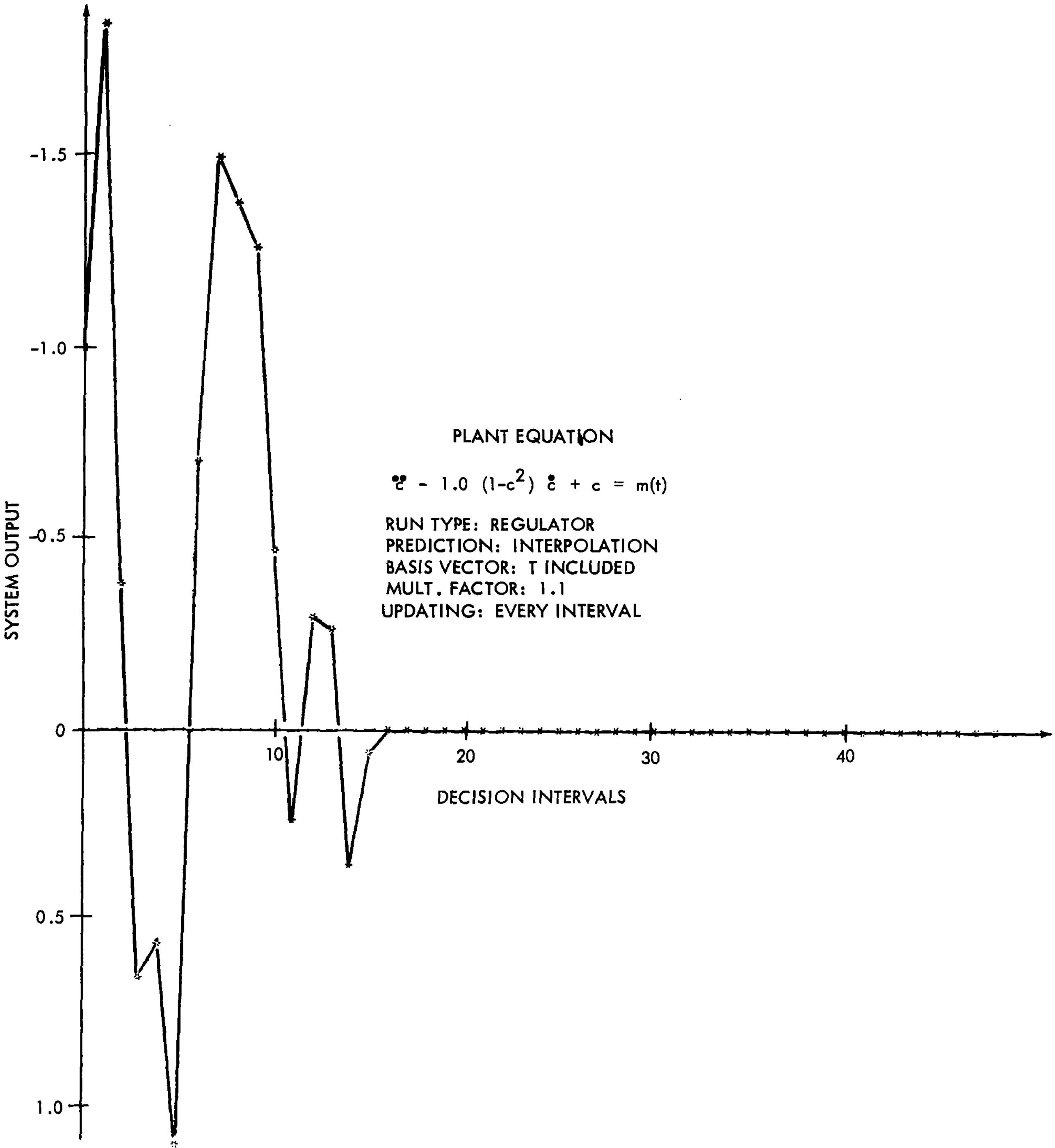


FIGURE 4-2 REGULATOR RUN FOR THE VAN DER POL NON-LINEAR SYSTEM WITH UPDATING

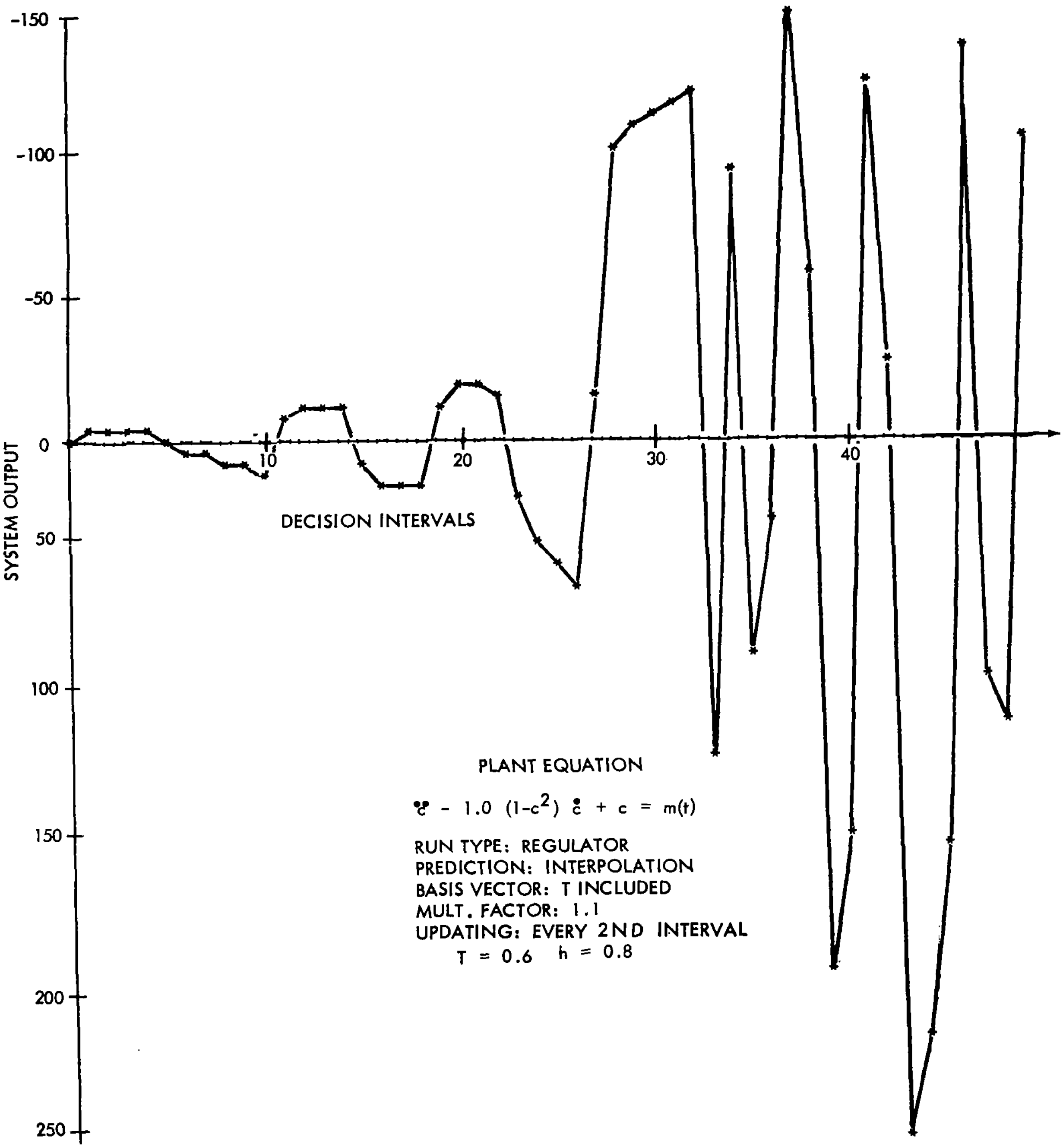


FIGURE 4-3 REGULATOR RUN FOR THE VAN DER POL NON-LINEAR SYSTEM WITH UPDATING

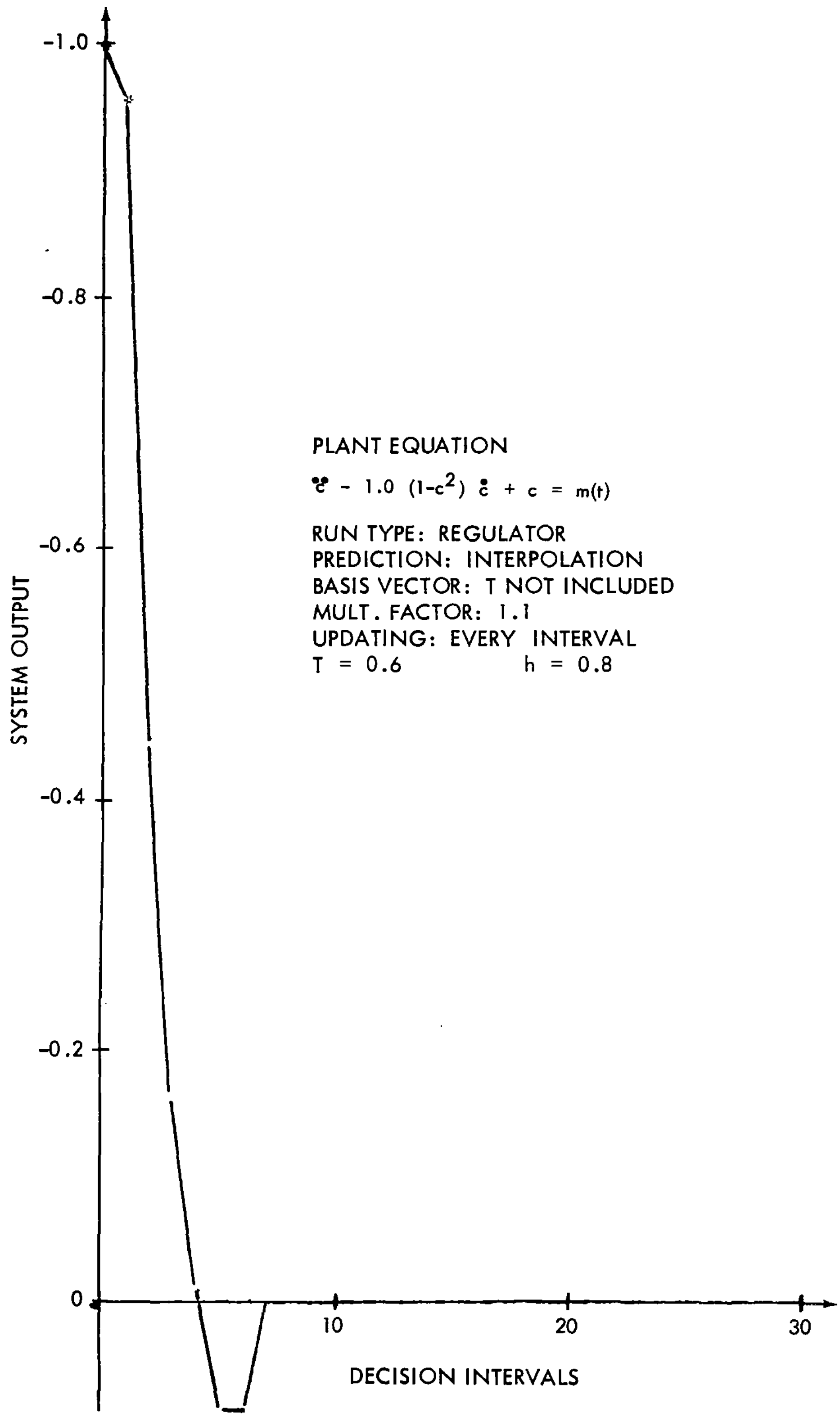


FIGURE 4-4 REGULATOR RUN FOR THE VAN DER POL NON-LINEAR SYSTEM WITH UPDATING

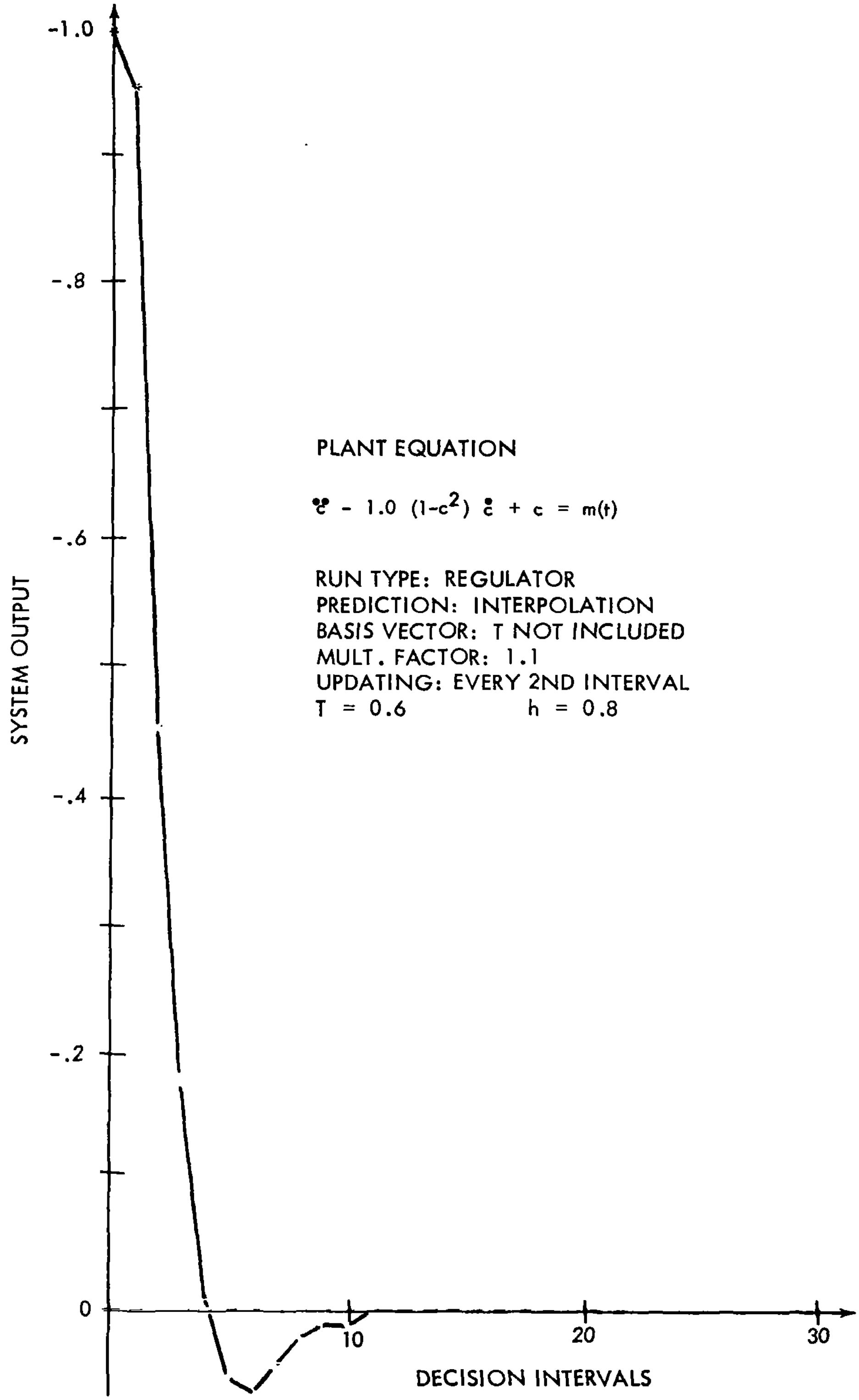


FIGURE 4-5 REGULATOR RUN FOR THE VAN DER POL NON-LINEAR SYSTEM WITH UPDATING

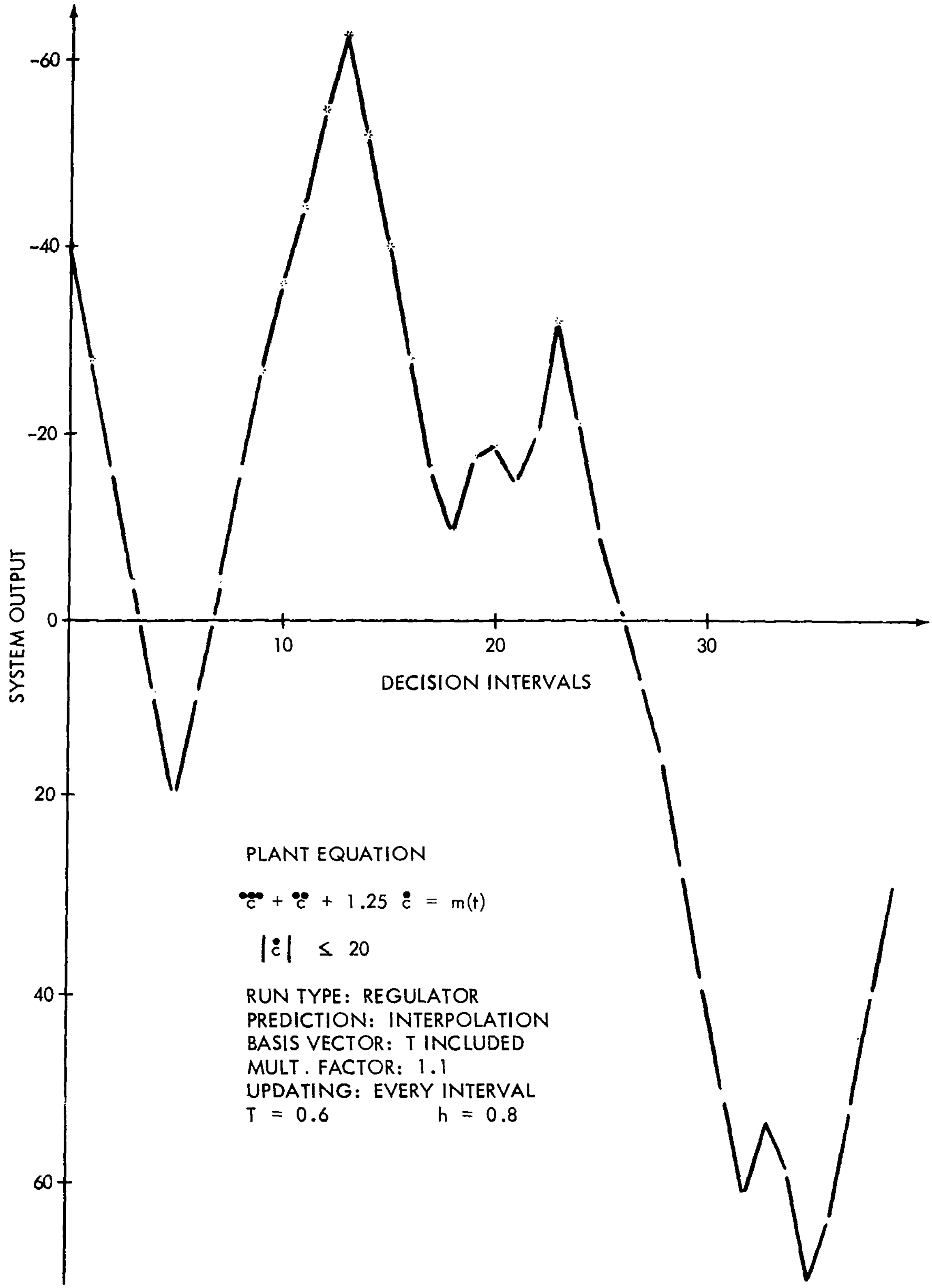


FIGURE 4-6 REGULATOR RUN FOR THE 3rd ORDER PLANT WITH VELOCITY SATURATION

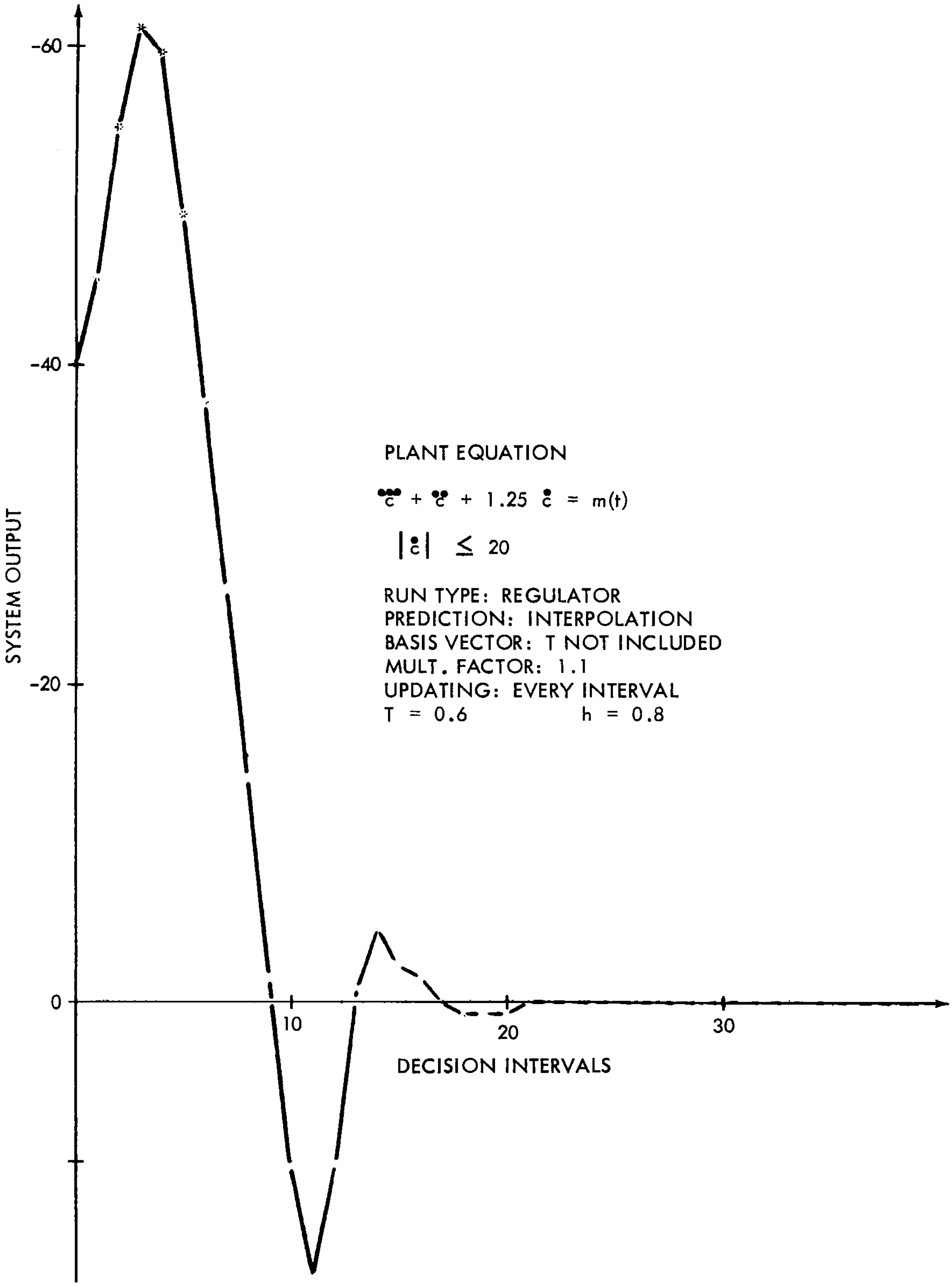


FIGURE 4-7 REGULATOR RUN FOR THE 3rd ORDER PLANT WITH VELOCITY SATURATION

REFERENCES

1. Volterra, V., Theory of Functionals and of Integral and Integro Differential Equations, Blackie and Sons, Ltd., London, 1930.

2. George, D. A., "Continuous Nonlinear Systems," Research Laboratory of Electronics, M. I. T., Cambridge, Report No. 355; 1959.

3. McFee, R. "Response of Nonlinear Systems to Arbitrary Inputs," AIEE Trans. on Applications and Industry, No. 56, September 1961, pp. 189-193.

4. Flake, R. H., "Volterra Series Representation of Nonlinear Systems," AIEE Trans. on Applications and Industry, No. 82, January 1963, pp. 330-335.

5. Flake, R.H., "Volterra Series Representation of Time Varying Nonlinear Systems," Proceedings of Second IFAC Congress, 1963; Butterworths, London.

6. Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identification," Proceedings of the JACC, 1964, pp. 366-375.

7. Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpolation and Lagrange Expansion for Functionals," Proceedings of the JACC, 1965, pp. 169-177.

8. Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Missouri, 1965.

SECTION 5

START-UP AND LEARNING INVESTIGATIONS

5.1 START-UP INVESTIGATION

Interpolation methods applicable to the running control of linear stationary, linear nonstationary, and non-linear plants have been described respectively in paragraphs 2.1, 3.1, and 4.1. As originally conceived, these methods require the prior accumulation of the results of a set of initial control actions equal in number to the assumed dimensionality of the basis vector before becoming applicable. In the language of numeric analysis, these methods are not "self starting".

Thus, their application requires an independent method of determining an initial sequence of control actions. This is termed the "start-up procedure".

Desirable Properties.-A desirable start-up procedure should have the following properties:

A simple computational algorithm, preferably a recursive form

An innate tendency (which may be only approximate) to realize the ultimate criterion of error norm minimization

No systematic tendency towards ill conditioning of the set of basis vectors being accumulated *

Preferably an analytic basis for the method.

*If the second and third criteria are incompatible, the third criterion must take precedence. The only known incompatible case in our studies is that of linear stationary plants in the regulator problem. Here, to the extent that the start-up procedure exactly realizes the second criterion, ill conditioning results. However, if a single control force is inexact, the tendency to ill conditioning is vitiated as shown in paragraph 2.2.

LINEAR STATIONARY PLANTS

In the case of a linear stationary but unknown plant, a start-up procedure meeting the above criteria exists in the Taylor Prediction control method described in paragraph 2.1, subject only to the condition that the very first control force is arbitrary. This procedure has been successfully demonstrated in numerous simulations, and is considered adequate but not necessarily optimum to the start-up of linear stationary plants.

NON-STATIONARY AND/OR NON-LINEAR PLANTS

No alternate elementary control method analogous to the Taylor procedure had been previously developed for these more general plants. In our simulation programs an artificial method of start-up involving the application of a prechosen arbitrary sequence of limited control forces was used. This method was known to be inadequate to a practical start-up situation, particularly in that it is totally deficient with respect to the second and fourth desirable properties previously enumerated. This is a serious practical shortcoming, since the initial period of control is often most critical. Even if this effect can be discounted, the errors accumulated during this learning time may be difficult to remove when the ultimate interpolation control method becomes operative.

Empirical Taylor Start-up.-A possibility initially entertained was the out-of-hand application of the Taylor start-up procedure to non-linear and/or nonautonomous plants. To the extent that the Taylor procedure had been successfully applied to the running control of a few non-linear and linear nonstationary plants, there was some empirical basis for this approach. Arrayed against it was the lack of an analytic foundation, with the resultant difficulty of establishing its possible utility by extensive simulatory experimentation.

Empirical Growing Basis Vector Start-up.-An alternate possibility was the concept of control during start-up by use of a basis vector growing in dimension with each step. Here as sufficient data becomes available to permit the solution at any given basis vector dimensionality, an inversion is made and interpolative control effected to that dimensionality.

Conceptually this procedure is straightforward. Practically it appears difficult to implement in that a computation procedure culminating in a full inversion must be executed at each step. The possibility of a recursive process exists partially vitiating the latter objection. However, there remain some conceptual problems as to which terms are to be included in the basis vector at each step. Stated loosely the available data for solution advances linearly with each step in start-up, while the basis vector augmentation logically occurs in blocks.

Lacking a more closely defined method, this one would have been investigated in further detail. Its known deficiencies are primarily procedural as outlined above. As compared with the preferred method next to be discussed, it has one further limitation. At any given stage of the start-up, it does not utilize all available data, higher order state variables being reserved until a sufficient set has been fully defined.

PSEUDOINVERSE START-UP PROCEDURE

In the closing stages of these contractual studies a start-up method based on the application of pseudo matrix inversion was suggested by Dr. John Zaborszky as a promising solution to the start-up problem.

An extensive study of this method has been independently made by Mr. Charles H. Wells in his Doctorate Dissertation "Minimum Norm Control of Discrete Systems" (reference 1), under the direction of Dr. John Zaborszky, Washington University, 1966. Mr. Wells has treated a number of other control topics besides the start-up procedure subsequently outlined here.

The remainder of this section will indicate only that form of the pseudoinverse matrix appropriate to start-up. Its pertinent properties will be described in qualitative terms and illustrated by contrived numeric examples. Appendix H is a more analytic but by no means rigorous exposition of both the pseudoinverse method and its application to start-up.

Analytic Identification of the Start-up Problem.-Before beginning the description of the pseudoinverse technique it is appropriate to identify the start-up problem analytically.

The interpolation methods of paragraphs 2.1, 3.1, and 4.1 reduce in principle to the multiple solution of several linear equation sets all expanded in components of the same basis vector. Thus, they can be decomposed into the following generic linear equation forms *:

$$x_i = a_{i1}\phi_1 + a_{i2}\phi_2 + \dots + a_{ij}\phi_j + \dots a_{im}\phi_m \tag{5-1}$$

In the running state use of interpolation methods, there exists one such equation for each predicted output state component x_i. The coefficients a_{ij} have been determined in the start-up process. All of the basis vector components ϕ_j except those involving the control force u are determined by sampling. Oversimplifying a bit, we might imagine u to also be determinate by some deus ex machina process. Then, each of the above linear equations is in principle solvable by direct substitution.

The superficial complexity of some of the equational developments of the interpolation method compared with the above description arises from two sources:

The composition of the entire set of equations, individually of the linear form stated above, into a formal system. This possibility exists in that all of the equations are linear forms of the same basis components.

The necessity of obtaining an optimal u in conjunction with and as a result of the predicted output state components x_i. This implicit computation replaces the above cited oversimplification.

The function of the start-up procedure then reduces to the determination of the linear coefficients a_{ij} of each of the basis vector expansions.

*Linearity of plant representation or restriction to the class of linear plants is not necessarily implied. For non-linear plants the basis vector components include quadratic terms in the state vector components x_i and in the control force u.

Conventionally, the leading alphabetic symbols a, b, ... are associated with the "knowns" or fixed parameters of a problem, and the trailing alphabetic symbols x, y, z denote "unknowns". Equation 5-1 has employed this convention as is fully appropriate in its running use.

However, in start-up (or in other fitting and interpolative processes) an inversion of the identities of "knowns" and "unknowns" occurs. To maintain the previously cited symbolic convention a redefinition of parameters is necessary. Accordingly, we now associate the symbol x_j with the indeterminate coefficients hitherto designated a_{ij}, and conversely the symbol $a_{()j}$ with particulate values of the basis vector components previously designated ϕ_j. We further remove the index i, which previously identified a particular linear equation set, by considering only one equation, with it being understood that by symmetry of form results obtained for one such equation can be extended to the entire set under appropriate identification. Finally we substitute the symbol $b_{()}$, for the state component hitherto symbolized by x_i.

With all of the above redefinitions the actual linear form appropriate to start-up is little changed:

$$b_{()} = a_{()1}\, x_1 + a_{()2}\, x_2 + \dots + a_{()j} x_j + \dots a_{()m} x_m \qquad (5\text{-}2)$$

The reserved index () will be used to identify the particular determination or experiment of the start-up sequence, numbering sequentially from the first step.

The start-up procedure is in one sense better than the running procedure. The implicit calculation problem, which complicated the "running" equation solution, does not exist. It can now be reasonably assumed that at any given determination, k, of start-up the complete set of coefficients b_k, a_{kj} is observable. Further, if we are willing to defer any attempt at solution for the unknowns $x_1, \dots x_j, \dots x_m$ until m determinations are complete, the resultant equation set has a classical closed form solution.

But the latter restriction is precisely inappropriate to an acceptable solution to the startup problem as previously defined. We specifically wish to initiate control procedures based on some knowledge of the unknowns $x_1 \ldots x_j \ldots x_m$ as early as possible, and certainly before the m control actions required by a classical solution. The role of the pseudoinverse matrix technique in start-up is to effect some "best estimate" of the ultimate solution at each step of start-up.

Introducing the usual matrix symbolism we require a solution for $\underline{x}$ in the equation set:

$$\underline{b} = \underline{A}\,\underline{x} \tag{5-3}$$

where $\underline{x}$ has constant dimensionality (m x 1), $\underline{b}$ grows in dimension from (1 x 1) at the completion of the first step to (m x 1) at the end of the start-up procedure, and correspondingly $\underline{A}$ grows in dimensionality (and rank) from (1 x m) at the completion of the first start-up step to (m x m) at start-up completion.

Pseudoinverse Matrix Solution.-Classifical linear equation theory has tended to ignore the solution of equation 5-3 until sufficient determinations have been made so as to render the matrix $\underline{A}$ square, and the resultant solution unique and exact. At this juncture it introduces a useful formalism, that of the inverse matrix defined by:

$$\underline{A}^{-1}\underline{A} = \underline{I} \tag{5-4}$$

$\underline{A}$ being restricted to the class of square nonsingular matrices. Under this restriction the solution of equation 5-3 becomes

$$\underline{x} = \underline{A}^{-1}\underline{b} \tag{5-5}$$

Recently Penrose, and Greville (references 2, 3, and 4) have considered the more general problem of a solution of equation 5-3 under variable rectangular dimensionality of $\underline{A}$. In our application to start-up we are interested in the growing rank or under specified case describable by:

$$\underline{b} = \underline{A}\,\underline{x} \tag{5-6}$$

where $\underline{x}$ is an unknown (n x 1) vector, $\underline{b}$ is a known (m x 1) vector, $\underline{A}$ is a known rectangular matrix of dimensionality (m x n), and $m < n$.

It is immediately recognizable that the problem as here described is not really that of finding solutions for $\underline{x}$, since at least a single infinity of exact solutions exist under the above inequality. Rather the problem is to identify a uniquely determined and operationally useful solution from among all these possible solutions.

Moore and Penrose have developed a method, that of matrix pseudoinversion, which solves the latter problem in a manner particularly useful to the present application. By analogy with equation 5-4 they define a pseudoinverse matrix, symbolized $\underline{A}^+$, which defines a "best approximate" solution $\tilde{\underline{x}}$ in the equation.

$$\tilde{\underline{x}} = \underline{A}^+\underline{b} \tag{5-7}$$

where $\tilde{\underline{x}}$ is a (n x 1) vector, $\underline{b}$ retains its former identify, and $\underline{A}^+$ is a (n x m) matrix uniquely derivable from the direct matrix $\underline{A}$ according to:

$$\underline{A}^+ = \underline{A}'(\underline{A}\,\underline{A}')^{-1} \tag{5-8}$$

The solution $\tilde{\underline{x}}$ is unique and has the useful property that of all possible solutions $\underline{x}$, its norm is minimum.

One further pertinent property of the pseudoinverse so defined is that in the limit m = n:

$$\underline{A}^+ = \underline{A}^{-1} \tag{5-9}$$

<u>Recursive Calculation of the Pseudoinverse Matrix</u>.-The operational description of the previous paragraph is sufficient for application of the pseudoinversion technique to the start-up problem. A practical refinement

is the application of recursive methods to its computation. The core computational problem occurs in the conventional (square) matrix inversion required by equation 5-8. The rank of this matrix grows linearly with each start-up step, and its direct inversion could present real-time calculation problems during start-up.

Two recursive methods for obviating this problem have been developed, one by Wells and another by the authors. They are analytically developed in Appendix H, and illustrated by numeric example in a following portion of this section.

QUALITATIVE DISCUSSION OF PSEUDOINVERSE START-UP

Putting aside the mathematical detail of the pseudoinverse method, the core concept in its application to start-up can be crudely stated: "In the lack of better information, choose that potential solution which is closest to the origin." Certainly in the extremum of no information whatsoever, this is reasonable strategy. Actually, plants requiring description in high dimensional phase spaces begin the start-up process in a situation which closely approximates this extremum case.

An additional subtlety also suggests the desirability of this strategy in the early stages of start-up when plant knowledge is minimal. The control algorithm utilizes the "best estimate" solution of the plant state to select the next control action. The norm minimization property inherent in the pseudoinverse estimation tends to initially "conservative" control actions, which are surely appropriate in the lack of substantive information.

In "mid start-up" the controlling factor increasingly becomes the "potential solution" property. Each preceding determination has effectively reduced the dimensionality of the space of potential solutions by one. While the pseudoinverse procedure does not explicitly exhibit this property, it appears tacitly in successive constraint of the locus of potential solutions.

The final control action of the start-up process renders the solution fully determinate. The important property here is that the pseudoinverse algorithm produces the "exact" inverse in this limiting case.

GRAPHIC-NUMERIC EXAMPLE OF PSEUDOINVERSE TECHNIQUES

The following contrived examples illustrate the principles and methods previously discussed. They are three dimensional to permit hand calculation and reader verification, and for graphical visualization. While the solutions coverge nicely, the parameters were not chosen for that purpose. Rather, one of our graphically oriented engineers was given the rudiments of the pseudoinverse method and asked to make sketches illustrating it. The actual equational parameters were post facto estimated from his sketches (Figures 5-1, 5-2, and 5-3).

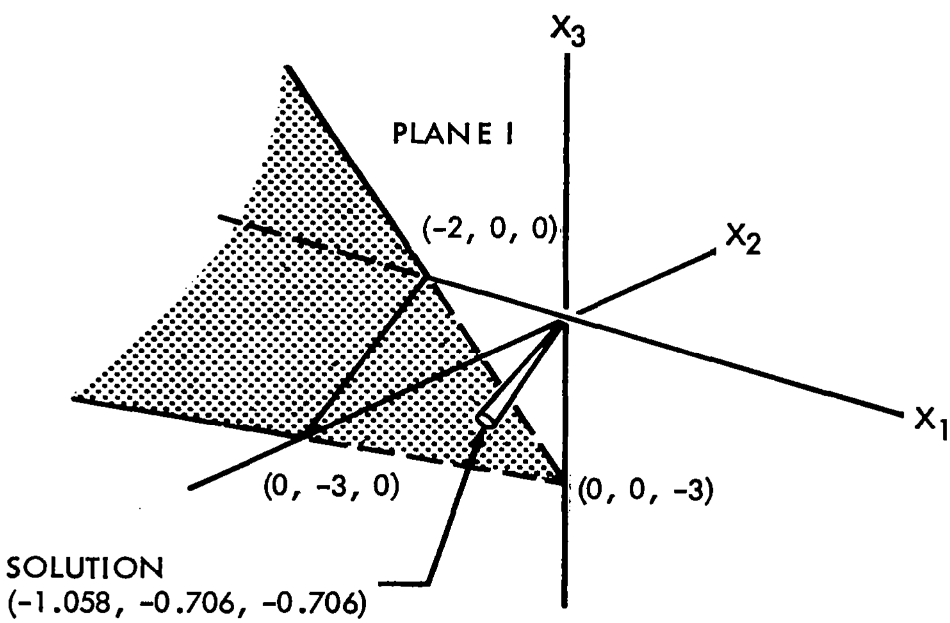


FIGURE 5-1 PSEUDOINVERSE SOLUTION OF SINGLE EQUATION

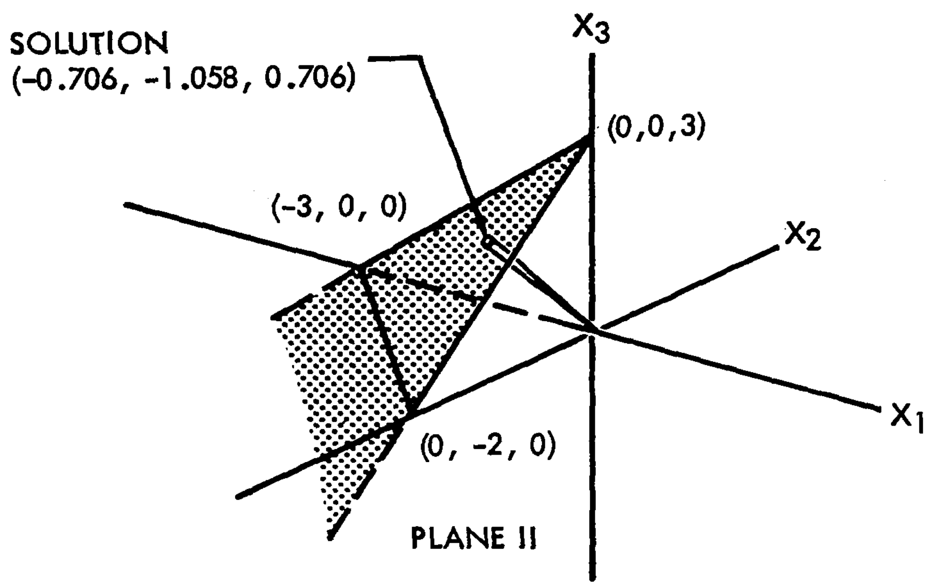


FIGURE 5-2 PSEUDOINVERSE SOLUTION OF SINGLE EQUATION

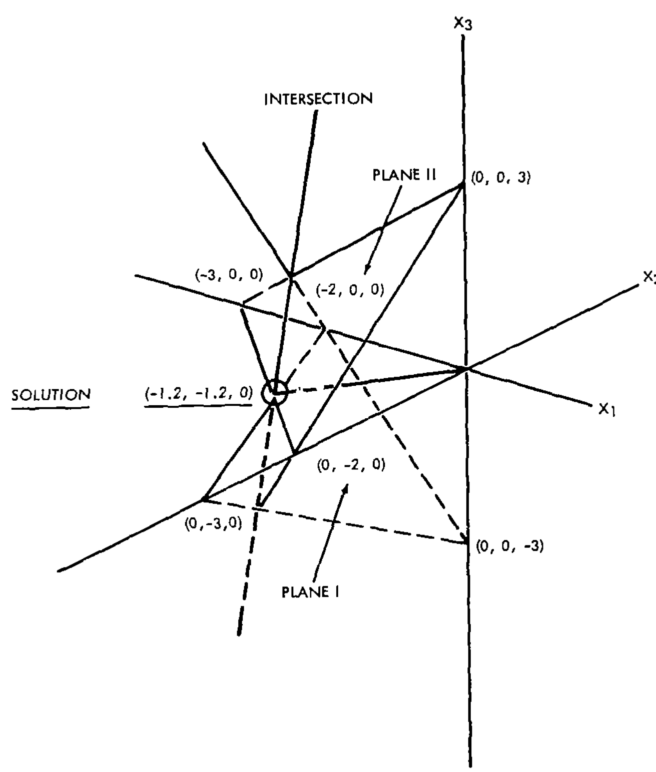


FIGURE 5-3 PSEUDOINVERSE SOLUTION OF TWO EQUATIONS

Example 1.- Consider an initial determination:

$$b_1 = 6 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = -3x_1 - 2x_2 - 2x_3 \tag{5-10}$$

In accordance with previous discussions in a start-up procedure the numeric 6 in the left hand side of the above equation is identifiable with the value of some measured output variable at the end of the first decision interval. Similarly, the three numerics of the right hand side (-3, -2, -2) are identifiable with known initial conditions and/or control forces applied at the start of the first decision interval. Equation 5-10 is graphed as plane I in Figure 5-1.

We now obtain a "best estimate" of the variables (x_1, x_2, x_3) by the pseudoinverse technique as follows:

$$\underline{A}^{+} = \underline{A}' \, (\underline{A}\,\underline{A}')^{-1} = \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \left(\begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17}$$

$$\underline{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \underline{A}^{+}\,\underline{b} = \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17}\, 6 = \begin{bmatrix} -18/17 \\ -12/17 \\ -12/17 \end{bmatrix} = \begin{bmatrix} -1.05\underline{8} \\ -0.70\underline{6} \\ -0.70\underline{6} \end{bmatrix}$$

Note that a unique solution has been obtained, and is indeed the point in Plane I lying on that normal to the plane passing through the origin.

The norm of $\underline{x}$ is minimal and of specific value:

$$\|\underline{\tilde{x}}\|^2 = \underline{\tilde{x}}'\,\underline{\tilde{x}} = 2.11\underline{6}$$

Example 2.-Alternately it could have been possible at the initial determination to have obtained a set of measurements defining the plane:

$$6 = -2x_1 - 3x_2 + 2x_3 \tag{5-11}$$

This locus is illustrated in Figure 5-2 as Plane II.

Assuming that this determination was the initial one and the sole available data, its pseudoinverse solution could be calculated as in the preceding example:

$$\underline{A}^{+} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \frac{1}{17}$$

$$\underline{\tilde{x}} = \begin{bmatrix} -12/17 \\ -18/17 \\ 12/17 \end{bmatrix} = \begin{bmatrix} -0.70\underline{6} \\ -1.05\underline{8} \\ 0.70\underline{6} \end{bmatrix}$$

$$\|\underline{\tilde{x}}\|^{2} = 2.11\underline{6}$$

Example 3.-Suppose now that the initial determination is that of Example 1 and that the determination of Example 2 is considered a subsequent second determination on the same system. For this case direct application of the pseudoinverse technique gives:

$$\underline{A}^{+} = \underline{A}' \, (\underline{A}\,\underline{A}')^{-1} = \begin{bmatrix} -3 & -2 \\ -2 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 17 & -8 \\ -8 & 17 \end{bmatrix}^{-1}$$

$$= \frac{1}{225} \begin{bmatrix} -35 & -10 \\ -10 & -35 \\ -50 & 50 \end{bmatrix}$$

$$\underline{\tilde{x}} = \underline{A}^{+} \underline{b} = \frac{1}{225} \begin{bmatrix} -35 & -10 \\ -10 & -35 \\ -50 & 50 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -1.2 \\ 0 \end{bmatrix}$$

$$\|\tilde{\underline{x}}\|^2 = 2.88$$

The graphical interpretation of this best approximate solution is given in Figure 5-3. The zero x_3 coordinate is an accident of the particular problem.

We can also here illustrate computation of the pseudoinverse matrix by both of the recursive algorithms.

By Wells' first formula for recursion:

$$\underline{A}_{k+1}^{+} = \left[\underline{A}_k^{+} - \underline{c}_k \, \underline{a}_k' \, \underline{A}_k^{+} \;\middle|\; \underline{c}_k \right]$$

$$\underline{A}_k^{+} = \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17}$$

and by definition, $\underline{a}_k'$ is the row added in the current determination, so that:

$$\underline{a}_k' = \begin{bmatrix} -2 & -3 & 2 \end{bmatrix}$$

by Wells' second formula:

$$\underline{c}_k = \left[\underline{a}_k' \left(\underline{I} - \underline{A}_k^{+} \underline{A}_k \right) \right]^{+}$$

$$\underline{c}_k = \left\{ \begin{bmatrix} -2 & -3 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{17} \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \right) \right\}^{+}$$

$$\underline{c}_k = \frac{1}{17} \begin{bmatrix} -10 & -35 & 50 \end{bmatrix}^{+}$$

$$\underline{c}_k = \frac{1}{225} \begin{bmatrix} -10 \\ -35 \\ 50 \end{bmatrix}$$

Back substituting this result into Wells' first formula:

$$\underline{A}_{k+1}^{+} = \left[\frac{1}{17}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} - \frac{1}{225}\begin{bmatrix} -10 \\ -35 \\ 50 \end{bmatrix}\begin{bmatrix} -2 & -3 & 2 \end{bmatrix}\frac{1}{17}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \;\vdots\; \frac{1}{225}\begin{bmatrix} -10 \\ -35 \\ 50 \end{bmatrix} \right]$$

$$= \frac{1}{225}\begin{bmatrix} -35 & -10 \\ -10 & -35 \\ -50 & 50 \end{bmatrix}$$

which is identical with the pseudoinverse previously obtained by direct substitution. While the recursion method is computatially disadvantageous in this primitive example, the converse is true as the rank increases

Finally, we demonstrate the bordering recursion as follows:

$$\underline{A}^{+} = \begin{bmatrix} \underline{A}_k' & \vdots & \underline{a}_k \end{bmatrix} \frac{1}{\alpha_k}\left[\begin{array}{c|c} \alpha_k \underline{B}_k^{-1} + \underline{B}_k^{-1}(\underline{A}_k\underline{a}_k)(\underline{a}_k'\underline{A}_k')\underline{B}_k^{-1} & -\underline{B}_k^{-1}(\underline{A}_k\underline{a}_k) \\ \hline -(\underline{a}_k'\underline{A}_k')\underline{B}_k^{-1} & 1 \end{array}\right]$$

$$\alpha_k = \underline{a}_k'\underline{a}_k - (\underline{a}_k'\underline{A}_k')\underline{B}_k^{-1}(\underline{A}_k\underline{a}_k)$$

From Example 1:

$$\underline{A}_k = \begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \qquad \underline{B}_k^{-1} = \frac{1}{17}$$

and by definition:

$$\underline{a}_k' = \begin{bmatrix} -2 & -3 & 2 \end{bmatrix}$$

Thus:

$$a_k = \begin{bmatrix} -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17} \begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \frac{225}{17}$$

$$\underline{A}^{+} = \begin{bmatrix} -3 & -2 \\ -2 & -3 \\ -2 & 2 \end{bmatrix} \frac{17}{225} \left[\begin{array}{c|c} \frac{225}{17}\,\frac{1}{17} + \frac{1}{17} \begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17} & -\frac{1}{17} \begin{bmatrix} -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \\ \hline -\begin{bmatrix} -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix} \frac{1}{17} & 1 \end{array} \right]$$

$$= \frac{1}{225} \begin{bmatrix} -35 & -10 \\ -10 & -35 \\ -50 & 50 \end{bmatrix}$$

which is again in agreement with the directly calculated result.

Example 4.-Consider that the two initial test responses of Example 3 are now augmented by a third sequential determination:

$$7 = -3x_1 - 2x_2 + 2x_3 \tag{5-12}$$

The equation set is now classifically complete and has a solution in the form:

$$\underline{x} = \underline{A}^{-1}\,\underline{b}$$

which under numeric substitution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -2 \\ -2 & -3 & 2 \\ -3 & -2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix}$$

First forming the "classical" inverse:

$$\underline{A}^{-1} = \frac{1}{20} \begin{bmatrix} -2 & 8 & -10 \\ -2 & -12 & 10 \\ -5 & 0 & 5 \end{bmatrix}$$

The resultant solution is:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1.7 \\ -0.7 \\ 0.25 \end{bmatrix}$$

with norm:

$$\|\underline{x}\|^2 = 3.44$$

We now demonstrate that the pseudoinverse procedure yields the classical inverse in this fully determined case.

$$\underline{A}^+ = \underline{A}' \, (\underline{A} \, \underline{A}')^{-1}$$

$$\underline{A}^+ = \begin{bmatrix} -3 & -2 & -3 \\ -2 & -3 & -2 \\ -2 & 2 & 2 \end{bmatrix} \left(\begin{bmatrix} -3 & -2 & -2 \\ -2 & -3 & 2 \\ -3 & -2 & 2 \end{bmatrix} \begin{bmatrix} -3 & -2 & -3 \\ -2 & -3 & -2 \\ -2 & 2 & 2 \end{bmatrix} \right)^{-1}$$

$$\underline{A}^+ = \frac{1}{20} \begin{bmatrix} -2 & 8 & -10 \\ -2 & -12 & 10 \\ -5 & 0 & 5 \end{bmatrix}$$

The desirability of this result over classical inversion methods is that it can be approached recursively from the successive underspecified solutions during start-up.

5.2 T-h LEARNING INVESTIGATION

In all of the procedures and results previously presented there runs a common thread, the primacy of the T-h control system parameters. To the extent that adaptivity has been demonstrated, it resides in the existence of a region of the T-h space where stable and effective control of a diversity of plants can be accomplished. For linear stationary plants, the extent of this region can be predicted by the Liapunov stability criterion. For other plant classes it must be determined by experiment.

In considering the application of formal learning techniques to this control, initial concentration on the T-h parameters seems natural. At the period when investigation of learning began, two problems requiring the determination of preferential T-h values were apparent from prior simulatory studies:

> For classes of plants where an appreciable region of the T-h plane was known to provide stable control, what specific T-h point was "optimal"?
>
> For those plants (largely high order) where any given plant had only a few isolated points of stability, how could such points be identified and utilized during the start-up procedure?

The following theoretical study deals exclusively with the first of these problems for several reasons:

> Since learning processes in the unknown plant context require experimentation with the actual plant control, the existence of a stability region in the first problem permits some freedom of action without catastrophic consequence.
>
> The second problem is patently more formidable. Also, as stated it must be solved in the start-up period. While start-up methods as described in Paragraph 5.1 had been devised, they had not been experimentally tested. Thus, a desirable background of pragmatic experience was lacking.

A RESTRICTED T-h LEARNING PROCEDURE

It is initially assumed that the control process has been successfully started and is operating in a stable fashion. The initial set of T-h values has been chosen from a boundary for the generic class of plants similar to and presumably containing the particular plant under control.

It is now proposed to "improve" the control by adjustment of T, h, or both under a learning procedure. This objective first requires the identification of a measurable and meaningful criterion of "improved" performance. The control policy is to reduce the norm of the difference between the actual state $\underline{x}(t)$ and the desired state $\underline{r}(t)$ on an interval-to-interval basis. Assuming this aim as realistic, a logical expression for the quality of control is the actually achieved norm averaged over a representative interval.

Averaged Norm Criterion (1).-Thus, a possible criterion is:

$$I_1 = \int_{\theta} \|\underline{x}(t) - \underline{r}(t)\|^2 dt \qquad \text{or} \qquad \sum_{\theta} \|\underline{x} - \underline{r}\|^2 \qquad (5\text{-}13)$$

where θ is an interval of representative length.

These forms differ only in whether or not the sampling period is sufficiently short to approach the continuum.

Control would be considered improved if under a permutation of T, h, or both, I_1 is reduced. However, for this use I_1 must be uniquely representative of the control policy and its parameters, rather than of changes in the desired input $\underline{r}$. A method of removing the undesired sensitivity to $\underline{r}$ is apparent only in a few special cases. The most practical such case is a randomly variant $\underline{r}$ applied to a linearly invariant system. Here I_1 should be proportional to the mean square $\underline{r}$ provided the integration interval θ is long enough, and provided the power spectra of $\underline{r}$ over each of the compared intervals are constant to the extent of scalar multiplication.

Maximal Excursion Criterion (2).-An alternate criterion, that of maximal excursion, might be preferable in certain practical applications.

$$I_2 = \max \|\underline{x}(t) - \underline{r}(t)\| \qquad 0 < t < \theta \qquad (5\text{-}14)$$

However, here the conditions for the statistical validity of $\underline{r}$ independence are more stringent than those stated for the previous case.

Accordingly, it is concluded that T-h parameter optimization by the use of either of the preceding criteria:

> Is not possible of implementation in the generality of plant treatment which has characterized the other studies
>
> Can be applied only with caution in those few cases where a priori applicability can be inferred. Particularly, it is necessary to establish statistical regularity of the input function in terms of an integration interval θ

Plant Model Criteria (3) and (4).-Hitherto these studies have deliberately excluded any use of any modeling concept. The rationale was simply that a model could not be a priori defined for an unknown plant. Postulation of some sort of plant model (preferably generic by plant class rather than specific to a given plant) is here treated for the following reasons:

> To the extent that a plant model can be considered as a filter, its output has greater statistical regularity than its input. Viewed in the frequency domain the power spectrum of its output is regularized.
>
> If the model is reasonably representative of the plant (class), it introduces the element of physical realizability into its output.

The first property alleviates (but does not totally eliminate) the problem of establishing statistical regularity of two sections of the reference function. Accordingly, let us modify the performance criteria as follows (3), (4):

$$I_3 = \int_\theta \|\underline{x}(t) - \underline{z}(t)\|^2 \, dt \qquad (5\text{-}15)$$

$$I_4 = \max \|\underline{x}(t) - \underline{z}(t)\| \qquad 0 < t < \theta \qquad (5\text{-}16)$$

where $\underline{z}(t)$ is the output of the model which is fed the same desired state as the system proper.

Note that prefiltering of the input to the actual control system is not implied, and that the model does not enter the control action computation per se. Its only function is in the search for improved control parameters T and h, which is a parallel function to the actual control. Note also that if the model is reduced to the identity operator, $I_1 = I_3$ and $I_2 = I_4$.

The parameter optimization procedure here postulated then is the evaluation of performance indices I_3 or I_4 over several successive equal duration periods, with concomitant perturbation of the actual control parameters T, h, or both. A subsequent search technique identifies the set producing the minimum values of the performance index I_3 or I_4. This procedure differs from a more common model exploratory technique, in that the experimentation is performed on the plant control, not on model parameters. Again the assumption of an unknown plant prohibits any close specification of the model, and its role is that of a filter in the parameter optimization loop and not as that of a reference standard.

Determination of the Integration Interval θ.-The first parameter to be determined in this approach is the value of a "sufficiently long" integral of integration θ. For the random stationary input case, the input power spectral density can be computed from sampled measurements or may be a priori known. An arbitrary cutoff frequency, characterized by the property that some large fraction of the input power lies at or below it, can be designated the cutoff frequency f_c. Then for $\theta > 1/2f_c$, all correlations introduced by band limiting have been included in the sample set, and the ensemble of sample functions has the same statistical expectations as the parent function.

A Search Procedure.-Let it now be assumed that two sections of equal length θ, but with perturbation between sections of one of the control parameters, say h, have been measured. The desired value $\underline{z}(t)$ is assumed to be random stationary over the interval of integration θ in each case. However, it may happen that the average value of $\|\underline{z}(t)\|$ is different between the two sections. Since a linear plant and control policy has been assumed in this study, this variation appears as a linear scaling factor in the performance index I_3. Thus:

$$I_3\ (h_1) = \int_\theta \|\underline{x}(t,h_1) - \underline{z}(t)\|^2 dt \tag{5-17}$$

$$I_3\ (h_2) = \frac{1}{\mu}\int_\theta \|\underline{x}(t',h_2) - \underline{z}(t')\|^2 dt' \tag{5-18}$$

where the introduction of the scaling factor

$$\mu = \int_\theta \|\underline{z}(t')\| dt' \Big/ \int_\theta \|\underline{z}(t)\| dt \tag{5-19}$$

removes this amplitude scaling effect and renders the computed values of the performance index compatible.

Once a sufficient number of such values is available a search for the minimum I_3 can be instituted by conventional methods. In such numerical search techniques, particularly for steepest descent or Newton-Raphson type methods, the components of the gradient can be approximated from the preceding by:

$$\frac{\partial I_3}{\partial h} \approx \frac{I_3(h_1) - I_3\ (h_2)}{h_1 - h_2} \tag{5-20}$$

Having locally optimized h, we can now proceed in a perfectly analogous way to the local optimization of T. A number of conventional methods of organizing the search procedure exist.

Evaluation of Method.-The advantages of this method include:

It is straightforward in concept and relatively simple of implementation.

Since the search for parameter optimization is conducted with the plant control per se, any optimization achieved has been directly verified before final commitment.

Its disadvantages include:

Since experimentation is conducted on-line, performance will be degraded during portions of the search.

A model concept has been introduced, albeit in an indirect and inexact manner. Particularly, it is not apparent that a "generalized" model for a class of plants as assumed here can indeed be synthesized.

It has been necessary to restrict the plant input to a class of functions exhibiting statistical regularity. In common with other statistical design methods, such properties are more characteristic of the ensemble of a large set of "similar" plants or of a long time history of control actions of a given plant, than of the limited samples of here-and-now control of a single system.

The search time may prove excessive. While the estimated time of a given determination appears reasonable, the requisite number of determinations has not been estimated.

A GENERALIZED T-h LEARNING PROCEDURE

The limitations of the previously described method arise largely from the assumption that the plant is totally unknown at the beginning of the parameter optimization procedure. The immediate results of this assumption are the restriction to linear plants and to random stationary input sequences.

The situation of complete lack of plant knowledge exists only at start-up under most of the control methods investigated. Only the pure Taylor control method does not learn in the control process. The mixed predictive method acquires an estimate of the plant forced response as control proceeds. The interpolation based methods build up knowledge of both the free and forced plant responses in the form of transition matrices.

The following learning method is based on the utilization of the closed loop transition equation. As here outlined it is based on the assumptions of:

Linear stationary plants

Parameter adjustment to a performance index based on optimality to solution of a standardized regulator problem.

These assumptions are arbitrary for analytical convenience and by no means necessary.

For clarity of presentation, the method is presented in outline form as follows:

1. The performance index is taken to be:

$$I_4 = \sum_{\theta} \| \underline{x}(nT) - \underline{z}(nT) \|^2 \qquad (5\text{-}21)$$

By choice of optimation to the regulator problem and removal of the model reference, the performance index reduces to:

$$I_4 = \sum_{\theta} \| \underline{x}(nT) \|^2 \qquad (5\text{-}22)$$

2. Let the actual control system be started with the initial parameter values T_1, h_1 chosen from a region of known stability for the generic plant class. At the conclusion of its start-up procedure the closed loop state transition matrix $\underline{\Psi}(T_1, h_1)$ has been determined satisfying:

$$\underline{x}\left[(N+1)T\right] = \underline{\Psi}(T_1, h_1)\ \underline{x}(NT) \qquad (5\text{-}23)$$

Note that for fixed T_1 the matrix $\underline{\Psi}(T_1, h_1)$ is a specific known function of h_1. By local linearization of the analytic form of $\underline{\Psi}(T_1, h_1)$ determine an expression for $\underline{\Psi}(T_1, h_1 + \Delta h)$.

3. Utilize this expression in conjunction with equations 5-21 and 5-23 to evaluate the performance index I_4 for values of $(h_1 + \Delta h)$ within the stability bounds of h. Two alternate possibilities exist for the optimization of h. The dependence of $\underline{\Psi}(T_1, h_1 + \Delta h)$ on h may be such as to permit analytic minization of I_4 with respect to h. If so, a preferential value of $(h_1 + \Delta h)$ designated h_2 can be computed in closed form. Lacking this a limited search in the set of computed values $\underline{\Psi}(T_1, h_1 + \Delta h)$, $\underline{\Psi}(T_1, h_1 + \Delta h')$, $\underline{\Psi}(T_1, h_1 + \Delta h'')$ ... can be performed to establish the value of h_2.

Note that the process of determining h_2 is an off-line computation, and that during its execution the system continues to operate with the initial parameters T_1 and h_1. Further note that while some complexity of computation may be involved, the process occurs after start-up, and in the running region where other computing requirements are minimal.

4. Introduce the locally optimal value of h_2 into the control process, which now operates with parameters T_1, h_2. Note that the approximate closed loop state transition matrix $\underline{\Psi}(T_1, h_2)$ is immediately available, since it has been calculated in the process of optimizing h.

5. Minimization of the performance index I_4 with respect to T at constant h_2 is next to be performed. Some possibility of analytically exhibiting the T dependence of $\underline{\Psi}(T_1+\Delta T, h_2)$ in a manner analogous to that invoked for h exists in the linear plant case. However, the dependence is assuredly more complicated. Here minimization by a search technique among I_4 computed values of argument

$$\underline{\Psi}(T_1+\Delta T,\ h_2),\quad \underline{\Psi}(T_1+\Delta T',\ h_2),\quad \underline{\Psi}(T_1+\Delta T'',\ h_2)$$

... is assumed. Designate the value of $(T_1+\Delta T)$ producing the minimum I_4 as T_2.

6. Introduce into the control the optimalized values (T_2, h_2). Here stability may be a somewhat more vexing problem since $\underline{\Psi}(T_2, h_2)$ is available only in interpolated form from the minimization search. Consequently if T_2 differs appreciably from T_1, it may be necessary to approach it by small steps.

7. This process is in principle iterative from step 3.

Evaluation Of Method.-A first conclusion is that the generalized T-h learning procedure is preferable to the restricted T-h learning procedure first described on several counts:

It permits off-line optimization of system parameters without interference with the control process.

It removes the restriction of inputs which limited the applicability of the first described procedure.

It utilizes available plant information, and dispenses with the modeling concept.

On the converse side the following limitations are recognized:

It has eliminated the undesired sensitivity of the performance index to the input function by substitution of optimization to a standardized input function. The permissable degree of generalization of this function beyond the regulator assumed in the preceding notes has not been established.

Neither has the detailed equational description of the performance index to the T-h parameters been made. Isolation of the h parameter sensitivity for linear systems can be predicted with some confidence. The T parameter determination and the possibility of extension to non-linear systems require investigation.

REFERENCES

1. Wells, C. H., "Minimum Norm Control of Discrete Systems," Doctoral Dissertation, Washington University, 1965.

2. Penrose, R., "A Generalized Inverse for Matrices," Proceedings of the Cambridge Philosophical Society, Vol. 51, 1956, pp. 406-413.

3. Penrose, R., "On Best Approximate Solutions of Linear Matrix Equations," Proceedings of the Cambridge Philosophical Society, Vol. 52, 1955, pp. 17-19.

4. Greville, T. N. E., "The Pseudoinverse of a Rectangular or Singular Matrix and Its Application to the Solution of Systems of Linear Equations," Society of Industrial and Applied Mathematics Review, Vol. 1, No. 1, January 1959, pp. 38-43.

SECTION 6

RECOMMENDATIONS FOR FURTHER STUDY

The theoretical and experimental research presented in this report has investigated in various depths of detail Emerson's control system performance for linear and non-linear time and nontime-varying systems. Throughout the course of these investigations certain areas noteworthy of further study have been recognized, and as such are summarized in this section. The areas of research considered to be the most desirable to study due to importance and/or productivity at modest cost are referred to as Primary Recommendations. Those areas of research, which to a large extent require rather intensive analytical work and/or programming, are denoted as Secondary Recommendations.

PRIMARY RECOMMENDATIONS

The primary recommendations are of considerable interest as well as importance, and as such are considered first. The three primary recommendations are:

Extension of Experimental Studies on linear time-varying and nontime-varying plants of higher order.

Extension of Experimental Studies on non-linear time-varying and nontime-varying plants.

Analytical study to develop the Volterra series (R=1) control equations applicable to linear time-varying and nontime-varying plants.

Each of these areas is considered in depth in the following paragraphs.

The extention of the experimental studies on linear time-varying plants to higher order is a logical step in determining an accurate performance assessment of our control system. Since the previous control experimentation on time-varying systems was restricted to second and third order plants having only one parameter vary as a function of time, the results were of use only for very limited conclusions. The main conclusion is that the Digital Adaptive Control System did show promising control capability for low order time-varying plants. Certain problem areas such as ill conditioning of the matrix of basis vectors were discovered, which need further analytical as well as experimental study. This particular problem was noted for nontime-varying as well as time-varying systems, and is a numerical problem directly associated with the Interpolation Prediction method. The area of research covered by the above discussion is rather broad, but each facet is of practical importance with regard to control system improvement for linear time-varying and nontime-varying systems. The experimental extensions will greatly utilize the existent analysis methods and computer programs.

The second recommendation for extension of experimental studies on non-linear time-varying and nontime-varying plants is of prime interest and importance. In the previous analytical studies two alternate control system approaches were developed, and the Interpolation Prediction method selected as the most desirable to program for experimental study. The only non-linear plant control experimentation during the previous research was conducted to determine if such plants could be adequately controlled with the linear control policy. The results of this limited experimentation were very good, and so investigation of control with the non-linear policy is recommended. As in the previous recommendation, analysis methods and computer programs already developed will be utilized to a great extent.

The last primary recommendation is to develop the Volterra series (R=1) control equations applicable to linear time-varying and nontime-varying plants. This requires an analytical study of a somewhat limited depth, since much of the work done in the R = 2 case will be of considerable help. The R = 1 equation development is of prime interest, since it will allow an alternate control approach for linear plants. Thus, a comparison, such as was done for the R = 2 Volterra series and the Interpolation methods, may be completed for the alternate linear plant control policies.

SECONDARY RECOMMENDATIONS

The following two recommendations are of great importance, and are only considered as secondary due to the large analytical and experimental work required for any depth of investigation:

Investigation of possible start-up procedures, such as the method presented in Section 5.

Investigation of possible learning or pattern recognition with regard to obtaining the T - h parameters for best possible control performance. This area of research was also discussed in Section 5.

Both of these areas require research which should and must be done before possible application of Emerson's control system may be meaningfully pursued. However, such investigations may be considered at a later date without causing serious effect on the primary recommendation areas of research.

APPENDIX A

DETAILED STUDY OF THE STATE VECTOR DISCONTINUITY PROBLEM

A.1 DERIVATION OF THE DISCRETE STATE EQUATION OF LINEAR STATIONARY PLANTS IN TERMS OF $t = kT^+$ INITIAL CONDITIONS

The mathematical description of the plant is expressed by the single equation

$$\dot{\underline{x}}(t) = \underline{H}\ \underline{x}(t) + \underline{G}\ \underline{u}(t) \tag{A-1}$$

where $\underline{H}$ and $\underline{G}$ are constant matrices. The vector $\underline{x}(t)$ is the plant state variable identified on a one to one basis with the plant output, c(t), and the first (n-1) derivatives of c(t). The quantity $\underline{u}(t)$ is the vector input to the plant identified on a one to one basis with the input, m(t), and the first m derivatives of m(t). A more complete discussion of this equation is presented in Section 2.1 of the text.

The solution of equation A-1 proceeds by considering the solution of the free or homogeneous equation

$$\dot{\underline{x}}(t) = \underline{H}\ \underline{x}(t) \tag{A-2}$$

By analogy with the corresponding scalar equation, it is possible to show the solution of A-2 is given by

$$\underline{x}(t) = e^{\underline{H}(t-t_o)}\ \underline{x}(t_o) \tag{A-3}$$

where

$$e^{\underline{H}\,t} = \sum_{i=0}^{\infty} \frac{\underline{H}^i\ t^i}{i!} \tag{A-4}$$

and $\underline{H}^o = \underline{I}$.

The series of equation A-4 can be shown to converge uniformly in any finite interval on the time axis and the sum function is a continuous function of t for all finite t. Equation A-3 is conveniently written in the form

$$\underline{x}(t) = \underline{F}(t-t_o)\ \underline{x}(t_o) \tag{A-5}$$

where $\underline{F}(t-t_o) = e^{\underline{H}(t-t_o)}$ is referred to as the transition matrix of the plant.

The solution to the forced (nonhomogeneous) equation is obtained by employing the variation of parameters technique and assuming the solution of equation A-1 is of the form

$$\underline{x}(t) = \underline{F}(t-t_o)\ \underline{w}(t) \tag{A-6}$$

Differentiating equation A-6 and solving for $\underline{w}(t)$ by comparison with equation A-1 simplifies to

$$\underline{w}(t) = \int_{t_o}^{t} \underline{F}^{-1}(\tau - t_o)\ \underline{G}\ \underline{u}(\tau)\ d\tau\ + \underline{c} \tag{A-7}$$

where $\underline{c}$ is the constant of integration. The existance of A-7 requires that $\underline{F}(t-t_o)$ possess an inverse for all t. The non-singular property of $\underline{F}(t-t_o)$ follows from the fact that

$$\left| e^{\underline{H}\,t} \right| = e^{t\ \mathrm{trace}\ \underline{H}} \tag{A-8}$$

Substituting the solution for $\underline{w}(t)$ into equation A-6 and making use of the fact that at $t=t_o$, $\underline{F}(t-t_o) = \underline{I}$ and $\underline{c} = \underline{x}(t_o)$ yields as the solution of equation A-1.

$$\underline{x}(t) = \underline{F}(t-t_o)\ \underline{x}(t_o) + \int_{t_o}^{t} \underline{F}(t-\tau)\ \underline{G}\ \underline{u}(\tau)\ d\tau \tag{A-9}$$

Equation A-9 is the general solution of the first order vector differential equation of the plant.

To place the solution of equation A-1 in a discrete form, consider $\underline{x}(t_o)$ to be the state of the plant at some time t = kT where T is the length of the sampling or decision interval in seconds. The solution for the plant state at the next sampling instant is given by

$$\underline{x}((k+1)T) = \underline{F}(T)\ \underline{x}(kT) + \int_{kT}^{(k+1)T} \underline{F}(t-\tau)\ \underline{G}\ \underline{u}(\tau)\ d\tau \tag{A-10}$$

Considering the plant input, m(t), to be constant over each sampling interval and defining the solution of the plant differential equation over the open interval $kT < t < (k+1)T$ yields

$$\underline{x}((k+1)T^-) = \underline{F}(T)\ \underline{x}(kT^+) + \int_{kT^+}^{(k+1)T^-} \underline{F}((k+1)T^- - \tau)\ \underline{G}\ \underline{u}(\tau)\ d\tau \tag{A-11}$$

where

$$\underline{u}'(t) = \begin{bmatrix} u_k & 0 & . & . & . & 0 \end{bmatrix} \qquad kT^+ \leq t \leq (k+1)T^- \tag{A-12}$$

Defining the solution equation A-11 over the open interval avoids the problem associated with discontinuities in the input, m(t), at sampling instants.

The form of the constant matrix $\underline{G}$ is given in Section 2.1 and is repeated here for convenience.

$$\underline{G} = \begin{bmatrix} & & \text{--- } 0 \text{ ---} & & \\ & & & & \\ B_0 & B_1 & . \ . \ . \ . & & B_m \end{bmatrix} \tag{A-13}$$

The product of $\underline{G}$ with $\underline{u}(t)$ is a column matrix of the form

$$[\underline{G}\ \underline{u}(t)]' = \begin{bmatrix} 0 & . & . & 0 & B_o u_k \end{bmatrix} \tag{A-14}$$

Let the last column of $\underline{F}(t)$ be defined as $\underline{f}(t)$. Making use of this definition and equations A-12 and A-14 permits equation A-11 to be written in the form

$$\underline{x}((k+1)T^-) = \underline{F}(T)\ \underline{x}(kT^+) + u_k B_o \int_{kT^+}^{(k+1)T^-} \underline{f}((k+1)T^- - \tau)\ d\tau \tag{A-15}$$

For a constant sampling rate (T constant), $\underline{F}(t)$ is a constant matrix and the integral on the right hand side of equation A-15 is a constant vector. An abbreviated notation for equation A-15 is

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^+) + \underline{a}\ u_k \tag{A-16}$$

where

$$\underline{a} = B_o \int_{kT^+}^{(k+1)T^-} \underline{f}((k+1)T^- - \tau)\ d\tau \tag{A-17}$$

and the evaluation of $\underline{F}(t)$ and equation A-17 for the length of the sampling interval, T, is tacitly implied.

Equation A-16 is a discrete state equation for linear, stationary plants and is the equation stated in the text (equation 2-12 of Section 2.1) as being stated in terms of $t = kT^+$ initial conditions. It is completely general in that the interval $kT^+ \leq t \leq (k+1)T^-$ may occur anywhere in the control time history, and the plant transfer function may contain zeroes as well as poles.

A.2 THE DISCONTINUITY VECTOR

The input to the plant is a sequence of piecewise constant control forces calculated by the control policy. The nature of the control action is such that the control force is constant over any one sampling or decision interval, but not continuous from interval to interval. The effect of the discontinuity in the control input at sampling instants is to cause corresponding discontinuities in a number of the state variable components which depends upon the plant configuration. The relationship between the values of the state variable components before and after the input discontinuity is expressible in the form

$$\underline{x}(kT^{+}) = \underline{x}(kT^{-}) + \underline{x}_d\ (u_k - u_{k-1}) \tag{A-18}$$

where the discontinuity vector, $\underline{x}_d$, is normalized to a unit magnitude discontinuity. The principle of superposition allows the evaluation of the discontinuity vector, $\underline{x}_d$, by considering $\underline{x}(kT^{-})$ to be the null vector $\underline{0}$, and the discontinuity to be unity in magnitude. Under these conditions

$$\underline{x}(kT^{+}) = \underline{x}_d \tag{A-19}$$

In order to determine the relationship between the plant configuration and the discontinuity vector, $\underline{x}_d$, a transfer function approach is employed. Assuming all initial conditions to be zero, a unit step function is applied to the plant and the Laplace transform of the state vector (the plant output and its first (n-1) derivatives) is obtained. By means of the initial value theorem of Laplace transform theory, the initial conditions are evaluated. These initial conditions constitute the discontinuity vector $\underline{x}_d$.

In order to be more quantitative, consider the general transfer function

$$G(s) = \frac{\sum_{i=0}^{n-1} B_i s^i}{\sum_{i=0}^{n} A_i s^i} = \frac{N(s)}{D(s)} \tag{A-20}$$

where $A_n = 1$ and at least B_o is non-zero.

The discontinuity vector is defined by

$$\underline{x}_d = \frac{\underline{x}(kT^{+}) - x(kT^{-})}{u_k - u_{k-1}}$$

or

$$\underline{x}_d' = \lfloor d_1\ d_2\ .\ .\ .\ d_n \rfloor \tag{A-21}$$

where, under the conditions outlined above

$$\underline{x}_d{}' = \underline{x}'(0^+) = \left\lfloor c(0^+) \quad \dot{c}(0^+) \quad . \quad . \quad . \quad \overset{n-1}{c}(0^+) \right\rfloor \tag{A-22}$$

The Laplace transform of the plant output due to a unit step is given by

$$C(s) = \frac{N(s)}{sD(s)} \tag{A-23}$$

The first component, d_1, of the discontinuity vector is given by

$$d_1 = c(0^+) = \lim_{s \to \infty} (sC(s)) = 0 \tag{A-24}$$

Thus, the output of the plant is continuous regardless of the plant configuration.

$$d_2 = \dot{c}(0^+) = \lim_{s \to \infty} (s^2 C(s)) = B_{n-1} \tag{A-25}$$

$$d_3 = \ddot{c}(0^+) = \lim_{s \to \infty} s(s^2 C(s) - d_2) = B_{n-2} - A_{n-1}\, d_2 \tag{A-26}$$

Similarly, the general term, d_i, may be expressed in the form

$$d_i = B_{n-i+1} - \sum_{j=1}^{i} A_{n-i+j}\, d_j \tag{A-27}$$

A convenient way of writing the solution for the $\underline{x}_d$ vector is in the form

$$\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 \\ A_{n-1} & 1 & 0 & . & . & . & 0 & 0 \\ A_{n-2} & A_{n-1} & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ A_1 & A_2 & A_3 & . & . & . & A_{n-1} & 1 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ B_{n-1} \\ B_{n-2} \\ . \\ B_1 \end{bmatrix} \tag{A-28}$$

It is important to note that as the degree of the numerator polynomial of the transfer function (the number of zeroes) lessens, more of the d_i's are zero. The nature of the discontinuity vector is such that the highest order state variable components are the ones in which the discontinuities occur. If the transfer function contains no zeroes the only non-zero coefficient is B_o and the discontinuity vector, $\underline{x}_d$, is the null vector $\underline{0}$. The complete state vector will then be continuous.

A.3 DERIVATION OF THE DISCRETE STATE EQUATION OF LINEAR STATIONARY PLANTS IN TERMS OF $t = kT^o$ INITIAL CONDITIONS

Figure A-1 defines the time instants kT^-, kT^o and kT^+. The relationship between the state vector $\underline{x}(kT^o)$ and $\underline{x}(kT^+)$ may be expressed in terms of the discontinuity vector derived in Section A.2 of this appendix.

$$\underline{x}(kT^+) = \underline{x}(kT^o) + \underline{x}_d\ u_k \tag{A-29}$$

Similarly

$$\underline{x}((k+1)T^-) = \underline{x}((k+1)T^o) + \underline{x}_d\ u_k \tag{A-30}$$

Substituting equations A-29 and A-30 into the state equation derived in Section A.1 (equation A-16) yields

$$\underline{x}((k+1)T^o) + \underline{x}_d\ u_k = \underline{F}\ (\ \underline{x}(kT^o) + \underline{x}_d\ u_k) + \underline{a}\ u_k \tag{A-31}$$

collecting terms

$$\underline{x}((k+1)T^o) = \underline{F}\ \ \underline{x}(kT^o) + \underline{\lambda}\ u_k \tag{A-32}$$

$$\underline{\lambda} = \underline{a} + \underline{F}\ \underline{x}_d - \underline{x}_d \tag{A-33}$$

Equation A-32 is the general form of the state equation in terms of $t = kT^o$ initial conditions.

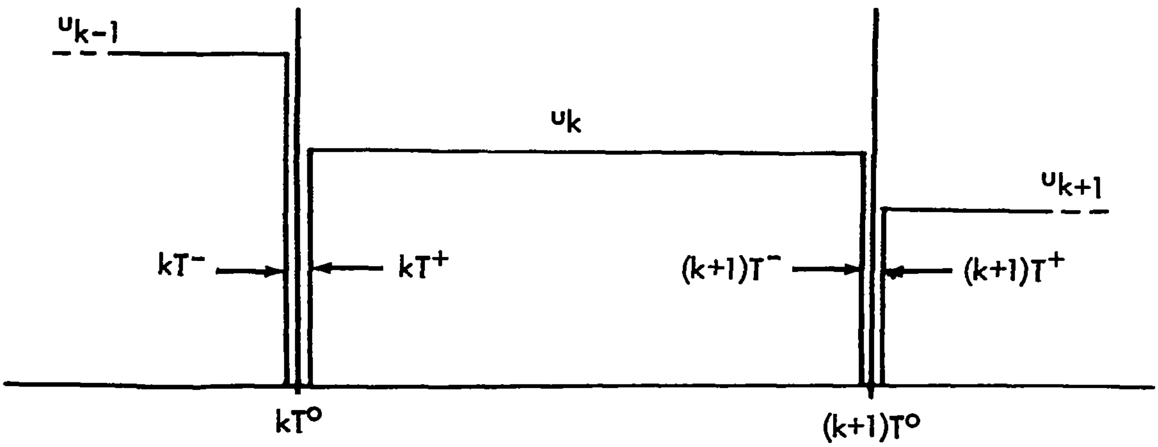


FIGURE A-1 A SEQUENCE OF CONTROL FORCES ILLUSTRATING THE DEFINITION OF kT°

A.4 SOME MISCELLANEOUS RELATIONSHIPS

The equations of this section are a collection of useful relationships which are used in various parts of the text and other appendices.

1. It was shown in Section A.2 that when a plant transfer function contains no zeroes the discontinuity vector, $\underline{x}_d$, becomes the null vector $\underline{0}$. Equation A-18 of Section A.2 therefore reduces to

$$\underline{x}(kT^-) = \underline{x}(kT^+) \qquad \text{(A-34)}$$

which means the state vector $\underline{x}(t)$ is continuous even though the input contains discontinuities. State equations A-16 and A-32 become identical as from equation A-33 when $\underline{x}_d = \underline{0}$

$$\underline{\lambda} = \underline{a} \qquad \text{(A-35)}$$

2. Two alternate expressions for the state equation A-16 in terms of kT^- initial conditions may be derived by substituting equation A-18 into equation A-16

$$\underline{x}((k+1)T^-) = \underline{F}\ [\underline{x}(kT^-) + \underline{x}_d\ (u_k - u_{k-1})] + \underline{a}\ u_k \qquad \text{(A-36)}$$

The first of these expressions is

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{b}_1\ u_k + \underline{b}_2\ u_{k-1} \qquad \text{(A-37)}$$

where

$$\begin{aligned} \underline{b}_1 &= \underline{a} + \underline{F}\ \underline{x}_d \\ \underline{b}_2 &= -\ \underline{F}\ \underline{x}_d \end{aligned} \qquad \text{(A-38)}$$

The second expression is only a slight rearrangement of the first

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{a}\ u_k + \underline{c}\ (u_k - u_{k-1}) \qquad \text{(A-39)}$$

where

$$\underline{c} = \underline{F}\ \underline{x}_d \tag{A-40}$$

Comparing equations A-38 and A-40 the following equalities exist

$$\begin{aligned} \underline{b}_1 &= \underline{a} + \underline{c} \\ \underline{b}_2 &= -\ \underline{c} \end{aligned} \tag{A-41}$$

Comparing equation A-38 with equation A-33 yields an alternate expression for $\underline{\lambda}$

$$\underline{\lambda} = \underline{b}_1 + \underline{F}^{-1}\ \underline{b}_2 \tag{A-42}$$

A.5 DERIVATION OF THE INTERPOLATION ESTIMATE OF THE DISCRETE STATE EQUATION FOR LINEAR TIME VARYING PLANTS IN TERMS OF $t = kT^{\circ}$ INITIAL CONDITIONS

Equation 3-19 of Section 3.1 is the interpolation estimate of the discrete state equation in terms of $t = kT^-$ initial conditions.

$$\underline{\tilde{x}}((k+1)T^-) = \underline{\theta}_1\ \underline{x}(kT^-) + \underline{\varphi}_1\ u_k + \underline{\varphi}_2\ u_{k-1} + \underline{\varphi}_3\ T \tag{A-43}$$

If for some interval $kT \leq t < (k+1)T$ the control force u_k is zero, then

$$\underline{\tilde{x}}((k+1)T^-) = \underline{\tilde{x}}((k+1)T^{o}) = \underline{\theta}_1\ \underline{x}(kT^-) + \underline{\varphi}_2\ u_{k-1} + \underline{\varphi}_3\ T \tag{A-44}$$

Since the change in $\underline{x}(t)$ over the interval $kT^{o} \leq t \leq (k+1)T^{o}$ is due entirely to the free response and the time variation of the plant,

$$\underline{\tilde{x}}((k+1)T^{o}) = \underline{\theta}_1\ \underline{x}(kT^{o}) + \underline{\varphi}_3\ T \tag{A-45}$$

Substituting equation A-45 into A-44 and rearranging terms yields

$$\underline{x}(kT^{o}) = \underline{x}(kT^-) + \underline{\theta}_1^{-1}\ \underline{\varphi}_2\ u_{k-1} \tag{A-46}$$

A similar relationship may be written for $\underline{x}((k+1)T^{o})$ and $\underline{x}((k+1)T^{-})$

$$\underline{x}((k+1)T^{o}) = \underline{x}((k+1)T^{-}) + \underline{\theta}_1^{-1} \underline{\varphi}_2 u_k \tag{A-47}$$

Substituting equations A-46 and A-47 into equation A-43

$$\begin{aligned}\underline{x}((k+1)T^{o}) - \underline{\theta}_1^{-1} \underline{\varphi}_2 u_k = \underline{\theta}_1 \left[\underline{x}(kT^{o}) - \underline{\theta}_1^{-1} \underline{\varphi}_2 u_{k-1}\right] \\ + \underline{\varphi}_1 u_k + \underline{\varphi}_2 u_{k-1} + \underline{\varphi}_3 T\end{aligned} \tag{A-48}$$

Rearranging equation A-48 and making the definition

$$\underline{\varphi}_e = \underline{\varphi}_1 + \underline{\theta}_1^{-1} \underline{\varphi}_2 \tag{A-49}$$

yields

$$\underline{x}((k+1)T^{o}) = \underline{\theta}_1 \underline{x}(kT^{o}) + \underline{\varphi}_e u_k + \underline{\varphi}_3 T \tag{A-50}$$

Equation A-50 exhibits the desired relationship.

APPENDIX B

DERIVATION OF GENERAL INTERPOLATION WORKING EQUATIONS

Gorman and Zaborszky (reference 1) have demonstrated the usefulness of interpolation as a particularly simple way of selecting a continuous functional, $\tilde{x}[u, \underline{\eta}](t)$, that coincides with the system functional at measured data points when no information is available regarding the dynamic relations of the plant. The data points take the form of a set of measured values for the system input u_m, initial conditions $\underline{\eta}_m$, and the system output x_m, m=1, 2, ...M. This set of measured items may include a variety of quantities on an optional basis. A sample list using m for the arbitrary past decision time could consist of:

(a) u_m the control force during $mT \leq t < (m+1)T$

(b) u_{m-1} the control force during $(m-1)T \leq t < mT$

(c) k=n-m the time distance from the present. Since the time interval is fixed, the time variation of the plant can be described by one parameter for each time interval.

(d) $\underline{\beta}_{mi^o} = \beta^{(i-1)}(mT^o)$ The initial conditions, i.e. the initial plant state.

θ_{mj} Any arbitrary conditions existing at t=mT which are known to uniquely influence the performance. For instance, dynamic pressure, Mach number, altitudes, may be used in this manner.

Ostfeld (reference 2) has considered in more detail the particular type of interpolation procedure which applies to the type of control function encountered in this report. Because the control inputs are constant over decision intervals T seconds in length, it is convenient to have the set of measured values take the form of the initial conditions $\underline{\eta}_m$ at the beginning of M decision intervals, the control forces, u_m (constants), during the intervals and the outputs x_m at the end of the intervals.

The method of solution for the approximating functional takes the form of solving the determinant equation:

$$\text{Det}\begin{bmatrix} \tilde{x}(u,\underline{\eta}) & x_1(u_1,\underline{\eta}_1) & \cdot & \cdot & \cdot & x_M(u_M,\underline{\eta}_M) \\ \underline{\phi}(u,\underline{\eta}) & \underline{\phi}_1(u_1,\underline{\eta}_1) & \cdot & \cdot & \cdot & \underline{\phi}_M(u_M,\underline{\eta}_M) \end{bmatrix} = 0 \qquad \text{(B-1)}$$

where x_1, x_2, ..., x_M are the measured outputs of the plant, $\underline{\phi}(u,\underline{\eta})$ is a vector of M linearly independent analytic base functionals $[u, \underline{\eta}]$, and $\underline{\phi}_1$, $\underline{\phi}_2$, ..., $\underline{\phi}_M$ are this vector (basis vector) evaluated at the measured data points, $u_1\underline{\eta}_1$, $u_2\underline{\eta}_2$, ..., $u_M\underline{\eta}_M$.

Equation B-1 may be expanded in terms of minors of the first column, yielding the following solution for $\tilde{x}(u,\underline{\eta})$

$$\tilde{x}(u,\underline{\eta}) = {}_D\underline{x}' \; \underline{\Phi}^{-1} \; \underline{\phi}(u,\underline{\eta}) \qquad \text{(B-2)}$$

where ${}_D\underline{x}$ is a vector consisting of the measured system output states:

$${}_D\underline{x}' = \underline{\; x_1 \quad x_2 \quad \cdot \quad \cdot \quad \cdot \quad x_M \;} \qquad \text{(B-3)}$$

and $\underline{\Phi}$ is an M x M matrix with elements $\underline{\phi}_i(u_i,\underline{\eta}_i)$ and will be referred to as the matrix of basis vectors: i.e.

$$\underline{\Phi} = \left[\underline{\phi}_1(u_1,\underline{\eta}_1) \quad \underline{\phi}_2(u_2,\underline{\eta}_2) \quad \cdot \quad \cdot \quad \cdot \quad \underline{\phi}_M(u_M,\underline{\eta}_M)\right] \qquad \text{(B-4)}$$

Due to the discrete nature of the interpolation equation it is convenient and simple to write equation B-2 in the following form:

$$\tilde{x}((n+1)T) = {}_D\underline{x}' \ \underline{\Phi}^{-1} \ \underline{\phi}(u_n, \underline{\eta}_n) \tag{B-5}$$

As noted before, $\tilde{x}$ is the approximated value of the output at (n+1)T due to initial conditions $\underline{\eta}_n$ at t=nT and a control force u_n applied over the decision interval $nT \leq t < (n+1)T$.

Equation B-5 may be generalized to estimate the state $\underline{x}((n+1)T)$ rather than just the output. This is accomplished by simply noting that the $x_m(u_m, \underline{\eta}_m)$ measured values in equation B-1 may be replaced by $\overset{i}{x}_m(u_m, \underline{\eta}_m)$ which would be a corresponding set of measured values for the ith derivative of the output. The dimension of the state vector $\underline{x}$ will correspond to either the assumed order of the system or the number of output state variables (derivatives) which can be measured. In practice this order will probably be determined by the number of state variables which may be established by measurement and/or estimation techniques. The interpolated approximation of the total state may now be written as:

$$\underline{\tilde{x}}((n+1)T) = {}_D\underline{X} \ \underline{\Phi}^{-1} \ \underline{\phi}(u_n, \underline{\eta}_n) \tag{B-6}$$

where ${}_D\underline{X}$ is a rectangular matrix of the form:

$${}_D\underline{X} = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_M \\ \dot{x}_1 & \dot{x}_2 & \cdot & \cdot & \cdot & \dot{x}_M \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overset{(p)}{x}_1 & \overset{(p)}{x}_2 & \cdot & \cdot & \cdot & \overset{(p)}{x}_M \end{bmatrix} \tag{B-7}$$

where p is the assumed system order.

REFERENCES

1. Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpolation and Lagrange Expansion for Functionals," Proceedings of the JACC, 1965, pp. 169-177.

2. Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm Through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Missouri, 1965.

APPENDIX C

SOME STABILITY CONSIDERATIONS

One of the basic features of the control algorithm is that the plant is assumed to be unknown and no attempt of identification is made in the sense of plant parameters. Under such an assumption no rigorous conclusions concerning stability during a control operation can be drawn. However, the control process can monitor whether an appropriate norm of the distance between the actual state, $\underline{x}(t)$, and the desired state, $\underline{r}(t)$, as measured in the n-dimensional manifold, is decreasing at least on the average. If it is observed that this is so, the control algorithm is operating in a stable manner.

The change in the error norm may be written

$$E_k = \left\| \underline{e}((k+1)T) \right\|^2 \underline{H} - \left\| \underline{e}(kT) \right\|^2 \underline{H} \tag{C-1}$$

where $\underline{e}(kT)$ is defined elsewhere in the text as

$$\underline{e}(kT) = \underline{r}(kT) - \underline{x}(kT) \tag{C-2}$$

and the positive definite matrix $\underline{H}$ defines the norm; i.e. if $\underline{H} = \underline{I}$ equation C-1 defines the Euclidean norm. Equation C-1 may be evaluated during individual runs where the concept of stability employed must necessarily depend upon the trajectory of the desired state.

In order to render the concept of stability more tractable to definitive conclusions, assume for the moment that $\underline{r}(t)$ is the null vector $\underline{0}$ for all t contained in the time interval (t_a, t_b) during which control of the plant is desired. In this case, the change in the error norm is expressible as

$$E_k = \left\|\underline{x}((k+1)T)\right\|^2_{\underline{H}} - \left\|\underline{x}(kT)\right\|^2_{\underline{H}} \tag{C-3}$$

Proving that E_k is negative for all $\underline{x}(kT)$ suffices to assure global asymptotic stability by Liapunov's second method and Krasovskii's theorem (reference 1) when the order of the state variable $\underline{x}(t)$ is equal to the actual order of the plant.

For study purposes, it is useful to evaluate equation C-3 assuming specific known plants. Such a study may be definitive as to the classes of plants for which stable operation of the control algorithm is possible.

If the plant is assumed to be linear, then equation C-3 may be written in the form

$$\begin{aligned} E_k = & \left[\underline{x}'(kT^o)\ \underline{F}'((k+1)T,\ kT) + u_k\ \underline{\lambda}'((k+1)T,\ kT)\right]\ \underline{H} \\ & \left[\underline{F}((k+1)T,\ kT)\ \underline{x}(kT^o) + \underline{\lambda}((k+1)T,\ kT)\ u_k\right] \\ & - \underline{x}'(kT^o)\ \underline{H}\ \underline{x}(kT^o) \end{aligned} \tag{C-4}$$

or when u_k is evaluated in terms of the control policy

$$\begin{aligned} E_k = & \left[\underline{x}'(kT^o)\ \underline{W}'((k+1)T,\ kT)\right]\ \underline{H}\ \left[\underline{W}((k+1)T,\ kT)\ \underline{x}(kT^o)\right] \\ & - \underline{x}'(kT^o)\ \underline{H}\ \underline{x}(kT^o) \end{aligned} \tag{C-5}$$

where

$$\underline{W}((k+1)T,\ kT) = \left[\underline{I} - \frac{\underline{\lambda}((k+1)T,\ kT)\ \underline{\lambda}'((k+1)T,\ kT)\ \underline{K}}{\underline{\lambda}'((k+1)T,\ kT)\ \underline{K}\ \underline{\lambda}((k+1)T,\ kT)} \right] \underline{F}((k+1)T,\ kT) \tag{C-6}$$

The form of equation C-5 is

$$E_k = - \underline{x}'(kT^o)\ \underline{M}((k+1)T,\ kT)\ \underline{x}(kT^o) \tag{C-7}$$

where if $\underline{M}$ can be shown to be positive definite for all kT, global asymptotic stability is assured.

In general, it is difficult if not impossible to obtain explicit expressions for $\underline{F}$ and $\underline{\lambda}$ if the plant is time varying. The linear stationary case for which expressions for $\underline{F}$ and $\underline{\lambda}$ are possible is discussed in detail in Section 2.1 of the text.

If the plant is nonlinear, the control policy equation is a nonlinear function of u_k for which no closed form solution is practical. For this reason, it is difficult to reach general conclusions as to the stability of the control operation using the nonlinear control algorithm.

REFERENCES

1. Kalman, R. E. and Bertram, J. E., "Control System Analysis and Design Via the Second Method of Lyapunov," ASME Trans. Ser. D, June, 1960, pp. 371-400.

APPENDIX D

REPRESENTATIVE SET OF LINEAR STATIONARY PLANTS

The set of plant transfer functions documented in this appendix is a subset of the total set of plant transfer functions considered for experimental study. The set listed here contains those used in the more complete simulation studies. The letter to the right of each transfer function refers to a brief discussion of the transfer function in following pages of this appendix.

D.1 SECOND ORDER TRANSFER FUNCTIONS

(1) $\dfrac{1}{\left[(s+0.5)^2+1\right]}$ (A)

(2) $\dfrac{1}{s^2+1}$ (A)

(3) $\dfrac{1}{\left[(s-0.5)^2+1\right]}$ (A)

(4) $\dfrac{1}{\left[(s-1)^2+1\right]}$ (A)

(5) $\dfrac{(s+1)}{\left[(s+0.5)^2+1\right]}$ (B)

(6) $\dfrac{(s+1)}{(s^2+1)}$ (B)

D.2 THIRD ORDER TRANSFER FUNCTIONS

(1) $\dfrac{1}{s\left[(s+0.5)^2+1\right]}$ (C)

(2) $\dfrac{1}{s(s^2+1)}$ (C)

(3) $\dfrac{1}{(s+1)\left[(s+0.1)^2+1\right]}$ (C)

(4) $\dfrac{1}{(s+0.2)\left[(s+0.5)^2+1\right]}$ (C)

(5) $\dfrac{(s+0.5)}{s\left[(s+0.5)^2+1\right]}$ (D)

(6) $\dfrac{(s+1)}{s(s^2+1)}$ (D)

(7) $\dfrac{(s+3)}{(s+1)\left[(s+0.2)^2+10^2\right]}$ (D)

D.3 FOURTH ORDER TRANSFER FUNCTIONS

(1) $\dfrac{1}{\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$ (E)

(2) $\dfrac{1}{\left[(s+0.5)^2+1\right]\left[(s+6)^2+5^2\right]}$ (E)

(3) $\dfrac{1}{\left[(s+0.5)^2+1\right]\left[(s+1)^2+5^2\right]}$ (E)

(4) $\dfrac{1}{(s+1)(s+5)\left[(s+0.5)^2+1\right]}$ (E)

(5) $\dfrac{1}{s(s+1)\left[(s+0.5)^2+1\right]}$ (E)

(6) $\dfrac{1}{s(s+5)\left[(s+0.5)^2+1\right]}$ (E)

(7) $\dfrac{1}{s(s+10)\left[(s+0.5)^2+1\right]}$ (E)

(8) $\dfrac{(s+2.8)}{s(s+11)\left[(s+1.25)^2+1\right]}$ (F)

(9) $\dfrac{(s+0.3)}{s(s+9)\left[(s+0.4)^2+1\right]}$ (G)

(10) $$\frac{(s+0.08)}{s(s+4.5)\left[(s+0.08)^2+1\right]}$$ (H)

(11) $$\frac{(s+0.5)(s+0.1)}{\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (I)

(12) $$\frac{(s+0.5)}{\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (I)

D.4 FIFTH ORDER TRANSFER FUNCTIONS

(1) $$\frac{1}{s\left[(s+0.5)^2+1\right]\left[(s+6)^2+5^2\right]}$$ (J)

(2) $$\frac{1}{s\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (J)

(3) $$\frac{1}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]}$$ (J)

(4) $$\frac{1}{s\left[(s+0.5)^2+1\right]\left[(s+2)^2+3^2\right]}$$ (J)

(5) $$\frac{1}{s(s+1)(s+5)\left[(s+0.5)^2+1\right]}$$ (J)

(6) $$\frac{1}{s(s+1)(s+10)\left[(s+0.5)^2+1\right]}$$ (J)

(7) $$\frac{1}{s(s+5)(s+5.1)\left[(s+0.1)^2+1\right]}$$ (J)

(8) $$\frac{1}{(s+10)\left[(s+0.2)^2+1\right]\left[(s+0.1)^2+10^2\right]}$$ (J)

(9) $$\frac{1}{(s+10)\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (J)

(10) $$\frac{1}{(s+5)\left[(s+0.5)^2+1\right]\left[(s+2)^2+3^2\right]}$$ (J)

(11) $$\frac{\left\{(s+3)^2+2^2\right\}}{s\left[(s+0.5)^2+1\right]\left[(s+6)^2+5^2\right]}$$ (K)

(12) $$\frac{(s+1)(s+3)}{s\left[(s+0.5)^2+1\right]\left[(s+6)^2+5^2\right]}$$ (K)

(13) $$\frac{(s+1)(s+3)}{s\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (K)

(14) $$\frac{\left\{(s+3)^2+2^2\right\}}{s\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (K)

(15) $$\frac{s+0.5}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]}$$ (K)

(16) $$\frac{\left\{(s+0.5)^2+3^2\right\}}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]}$$ (K)

(17) $$\frac{s+0.15}{(s+10)\left[(s+0.2)^2+1\right]\left[(s+0.1)^2+10^2\right]}$$ (K)

(18) $$\frac{\left\{(s+0.15)^2+0.5^2\right\}}{(s+10)\left[(s+0.2)^2+1\right]\left[(s+0.1)^2+10^2\right]}$$ (K)

(19) $$\frac{\left\{(s+1)^2+8^2\right\}}{(s+10)\left[(s+1)^2+1\right]\left[(s+0.1)^2+10^2\right]}$$ (K)

D.5 SIXTH ORDER TRANSFER FUNCTIONS

(1) $$\frac{1}{s(s+10)\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (L)

(2) $$\frac{1}{s(s+10)\left[(s+1)^2+1\right]\left[(s+0.1)^2+10^2\right]}$$ (L)

(3) $$\frac{1}{s(s+10)\left[(s+1)^2+1\right]\left[(s+2)^2+3^2\right]}$$ (L)

(4) $$\frac{1}{(s+10)(s+10.5)\left[(s+0.1)^2+5^2\right]\left[(s+5)^2+1\right]}$$ (L)

(5) $$\frac{1}{(s+5)(s+10)\left[(s+1)^2+1\right]\left[(s+0.5)^2+3^2\right]}$$ (L)

(6) $$\frac{1}{(s+10)(s+20)\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (L)

(7) $$\frac{1}{\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (L)

(8) $$\frac{1}{\left[(s+1)^2+1\right]\left[(s+0.5)^2+6^2\right]\left[(s+3)^2+4^2\right]}$$ (L)

(9) $$\frac{(s+0.02)(s+0.36)}{s(s+12)\left[(s+0.5)^2+16^2\right]\left[(s+0.37)^2+1\right]}$$ (M)

(10) $$\frac{(s+1)(s+3)}{s(s+10)\left[(s+2)^2+4^2\right]\left[(s+0.2)^2+1\right]}$$ (N)

(11) $$\frac{(s+0.02)(s+0.36)}{s(s+12)\left[(s+0.5)^2+4^2\right]\left[(s+0.37)^2+1\right]}$$ (N)

(12) $$\frac{\left\{(s+0.3)^2+0.5^2\right\}}{s(s+10)\left[(s+0.5)^2+1\right]\left[(s+0.5)^2+3^2\right]}$$ (N)

(13) $$\frac{(s+6)\left\{(s+0.3)^2+0.5^2\right\}}{s(s+10)\left[(s+0.5)^2+1\right]\left[(s+0.2)^2+3^2\right]}$$ (N)

(14) $$\frac{(s+5)\left\{(s+1)^2+2^2\right\}}{(s+10)(s+10.5)\left[(s+0.1)^2+5^2\right]\left[(s+5)^2+1\right]}$$ (N)

(15) $$\frac{(s+6)\left\{(s+1)^2+2^2\right\}}{(s+3)(s+10)\left[(s+0.1)^2+5^2\right]\left[(s+5)^2+1\right]}$$ (N)

(16) $$\frac{(s+2)(s+0.75)}{\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (N)

D.6 SEVENTH ORDER TRANSFER FUNCTIONS

(1) $$\frac{1}{s(s+4)(s+10)\left[(s+0.2)^2+1\right]\left[(s+4)^2+5^2\right]}$$ (P)

(2) $$\frac{1}{s(s+4)(s+10)\left[(s+1)^2+1\right]\left[(s+2)^2+3^2\right]}$$ (P)

(3) $$\frac{1}{s(s+10)(s+10.5)\left[(s+0.1)^2+5^2\right]\left[(s+5)^2+1\right]}$$ (P)

(4) $$\frac{1}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (P)

(5) $$\frac{1}{(s+10)\left[(s+6)^2+1\right]\left[(s+0.5)^2+2^2\right]\left[(s+0.1)^2+10^2\right]}$$ (P)

(6) $$\frac{1}{(s+10)\left[(s+6)^2+1\right]\left[(s+0.5)^2+10^2\right]\left[(s+0.1)^2+2^2\right]}$$ (P)

(7) $$\frac{(s+5)(s+8)\left\{(s+0.4)^2+0.8^2\right\}}{s(s+4)(s+10)\left[(s+0.2)^2+1\right]\left[(s+4)^2+5^2\right]}$$ (R)

(8) $$\frac{\left\{(s+1)^2+4^2\right\}\left\{(s+1)^2+8^2\right\}}{(s+3)(s+6)(s+10)\left[(s+1)^2+2^2\right]\left[(s+3)^2+6^2\right]}$$ (R)

(9) $$\frac{(s+0.75)(s+2)}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (R)

(10) $$\frac{(s+0.75)\left\{(s+0.1)^2+2^2\right\}}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (R)

(11) $$\frac{\left\{(s+0.1)^2+2^2\right\}}{s\left[(s+1)^2+1\right]\left[(s+0.2)^2+10^2\right]\left[(s+0.5)^2+6^2\right]}$$ (R)

(12) $$\frac{\left\{(s+1)^2+1\right\}}{(s+10)\left[(s+6)^2+1\right]\left[(s+0.5)^2+10^2\right]\left[(s+0.1)^2+2^2\right]}$$ (R)

D.7 EIGHTH ORDER TRANSFER FUNCTIONS

(1) $$\frac{1}{s(s+10)\left[(s+6)^2+1\right]\left[(s+0.5)^2+2^2\right]\left[(s+0.1)^2+10^2\right]}$$ (P)

(2) $$\frac{1}{s(s+10)\left[(s+6)^2+1\right]\left[(s+0.5)^2+10^2\right]\left[(s+0.1)^2+2^2\right]}$$ (P)

(3) $$\frac{1}{s(s+10)\left[(s+1)^2+10^2\right]\left[(s+6)^2+1\right]\left[(s+2)^2+3^2\right]}$$ (P)

(4) $$\frac{1}{\left[(s+1)^2+1\right]\left[(s+5)^2+2^2\right]\left[(s+0.5)^2+5^2\right]\left[(s+0.1)^2+10^2\right]}$$ (P)

D.8 NINTH ORDER TRANSFER FUNCTIONS

(1) $$\frac{1}{s\left[(s+1)^2+1\right]\left[(s+0.1)^2+10^2\right]\left[(s+0.5)^2+5^2\right]\left[(s+5)^2+2^2\right]}$$ (P)

(2) $$\frac{1}{(s+4)\left[(s+1)^2+1\right]\left[(s+0.1)^2+10^2\right]\left[(s+0.5)^2+5^2\right]\left[(s+5)^2+2^2\right]}$$ (P)

(3) $$\frac{1}{(s+10)\left[(s+1)^2+1\right]\left[(s+0.1)^2+10^2\right]\left[(s+0.5)^2+5^2\right]\left[(s+5)^2+2^2\right]}$$ (P)

D.9 BRIEF DISCUSSION OF TRANSFER FUNCTIONS

(A) These second order transfer functions are typical of those considered in Emerson's previous work, (reference 1). They are included to provide continuity in the research effort, and to study the effect of new types of prediction as compared with some of the previously studied types. The set of four plants range from a reasonable well damped oscillatory pole pair (1) to no damping (2) and finally to a pair of unstable plants (3,4).

(B) These second order transfer functions possess denominators which are typical of those considered in Emerson's previous work (reference 1) with no zeroes. They are included to study the effect of adding zeroes to a low order transfer function for a relatively uncomplicated starting point.

(C) These third order transfer functions are typical of those considered in Emerson's previous work (reference 1). They are included to provide continuity in the research effort, and to study the effect of new types of prediction as compared with some of the previous types used.

(D) These third order transfer functions possess denominators which are typical of those considered in Emerson's previous work (reference 1) with no zeroes. They are included to study the effect of adding zeroes to transfer functions only slightly more involved than second order transfer functions.

(E) These fourth order transfer functions are of three basic types with each possessing the same predominant pole pair ($-0.5\pm j1$). Transfer functions (1), (2), and (3) contain a second complex pole pair in various parts of the complex plane in order to study the effect of different positions of the second pole pair. Transfer function (4) contains two real poles neither of which is at the origin. Transfer functions (5), (6), and (7) are included to study the effect of moving a real pole progressively further out on the real axis.

(F) This transfer function is a typical short term approximation to the pitch angle control system for a conventional transport flying at 150 mph at sea level, (reference 2).

(G) This fourth order transfer function is a typical short term approximation to the pitch angle control system for a jet transport flying at 600 ft/sec at 40,000 ft., (reference 2).

(H) This fourth order transfer function is representative of the pitch angle control system for the X-15 aircraft at Mach 6 at 60,000 ft., (reference 3).

(I) These fourth order transfer functions are included to study the effect of adding zeroes to transfer functions considered in previous Emerson work (reference 1) which were observed to be among the most difficult to control.

(J) These fifth order transfer functions are essentially of three types. The first set (1), (2), (3), and (4) consist of those having a pole at the origin, a relatively predominant pole pair, and a second pole pair in various parts of the plane. The second set (5), (6), and (7) consists of those having a pole at the origin, a predominant pole pair, and two real roots in various configurations. The third set (8), (9), and (10) consist of two pole pairs with one being relative predominant to different degrees, and a pole on the real axis.

(K) These fifth order transfer functions with zeroes are included to study the effect of adding zeroes in various configurations to pole configurations similar to those discussed in (J). They represent no known physical plant but do cover a wide spectrum of possible configurations.

(L) These sixth order transfer functions are essentially of three types. The first set (1), (2), and (3) consist of a pole at the origin, a real root, and two complex pole pairs with one being relatively predominant to varying degrees. The second set (4), (5), and (6) is similar to the first except there are two poles on the real axis and the effect of various locations of these two poles may be studied. The third set (7), and (8) consist of three complex pole pairs with one being slightly more well behaved than the other.

(M) This sixth order transfer function is a typical long term approximation to the pitch angle control system for a jet transport flying at 600 ft/sec. at 40,000 ft., (reference 2).

(N) These sixth order transfer functions with zeroes are included to study the effect of adding zeroes in various configurations to pole configurations similar to those discussed in (L). They represent no known physical plants, but do cover a wide spectrum of possible configurations.

(P) These seventh, eighth and ninth order transfer functions are not representative of any known physical plants. They do cover a wide spectrum of possible pole configurations.

(R) These seventh order transfer functions with zeroes are included to study the effect of adding zeroes in various configurations to similar 'poles-only' seventh order transfer functions.

REFERENCES

1. "Digital Adaptive Control System Research, Summary Report," Emerson Electric Report No. 1544-5, March 31, 1964.

2. Blakelock, J. H., Automatic Control of Aircraft and Missiles, John Wiley & Sons, Inc., 1965, Chapter 2.

3. Serex, H. M. and Holt, J. N., "Comparison of High-Gain-Linear and Self Adaptive Flight Control Systems with Nonlinearities Introduced," Master of Science Thesis, Air Force Institute of Technology, Thesis GGC/EE/64-11.

APPENDIX E

THE ABILITY OF THE CONTROL POLICY TO FOLLOW DESIRED TRAJECTORIES USING THE DIFFERENT TYPES OF PREDICTION

The purpose of this appendix is to collect a set of analytical studies necessary to correlate some experimental observations with what would be predicted from an analytical viewpoint. The studies here are after the fact' in the sense that certain trends were observed in the experimental data, and analytical studies were made to explain these trends. The studies are not intended to exhaust the entire area as time permitted only a limited 'scratching of the surface .

E.1 SINGULARITY OF A CERTAIN MATRIX

The analytical studies presented in the sequel make use of the singular nature of a particular matrix form. A general proof of the singularity of the matrix form is given here.

The particular form in question is

$$\underline{\Gamma} = \frac{\underline{\alpha}\ \underline{\alpha}'\ \underline{D}}{\underline{\alpha}'\ \underline{D}\ \underline{\alpha}} - \underline{I} \tag{E-1}$$

where $\underline{\alpha}$ is a column matrix, $\underline{D}$ is a nonsingular diagonal matrix, $\underline{I}$ is the the identity matrix, and $\underline{\alpha}$, $\underline{D}$ and $\underline{I}$ are of conformal order.

Since $\underline{\alpha}$ is a column matrix and therefore has a rank of one, the square symmetric matrix, $\underline{\alpha}\ \underline{\alpha}'$, also has a rank of one. The square matrix resulting from the product of $\underline{\alpha}\ \underline{\alpha}'$ with the nonsingular matrix $\underline{D}$ also has a rank of one. It will be symmetric only if $\underline{D}$ is of diagonal form.

$$\underline{\alpha}\ \underline{\alpha}'\ \underline{D} \rightarrow \text{rank of one} \tag{E-2}$$

It is easily shown that

$$\underline{\alpha}' \, \underline{D} \, \underline{\alpha} \; = \; \text{trace} \; (\, \underline{\alpha} \, \underline{\alpha}' \, \underline{D} \,) \tag{E-3}$$

Since $\underline{\alpha}\,\underline{\alpha}'\,\underline{D}$ has a rank of one, it possesses only one non-zero eigenvalue so that by equation E-3, the scalar quantatity $\underline{\alpha}'\,\underline{D}\,\underline{\alpha}$ is equal to that eigenvalue, since the trace of a matrix is equal to the sum of the eigenvalues of the matrix.

Let $\underline{T}$ be the matrix which transforms $\underline{\alpha}\,\underline{\alpha}'\underline{D}$ into its diagonal form (or Jordan canonical form).

$$\underline{T}^{-1} (\, \underline{\alpha} \, \underline{\alpha}' \, \underline{D} \,) \underline{T} = \underline{\Lambda} \tag{E-4}$$

Because $\underline{\Lambda}$ has only one non-zero eigenvalue, λ_1, and the trace of a matrix is invarient under a similarity transformation:

$$\underline{\alpha}' \, \underline{D} \, \underline{\alpha} \; = \lambda_1 \tag{E-5}$$

Applying the same similarity transformation to $\underline{\Gamma}$

$$\underline{\Gamma}_d \; = \; \underline{T}^{-1} \left(\frac{\underline{\alpha} \, \underline{\alpha}' \, \underline{D}}{\underline{\alpha}' \, \underline{D} \, \underline{\alpha}} - \underline{I} \right) \underline{T} \; = \; \frac{1}{\lambda_1} \underline{\Lambda} - \underline{I} \tag{E-6}$$

Thus, $\underline{\Gamma}$ is similar to $\underline{\Gamma}_d$

$$\underline{\Gamma}_d \; = \; \frac{1}{\lambda_1} \underline{\Lambda} - \underline{I} = \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} \tag{E-7}$$

where $\underline{\Gamma}_d$ is obviously singular with a rank equal to one less than its order. Since similar matrices have the same set of eigenvalues, $\underline{\Gamma}$ is also singular with a rank of one less than its order. Q.E.D.

E.2 THE POLES ONLY CASE

When a plant has a transfer function containing only poles, the state equation may be written in the form

$$\underline{x}((k+1)T) = \underline{F}\,\underline{x}(kT) + \underline{a}\,u_k \tag{E-8}$$

where the control force, u_k, is given by the control policy equation:

$$u_k = \frac{\underline{a}_a{}'\,\underline{K}\left[\underline{r}((k+1)T) - \underline{F}_a\,\underline{x}(kT)\right]}{\underline{a}_a{}'\,\underline{K}\,\underline{a}_a} \tag{E-9}$$

where $\underline{a}_a$ and $\underline{F}_a$ are the approximations to $\underline{a}$ and $\underline{F}$ according to what type of prediction is used. (i.e. Exact, Taylor, etc.)

Make the definition

$$\underline{\Gamma}_a = \frac{\underline{a}\,\underline{a}_a{}'\,\underline{K}}{\underline{a}_a{}'\,\underline{K}\,\underline{a}_a} \tag{E-10}$$

Substitution of equations E-10 and E-9 into E-8 yields

$$\underline{x}((k+1)T) = \left[\underline{F} - \underline{\Gamma}_a\,\underline{F}_a\right]\underline{x}(kT) + \underline{\Gamma}_a\,\underline{r}((k+1)T) \tag{E-11}$$

<u>Following A Step</u> - Assume that for $t \geq t_o$, where t and t_o are contained in the open interval (t_a, t_b) during which control of the plant is desired, the desired output is given by:

$$\underline{r}'(t) = \underline{r}'_{ss} = \underline{\begin{matrix} R & 0 & \cdots & 0 \end{matrix}} \tag{E-12}$$

where R is the magnitude of the desired position.

Assume that the actual steady state output is some constant times the desired output (the assumption is made that the control policy is 'operating' at a stable T - h point).

$$\underline{r}_{ss} = \underline{E}\,\underline{x}_{ss} \tag{E-13}$$

where

$$\underline{E} = \begin{bmatrix} b & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

$$\underline{x}_{ss} = \underline{x}((k+1)T) = \underline{x}(kT) \qquad k \geq q \tag{E-14}$$

and q is sufficiently large so that initial transients in the output may be assumed to have died out.

Substituting equations E-13 and E-14 into equation E-11 and collecting terms yields

$$\underline{x}_{ss} = [\underline{F} - \underline{\Gamma}_a\,\underline{F}_a + \underline{\Gamma}_a\,\underline{E}]\,\underline{x}_{ss} \tag{E-15}$$

Equation E-15 may be recognized as a special case of the more general eigenvalue problem

$$\underline{A}\,\underline{z} = \lambda\,\underline{z} \tag{E-16}$$

where in this case

$$\underline{A} = \underline{F} - \underline{\Gamma}_a \underline{F}_a + \underline{\Gamma}_a \underline{E}$$
$$\underline{z} = \underline{x}_{ss} \qquad \text{(E-17)}$$

For equation E-15 to be valid, $\underline{A}$ as defined by E-17 must possess an eigenvalue of unity for some value of b in which case $\underline{x}_{ss}$ will exist and will be the eigenvector of $\underline{A}$ corresponding to the unity eigenvalue.

The desired value of b is unity, as the actual output will then be equal to the desired output and no steady state error will exist. If $b \neq 1$ then some steady state error will exist corresponding to the value of b for which $\underline{A}$ possesses a unity eigenvalue.

Exact Prediction - To establish a reference standard against which to compare other types of prediction, the exact values of $\underline{F}$ and $\underline{a}$ will be assumed to be known in which case:

$$\underline{F}_a = \underline{F}$$
$$\underline{a}_a = \underline{a} \qquad \text{(E-18)}$$
$$\underline{\Gamma}_a = \underline{\Gamma}_e = \frac{\underline{a}\ \underline{a}'\ \underline{K}}{\underline{a}'\ \underline{K}\ \underline{a}}$$

Equation E-15 may then be written in the form

$$\underline{x}_{ss} = \left[\underline{F} - \underline{\Gamma}_e \underline{F} + \underline{\Gamma}_e \underline{E}\right] \underline{x}_{ss} \qquad \text{(E-19)}$$

If $\underline{x}_{ss}$ exists as given by equation E-19, then for some value of b

$$\text{Det}\left[\underline{F} - \underline{\Gamma}_e \underline{F} + \underline{\Gamma}_e \underline{E} - \underline{I}\right] = 0 \tag{E-20}$$

If the actual and desired output states are to be identical, then $\underline{E} = \underline{I}$ (b = 1) and it must be possible to write the statement:

$$\text{Det}\left[\underline{F} - \underline{\Gamma}_e \underline{F} + \underline{\Gamma}_e - \underline{I}\right] = 0 \tag{E-21}$$

Factoring E-21

$$\text{Det}\left[(\underline{\Gamma}_e - \underline{I})\cdot(\underline{I} - \underline{F})\right] = 0 \tag{E-22}$$

Equation E-22 is a valid statement if either $\underline{\Gamma}_e-\underline{I}$ or $\underline{I}-\underline{F}$ is singular. $\underline{I}-\underline{F}$ will be singular only if the plant transfer function contains at least one pole at the origin, as in that event the first column of $\underline{F}$ is identical to that of $\underline{I}$. This, of course, depends on the specific nature of the plant and in general cannot be assumed. The matrix $\underline{\Gamma}_e-\underline{I}$ has been shown to be singular (Section E.1) regardless of the specific form of $\underline{a}$ with a rank one less than the order of $\underline{a}$ so that E-22, hence E-21, are valid statements and the actual and desired steady state outputs will be identical.

Conclusion - Assuming that the control policy is operated at a stable T-h point and Exact Prediction is used, no steady state output error will exist for a desired state of a constant output position regardless of the plant configuration.

Taylor Prediction - When the control policy employs Taylor Prediction, the estimates of $\underline{F}$ and $\underline{a}$ will be $\underline{F}_T$ and $\underline{a}_T$, respectively, as defined in Section 2.1 and $\underline{\Gamma}_a=\underline{\Gamma}_T$ as defined by equation E-23.

$$\underline{\Gamma}_T = \frac{\underline{a}\,\underline{a}_T'\,\underline{K}}{\underline{a}_T'\,\underline{K}\,\underline{a}_T} \tag{E-23}$$

Equation E-15 may then be written in the form

$$\underline{x}_{ss} = \left[\underline{F} - \underline{\Gamma}_T \underline{F}_T + \underline{\Gamma}_T \underline{E} \right] \underline{x}_{ss} \tag{E-24}$$

For the actual and desired output states to be identical it must be shown that $\underline{E}=\underline{I}$ and:

$$\text{Det} \left[\underline{F} - \underline{\Gamma}_T \underline{F}_T + \underline{\Gamma}_T - \underline{I} \right] = 0 \tag{E-25}$$

Factoring and rearranging E-25 yields

$$\text{Det} \left[\underline{F} - \underline{I} + \underline{\Gamma}_T (\underline{I} - \underline{F}_T) \right] = 0 \tag{E-26}$$

The first column of the product $\underline{\Gamma}_T$ $(\underline{I}-\underline{F}_T)$ will be a column of zeroes because the first columns of $\underline{I}$ and $\underline{F}_T$ are identical. If $\underline{F}$ can also be assumed to possess a first column identical to that if $\underline{I}$ then the singularity of the matrix of the determinant E-26 is assured as the first column of the matrix will consist of a column of zeroes. This will be true only if the plant transfer function has at least one pole at the origin.

<u>Conclusion</u> - Assuming that the control policy is operated at a stable T-h point and Taylor Prediction is used, no steady state output error will exist for a desired state of a constant output position in general only if the plant transfer function possesses at least one pole at the origin. If the plant transfer function possesses no pole at the origin, a steady error will exist corresponding to the value of b for which the matrix of equation E-24 is singular and the matrix determinant therefore zero.

<u>Interpolation Prediction</u> - No analytical study has been made assuming Interpolation Prediction. It would be expected that as $\underline{F}_I \rightarrow \underline{F}$ and $\underline{a}_I \rightarrow \underline{a}$ the results would be equivalent to those obtained using exact prediction.

Following A Ramp - Assume that for $t \geq t_o$ where t and t_o are contained in the open interval (t_a, t_b) during which control of the plant is desired, the desired output is given by:

$$\underline{r}'(kT) = \lfloor R{\cdot}(kT) \quad R \quad 0 \quad . \quad . \quad 0 \rfloor \tag{E-27}$$

where R is the magnitude of the desired output velocity.
Define a vector

$$\underline{\Delta r}_{ss} = \underline{r}'((k+1)T) - \underline{r}'(kT) = \lfloor RT \quad 0 \quad . \quad . \quad . \quad 0 \rfloor \qquad k = 1, 2, \ldots, n \tag{E-28}$$

In terms of the difference between two successive desired states the desired output vector $\underline{\Delta r}_{ss}$ takes the form of a desired rate of change and is a constant vector for a constant T.

Similarly, two successive state equations may be written

$$\begin{aligned} \underline{x}((k+1)T) &= \underline{F}\ \underline{x}(kT) + \underline{a}\ u_k \\ \underline{x}(kT) &= \underline{F}\ \underline{x}((k-1)T) + \underline{a}\ u_{k-1} \end{aligned} \tag{E-29}$$

Subtracting one of the state equations from the other yields

$$\underline{x}((k+1)T) - \underline{x}(kT) = \underline{F}\left[\underline{x}(kT) - \underline{x}((k-1)T\right] + \underline{a}\ (u_k - u_{k-1}) \tag{E-30}$$

Assume that the actual steady state output approaches a constant rate of change after the initial transients have died out

$$\underline{\Delta x}_{ss} = \underline{x}((k+1)T) - \underline{x}(kT) = \underline{x}(kT) - \underline{x}((k-1)T) \qquad k \geq q \tag{E-31}$$

where the initial transients have died out by the time t=qT.

The control policy equation will yield as solutions for u_k and u_{k-1}

$$u_k = \frac{\underline{a}_a{}' \, \underline{K} \left[\underline{r}((k+1)T) - \underline{F}_a \, \underline{x}(kT) \right]}{\underline{a}_a{}' \, \underline{K} \, \underline{a}_a}$$

$$u_{k-1} = \frac{\underline{a}_a{}' \, \underline{K} \left[\underline{r}(kT) - \underline{F}_a \, \underline{x}((k-1)T) \right]}{\underline{a}_a{}' \, \underline{K} \, \underline{a}_a} \tag{E-32}$$

Again make the definition

$$\underline{\Gamma}_a = \frac{\underline{a} \, \underline{a}_a{}' \, \underline{K}}{\underline{a}_a{}' \, \underline{K} \, \underline{a}_a} \tag{E-33}$$

Substituting equations E-28, E-31, E-32 and E-33 into equation E-30 yields

$$\underline{\Delta x}_{ss} = \underline{F} \, \underline{\Delta x}_{ss} + \underline{\Gamma}_a \, \underline{\Delta r}_{ss} - \underline{\Gamma}_a \, \underline{F}_a \, \underline{\Delta x}_{ss} \tag{E-34}$$

Assume a steady state error exists of the form

$$\underline{\Delta r}_{ss} = \underline{E} \, \underline{\Delta x}_{ss} \tag{E-35}$$

where

$$\underline{E} = \begin{bmatrix} b & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{E-36}$$

Substituting E-25 into E-34 and collecting terms yields

$$\underline{\Delta x}_{ss} = \left[\underline{F} - \underline{\Gamma}_a \underline{F}_a + \underline{\Gamma}_a \underline{E} \right] \underline{\Delta x}_{ss} \tag{E-37}$$

Equation E-37 is of the same form as equation E-15 and similar solutions will exist for values of $\underline{E}$ for which unity eigenvalues exist.

<u>Exact Prediction</u> - To establish a reference standard against which to compare other types of prediction, the exact values of $\underline{F}$ and $\underline{a}$ will be assumed to be known in which case:

$$\begin{aligned} \underline{F}_a &= \underline{F} \\ \underline{a}_a &= \underline{a} \\ \underline{\Gamma}_a &= \underline{\Gamma}_e = \frac{\underline{a}\ \underline{a}'\ \underline{K}}{\underline{a}'\ \underline{K}\ \underline{a}} \end{aligned} \tag{E-38}$$

Equation E-37 may then be written in the form

$$\underline{\Delta x}_{ss} = \left[\underline{F} - \underline{\Gamma}_e \underline{F} + \underline{\Gamma}_e \underline{E} \right] \underline{\Delta x}_{ss} \tag{E-39}$$

For the actual and desired output rates to be the same $\underline{E}=\underline{I}$ (b=1) and the determinant of equation E-40 must be zero.

$$\mathrm{Det} \left[\underline{F} - \underline{\Gamma}_e \underline{F} + \underline{\Gamma}_e - \underline{I} \right] = 0 \tag{E-40}$$

Factoring E-40

$$\mathrm{Det} \left[(\underline{\Gamma}_e - \underline{I}) \cdot (\underline{I} - \underline{F}) \right] = 0 \tag{E-41}$$

That equation E-41 is a valid statement (hence E-40 is valid) is evident regardless of the plant configuration as it has been shown (Section E.1) that $\underline{\Gamma}_e - \underline{I}$ is singular.

Conclusion - Assuming that the control policy is operated at a stable T-h point and Exact Prediction is used, no steady state output rate error will exist for a desired state of a constant rate regardless of the plant configuration. The actual state can therefore differ at most from the desired state by a constant positional error.

Taylor Prediction - When the control policy employs Taylor Prediction, the estimates of $\underline{F}$ and $\underline{a}$ will be $\underline{F}_T$ and $\underline{a}_T$ and $\underline{\Gamma}_a = \underline{\Gamma}_T$ as defined by equation E-42

$$\underline{\Gamma}_T = \frac{\underline{a}\ \underline{a}_T'\ \underline{K}}{\underline{a}_T'\ \underline{K}\ \underline{a}_T} \qquad \text{(E-42)}$$

Equation E-37 may then be written in the form:

$$\underline{\Delta x}_{ss} = \left[\underline{F} - \underline{\Gamma}_T\ \underline{F}_T + \underline{\Gamma}_T\ \underline{E} \right] \underline{\Delta x}_{ss} \qquad \text{(E-43)}$$

For the actual and desired output states to be identical it must be shown that $\underline{E} = \underline{I}$ and

$$\text{Det} \left[\underline{F} - \underline{\Gamma}_T\ \underline{F}_T + \underline{\Gamma}_T - \underline{I} \right] = 0 \qquad \text{(E-44)}$$

Factoring and rearranging E-44 yields

$$\text{Det} \left[\underline{F} - \underline{I} + \underline{\Gamma}_T\ (\underline{I} - \underline{F}_T) \right] = 0 \qquad \text{(E-45)}$$

The first column of $\underline{\Gamma}_T$ $(\underline{I} - \underline{F}_T)$ will be a column of zeroes because the first columns of $\underline{I}$ and $\underline{F}_T$ are identical. If $\underline{F}$ can also be assumed to possess a first column identical to that of $\underline{I}$, then the singularity of the matrix of the determinant E-45 is assured as the first column of the matrix will consist of a column of zeroes. This will be true only if the plant transfer function has at least one pole at the origin.

Conclusion - Assuming that the control policy is operated at a stable T-h point and Taylor Prediction is used, no steady state output rate error will exist for a desired state of a constant rate if the plant transfer function possesses at least one pole at the origin. The actual state can, therefore, differ at most from the desired state by a constant positional error. If the plant transfer function does not contain a pole at the origin, a steady state rate error will exist corresponding to the value of b of matrix $\underline{E}$ for which the matrix of equation E-43 is singular. The actual and desired output positions will then diverge at a constant rate depending on the value of b.

E.3 THE POLE-ZERO CASE

When a plant has a transfer function containing both poles and zeroes, the state equation may be written in the form:

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{b}_1\ u_k + \underline{b}_2\ u_{k-1} \tag{E-46}$$

or, as has been shown in Appendix A, in the alternate form:

$$\underline{x}((k+1)T^-) = \underline{F}\ \underline{x}(kT^-) + \underline{a}\ u_k + \underline{c}\ (u_k - u_{k-1}) \tag{E-47}$$

The alternate form E-47 is the most useful form for the analytical studies in this appendix so that when the state equation is referred to, E-47 will be tacitly implied.

Following a Step - Assume that for $t \geq t_o$ where t and t_o are contained in the open interval (t_a, t_b) during which control of the plant is desired, the desired output is given by:

$$\underline{r}'(kT) = \underline{r}'_{ss} = \lfloor R \quad 0 \quad . \quad . \quad . \quad 0 \rfloor \tag{E-48}$$

where R is the magnitude of the desired position.

Assume that the actual steady state output is some constant times the desired output (the assumption is made that the control policy is operating at a stable T-h point).

$$\underline{r}_{ss} = \underline{E}\,\underline{x}_{ss}$$

$$\underline{E} = \begin{bmatrix} b & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{E-49}$$

$$\underline{x}_{ss} = \underline{x}((k+1)T^{-}) - \underline{x}(kT^{-}) \qquad k \geq q \tag{E-50}$$

where q is sufficiently large so that initial transients in the output may be assumed to have died out.

It can be easily shown by considering the differential equation of the plant that the control force will be a constant value when steady state conditions have been reached so that the steady state control force will be given by the control policy equation as:

$$u_{ss} = \frac{\underline{\lambda}'_a \,\underline{K} \left[\underline{r}_{ss} - \underline{F}_a \,\underline{x}_{ss} + \underline{c}_a \, u_{ss} \right]}{\underline{\lambda}'_a \,\underline{K}\, \underline{\lambda}_a} \tag{E-51}$$

which when solved for an explicit value of u_{ss} yields:

$$u_{ss} = \frac{\underline{\lambda}'_a \,\underline{K} \left[\underline{r}_{ss} - \underline{F}_a \,\underline{x}_{ss} \right]}{\underline{\lambda}'_a \,\underline{K}\, (\underline{\lambda}_a - \underline{c}_a)} \tag{E-52}$$

Substituting equations E-52, E-50, and E-49 into equation E-47 and collecting terms yeilds

$$\underline{x}_{ss} = \left[\underline{F}^{*} - \underline{\Gamma}_a \,\underline{F}_a + \underline{\Gamma}_a \,\underline{E} \right] \underline{x}_{ss} \tag{E-53}$$

where

$$\underline{\Gamma}_a = \frac{\underline{a}\ \underline{\lambda}_a'\ \underline{K}}{\underline{\lambda}_a'\ \underline{K}\ (\underline{\lambda}_a - \underline{c}_a)} \tag{E-54}$$

If the actual and desired output states are to be identical, then $\underline{E} = \underline{I}$ (b=1), and it must be possible to write the statement

$$\text{Det}\left[\underline{F} - \underline{\Gamma}_a\ \underline{F}_a + \underline{\Gamma}_a - \underline{I}\right] = 0 \tag{E-55}$$

Exact Prediction - To establish a reference standard against which to compare other types of prediction, the exact values of $\underline{F}$, $\underline{\lambda}$, and $\underline{c}$ will be assumed to be known in which case:

$$\begin{aligned} \underline{F}_a &= \underline{F} \\ \underline{\lambda}_a &= \underline{\lambda} \qquad\qquad \underline{\Gamma}_a = \underline{\Gamma}_e = \frac{\underline{a}\ \underline{\lambda}'\ \underline{K}}{\underline{\lambda}'\ \underline{K}\ (\underline{\lambda} - \underline{c})} \\ \underline{c}_a &= \underline{c} \end{aligned} \tag{E-56}$$

The determinant equation E-55 may then be written in the form

$$\text{Det}\left[(\underline{\Gamma}_e - \underline{I})\cdot(\underline{I} - \underline{F})\right] = 0 \tag{E-57}$$

Unfortunately $\underline{\Gamma}_e$ is not of the form which will assure that $\underline{\Gamma}_e - \underline{I}$ will be singular. This, of course, is different from the case for poles only in which $\underline{\Gamma}_e - \underline{I}$ was singular. For E-57 to be a valid statement $\underline{F} - \underline{I}$ must be singular which requires that the plant transfer function must have at least one pole at the origin.

If the plant does not possess a pole at the origin then there will be some steady state error corresponding to the value of b of the matrix $\underline{E}$ for which equation E-58 is valid

$$\text{Det}\left[\underline{F} - \underline{\Gamma}_e\ \underline{F} + \underline{\Gamma}_e\ \underline{E} - \underline{I}\right] = 0 \tag{E-58}$$

Conclusion - Assuming that the control policy is operated at a stable T-h point and Exact Prediction or Interpolation Prediction is used in which the interpolation provides good estimates of $\underline{F}$, $\underline{\lambda}$, and $\underline{c}$, no steady state output error will exist for a desired state of a constant output poisition in general only if the plant transfer function has at least one pole at the origin. If it does not, a steady state error will exist corresponding to the value of b of matrix $\underline{E}$ for which the matrix of equation E-58 is singular and the matrix determinant therefore zero.

Following a Ramp - Assume that for $t \geq t_o$ where t and t_o are contained in the open interval (t_a, t_b) during which control of the plant is desired, the desired output is given by:

$$\underline{r}'(kT) = \lfloor R\cdot(kT) \quad R \quad 0 \quad . \quad . \quad 0 \rfloor \tag{E-59}$$

where R is the magnitude of the desired output velocity.

Define a vector

$$\underline{\Delta r}_{ss} = \underline{r}'((k+1)T) - \underline{r}'(kT) = \lfloor RT \quad 0 \quad . \quad . \quad . \quad 0 \rfloor \tag{E-60}$$

$$k = 1, 2, \quad . \quad . \quad . \quad n$$

In terms of the difference between two successive desired states the desired output vector $\underline{\Delta r}_{ss}$ takes the form of a desired rate of change and is a constant vector for a constant T.

Similarly, two successive state equations may be written

$$\begin{aligned} \underline{x}((k+1)T^-) &= \underline{F}\, \underline{x}(kT^-) + \underline{a}\, u_k + \underline{c}\, (u_k - u_{k-1}) \\ \underline{x}(kT^-) &= \underline{F}\, \underline{x}((k-1)T^-) + \underline{a}\, u_{k-1} + \underline{c}\, (u_{k-1} - u_{k-2}) \end{aligned} \tag{E-61}$$

One of the state equations may be subtracted from the other yielding equation E-62.

$$\begin{aligned} \underline{x}((k+1)T^-) - \underline{x}(kT^-) = \underline{F}\left[\underline{x}(kT^-) - \underline{x}((k-1)T^-)\right] &+ \underline{a}\, (u_k - u_{k-1}) \\ &+ \underline{c}\, (u_k - u_{k-2}) \end{aligned} \tag{E-62}$$

Assume that the actual steady state output approaches a constant rate of change after the initial transients have died out

$$\underline{\Delta x}_{ss} = \underline{x}((k+1)T^-) - \underline{x}(kT^-) = \underline{x}(kT^-) - \underline{x}((k-1)T^-) \qquad k \geq q \tag{E-63}$$

where the initial transients have died out by the time $t = qT$.

Assume a steady state error exists of the form

$$\underline{\Delta r}_{ss} = \underline{E}\, \underline{\Delta x}_{ss} \tag{E-64}$$

where operation at a stable T-h point is assumed.

It can easily be shown by considering the differential equation of the plant that the control force will approach a constant rate of change when steady state conditions have been reached. Any particular control force is given by the control policy equation

$$u_k = \frac{\underline{\lambda}_a' \underline{K} \left[\underline{r}((k+1)T) - \underline{F}\, \underline{x}(kT^-) + \underline{c}_a u_{k-1}\right]}{\underline{\lambda}_a' \underline{K}\, \underline{\lambda}_a} \tag{E-65}$$

The steady state rate of change of the control force for steady state conditions will be given by the difference between two successive control policy equations

$$\Delta u_{ss} = \frac{\underline{\lambda}_a' \underline{K} \left[\underline{\Delta r}_{ss} - \underline{F}\, \underline{\Delta x}_{ss} + \underline{c}_a \Delta u_{ss}\right]}{\underline{\lambda}_a' \underline{K}\, \underline{\lambda}_a} \tag{E-66}$$

which when solved for an explicit value of Δu_{ss} yields

$$\Delta u_{ss} = \frac{\underline{\lambda}_a' \underline{K} \left[\underline{\Delta r}_{ss} - \underline{F}\, \underline{\Delta x}_{ss}\right]}{\underline{\lambda}_a' \underline{K} (\underline{\lambda}_a - \underline{c}_a)} \tag{E-67}$$

Substituting equations E-67, E-64, and E-63 into equation E-62 and collecting terms yields

$$\underline{\Delta x}_{ss} = \left[\underline{F} - \underline{\Gamma}_a \underline{F} + \underline{\Gamma}_a \underline{E}\right] \underline{\Delta x}_{ss} \tag{E-68}$$

where

$$\underline{\Gamma}_a = \frac{\underline{a}\, \underline{\lambda}_a' \underline{K}}{\underline{\lambda}_a' \underline{K} (\underline{\lambda}_a - \underline{c}_a)} \tag{E-69}$$

If the actual and desired output rates are to be identical, then $\underline{E} = \underline{I}$ (b=1), and it must be possible to write the statement

$$\mathrm{Det} \left[\underline{F} - \underline{\Gamma}_a \underline{F}_a + \underline{\Gamma}_a - \underline{I}\right] = 0 \tag{E-70}$$

Exact Prediction - To establish a reference standard against which to compare other types of prediction, the exact values of $\underline{F}$, $\underline{\lambda}$, and $\underline{c}$ will be assumed to be known in which case:

$$\begin{aligned} \underline{F}_a &= \underline{F} \\ \underline{\lambda}_a &= \underline{\lambda} \\ \underline{c}_a &= \underline{c} \\ \underline{\Gamma}_a &= \underline{\Gamma}_e = \frac{\underline{a}\ \underline{\lambda}'\ \underline{K}}{\underline{\lambda}'\ \underline{K}\ (\underline{\lambda} - \underline{c})} \end{aligned} \tag{E-71}$$

The determinant equation E-70 may then be written in the form

$$\text{Det}\left[(\underline{\Gamma}_e - \underline{I})\cdot(\underline{I} - \underline{F})\right] = 0 \tag{E-72}$$

Again, $\underline{\Gamma}_e$ is not of the form which will assure that $\underline{\Gamma}_e - \underline{I}$ will be singular. For E-72 to be a valid statement $\underline{F}$-$\underline{I}$ must be singular which requires that the plant transfer function must have at least one pole at the origin.

If the plant does not possess a pole at the origin, then there will be some steady state error corresponding to the value of b of matrix $\underline{E}$ for which equation E-73 is valid.

$$\text{Det}\left[\underline{F} - \underline{\Gamma}_e\,\underline{F} + \underline{\Gamma}_e\,\underline{E} - \underline{I}\right] = 0 \tag{E-73}$$

Conclusion - Assuming that the control policy is operated at a stable T-h point and Exact Prediction or Interpolation Prediction is used in which the interpolation provides good estimates of $\underline{F}$, $\underline{\lambda}$, and $\underline{c}$, no steady state output rate error will exist for a desired state of a constant rate if the transfer function possesses at least one pole at the origin. The actual state can, therefore, differ at most from the desired state by a constant positional error. If the plant transfer function does not contain a pole at the origin, a steady state rate error will exist corresponding to the value of b of matrix $\underline{E}$ for which the matrix of equation E-73 is singular and the matrix determinant therefore zero.

APPENDIX F

STUDY ON SINGULARITY PROBLEM WITH THE INTERPOLATION METHOD

In the present method, the interpolation is between a group of quantities $\underline{\alpha}_m$ at $t = mT^o$, and a set of quantities $\underline{\beta}_m$ one decision interval later at $t = (m+1)T°$.

The set $\underline{\alpha}_m$ ($\underline{\alpha}$ measured at $t = mT$) may include a variety of quantities on an optional basis.

In our application of the interpolation method the $\underline{\alpha}_m$ set of measured quantities consists of the initial conditions $\underline{\eta}_m$ at the beginning of M decision intervals, and the control forces, u_m (constants), applied during the intervals. The $\underline{\beta}_m$ set is composed of the outputs x_m at the end of the M decision intervals.

Based on the above information the method of solution for the system approximating functional, $\tilde{x}(u,\underline{\eta})$, means solving the following determinant equation:

$$\text{Det}\begin{bmatrix} \tilde{x}(u,\underline{\eta}) & x_1(u_1,\underline{\eta}_1) & \cdots & x_M(u_M,\underline{\eta}_M) \\ \underline{\phi}(u,\underline{\eta}) & \underline{\phi}_1(u_1,\underline{\eta}_1) & \cdots & \underline{\phi}_M(u_M,\underline{\eta}_M) \end{bmatrix} = 0 \qquad \text{(F-1)}$$

where $x_1, x_2, \ldots x_M$ are the measured plant outputs at the end of M decision intervals, and $\underline{\phi}(u,\underline{\eta})$ is a vector of M linearily independent analytic base functionals $(u,\underline{\eta})$ and $\underline{\phi}_1, \underline{\phi}_2, \ldots \underline{\phi}_M$ are this vector evaluated at the measured data points $u_1\underline{\eta}_1, u_2\underline{\eta}_2, \ldots u_M\underline{\eta}_M$.

The determinant is readily expanded for a solution of $\tilde{x}(u,\underline{\eta})$; (See Appendix B for a detailed equational derivation):

$$\tilde{x}(u,\underline{\eta}) = {}_D\underline{x}'\ \underline{\Phi}^{-1}\ \underline{\phi}(u,\underline{\eta}) \qquad \text{(F-2)}$$

Because of the discrete nature of the interpolation equation it is convenient to write Equation F-2 as;

$$\tilde{x}((n+1)T) = {}_D\underline{x}' \; \underline{\Phi}^{-1} \; \underline{\phi}(u_n, \underline{\eta}_n) \tag{F-3}$$

where $\tilde{x}$ is the approximate value of the output at (n+1)T due to $\underline{\eta}_n$ initial conditions at t=nT and a control force u_n applied over the decision interval $nT \leq t < (n+1)T$.

If several derivatives of the output are measured, the determinant equation (like Equation F-1) can be set up for each one of them. These equations may then be combined into a single equation;

$$\underline{\tilde{x}}((n+1)T) = {}_D\underline{X} \; \underline{\Phi}^{-1} \; \underline{\phi}(u_n, \underline{\eta}_n) \tag{F-4}$$

It may be noted that as stated this is still just several single variable interpolations concisely stated rather than multi-input-output interpolation. All depend on the nonsingularity of $\underline{\Phi}$, which brings us to the topic of this appendix.

F.1 SINGULARITY OF $\underline{\Phi}$

In such a situation two types of conditions which could possibly make $\underline{\Phi}$ singular are of basic concern. These are:

(a) Inherent singularity

(b) Accidential singularity

INHERENT SINGULARITY

Inherent singularity implies something basic in the system and the proposed process which would make the desired parameters unobservable. An obvious fact is that $\underline{\Phi}$ depends only on the input $\underline{\phi}$ and not on the output $\underline{x}$. In other words, the only requirements for the nonsingularity of $\underline{\Phi}$ is that the vectors $\underline{\phi}$ be linearly independent. This is similar to curve fitting in two dimensions. For instance, if a parabola is to be fitted it will be enough to select three abscissa (x) points which are different. The curve y itself only needs to have finite values at these points for a parabola to be defined. The nature of the curve has no bearing on the existance of the parabola although it obviously very strongly affects the quality of the approximation to the curve by the parabola. The existance of the parabola will, however, depend on the selection of the three abscissa points. If any two of them should coincide, the problem becomes singular. If two points are close together, the problem may become ill conditioned.

Our problem is the multivariable version of the simple parabola fitting, so it is not surprising that the singularity of $\underline{\Phi}$ hinges only on the inputs or its components $\underline{\eta}$ and u, and is independent of the system proper. Since the singularity is not influenced by the system there is no possibility of inherent singularity or unobservability.

ACCIDENTAL SINGULARITY

The singularity of $\underline{\Phi}$ depends solely on the selection of the $\underline{\phi}$ vectors which, however, contain among other items the state and consequently do not represent an arbitrary choice. Accidential singularity might occur if the columns or the rows of $\underline{\Phi}$ are linearly dependent.

The possibility of linear dependence of the columns is considered first. The simplest way trouble may develop would be when two or more columns are proportional. However, this is not feasible since at least a section of the columns will consist of the state and (unless it be a constant or zero state) the states at consecutive decision points will not be proportional.

In the application of the interpolation method to the linear systems, the $\underline{\phi}$ vector is composed of the control force, u, and the initial state, $\underline{\eta}$ That is:

$$\underline{\phi}(u_n,\underline{\eta}_n) = \begin{bmatrix} u_n \\ \underline{x}(nT) \end{bmatrix} \qquad \text{(F-5)}$$

Thus, for $\underline{\phi}(u_n, \underline{\eta}_n)$ assumed to have e components the singularity would require:

$$\text{Det} \begin{bmatrix} u_{n-e} & u_{n-e+1} & \cdot & \cdot & \cdot & u_{n-1} \\ \underline{x}_{n-e} & \underline{x}_{n-e+1} & \cdot & \cdot & \cdot & \underline{x}_{n-1} \end{bmatrix} = 0 \qquad \text{(F-6)}$$

or

$$u_{n-1} = \underline{x}'_{n-1} \begin{bmatrix} \underline{x}_{n-e} & \cdot & \cdot & \cdot & \underline{x}_{n-2} \end{bmatrix}^{-1} \bullet \begin{bmatrix} u_{n-e} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-2} \end{bmatrix} \qquad \text{(F-7)}$$

In other words, there will exist a $\mathbf{u}_{n-1}$ which will make $\underline{\Phi}$ singular provided

$$\text{Det} \begin{bmatrix} \underline{x}_{n-e} & \cdot & \cdot & \cdot & \underline{x}_{n-2} \end{bmatrix} \neq 0 \qquad \text{(F-8)}$$

or

$$\text{Det} \begin{bmatrix} (\underline{F}\,\underline{x}_{n-e} + u_{n-e}\,\underline{a}) & \cdot & \cdot & \cdot & (\underline{F}\,\underline{x}_{n-3} + u_{n-3}\,\underline{a}) \end{bmatrix} \neq 0 \qquad \text{(F-9)}$$

where $\underline{F}$ is the state transition matrix and $\underline{a}$ is the sensitivity vector.

Since this depends on the control force sequence, there should be some u_k sequence where this is nonsingular. Then a final u_{n-1} control force as defined above should exist which makes $\underline{\Phi}$ singular. The occurance of this should be a rare event, however, which could be remedied by simply skipping the inversion of $\underline{\Phi}$ for one decision interval.

There still remains the question of whether the particular control policy which is used in our control system does not lead to just such a singular sequence of control forces. The control force equation is:

$$u_n = \frac{\underline{a}'\,\underline{K}\,\left[\underline{r}((n+1)T) - \underline{F}\,\underline{x}(nT)\right]}{\underline{a}'\,\underline{K}\,\underline{a}} \qquad \text{(F-10)}$$

where $\underline{a}$ is the sensitivity vector, $\underline{K}$ the weighting matrix, $\underline{F}$ the state transition matrix, $\underline{r}$(n+1)T) the desired state at t = ((n+1)T), and $\underline{x}$(nT) the output state at t = nT.

Considering the case where $\underline{r}((n+1)T)$ is zero or constant the control equation may be rewritten as:

$$u_n = \underline{p}' \, \underline{x}(nT) \tag{F-11}$$

where $\underline{p}$ is a fixed vector defined for zero desired case as:

$$\underline{p}' = \frac{-\underline{a}' \, \underline{K} \, \underline{F}}{\underline{a}' \, \underline{K} \, \underline{a}} \tag{F-12}$$

For this situation $\underline{\Phi}$ may be written

$$\underline{\Phi} = \begin{bmatrix} \underline{p}' \, \underline{x}_{n-e} & \underline{p}' \, \underline{Q} \, \underline{x}_{n-e} & \underline{p}' \, \underline{Q}^2 \, \underline{x}_{n-e} & \cdots & \underline{p}' \, \underline{Q}^{e-2} \, \underline{x}_{n-e} \\ \underline{x}_{n-e} & \underline{Q} \, \underline{x}_{n-e} & \underline{Q}^2 \, \underline{x}_{n-e} & \cdots & \underline{Q}^{e-2} \, \underline{x}_{n-e} \end{bmatrix} \tag{F-13}$$

where

$$\underline{Q} = \underline{F} + \underline{a} \, \underline{p}' = \underline{F} - \frac{\underline{a} \, \underline{a}' \, \underline{K} \, \underline{F}}{\underline{a}' \, \underline{K} \, \underline{a}} = \left[\underline{I} - \frac{\underline{a} \, \underline{a}' \, \underline{K}}{\underline{a}' \, \underline{K} \, \underline{a}} \right] \underline{F} \tag{F-14}$$

This matrix is inherently singular. Furthermore, it will be singular whatever is selected for $\underline{p}$ as long as it is constant. Even inclusion of other variables in $\underline{\Phi}$ besides u and $\underline{x}$ will not change the trouble. The basic conclusion is that for <u>linear systems</u> any policy which results in using a linear combination of the state variables for determining the control force makes the matrix $\underline{\Phi}$ needed singular. This condition exists for our linear control policy <u>only</u> when the desired output state is either zero or constant. If the desired output state $\underline{r}((n+1)T)$ consists of a nonconstant trajectory, the $\underline{p}$ <u>will</u> <u>not</u> be constant, and the matrix <u>will</u> <u>not</u> be inherently singular due to the control force equation.

Also, of importance is the conclusion that for the nonlinear case the situation leading to $\underline{\Phi}$ being inherently singular can not occur. Since the control force equation for the nonlinear control policy is a cubic equation, the linearity of selecting the control force is absent.

Now let the chance of singularity of $\underline{\Phi}$ because of linear dependence of the rows be investigated. The critical situation here is that if any of the components of $\underline{\Phi}$ such as the control force (or altitude or Mach number should they be included) stays constant over a number of intervals, then there will be two or more rows in $\underline{\Phi}$ consisting of the same numbers. Therefore, these rows are proportional to each other, and the matrix is singular. Such a condition could easily occur as a result of prolonged use of the maximum available control force. Also, in the nonlinear case the $\underline{\phi}$ vector includes such items as u and u^2; and so the possibility of such singularity is more likely to occur.

With the above qualifications, it seems that the matrix $\underline{\Phi}$ should be counted on to be nonsingular except in cases of accidental coincidences in the coefficients. This event should be somewhat rare.

APPENDIX G

DERIVATION OF THE SECOND ORDER VOLTERRA SERIES WORKING EQUATIONS

The purpose of this appendix is to present the general theory of the Volterra series plant representation and the equational development of a truncated series approximation. The method is termed the Volterra series approximation as opposed to the interpolation approximation discussed in Appendix B. Although both procedures rely on the existance of a Volterra series representation of the plant, the approximation procedure developed in this appendix follows more directly from the actual series. Relegation of the discussion of the Volterra series procedure to an appendix should not be interpreted as assigning it secondary importance as compared with the interpolation procedure. As is discussed in Section 4.2 of the text, the choice of the interpolation procedure for more extensive investigation was largely based on expediency, rather than any superior features.

The Volterra series procedure has been thoroughly documented by Zaborszky and Humphrey (reference 1) but, in order to make this report self contained, Section G.1 is a summary of the pertinent sections of their paper.

G.1 CONTROL WITHOUT MODEL OR PLANT IDENTIFICATION

The assumptions concerning the controlled plant are very general. They are valid for almost all physical equipment. Specifics like assumption of a particular order, linearity or of slow variation of the system are avoided. Information about the plant's behavior is derived solely from potentially noisy measurements. The output quantity and the control variable (including possibly a few derivatives) are available for measurement, but the complete "state" vector is not. Under the assumptions which are made here, its dimensionality is unknown.

ASSUMPTIONS

Application of the control method described in the paper is restricted to systems which produce continuous and bounded outputs, x(t), when excited by continuous and bounded control inputs, u(t). The input-output relationship of such a system is a functional which maps the Banach space of continuous functions over an interval onto itself. If such a functional is continuous, it can be approximated over finite time intervals arbitrarily closely by a finite functional polynomial (reference 2) of the form

$$x(t) = y(t) + \sum_{j=1}^{J} \int_0^t \cdots \int_0^t h_j(t,\tau_1, \ldots, \tau_j)\, u(\tau_1) \ldots u(\tau_j)\, d\tau_1 \ldots d\tau_j \tag{G-1}$$

where h_j are the kernels of the functional polynomial fit.

If the functional is analytic, it can be represented by an infinite series ($J=\infty$) of the type used in equation G-1, a "Volterra" (reference 3) or functional Taylor series; h_j are then the Volterra kernels. In equation G-1, y(t) represents the free response which would occur in the absence of any control input, u(t). It must be remembered, however, that equation G-1 does not imply superposition because the h_j kernels are not unique. They depend on **y**(t), just as the coefficients of an ordinary Taylor series depend on the point around which the expansion is obtained.

Note that only the existence of a relation in the form of equation G-1 is assumed. Any knowledge of the kernels, or an intention to identify them, is not assumed. This encompasses a broad class of systems. Continuous nonlinearities and time variations are permitted without the assumption of any particular order for the differential equations or of any knowledge concerning either the speed of variation or the existence of nonlinearity. Discontinuous nonlinearities, such as relays in the plant, are about the only features excluded. Of course, if there are any relays in a control system, they are not likely to be in the plant. Discontinuous time variations are permissible if their occurrences can be easily recognized as, for instance, in the staging of a missile. Extensions to more than one output or control variable are direct.

REPRESENTATION OF THE RESPONSE OF THE PLANT AND ITS SENSITIVITY TO CONTROL ACTION

The specific control variable functions considered in this study are piecewise constant.

$$u(t) = u_k \qquad kT \leq t < (k+1)T \qquad \text{and} \qquad |u_k| \leq U \tag{G-2}$$

This form of control variable is almost inherent in any control which relies on the on-line digital computer.

The present time will be t=nT; an nT second length section of the latest signals x(t) and u(t) will be kept in the computer memory. Then, for $t \geq 0$, with a functional power series of the type of equation G-1 for the interval $0 \leq t < nT$ and substituting equation G-2, the following equation results:

$$x(t) = y(t) + \sum_{j=1}^{J} \sum_{k_1=0}^{n-1} \cdots \sum_{k_j=0}^{n-1} A_{k_1 \ldots k_j}(t)\, u_{k_1} \cdots u_{k_j} \tag{G-3}$$

where by equations G-1, 2, 3 (G-4)

$$A_{k_1 \cdots k_j}(t) = \begin{cases} 0 & t < XT \\ \displaystyle\int_{k_1 T}^{K} \cdots \int_{k_j T}^{K} h_j(t, \tau_1, \ldots, \tau_j)\, d\tau_1 \ldots d\tau_j & XT \leq t < (X+1)T \\ \displaystyle\int_{k_1 T}^{(k_1+1)T} \cdots\cdots \int_{k_j T}^{(k_j+1)T} h_j(t, \tau_1, \ldots, \tau_j)\, d\tau_1 \ldots d\tau_j & (X+1)T \leq t \end{cases}$$

with $X = \max\{k_1, k_2, \ldots, k_j\}$

and $K = \begin{cases} (k_i+1)T & \text{for } k_i < X \\ t & \text{for } k_i = X \end{cases}$

Equation G-3 can be rearranged as shown following:

$$x(t) = y(t) + \sum_{k=0}^{n-1} \sum_{r=1}^{R} a_{k^r}(t)\, u_k^{\,r} \qquad \text{(G-5)}$$

where usually R=J, $a_{k^r}(t)$ stands for $a_{k\ldots k}$, with k having been repeated r times.

From equations G-3 and G-4, with the symmetry of the kernels having been considered:

$$a_{k^r}(t) = A_{k^r}(t) + \sum_{i=1}^{k} (r+1)\, A_{k^r,k-i}(t)\, u_{k-i} \qquad \text{(G-6)}$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{2-\delta_{ij}}{2} \times \frac{(r+2)!}{r!} A_{k^r,k-i,k-j}(t)\, u_{k-i}\, u_{k-j} + \ldots$$

where δ_{ij} is the Kronecker delta and, with reference to equation G-4, $A_{k^r,\,k-i,\,k-j}$ stand for $A_{k\ldots k(k-i)\,(k-j)}$ with k repeated r times.

More generally,

$$a_{k^r}(t) = \frac{1}{u_{k^{r_1}}} \sum_{<k>} A_{<k>}\, m_{<k>}\, U_{<k>} \tag{G-7}$$

$$m_{<k>} = \sum_{r_2=0}^{R-r_1} \sum_{r_3=0}^{R-r_1-r_2} \cdots \sum_{r_R=0}^{R-r_1\cdots-r_{R-1}} \frac{\left(\sum_{i=1}^{R} r_i\right)!}{\prod_{i=1}^{R} r_i!}$$

where, with reference to equation G-4, $A_{\langle k\rangle}=A_{k^{r_1}\,(k-1)^{r_2}\cdots(k-R)^{r_R}}(t)$ denotes one of the $A_{k_1\ldots k_j}(t)$ with k repeated r_1 times; (k-1) r_2 times, etc. (Note that r_1=r when equation G-5 is used); $m_{\langle k\rangle}$ is the multiplicity of occurrence of the term $A_{\langle k\rangle}$, as defined by equation G-7; $\sum_{\langle k\rangle}$ denotes summation over all the different $A_{\langle k\rangle}$ which have significant contributions; and

$$U_{<k>} = u_k^{\,r_1}\; u_{k-1}^{\,r_2}\; \cdots\; u_{k-R}^{\,r_R} \tag{G-8}$$

Defining vectors

$$\underline{x} = x^{(i)}\Big]_{NX1} \qquad \underline{y} = y^{(i)}\Big]_{NX1} \qquad \underline{u}_k = u_k^{\,r}\Big]_{RX1} \tag{G-9}$$

$$\underline{A}_k = a_{k^r}^{(i)}(t)\Big]_{NXR} \qquad \begin{array}{l} i = 0,\ 1,\ \ldots\ ,\ n-1 \\ r = 1,\ 2,\ \ldots\ ,\ R \end{array}$$

where $\underline{x}$ can be a state vector, if N is the order of the system; then, from equation G-5

$$\underline{x}(t) = \underline{y}(t) + \sum_{k=0}^{n} \underline{A}_k(t)\ \underline{u}_k \qquad t \geq 0 \qquad \text{(G-10)}$$

Such a representation is possible for the class of plants considered because, for this class, functions y(t) and $A_{\langle k \rangle}(t)$ will be continuous and repeatedly differentiable with respect to t, where $\langle k \rangle$ indicates any of the ordered sets of subscripts used in equation G-6. The possible exception is at t = iT, with i an integer. Here, higher derivatives of $A_{\langle k \rangle}(t)$ will be discontinuous. Then, truncated Taylor series representations can be found for y(t) and $A_{\langle k \rangle}(t)$, respectively.

$$y^{(i)}(t) = \sum_{p=i}^{P} y_p \frac{p!}{(p-i)!} t^{p-i} \qquad \text{(G-11)}$$

$$A_k{}^{(i)}(t) = \begin{cases} 0 & t < kT \\ \displaystyle\sum_{p=i}^{P} A_{<k>p-} \frac{p!}{(p-i)!} (t-kT)^{p-i} & kT \leq t < (k+1)T \\ \displaystyle\sum_{p=i}^{P} A_{<k>p+} \frac{p!}{(p-i)!} (t-kT)^{p-i} & (k+1)T \leq t \end{cases} \qquad \text{(G-12)}$$

Finally, for a continuously, if arbitrarily, time-varying plant

$$A_{<k>p\pm} = A_{<n-h>p\pm} = \sum_{s=0}^{S} A_{<n>ps\pm} (-hT)^{s} \qquad \text{(G-13)}$$

provided that

$$<k-h> = <k_1-h,\ k_2-h,\ .\ .\ .\ .\ ,k_j-h> \qquad \text{(G-13A)}$$

$$\text{if}\quad <k> = <k_1,\ k_2,\ .\ .\ .\ ,k_j> \quad \text{and} \quad k_i-h \geq 0$$

Probably, S=1 is sufficient for most plants.

Equation G-10 can be rewritten for $t \geq nT$ as

$$\underline{x}(t) = \underline{x}_n(t) + \underline{A}_n(t)\ \underline{u}_n \qquad \text{(G-14)}$$

where

$$\underline{x}_n(t) = \underline{y}(t) + \sum_{k=0}^{n-1} \underline{A}_k(t)\ \underline{u}_k \qquad \text{(G-15)}$$

represents the <u>current response</u> of the system at $t \geq nT$, resulting from its initial state at t=0 (the term y(t)) and the u_k, k=0, 1, . . ., n-1 control steps applied during $0 \leq t < nT$.

The last term in equation G-15 identifies the effect of the control variable u_n, which will be applied $nT \leq t < (n+1)T$; $\underline{A}_n$ is then the <u>current sensitivity</u> of the system to this force. Of course, $\underline{A}_n\underline{u}_n$ is a column of polynomials in u_n. Note that, in spite of its form, equation G-14 does not represent superposition since the sensitivity $\underline{A}_n(t)$ is not a unique constant of the plant, but a function of past states, $\underline{y}(t)$, and of past control forces, u_k, applied to the plant. Equations G-14 and G-15 represent a kind of "canonical equations" which describe the current expected behavior of the plant which may be linear or nonlinear, stationary of time varying. These canonical or standard equations define the current behavior of the general class of plants when controlled digitally with a zero order hold. The coefficients of these canonical equations can be computed if the plant is known, or they can be determined from the signals of the immediate past, as has been proposed here.

Both current response $\underline{x}_n(t)$ and current sensitivity $\underline{A}_n(t)$ are fully determined by equations G-6-15 when the present parameters $A_{\langle n \rangle ps\pm}$ and y_p are identified. This represents identification of the current response and sensitivity to the next input step, but it does not, however, identify the plant in the normal sense.

A plant is identified when a relationship (differential equation, transfer function, Volterra series, etc.) is established (references 4, 5, 6) which permits computation of the plant output for an arbitrary input and an arbitrary initial state. What is identified in this study permits only a prediction of the response for the existing conditions of state and control forces applied in the past and under the influence of the control step ahead which is of a strongly limited nature. In this sense, it is not plant identification but identification of current response and current sensitivity to control force.

DETERMINING THE COEFFICIENTS

From equations G-11 and G-12, equation G-10 can be rewritten in the form

$$\underline{x}(t) = \sum_{e=0}^{P} \underline{g}_{ke}\; t^{e} \qquad kT \leq t < (k+1)T \qquad \text{(G-16)}$$

This is simply a Taylor series expansion about some convenient point (ideally $P=\infty$) of the output and its derivatives. The coefficients $\underline{g}_{ke}$ are, by equations G-11-13, linear combinations of the $A_{\langle n \rangle ps\pm}$ and the y_p coefficients. A different combination will arise for every interval, unless $u_k=u_i$ for all k and i. Consequently, there will be a separate series of the form of G-16 for every interval T.

If it is assumed that the signal x(t) can be measured exactly, then a definite set of g_{ke} can be established for each interval $kT \leq t < (k+1)T$. When these are equated to the expressions for g_{ke} obtained from equations G-6-13, a set of simultaneous linear equations results which uniquely determines the $A_{\langle n\rangle ps\pm}$ and y_p coefficients, provided that the number of coefficients and intervals is properly coordinated.

Specifically, for the first term in g_{ke}

$$g_{ke} = y_e + \sum_{s=0}^{S} \sum_{p=0}^{P} \binom{p}{e} \sum_{<n>} m_{<n>} \bullet \tag{G-17}$$

$$\left[\sum_{h=n-k+1}^{n} \left\{ U_{<n-h>} A_{<n>ps+} (-hT)^{s} \left[-(n-h)T\right]^{p-e} \right\} + U_{<k>} A_{<n>ps-} \left[-(n-k)T\right]^{s} (-kT)^{p-e} \right]$$

$k = 0,1,2, \ldots ,n$

$e = 0,1,2, \ldots ,P$

where notations $\sum_{\langle n\rangle} m_{\langle n\rangle}$ and $U_{\langle n-h\rangle}$ were defined in conjunction with equations G-7 and G-8, and $\langle n-h\rangle$ is defined in equation G-13A.

This will yield a sufficient number of equations if

$$n = 2\mu(S+1) + 1 \tag{G-18}$$

where μ is the number of the $\langle n \rangle$ sets considered significant and the determination of which is desired.

Equations G-17 will be independent if all the u_k are not identical as they would be, for instance, when the limit U of u_k is requested continually When this situation arises, it still would be possible to determine combination coefficients ($g_{ke}=g_e$) which will predict the response as long as $|u_k|=U$ is maintained. However, any evaluation of the sensitivity to the choice of u_k would be lost. Basically, a different control policy from the one considered in this study is necessary when the available control force is so limited that $|u_k| = U$ is used most of the time. Although assumption of an exact noise free measurement of $\underline{x}(t)$ is unrealistic, it is no less realistic than the assumption of a perfectly identified plant and a perfectly identified state vector, which are the bases of the major part of the extensive optimal control literature. In both cases, these idealized assumptions have value because of their establishing idealized reference points.

G.2 SOME SIMPLIFICATIONS

In the broad class of systems considered, the higher order terms in the working equations become progressively smaller as T is reduced so that eventually only the first order terms are significant. It appears that the practical cases would tend to be limited to R=1 and R=2 where R is the upper limit of the summation in equation G-5 and is usually equal to J, the order of the truncated Volterra series. Beyond R=2, the number of terms begins to poliferate prohibitively.

A rudimentary form of the R=1 case was presented in Section 2 of the text where a truncated Taylor series was used as an estimate of the state transition matrix. A more formalistic R=1 procedure is among those discussed in Section 6 under recommendations for further study.

The R=2 case was investigated in some detail by considering specific cases which are outlined in the sequel. In this way, the characteristics of the R=2 case are accessed in terms of the degree of complexity of the equations and the amount of computation involved. In order to make the R=2 case more tractable to implementation, some practical limits must be imposed. These limits take the form of specific values for the number of μ sets considered significant, the point at which the Taylor series for y(t) and the $A_{\langle k\rangle}(t)$ are truncated, and the upper limit, S, of the series expansions of the Taylor series coefficients y_p and $A_{\langle k\rangle p\pm}$ which account for time variation of the plant.

It should be sufficient to consider only a few intervals, I, of the immediate past as the contribution of prior intervals will become increasingly negligible. In this way, the number of μ sets is limited. Assuming the Taylor series for y(t) and $A_{\langle k\rangle}(t)$ are utilized only in the immediate vicinity of the expansion point (the decision interval, T, is reasonably short), truncation after a few terms should be feasible. Finally, a linear approximation to the time variation of the plant should be sufficient in most cases (S=1) and in many instances S=0 may give sufficient accuracy if the plant is relatively slowly time varying.

The definition of a particular R=2 case, therefore, takes the form of specifying

1. I - the number of past intervals which contribute significantly to the present response.
2. P - the point at which the Taylor series for y(t) and $A_{\langle k\rangle}(t)$ are truncated.
3. S - the degree of the polynomial fit accounting for plant time variation.

EXAMPLE CASE ONE

In this case, the R=2 working equations are developed with the following assumptions

1. I = 1
2. P = 2
3. S = 0

Under the above assumptions, equation G-17 reduces to

$$g_{ke} = y_e + \sum_{p=0}^{2} \binom{p}{e} \Big\{ m_{\langle k-1\rangle} U_{\langle k-1\rangle} A_{\langle k-1\rangle p+} \left[-(k-1)T\right]^{p-e} \qquad \text{(G-19)}$$

$$+ m_{\langle k\rangle} U_{\langle k\rangle} A_{\langle k\rangle p-} (-kT)^{p-e} \Big\}$$

where from equation G-6 or G-7 the significant μ sets is 3. The number of past intervals of data which are necessary to determine the coefficients is given by equation G-18 to be 7.

The coefficients for which values are needed during the interval $nT \leq t < (n+1)T$ are

$$\left.\begin{array}{l} A_{np-} \\ A_{nnp-} \\ A_{n(n-1)p-} \\ A_{np+} \\ A_{nnp+} \\ A_{n(n-1)p+} \\ y_p \end{array}\right\} \quad p = 0,\ 1,\ 2 \qquad \text{(G-20)}$$

which gives a total of 21 unknowns. A set of 21 equations of the form G-19 must be formulated in order to evaluate these coefficients. To obtain the 21 equations, the g_{ke} coefficients are measured in the form of equation G-16 where the expansion point is assumed to be absorbed in the coefficient. These measurements are made during seven intervals of the immediate past and the equations will be formed by equating the measured $\underline{g}_{ke}$ coefficients to the unknown coefficients through equation G-19.

In this example, these equations for the interval $kT \leq t < (k+1)T$ takes the form shown in equations G-21-23 where, for the sake of notational brevity, $q = k-1$.

$$
\begin{aligned}
g_{ko} = y_o &+ A_{q(0+)}\, u_q + A_{qq(0+)}\, u_q^{\,2} + 2\, A_{q(q-1)(0+)}\, u_q\, u_{q-1} \\
&+ \left\{ A_{q(1+)}\, u_q + A_{qq(1+)}\, u_q^{\,2} + 2\, A_{q(q-1)(1+)}\, u_q\, u_{q-1} \right\} (-qT) \\
&+ \left\{ A_{q(2+)}\, u_q + A_{qq(2+)}\, u_q^{\,2} + 2\, A_{q(q-1)(2+)}\, u_q\, u_{q-1} \right\} (-qT)^2 \\
&+ \quad A_{k(0-)}\, u_k + A_{kk(0-)}\, u_k^{\,2} + 2\, A_{kq(0-)}\, u_k\, u_q \\
&+ \left\{ A_{k(1-)}\, u_k + A_{kk(1-)}\, u_k^{\,2} + 2\, A_{kq(1-)}\, u_k\, u_q \right\} (-kT) \\
&+ \left\{ A_{k(2-)}\, u_k + A_{kk(2-)}\, u_k^{\,2} + 2\, A_{kq(2-)}\, u_k\, u_q \right\} (-kT)^2
\end{aligned}
\tag{G-21}
$$

$$g_{k1} = y_1 + A_{q(1+)} + A_{qq(1+)} u_q^2 + 2 A_{q(q-1)(1+)} u_q u_{q-1}$$
$$+ 2 \left\{ A_{q(2+)} u_q + A_{qq(2+)} u_q^2 + 2 A_{q(q-1)(2+)} u_q u_{q-1} \right\} (-qT)$$
$$+ A_{k(1-)} u_k + A_{kk(1-)} u_k^2 + 2 A_{kq(1-)} u_k u_q$$
$$+ 2 \left\{ A_{k(2-)} u_k + A_{kk(2-)} u_k^2 + 2 A_{kq(2-)} u_k u_q \right\} (-kT) \tag{G-22}$$

$$g_{k2} = y_2 + A_{q(2+)} u_q + A_{qq(2+)} u_q^2 + 2 A_{q(q-1)(2+)} u_q u_{q-1}$$
$$+ A_{k(2-)} u_k + A_{kk(2-)} u_k^2 + 2 A_{kq(2-)} u_k u_q \tag{G-23}$$

A total of 21 equations is obtained if k assumes seven values corresponding to seven intervals of the immediate past. The corresponding $A_{\langle k \rangle p\pm}$ coefficients of the different intervals may be set equal to each other and to $A_{\langle n \rangle p\pm}$ which gives a total of 21 equations and 21 unknowns In matric form the set of 21 equations can be expressed compactly as

$$\underline{M}\ \underline{a} = \underline{g} \tag{G-24}$$

where

$$\underline{g}' = \lfloor g_{k_1 0} \quad g_{k_1 1} \quad g_{k_1 2} \quad \cdot \quad \cdot \quad \cdot \quad g_{k_7 0} \quad g_{k_7 1} \quad g_{k_7 2} \rfloor \tag{G-25}$$

and

$$\underline{\alpha}' = \lfloor y_0 \; y_1 \; y_2 \; A_{n(0+)} \qquad A_{n(n-1)(2+)} \; A_{n(0-)} \qquad A_{n(n-1)(2-)} \rfloor \tag{G-26}$$

The matrix $\underline{M}$ is a 21 x 21 square matrix consisting of the known constants and control forces of equations G-21-23. It is possible to partition $\underline{M}$ in such a way that inversion of the full matrix is not required. Instead, a partial solution is obtained from the equation obtained from the partitioned $\underline{M}$.

$$\underline{\hat{M}}\,\underline{\hat{\alpha}}_1 = \underline{\hat{g}} \tag{G-27}$$

where

$$\underline{\hat{g}}' = \lfloor g_{k_1 2} \; g_{k_2 2} \; g_{k_3 2} \; \cdot \; \cdot \; \cdot \; g_{k_7 2} \rfloor \tag{G-28}$$

and

$$\underline{\hat{\alpha}}_1' = \lfloor y_2 \; A_{n(2+)} \; A_{nn(2+)} \; A_{n(n-1)(2+)} \; A_{n(2-)} \; A_{nn(2-)} \; A_{n(n-1)(2-)} \rfloor \tag{G-29}$$

By inverting the 7 x 7 matrix $\underline{\hat{M}}$, a solution is obtained for the unknowns contained in $\underline{\hat{\alpha}}_1$.

$$\underline{\hat{\alpha}}_1 = \underline{\hat{M}}^{-1}\,\underline{\hat{g}} \tag{G-30}$$

Having found values for seven of the unknowns, a solution for another seven is obtained in the form

$$\underline{\hat{\alpha}}_2 = \underline{\hat{M}}^{-1}\,\underline{\beta}_1 \tag{G-31}$$

where $\underline{\hat{M}}$ is the same matrix as in equation G-30 and

$$\underline{\hat{\alpha}}_2' = \lfloor y_1 \; A_{n(1+)} \; A_{nn(1+)} \; A_{n(n-1)(1+)} \; A_{n(1-)} \; A_{nn(1-)} \; A_{n(n-1)(1-)} \rfloor \tag{G-32}$$

$\underline{\beta}_1$ is a function of the $g_{k_i 1}$ and the quantities contained in $\hat{\underline{\alpha}}_1$.

Similarly, a solution is obtained for the remaining seven unknowns

$$\hat{\underline{\alpha}}_3 = \hat{\underline{M}}^{-1} \underline{\beta}_2 \tag{G-33}$$

where $\hat{\underline{M}}$ is again the same matrix as in equation G-30 and

$$\hat{\underline{\alpha}}_3' = \lfloor y_0 \;\; A_{n(0+)} \;\; A_{nn(0+)} \;\; A_{n(n-1)(0+)} \;\; A_{n(0-)} \;\; A_{nn(0-)} \;\; A_{n(n-1)(0-)} \rfloor \tag{G-34}$$

$\underline{\beta}_2$ is a function of the $g_{k_i 0}$ and the previously determined unknowns in $\hat{\underline{\alpha}}_1$ and $\hat{\underline{\alpha}}_2$.

The specific form of $\hat{\underline{M}}$ in this example case is

$$\hat{\underline{M}} = \begin{bmatrix} 1 & u_{k_1-1} & u_{k_1-1}^2 & (2u_{k_1-1}\, u_{k_1-2}) & u_{k_1} & u_{k_1}^2 & (2u_{k_1}\, u_{k_1-1}) \\ 1 & u_{k_1} & u_{k_1}^2 & (2u_{k_1}\, u_{k_1-1}) & u_{k_2} & u_{k_2}^2 & (2u_{k_2}\, u_{k_1}) \\ 1 & u_{k_2} & u_{k_2}^2 & (2u_{k_2}\, u_{k_1}) & u_{k_3} & u_{k_3}^2 & (2u_{k_3}\, u_{k_2}) \\ 1 & u_{k_3} & u_{k_3}^2 & (2u_{k_3}\, u_{k_2}) & u_{k_4} & u_{k_4}^2 & (2u_{k_4}\, u_{k_3}) \\ 1 & u_{k_4} & u_{k_4}^2 & (2u_{k_4}\, u_{k_3}) & u_{k_5} & u_{k_5}^2 & (2u_{k_5}\, u_{k_4}) \\ 1 & u_{k_5} & u_{k_5}^2 & (2u_{k_5}\, u_{k_4}) & u_{k_6} & u_{k_6}^2 & (2u_{k_6}\, u_{k_5}) \\ 1 & u_{k_6} & u_{k_6}^2 & (2u_{k_6}\, u_{k_5}) & u_{k_7} & u_{k_7}^2 & (2u_{k_7}\, u_{k_6}) \end{bmatrix} \tag{G-35}$$

where it can be seen that $\hat{\underline{M}}$ is a function of the past control forces only. The solution for the 21 unknowns involves the inversion of this 7 x 7 matrix and three matrix multiplications which is simpler than inverting the full 21 x 21 matrix of equation G-24.

Assuming a second order system, the output state is given by

$$\left.\begin{aligned} x(t) &= \sum_{e=0}^{2} g_{ne}\, t^{e} \\ \dot{x}(t) &= \sum_{e=0}^{2} e\, g_{ne}\, t^{e-1} \end{aligned}\right\} \qquad nT \leq t < (n+1)T \qquad \text{(G-36)}$$

where the g_{ne} coefficients are obtained by substituting the solution values of the $A_{\langle n\rangle p\pm}$ coefficients and the control forces u_n, u_{n-1} and u_{n-2} into equations G-21 through G-23.

The vector $\underline{x}(t)$ can be represented during the interval $nT \leq t < (n+1)T$ by an equation of the form

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} x_{n1}(t) \\ x_{n2}(t) \end{bmatrix} + \begin{bmatrix} a_{n1} & a_{nn1} \\ a_{n2} & a_{nn2} \end{bmatrix} \bullet \begin{bmatrix} u_n \\ u_n^{2} \end{bmatrix} \qquad \text{(G-37)}$$

where equation G-37 follows from G-36 with the proper division of the terms into those which contain u_n, those that contain u_n^2 and those which contain neither u_n or u_n^2.

Equation G-37 is of a form suitable for application of the control policy as presented in Section 4.1 of the text. The matrix $\hat{\underline{M}}$ of equations G-30, 31, 33 must be 'updated' every interval by shifting the most recent control force functions into the last row (equation G-35) and shifting the remaining rows up one thereby shifting the row containing the functions of the 'oldest' control forces out of the matrix. New $A_{\langle k\rangle p\pm}$ coefficients are therefore determined every interval.

EXAMPLE CASE TWO

In this case, the degree of the Taylor series expansions of the $A_{\langle k\rangle}(t)$ terms is raised by one.

1. I = 1
2. P = 3
3. S = 0

Under the above assumptions, equation G-17 reduces to

$$g_{ke} = y_e + \sum_{p=0}^{3} \binom{p}{e} \Big\{ m_{\langle k-1\rangle} U_{\langle k-1\rangle} A_{\langle k-1\rangle p+} \left[-(k-1)T\right]^{p-e} + m_{\langle k\rangle} U_{\langle k\rangle} A_{\langle k\rangle p-} (-kT)^{p-e} \Big\} \qquad \text{(G-38)}$$

where from equation G-6 or G-7 the significant μ sets is again 3. The number of past intervals of data which are necessary to determine the coefficients is given by equation G-18 to be 7.

The unknown coefficients to be determined are

$$\left.\begin{matrix} A_{np+} & \\ & A_{np-} \\ A_{nnp+} & \\ & A_{nnp-} \\ A_{n(n-1)p+} & \\ & A_{n(n-1)p-} \\ y_p & \end{matrix}\right\} \quad p = 0,\ 1,\ 2,\ 3 \qquad \text{(G-39)}$$

where in this case the number of unknowns is 28. The solution proceeds in a manner identical with that of the first example case except for the inclusion of the additional terms corresponding to p=3. The matrix $\hat{\underline{M}}$ is again 7 x 7 and the solution for the 28 unknowns requires four matrix multiplications of the form of equations G-30, 31, 33.

EXAMPLE CASE THREE

In this case, a time varying plant is assumed.

1. I = 1
2. P = 2
3. S = 1

Under the above assumptions, equation G-17 reduces to

$$g_{ke} = y_e + \sum_{s=0}^{1} \sum_{p=0}^{2} \binom{p}{e} \Big\{ m_{<k-1>} U_{<k-1>} A_{<k-1>ps+} \left[-(n-k+1)T\right]^{s} \left[-(k-1)T\right]^{p-e} + m_{<k>} U_{<k>} A_{<k>} \left[-(n-k)T\right]^{s} (-kT)^{p-e} \Big\} \tag{G-40}$$

where from equation G-6 or G-7 the significant μ sets is again 3. The number of past intervals of data which are necessary to determine the coefficients is given by equation G-18 to be 13.

The unknown coefficients to be determined are

$$\left.\begin{array}{ll} A_{nps+} & \\ & A_{nps-} \\ A_{nnps+} & \\ & A_{nnps-} \\ A_{n(n-1)ps+} & \\ & A_{n(n-1)ps-} \\ y_p & \end{array}\right\} \quad \begin{array}{l} p = 0,\ 1,\ 2 \\ s = 0,\ 1 \end{array} \tag{G-41}$$

where in this case the number of unknowns is 39. The solution proceeds in a manner very similar to that of example case one except that there are now two $A_{\langle k \rangle p}$ terms on the right hand sides of the equations equivalent to G-21-23 due to the fact that s takes on values of 0 and 1 whereas before only those terms corresponding to s=0 were considered. The matrix $\underline{\hat{M}}$ in this case will be 13 x 13 due to the fact that for a given value of p there are now 13 unknowns as opposed to only seven given in example case one. Once $\underline{\hat{M}}$ is inverted, it can be used repeatedly in the three matrix multiplications needed to solve for the 39 unknowns. The procedure for placing the equations in the proper form is identical to that of example one where an equation of the form of G-37 is obtained.

G.3 CONCLUSIONS

The R=2 Volterra series method of estimation of the current plant response and current sensitivity involves quite a few unknowns. The matrix which must be inverted to solve for these unknowns is not equal to the number of unknowns, fortunately, as inverting a 39 x 39 matrix could not be considered practical. The matrix which is inverted is a function of the control forces which can in most cases be known with greater accuracy than the state variables which make up the inverted matrix in the interpolation procedure.

A summary of some important equations is given below.

1. The number of intervals of data required to identify the coefficients

$$n = 2\mu\,(S+1) + 1 \qquad \text{(G-42)}$$

2. The number of unknowns

$$\text{unknowns} = (P + 1)\ \left\{2\mu\,(S+1) + 1\right\} \qquad \text{(G-43)}$$

3. The size of the matrix which must be inverted to identify the unknown coefficients

$$\text{size} = 2\mu\,(S+1) + 1 \qquad \text{(G-44)}$$

4. The number of matric multiplications which must be performed to identify the unknown coefficients

$$\text{multiplications} = P + 1 \qquad \text{(G-45)}$$

Note that equations G-42 and G-44 are identical and that the size of the matrix which must be inverted is independent of the order of the Taylor series for the $A_{\langle k \rangle}(t)$ terms. The number of Taylor coefficients reflects in the number of matrix multiplications which must be made which is a much easier operation than matric inversion.

REFERENCES

1. Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identification," Proceedings of the JACC, 1964, pp. 366-375.

2. Volterra, V., Theory of Functionals and of Integral and Integro Differential Equations, Blackie and Sons, Ltd., London, 1930.

3. Blackman, J., "The Representation of Non-Linear Networks," Report No. 81560, Syracuse University Research Institute, undertaken for the Air Force Cambridge Research Center, Bedford, Mass., 1956.

4. Zaborszky, J. and Berger, R. L., "An Integral Square Self-Optimalyzing Adaptive Control," AIEE Trans. on Applications and Industry, Vol. 81, November 1962, pp. 256-262.

5. Zaborszky, J. and Berger, R. L., "Multivariable Adaptive Computer Control by Identification Feedback," IEEE Trans. on Applications and Industry, Vol. 82, September 1963, pp. 278-286.

6. Zaborszky, J. and Flake, R. H., "On Fitting With Functional Polynomials", Preprint Vol. of JACC for 1966; to be published in ASME Trans. on Basic Engineering.

APPENDIX H

PSEUDOINVERSE OF A RECTANGULAR MATRIX

The concept of the pseudoinverse of a rectangular matrix is apparently first due to Moore, and was later rediscovered by Penrose with extension by Greville. (References 1,2, and 3).

The basics of Penrose's development with respect to the present problem are as follows:
Consider the representation of a real linear equation set by the conformable matrix equation:

$$\underline{A}\ \underline{x} = \underline{b} \tag{H-1}$$

where $\underline{A}$ is a known generally rectangular matrix of dimension (m×n), $\underline{b}$ is a known (m×1) vector and $\underline{x}$ is an unknown (n×1) vector.

The equation set H-1 can conventionally be described as:

Underspecified	if	$m < n$
Fully specified		$m = n$
Overspecified		$m > n$

Then a non-unique generalized inverse of $\underline{A}$ designated $\underline{A}^+$ is defined by the condition:

$$\underline{A}\ \underline{A}^+\ \underline{A} = \underline{A} \tag{H-2}$$

If $\underline{A}^+$ is further constrained by:

$$\underline{A}^+\ \underline{A}\ \underline{A}^+ = \underline{A}^+ \tag{H-3}$$

$$(\underline{A}\,\underline{A}^{+})' = \underline{A}\,\underline{A}^{+} \tag{H-4}$$

$$(\underline{A}^{+}\,\underline{A})' = \underline{A}^{+}\,\underline{A} \tag{H-5}$$

Penrose has shown that the generalized inverse satisfying all these conditions is unique and termed it the pseudoinverse $\underline{A}^{+}$.

In the familiar fully specified case where $\underline{A}$ is square, if it also is nonsingular

$$\underline{A}^{+} = \underline{A}^{-1} \tag{H-6}$$

where $\underline{A}^{-1}$ is the conventionally defined inverse of $\underline{A}$. Thus the successive solution of an underspecified growing set of linear equations in a manner to be described converges on the exact solution as it reaches full specification.

H.1 BEST APPROXIMATION PROPERTY OF PSEUDOINVERSE SOLUTION

The particular utility of the pseudoinverse to the present problem is based on its ability to yield a best approximate solution for $\underline{x}$ according to the following definition:

Definition — $\underline{\tilde{x}}$ is defined as the best approximate solution of $\underline{x}$ in the sense that:
either,

$$(\underline{A}\,\underline{\tilde{x}} - \underline{b})'(\underline{A}\,\underline{\tilde{x}} - \underline{b}) < (\underline{A}\,\underline{x} - \underline{b})'(\underline{A}\,\underline{x} - \underline{b}) \quad \text{for all} \quad \underline{x} \neq \underline{\tilde{x}} \tag{H-7}$$

or

$$(\underline{A}\,\underline{\tilde{x}} - \underline{b})'(\underline{A}\,\underline{\tilde{x}} - \underline{b}) = (\underline{A}\,\underline{x} - \underline{b})'(\underline{A}\,\underline{x} - \underline{b}) \quad \text{and} \quad \underline{\tilde{x}}'\underline{\tilde{x}} < \underline{x}'\underline{x} \tag{H-8}$$

The inequality of the first stated best approximation criterion will be recognized as a form of the "least squares" fitting criterion, and will subsequently be so identified in the overspecified case. The inequality of the second stated criterion is simply the condition that the solution $\underline{\tilde{x}}$ have minimum Euclidean norm (which also implies minimum energy in physical phase spaces). Note that the equality of the second stated criterion can generally be satisfied only by underspecified equation sets.

Theorem — The following theorem due to Penrose asserts:
$\underline{\tilde{x}} = \underline{A}^{+}\underline{b}$ is the unique best approximate solution of the equation set $\underline{A}\ \underline{x} = \underline{b}$.

The utility of the pseudoinverse matrix having been thus established, it remains to find forms for its calculation. Two special cases follow.

PSEUDOINVERSE MATRIX OF AN UNDERSPECIFIED EQUATION SET

The pseudoinverse matrix for the underspecified case can be formalistically exhibited as follows:

Consider the set of m linear equations representing the underspecified case

$$\underline{A}_{(m\times n)}\ \underline{x}_{(n\times 1)} = \underline{b}_{(m\times 1)} \qquad m<n \qquad \text{(H-9)}$$

Introduce a reduced set of variables $\underline{y}$ defined by:

$$\underline{x}_{(n\times 1)} = \underline{A}'_{(n\times m)}\ \underline{y}_{(m\times 1)} \qquad \text{(H-10)}$$

Premultiply by $\underline{A}$ and form the solution for $\underline{y}$ by inversion.

$$\underline{A}\ \underline{A}'\ \underline{y} = \underline{b} \qquad \text{(H-11)}$$

$$\underline{y} = (\underline{A}\ \underline{A}')^{-1}\ \underline{b} \qquad \text{(H-12)}$$

Back substitute the formal solution for $\underline{y}$ and identify the resultant product as the right pseudoinverse $\underline{A}^+$.

$$\underline{\tilde{x}} = \underline{A}' \ (\underline{A} \ \underline{A}')^{-1} \ \underline{b} = \underline{A}^+ \ \underline{b} \tag{H-13}$$

$$\underline{A}^+ = \underline{A}' \ (\underline{A} \ \underline{A}')^{-1} \tag{H-14}$$

which is the desired pseudoinverse in the underspecified case. This development presupposes that $\underline{A}$ is of maximal rank, so that the rows of $\underline{A}$ are linearly independent, and $(\underline{A} \ \underline{A}')$ is nonsingular. It can be shown to satisfy the second stated best approximation criterion:

$$\begin{gathered} (\underline{A} \, \underline{\tilde{x}} - \underline{b})' \ (\underline{A} \, \underline{\tilde{x}} - \underline{b}) = (\underline{A} \, \underline{x} - \underline{b})' \ (\underline{A} \, \underline{x} - \underline{b}) \\ \text{and} \quad \| \underline{\tilde{x}} \| = \underline{\tilde{x}}' \underline{\tilde{x}} < \underline{x}' \underline{x} = \| \underline{x} \| \end{gathered} \tag{H-15}$$

This underspecified case is that applicable to the startup problem, and is seen to satisfy the condition of a minimum norm of the solution $\underline{\tilde{x}}$.

It should be recognized that the preceding development is formal only, and does not constitute a proof Particularly the conditions for existance and uniqueness of the indicated forms are to be found in the references.

PSEUDOINVERSE MATRIX OF AN OVERSPECIFIED EQUATION SET

The corresponding form of the pseudoinverse matrix for the overspecified case can be displayed even more directly as follows.

Consider the set of m linear equations representing the overspecified case

$$\underline{A}_{(m \times n)} \ \underline{x}_{(n \times 1)} = \underline{b}_{(m \times 1)} \qquad m > n \tag{H-16}$$

Note that $\underline{A}$ is of maximal rank n.

Left multiply by the transpose of $\underline{A} = \underline{A}'$ and effect a formal solution by inversion of the resultant product matrix.

$$\underline{A}' \; \underline{A} \; \underline{x} = \underline{A}' \; \underline{b} \tag{H-17}$$

$$(\underline{A}' \; \underline{A})^{-1} (\underline{A}' \; \underline{A}) \; \underline{\tilde{x}} = \underline{I} \; \underline{\tilde{x}} = \underline{\tilde{x}} = (\underline{A}' \; \underline{A})^{-1} \; \underline{A}' \; \underline{b} = \underline{A}^{+} \; \underline{b} \tag{H-18}$$

Here the left pseudoinverse is identified with the matric product

$$\underline{A}^{+} = (\underline{A}' \; \underline{A})^{-1} \; \underline{A}' \tag{H-19}$$

Again the preceding development is formalistic, and presupposes that $\underline{A}$ is of maximal rank so that the product $(\underline{A}'\underline{A})$ is positive definite and therefore nonsingular.

This "overspecified" case is not per se pertinent to the "startup problem". It is presented here however for completeness, and because it may ultimately be found useful in an updating procedure. The formal similarity of the "overspecified" to the "underspecified" case is striking, and suggests a closer parallism of these extremum cases than is superficially apparent.

H.2 TWO RECURSIVE ALGORITHMS FOR THE PSEUDOINVERSE

In principle equation H-14 is directly applicable at each stage of the startup process. However its direct utilization requires the inversion of the square matrix $(\underline{A} \; \underline{A}')$, which grows in rank at each successive step. The fact that the matrix $\underline{A}$ grows only by a single row at each step of the startup process with all previously determined elements unchanged suggests that a recursive algorithm avoiding full rank inversion at each step may be found.

WELLS' RECURSIVE ALGORITHM (Reference 4) — Wells' discovered one such algorithm as follows:

Given a matrix $\underline{A}_k$ extant at the k'th step of the starting procedure, at the next step the matrix $\underline{A}_{k+1}$ is of the form:

$$\underline{A}_{k+1} = \begin{bmatrix} \underline{A}_k \\ \hline \underline{a}_k' \end{bmatrix} \qquad \text{(H-20)}$$

where $\underline{a}_k'$ denotes the added row.
Remembering that the pseudoinverse is conformable, a corresponding form of the pseudoinverse is:

$$\underline{A}_{k+1}^{+} = \left[\underline{C}_k' \;\middle|\; \underline{c}_k \right] \qquad \text{(H-21)}$$

where $\underline{c}_k$ denotes an added column.

By introduction of some additional transformations, Wells was able to identify the foregoing process of row augmentation of the direct matrix with general columnar augmentation as previously analyzed by Greville. Wells obtained pertinent results both in the case where the direct matrix was stationary in rank and where it grows in rank under the row augmentation.

Only the latter case is pertinent to the startup procedure, and for it Wells result is:

$$\underline{A}_{k+1}^{+} = \left[\underline{A}_k^{+} - \underline{c}_k \left(\underline{a}_k' \underline{A}_k^{+} \right) \;\middle|\; \underline{c}_k \right] \qquad \text{(H-22)}$$

where:

$$\underline{c}_k = \left[\underline{a}_k' \left(\underline{I} - \underline{A}_k^{+} \underline{A}_k \right) \right]^{+} = \underline{Q}_k \, \underline{a}_k \left(\underline{a}_k' \underline{Q}_k \underline{a}_k \right)^{-1} \qquad \text{(H-23)}$$

and:

$$\underline{Q}_k = \underline{I} - \underline{A}_k^{+} \underline{A}_k \tag{H-24}$$

Note that the only matrix inversion required at each step by Wells recursive algorithm is the trivial inversion of a scalar.

<u>ANOTHER RECURSIVE ALGORITHM</u> — In the course of preparing this report another recursive algorithm for the running computation of the pseudoinverse was developed based on the application of matrix inversion by bordering. The development is as follows:

From the definition of the pseudoinverse matrix for the underspecified case, at the k'th step of startup it is given by:

$$\underline{A}_k^{+} = \underline{A}_k{}' \left(\underline{A}_k \underline{A}_k{}' \right)^{-1} = \underline{A}_k{}' \underline{B}_k^{-1} \tag{H-25}$$

where: $\underline{B}_k = \underline{A}_k \underline{A}_k{}'$

At the succeeding step it is given by:

$$\begin{aligned}
\underline{A}_{k+1}^{+} &= \left[\underline{A}_k{}' \,\vdots\, \underline{a}_k \right] \left(\begin{bmatrix} \underline{A}_k \\ \hline \underline{a}_k{}' \end{bmatrix} \left[\underline{A}_k{}' \,\vdots\, \underline{a}_k \right] \right)^{-1} \\
&= \left[\underline{A}_k{}' \,\vdots\, \underline{a}_k \right] \left[\begin{array}{c|c} \underline{A}_k \underline{A}_k{}' & \underline{A}_k \underline{a}_k \\ \hline \underline{a}_k{}' \underline{A}_k{}' & \underline{a}_k{}' \underline{a}_k \end{array} \right]^{-1} \\
&= \left[\underline{A}_k{}' \,\vdots\, \underline{a}_k \right] \begin{bmatrix} \underline{B}_k & \underline{d}_k \\ \underline{e}_k{}' & f_k \end{bmatrix}^{-1}
\end{aligned} \tag{H-26}$$

where:

$$\underline{d}_k = \underline{A}_k \underline{a}_k \qquad \underline{e}_k{}' = \underline{a}_k{}' \underline{A}_k{}' \qquad f_k = \underline{a}_k{}' \underline{a}_k \tag{H-27}$$

Note that under this partioning $\underline{d}_k$ is a row vector, $\underline{e}'_k$ is a column vector, and f_k is a scalar.

But the required inverse is readily recognized as that of a bordered matrix, with a well known solution which becomes in the present symbology: (Reference 5)

$$\begin{bmatrix} \underline{B}_k & \underline{d}_k \\ \underline{e}_k{}' & f_k \end{bmatrix}^{-1} = \frac{1}{\alpha_k} \left[\begin{array}{c|c} \alpha_k \underline{B}_k^{-1} + \underline{B}_k{}^{-1} \underline{d}_k \underline{e}_k{}' \underline{B}_k{}^{-1} & -\underline{B}_k{}^{-1} \underline{d}_k \\ \hline -\underline{e}_k{}' \underline{B}_k{}^{-1} & 1 \end{array} \right] \tag{H-28}$$

$$\text{where:} \quad \alpha_k = f_k - \underline{e}_k{}' \underline{B}_k{}^{-1} \underline{d}_k \tag{H-29}$$

Since by supposition in a recursive process the only inverse requisite for this determination $\underline{B}_k{}^{-1}$ has been calculated in the preceding step, this algorithm permits running calculation of the pseudoinverse, again without growing rank inversion.

REFERENCES

1. Penrose, R., "A Generalized Inverse for Matrices," Proceedings of the Cambridge Philosophical Society, Vol. 51, 1956, pp. 406-413.

2. Penrose, R., "On Best Approximate Solutions of Linear Matrix Equations," Proceedings of the Cambridge Philosophical Society, Vol. 52, 1955, pp. 17-19.

3. Greville, T. N. E., "The Pseudoinverse of a Rectangular or Singular Matrix and Its Application to the Solution of Systems of Linear Equations,: Society of Industrial and Applied Mathematics Review, Vol. 1, No. 1, Jan., 1959, pp. 38-43.

4. Wells, C. H., "Minimum Norm Control of Discrete Systems," Doctoral Dissertation, Washington University, St. Louis, Missouri, 1966.

5. Faddeeva, V. N., <u>Computational Methods of Linear Algebra</u>, Dover Publications, Inc., New York, 1959, Chapter 2.

APPENDIX I

COMPUTATION REQUIREMENTS

Practical application of the various control methods of this report depends directly on the requirements they impose on the on-line digital control computer. It is the purpose of this appendix to identify these computational requirements.

The following analysis deals with the 'Interpolation Prediction' method of plant representation for several reasons:

(1) 'Interpolation Prediction' has been identified as the preferred method of linear stationary plant control, based on generality of application. For linear nonstationary plants, its formalism is innately suitable. For nonlinear and/or nonstationary plants, it is tentatively preferred, partially because of computation requirements.

(2) The computational requirements of the other methods of linear stationary plant control have been previously estimated in reference 1.

(3) The 'Interpolation Prediction' method exhibits all of the known problems of computer use in at least moderately complex form.

I.1 METHODS OF ESTIMATING COMPUTER REQUIREMENTS

A superfically simple way of establishing the computing requirements is to print out the elapsed clock times of execution of 'appropriate' portions of the IBM 7094 simulation programs. This method has not been used for two reasons:

(1) The complexity of branching and looping operations of the simulation programs (which were designed to be highly flexible)

makes the 'appropriate portions' difficult to identify.

(2) The extensive use of 'canned' general subroutines suggested that such clock times would be so highly conditioned as to mask the desired inherent times of calculation.

We particularly found this method to be prohibitively difficult and tedious to apply to already existant programs.

The following analyses then were made by estimation directly from the control equations, assuming conventional methods of computation.

SPECIFIC ESTIMATION METHODS

Execution Time - The following analyses will show that three basic operations dominate the use of the 'Interpolation Prediction' method:

(1) Vector addition

(2) Matrix multiplication of two types:

 (a) Vector by vector multiplication

 (b) Multiplication of a vector by a conformable square matrix

(3) Inversion of a nonsingular square matrix.

The computing time required for operation (1) is simply:

$$T_1 = qA$$

where q is the vector dimension and A is the computer add time including operand access.

Operation (2a) requires an execution time of:

$$T_{2a} = q(A + M)$$

where M is the computer multiplication time and the composite operation requires access to 2q operands. Following reference 2 we shall alternately use:

$$T_{2a} = 2.75q\ M$$

The leading numeric factor represents an average observed factor appropriate to one-to-one combined operations of multiplication and addition including access time, arbitrarily increased by 10% to absorb trivial operations and bookkeeping.

Operation (2b) using the same estimating factor has execution time:

$$T_{2b} = 2.75q^2 \; M$$

Assuming a fully developed matrix and a minimum operation method such as the Gauss-Jordan reduction, reference 2 gives a characteristic time for execution of operation (3) of:

$$T_3 = 2.75q^3 \; M$$

If additionally r solutions of the simultaneous set of equations partially described by the matrix under inversion is desired, the execution time becomes:

$$T_3 = 2.75 \; (q^3 + rq^2)M$$

The following time estimates then will be based on identification of the aforedescribed operations.

Instruction Storage Requirements - Subroutine programming of these basic operations is assumed with estimated instruction storage requirements as given in Table I.1.

Operation	Instruction Words
(1) Vector addition	10
(2a) Vector multiplication	20
(2b) Vector by matrix multiplication	50
(3) Matrix inversion and/or multiple solution	400 (reference 2)

Table I.1

STORAGE REQUIREMENTS

I.2 LINEAR STATIONARY PLANTS

Running Computation - The final closed form solution for the running value of the control force is given in matric form in Section 2.1 Table 2.1 as:

$$u_k = \frac{\underline{\lambda}_I{}' \; \underline{K} \left[\underline{r}((k+1)T) - \underline{F}_I \; x(kT^-) - \underline{b}_{2I} \; u_{k-1}\right]}{\underline{\lambda}_I{}' \; \underline{K} \; \underline{\lambda}_I} \tag{I-1}$$

Once the initialization process is complete, the matrices $\underline{\lambda}_I$, $\underline{\lambda}_I'$, $\underline{b}_{2_I}$, $\underline{F}_I$ are fully determined. Further the positive definite weighting matrix $\underline{K}$ is an a priori choice. Thus all of the matric coefficient products can be evaluated and stored at the conclusion of initialization.

Under this assumption and appropriate identification of the stored coefficient matrices in the running calculation equation I-1 takes the form:

$$u_k = \underline{a}'\underline{r}((k+1)T) + \underline{b}'\underline{x}(kT^-) + c\ u_{k-1} \qquad \text{(I-2)}$$

where the definitions of the (lxp) coefficient row vectors $\underline{a}'$ and $\underline{b}'$ and the scalar c are obvious by comparison with equation I-1, p being the plant order.

The running computation of equation I-2 requires two multiplications of p vectors, utilizing (2p + 2) prestored operands. In accordance with our general time estimation policy, the running computation time/decision interval is:

$$T_{multiply} = 5.5\ p\ M$$

Using the M = 12 μs multiplication time of the Honeywell H-387 airborne control computer as typical, results in an estimated computation time/decision interval of 0.33 ms for a fifth order plant. The preceding estimate is for computation only, and does not include input-output transfer time to the computer. A reasonable estimate of such time for a fifth order plant is of the order of 0.07 ms based on the manufacturer's (fragmentary) specifications.

Thus the total computation time/decision interval in the running state is:

$$T_{running} = 0.4\ ms$$

The preceding time estimates presume the use of the fixed point arithmetic inherent in the machine design. We have successfully implemented

DACS control in fixed point computation in previous studies, but cannot conscionably recommend it in the high order unknown plant context. Typically internal machine conversion to floating point computation slows calculation by a factor of approximately ten; and requires storage of approximately 200 instruction words for this purpose.

The calling routine for execution of equation I-2 should require approximately 20 instruction words.

STARTUP AND INITIALIZATION

The startup process occurs during the first (p + 2) control actions, p being the system order. During this startup period, a growing basis of vectors representing the results of the control actions are read in and stored in the computer memory. It is here assumed that the necessary computations with this data, here termed initialization, is deferred until the complete set of (p + 2) startup actions is complete.

This policy has the effect of making the computing requirements during startup and running use commensurate. However, a sharply peaked computing load occurs between transition. A better approach is exemplified in the recursion forms of the pseudo-inverse startup procedure, where the initialization calculation builds up concurrently with data availability. Recognizing this possibility for amelioration, we proceed to identify the requirements separately.

Start-up Computation - With the single exception that all data read into the computer is stored, the startup control computation can proceed quite analogously to the process previously described for the running computation in 'Interpolation Prediction .

The time of computation at each startup step should be closely approximate to that previously computed for the running computation augmented by something like 2p (A + S) where S is the memory write time. Again using the example of a fifth order system controlled by the Honeywell H-387 computer (p = 5, A = 2 μs, S = 4 μs), the computation time at a startup step is:

$$T_{startup} = 0.46 \text{ ms}$$

inclusing control force calculation, inputting, and data storage. Again fixed point computation has been assumed.

Initialization Computation - At the conclusion of the (p + 2) steps of the startup procedure, a square matrix $\underline{\Phi}$ comprising the (p + 2) particulate values of the base functional vectors $\underline{\phi}_k$ recorded in startup is available.

$$\underline{\Phi} = \left[\underline{\phi}_1 \,\vdots\, \underline{\phi}_2 \,\vdots\, \cdots \,\vdots\, \underline{\phi}_k \,\vdots\, \cdots \,\vdots\, \underline{\phi}_{(p+2)} \right] \tag{I-3}$$

$$\underline{\phi}_k{}' = \underline{\underline{x}'(kT) \quad u_k \quad u_{k-1}} \tag{I-4}$$

By the existance of the finite difference equation

$$\underline{x}((k+1)T^-) = \underline{\Theta}_1\, \underline{x}(kT^-) + \underline{\varphi}_1\, u_k + \underline{\varphi}_2\, u_{k-1} \tag{I-5}$$

the recorded values of the state vector $\underline{x}$ in the $\underline{\Phi}$ matrix have a dual interpretation as follows. Assume that the recorded values are representative of consecutive control actions. Then the $\underline{x}$ vector identified with the left hand side of equation I-5 for a given control action is identical to the $\underline{x}$ vector of the right hand side of the succeeding control action.

By utilization of this property, it is possible to select from the matrix I-3 the 'delayed' set of $\underline{x}$ vectors comprising the p x (p + 2) dimension matrix $_D\underline{X}$.*

$${}_D\underline{X} = \left[\underline{x}(2T^-) \,\vdots\, \cdots \,\vdots\, \underline{x}((k+1)T^-) \,\vdots\, \cdots \,\vdots\, \underline{x}((p+3)T^-) \right] \tag{I-6}$$

*As written equation I-6 assumes that the $\underline{x}'(kT)$ vectors of equation I-4 represent the state components existant at the initiation of the k'th control action. Thus all components of $_D\underline{X}$ except the last column are contained in the $\underline{\Phi}$ matrix. The last column is determinate at start-up termination but requires separate storage.

With the identification of equations I-3, I-5, and I-6, the set of linear equations existant at startup termination has the form:

$$ {}_{D}\underline{X} = \underline{B}\,\underline{\Phi} \tag{I-7} $$

Where $\underline{B}$ is a matrix of unknowns of dimension p x (p+2). The solution for the elements of $\underline{B}$ amounts to the p fold solution of the linear equation set I-7 of common basis $\underline{\Phi}$.

Invoking the previously cited formula for the execution time of this operation, with q = p+2 and r = p:

$$ T_{solution} = 2.75\ (2p^3 + 6p^2 + 12\ p + 8)\ M $$

Again using the 12 μs multiplication time of the Honeywell H-387 and a fifth order equation as typical:

$$ T_{solution} = 15.5 \text{ ms} $$

In principle $(p + 2)^2$ data words are required for storage of the direct $\underline{\Phi}$ matrix and p additional words are required for the single $\underline{x}$ vector not contained in $\underline{\Phi}$. Depending on the convenience of abstracting subarrays from $\underline{\Phi}$, it may be desirable to store the p(p + 2) elements of ${}_{D}\underline{X}$ separately in spite of their high redundance in $\underline{\Phi}$.

Further if it is desired to maintain the identity of the direct matrix $\underline{\Phi}$ for updating use as many as $(p + 2)^2$ additional data words could be required.

With the basic solution of equation I-7 accomplished, it remains to manipulate appropriate subsets of the $\underline{B}$ matrix into the forms required by the running equations I-1 and I-2. Using the notation of those equations, the matrix $\underline{B}$ can be partitioned as follows:

$$ \underline{B} = \left[\, \underline{F}_I \;\vdots\; \underline{b}_{1_I} \;\vdots\; \underline{b}_{2_I} \,\right] \tag{I-8} $$

where $\underline{F}_I$ is a p x p matrix, and $\underline{b}_{1_I}$ and $\underline{b}_{2_I}$ are p x 1 vectors.

An auxiliary definition based on the developments of Appendix A defines:

$$\underline{\lambda}_I = \underline{b}_{1_I} + \underline{F}_I^{-1} \underline{b}_{2_I} \tag{I-9}$$

The p order inversion of equation I-9 by a previously cited estimation formula requires an execution time:

$$T_3 = 2.5\ p^3\ M$$

Completing the calculation of $\underline{\lambda}_I$ requires a matrix by vector multiplication and a vector addition. Thus the total computation time for $\underline{\lambda}_I$ as a function of system order is:

$$T_{\underline{\lambda}_I} = 2.75\ (p^3 + p^2)\ M + p\ A$$

In the standard fifth order example here being developed this becomes:

$$T_{\underline{\lambda}_I} = 5\ \text{ms}$$

An additional data storage requirement of p^2 words developes for the calculation and storage of $\underline{F}_I^{-1}$. Since all operations are repetitions of previously estimated subroutines, ~20 additional instruction words are required.

The final initialization computation is of the several matric products required for the conversion of equation I-1 to I-2. First the column vector $\underline{\lambda}_I' \underline{K}$ is calculated. Because $\underline{K}$ is diagonal the appropriate time is that of a vector-vector product. After its calculation, two actual vector-vector products, $\underline{\lambda}_I' \underline{K}\ \underline{\lambda}_I$ and $\underline{\lambda}_I' \underline{K}\ \underline{b}_{2_I}$ follow. Finally the matrix-vector product $\underline{\lambda}_I' \underline{K}\ \underline{F}_I$ is calculated. Thus the total time of execution of these clean-up operations is:

$$T_{\text{products}} = 2.75\ (3p + p^2)\ M$$

Which for the example fifth order case yields:

$$T_{\text{products}} = 1.3\ \text{ms}$$

A calling routine of 20 instruction words should suffice to control this calculation.

The total time for the complete initialization calculation is obtained by summing as follows:

$$T_{initialization} = T_{solution} + T_{\underline{\lambda}_I} + T_{products}$$

$$= 2.75\ (3p^3 + 8p^2 + 15p + 8)\ M + p\ A$$

As evaluated for a fifth order plant using Honeywell H-387 parameters:

$$T_{initialization} = 21.8 \text{ ms}$$

Comparison of this result with the computation time of a startup step $T_{startup} = 0.46$ ms indicates a ratio approaching 50:1. This indicates a consistency of computation load as follows. It is assumed that the characteristic time for a startup or running calculation should be a small fraction of a decision interval, not more than say one-tenth. The (one shot) initialization calculation can thus be intermeshed with something like five additional startup steps, assuming full time computer utilization during this time only.

I.3 COMPUTER ACCURACY

The accuracy of digital computation is ultimately limited by the propagation of roundoff errors into the desired computational result. Since such cumulative errors are generally unpredictable, conventional practice is to provide internal representation of operands to sufficient precision that the most significant digits of the desired result are stable against such errors. Thus accuracy of computation is reflected into computer requirements directly as the computer data word length.

The identification of the requisite control computer accuracy was not a formal objective of the current study. However the following indirect conclusions can be inferred:

(1) Since the sole computational output is the value of a command control force which can be practically implemented to an

accuracy $\sim 0.1\%$, a valid 10 bit output word is adequate. This conclusion was verified in our previous hybrid simulations.

(2) Similarly available input sensors are typically of the same order of absolute accuracy. Ten bit input quantization was used in our previous hybrid simulations, with good correlation to digitally computed results of considerably higher precision.

(3) The most extensive and critical computation process involved in this method is matrix inversion. With a 36 bit word length our IBM 7094 simulations have routinely shown errors smaller than 10^{-6} in all elements of the approximate identity matrix formed by multiplication of the direct matrix by its calculated inverse. Inadequate control was observed, when by ill conditioning, these errors exceeded 10^{-3}. Assuming similar behavior for errors induced by less precise data representation, a minimum control computer word length of 26 bits is implied.

I.4 AVAILABLE SMALL AIRBORNE COMPUTERS

Preceding portions of this appendix have been based on the characteristics of the Honeywell H-387 airborne control computer. Unique suitability to the present problem is not implied. Rather it was chosen as a study vehicle by the following considerations.

A recent Emerson contractual study (reference 3) compared the characteristics of the following MIL E-5400 Class I airborne computers:

GE A-212; Honeywell H-387; Kearfott L90-1; Litton L304, L305, L306; Univac 1824; AC Sparkplug Magic III N; CDC MICC

The here pertinent requirements of the reference study were:

(1) A maximum basic cycle time (instruction and operation) not greater than 8 μs.

(2) In addition to standard operation codes, the computer should have a "Wait for Interrupt" instruction; and the instruction word should have the possibility of including an "inhibit" bit which would prevent the computer from servicing an interrupt

if the computer were performing high priority operations.

(3) Associated with the "Load Output Register" instruction and "Read Input Register" instruction there should be a capability of selecting registers external to the computer by appropriate strobe pulses.

(4) The computer memory size should be expandable to at least 16K 12-bit words.

(5) The computer weight must be less than 50 (fifty) pounds. The size should be less than 0.75 cubic feet.

(6) The computer must satisfy MIL E-5400 Class I.

(7) A 24-bit minimum precision of the data word is required, and greater precision to a maximum of 36-bits is desirable.

Several other requirements, peculiar to the reference study application but not germane to DACS usage, were included in the original specification. However it was found that their relinquishment does not add to the forestated set of computers satisfying requirements (1) through (7).

All of the listed machines satisfy the stated requirements with occasional minor deviations. The previous analyses have shown that two properties are particularly important to the DACS application:

(1) Data word length

(2) Multiplication execution time

Available data indicated that, while these characteristics were not necessarily reciprocally correlated, neither did a uniquely preferable combination exist. The Honeywell H-387 was somewhat arbitrarily chosen as satisfactory in the second property, but marginal in the first.

REFERENCES

1. "Digital Adaptive Control System Research, Summary Report," Emerson Electric Report No. 1544-5, March 31, 1964.

2. Ralston, A. and Wilf, H. S., ed., Mathematical Methods for Digital Computers, Wiley, 1960, Part II-2.

3. "Study and Design of an Integrated Radome Antenna and RF Circuitry (RARF)," Technical Report (AF 33(615) -3289) Vol. I, Emerson Electric Co., Electronics and Space Div., Feb. 1966 (SECRET).

BIBLIOGRAPHY

Aseltine, J. A., Transform Method In Linear System Analysis, McGraw-Hill, Inc., New York, 1958.

Bellman, R., Stability Theory of Differential Equations, McGraw-Hill, Inc. New York, 1953.

Blackman, J., "The Representation of Nonlinear Networks," Report No. 81560, Syracuse University Research Institute, Undertaken for the Air Force Cambridge Research Center, Bedford, Mass., 1956.

Blakelock, J. H., Automatic Control of Aircraft and Missiles, John Wiley and Sons, Inc., 1965.

Chidambara, M. R. and Wells, C. H., "Determination of State Variables for Digital Control," IEEE Trans., Vol. AC-11, No. 2, April 1966.

Chidambara, M. R., "A Method of Determination of State Variables for Known Plants," Doctoral Dissertation, Washington University, St. Louis, Mo., 1964.

Coddington, E. A. and Levinson, N., Theory of Ordinary Differential Equations, McGraw-Hill, Inc., New York, 1955.

Cunningham, W. J., Introduction To Nonlinear Analysis, McGraw-Hill, Inc., New York, 1958.

"Digital Adaptive Control System Research, Quarterly Report No. 1," Emerson Electric Report No. 1544-1, May 10, 1963.

"Digital Adaptive Control System Research, Quarterly Report No. 2," Emerson Electric Report No. 1544-2 August 20, 1963.

"Digital Adaptive Control System Research, Quarterly Report No. 3," Emerson Electric Report No. 1544-3, November 15, 1963.

"Digital Adaptive Control System Research, Quarterly Report No. 4," Emerson Electric Report No. 1544-4, February 21, 1964.

"Digital Adaptive Control System Research, Summary Report," Emerson Electric Report No. 1544-5, March 31, 1964.

Faddeeva, V. N., Computational Methods of Linear Algebra, Dover Publications, Inc., New York, 1959.

Flake, R. H., "Volterra Series Representation of Nonlinear Systems," AIEE Trans. on Applications and Industry, No. 82, January, 1963.

Flake, R. H., "Volterra Series Representation of Time-varying Non-linear Systems," Proceedings of Second IFAC Congress, 1963, Butterworths, London.

Flake, R. H., "Volterra Series Representation of Nonlinear Systems," Doctorial Dissertation, Washington University, St. Louis, Mo., 1962.

Gantmacher, F. R., The Theory of Matrices,Vol. I and II, Chelsea Publishing Company, New York, 1959.

Gardner, M. F., and Barnes, J. L., Transients In Linear Systems, John Wiley and Sons, Inc., New York, 1942.

George, D. A., "Continuous Nonlinear Systems," Research Laboratory of Electronics, M. I. T., Cambridge, Report No. 355; 1959.

Gibson, J. E. and Meditch, J. S., "Adaptive Control Review," Technical Document Report No. ASD-TDR-61-28, Vol. I, September 1965.

Gibson, J. E., Nonlinear Automatic Control, McGraw-Hill, Inc., New York, 1963.

Gorman, D. and Zaborszky, J., "Functional Representation of Nonlinear Systems, Interpoaltion and Lagrange Expansion for Functionals," Proceedings of the JACC, 1965.

Greville, T. N. E., "The Pseudoinverse of a Rectangular or Singular Matrix and Its Application to the Solution of Systems of Linear Equations," Society of Industrial and Applied Mathematics Review, Vol. 1, No. 1, January 1959.

Guillemin, E. A., The Mathematics of Circuit Analysis, John Wiley and Sons, Inc., New York, 1949.

Hildebrand, F. B., Introduction To Numerical Analysis, McGraw-Hill, Inc., New York, 1956.

Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," Journal of the Society of Industrial and Applied Mathematics, Ser. A: On Control, Vol. 1, No. 2.

McFee, R., "Response of Nonlinear Systems to Arbitrary Inputs," AIEE Trans. on Applications and Industry, No. 56, September, 1961.

Ostfeld, D. M., "A Suboptimal Nonlinear Control Algorithm Through Multivariable Interpolation," Master of Science Thesis, Washington University, St. Louis, Mo., 1965.

Penrose, R., "A Generalized Inverse for Matrices," Proceedings of the Cambridge Philosophical Society, Vol. 51, 1956.

Penrose, R., "On Best Approximate Solutions of Linear Matrix Equations," Proceedings of the Cambridge Philosophical Society, Vol. 52, 1955.

Ralston, A. and Wilf, H. S., Ed., Mathematical Methods for Digital Computers, John Wiley and Sons, Inc., 1960.

Serex, H. M. and Holt, J. N., "Comparison of High-Gain-Linear and Self Adaptive Flight Control Systems with Nonlinearities Introduced," Master of Science Thesis, Air Force Institute of Technology, Thesis GGC/EE/64-11.

"Study and Design of an Integrated Radome Antenna and RF Circuitry (RARF)," Technical Report (AF33(615) - 3289), Vol. I, Emerson Electric Co., Electronics and Space Division, February 1966, (Secret).

Tou, J. T., Modern Control Theory, McGraw-Hill, Inc., New York, 1964.

Volterra, V., Theory of Functionals and of Integral and Integro-Differential Equations, Blackie and Sons, Lts., London, 1930.

Wells, C. H., "Minimum Norm Control of Discrete Systems," Doctorial Dissertation Washington University, St. Louis, Mo., 1966.

Zaborszky, J. and Berger, R. L., "An Integral Square Self-Optimalyzing Adaptive Control," AIEE Trans. on Applications and Industry, Vol. 81, November, 1962.

Zaborszky, J. and Berger, R. L., "Multivariable Adaptive Computer Control by Identification Feedback," IEEE Trans. on Applications and Industry, Vol. 82, September, 1963.

Zaborszky, J. and Flake, R. H., "On Fitting With Functional Polynomials," Preprint Vol. of JACC for 1966; to be published in ASME Trans. on Basic Engineering.

Zaborszky, J. and Humphrey, W. L., "Control Without Model or Plant Identif cation," Proceedings of the JACC, 1964.

Zadeh, L. A. and Desoer, C. A., Linear System Theory, McGraw-Hill, Inc., New York, 1963.

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