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Prepared by
ARIZONA STATE UNIVERSITY
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for

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ABSTRACT

The dynamic characteristics of a variable-mass, slender elastic body have been investigated. The mathematical model, simulating a solid-fuel rocket, comprises of a cylindrical elastic shell, open at one end and closed at the other end, subjected to internal gas flow due to fuel burning. Most other investigators of the problem follow two major approaches. One approach regards the rocket as an elastic shell of constant mass with the burning of fuel effect in the form of a jet thrust applied at the open end. The other ignores the elasticity of the rocket but considers the mass variations. The unique feature of this work lies in the fact that it uses a more realistic mathematical model by considering simultaneously the mass variation and both the axial and flexural elasticity of the boost vehicle. A distinct objective of this investigation is to examine the meaning of normal mode vibration for mass-varying systems. The investigation was initiated by the first author during his work as a Research Fellow in the Dynamic Loads Division, NASA, Langley Research Center in the summer of 1964.

The motion of the vehicle is assumed to be planar and consisting of two rigid-body translations, one rigid-body rotation, one axial elastic displacement and one lateral elastic displacement. Since the interest lies in the dynamic characteristics of the vehicle rather than its absolute position in space, body axes and not inertial axes are used. The mathematical formulation is effected by first deriving the differential equation of motion in vector form for a mass-varying element of case and then combining

it with the vector differential equation of a fluid element. The latter is derived by using a noninertial control volume. This entirely novel approach provides a large degree of mathematical rigor. A variational principle transforms the single vector equation into three scalar ordinary and two scalar partial differential equations with the associated boundary conditions. The five coupled equations of motion are highly nonlinear and, in addition, contain terms due to the internal gas flow.

An investigation of the forcing functions resulting from the mass rate of flow, as reflected in terms due to internal pressure and velocity distributions, is undertaken and the effect of considering a noninertial control volume examined. Suitable approximations concerning the internal fluid flow are introduced and their validity verified.

To study the dynamic characteristics of the boost vehicle, the case of the vertical, upright flight was solved. The closed form solution was effected by regarding the vertical rigid-body flight of the vehicle as a "primary motion." The remaining two rigid-body motions and two elastic displacements are considered as perturbations from the primary motion forming a "secondary motion." Although as a result of this assumption the equations of motion become linear, one should recall that they involve nonperiodic time-dependent coefficients for which there is no general method of solution. The partial differential equations are reduced to two coupled sets of ordinary differential equations with time-dependent coefficients by expanding the elastic displacements in series of eigenfunctions corresponding to the associated constant-mass system. Assuming that the mass distribution is uniform, although time-dependent, the set of equations for the axial motion becomes uncoupled and a solution can be obtained in terms of Bessel functions. By limiting the number of terms in both series, one can obtain a solution for the lateral motion by the method of Frobenius.

No such solutions are known to have been obtained before. Although in effecting a solution, the eigenfunctions of the corresponding constant-mass system have been used, the elastic motion is by no means normal mode vibration since both the frequency of oscillation and the amplitudes vary with time. Hence, one cannot speak of natural frequencies and normal modes, in an ordinary sense, and should regard the use of eigenfunctions strictly as a mathematical convenience with no particular physical significance attached. For no initial lateral displacement or velocity no lateral vibration is excited.

As a verification of the assumptions necessary in a perturbation solution, a numerical solution for the coupled equations of motion is also obtained. The equations are written in finite difference form and are programmed for solution on a digital computer by utilizing an iterative procedure to cope with the nonlinearity of the equations.

An excellent correlation between the two solutions is obtained, with the variation increasing slightly with time. The assumption of rectilinear, rigid-body translation is confirmed, as well as the lack of excitation for the lateral elastic motion. The effect of the fluid flowing through the case is sufficient to prevent rigid-body rotation, or tumbling, and only a very slight rigid-body drift normal to the line of flight is indicated. The internal pressure, acting in the axial direction on both ends, subjects the case to an axial tension which tends to stiffen the case and reduce the lateral vibration, especially at the forward end.

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I. INTRODUCTION

1.1 Statement of the Problem

This investigation is concerned with the dynamic characteristics of a variable-mass body moving through space. The mathematical model, simulating a solid-fuel rocket, consists of a slender, elastic case closed at the end $x = L$ and open through a nozzle at the end $x = 0$ (see Fig. 1.1). The products of combustion, treated as a gas flowing relative to the case, are expelled through the end $x = 0$. Of primary interest is the effect of mass variation and axial thrust upon the system stability as well as the question of existence of normal mode vibration for variable-mass systems.

1.2 State of the Art

Most research in the area of dynamic characteristics of a boost vehicle seems to concentrate on either a rigid-body of time-dependent mass or an elastic body of constant mass. In the latter some investigators allow the mass to shift to simulate sloshing.

The treatment of the missile as a rigid-body of time-dependent mass has been adequately covered by many writers, including Grubin[1]* and Dryer[2] among others, and by Leitmann[3] and Meriam,[4] who also consider the effect of a relative shift in the center of gravity of the body. The ballistic trajectories of spin- and fin-stabilized rigid bodies are treated in a publication by Davis, Follin, and Blitzer,[5] and the effect of jet damping has been treated for both bodies of constant as well as variable mass by a number of investigators, including Gilmore and Keller,[6] Barton,[7,8] and Leon.[9] All these authors, as well as a great many more, ignore the elasticity of the vehicle shell.

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* Numbers refer to publications listed in References

On the other hand, a considerable amount of effort has been expended on the analysis of an elastic body subjected to longitudinal acceleration. Seide[10] for instance, has treated the effect of both a compressive (thrust) force and a tensile force on the frequencies and mode shapes of lateral vibration of a continuous slender body. Silverberg[11] and Cox,[12,13] among others, have investigated a missile as a lumped system. Others, such as Beal,[14] have been concerned with the problem of buckling instabilities of a uniform beam subjected to an end thrust and the change in the natural frequencies of such a system. In contrast to the investigations mentioned earlier, however, these studies all neglect the change of mass of the body as a function of time.

A series of reports by Miles, Young and Fowler[15] offers a comprehensive treatment of a wide range of subjects including fuel sloshing. Again the mass variation is not accounted for.

Attempts have been made to consider the mass variation and body elasticity simultaneously. In this connection one should mention Birnbaum[16] and Edelen[17] who treated the variation of mass together with the flexural elasticity of the missile case. Unfortunately both investigators are rather vague in their treatment of the fluid flow problem. Neither of these two investigators attempts to include the axial elasticity of the missile shell, although its influence upon the system may be significant. Both are concerned with solid-fuel rockets and assume that the thrust is concentrated at the nozzle end, whereas any effect of internal pressure on the closed end is not clearly stated, such that a shell in axial compression rather than tension results.

In later studies, the effect of both axial and lateral elasticity of a missile of time-dependent mass moving through space has been considered by Meirovitch.[18] This work was further extended by utilizing a different formulation for the momentum of the internal fluid as well as a body axes system instead of inertial axes.[19,20] The work presented here gives a summary of the results obtained in References 19 and 20 and further numerical results not reported there.

1.3 Description of the System and Assumptions

The missile case is considered to be capable of both axial and flexural elastic deformation, as well as rigid-body motion. Whereas the mass per unit length of the case does change with time, it is assumed that the unburned fuel does not contribute to the strength of the missile case so that the axial and lateral stiffness remain unchanged with time. Simple beam theory is permitted by the assumption of a slender missile, together with small elastic deflections and by the exclusion of spin-stabilized missiles.

Such influences as result from unstable burning and the effect of pressure and temperature gradients, as well as from acceleration forces on the burning rate and the inclusion of solids and burning particles in the internal fluid, are considered to have little effect on the dynamic characteristics of the missile itself and are not treated in the analysis. More vital to the behavior of the missile are the internal pressure and relative velocity distributions, as well as the effect of friction on the flow and the possibility of a gimbal angle at the nozzle; these factors are treated in some detail.

The purpose of this investigation is to study the dynamic characteristics of the missile rather than to calculate the exact position of the missile at any time, so that body axes are used rather than inertial axes, with the result of a substantial reduction in the complexity of the final equations of motion.

Figure 1.1 represents the vehicle and the various systems of coordinates.

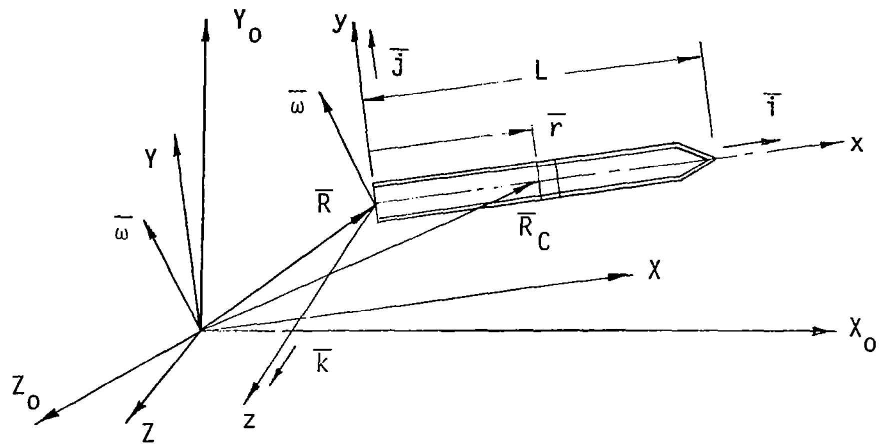


Figure 1.1

Body and Inertial Axes System

The quantities that appear in Fig. 1.1 are defined as follows:

X_0, Y_0, Z_0 - inertial axes

X, Y, Z - noninertial axes rotating with respect to the X_0, Y_0, Z_0 axes

x,y,z - noninertial, body axes translating with respect to the X,Y,Z axes (hence, translating and rotating with respect to the X_0. Y_0, Z_0 axes)

$\bar{i}$, $\bar{j}$, $\bar{k}$ - unit vectors along the x,y,z axes (hence, also along the X,Y,Z axes)

$\bar{R}$ - absolute position of the origin of the axes x,y,z

$\bar{r}$ - position of a point on the rocket case relative to the axes x,y,z

$\overline{R}_C$ - absolute position of a point on the rocket case

$\overline{\omega}$ - angular velocity of the axes x,y,z (hence, also of the axes X,Y,Z)

where the bar above any symbol denotes a vector quantity.

Additional assumptions are introduced as needed during the course of the investigation. These assumptions relate to the nature of the fluid flow and the order of magnitude of various factors resulting from the fluid flow as well as the relative magnitude of the rigid-body motions and elastic displacements.

1.4 General Discussion of the Present Investigation

Sections II and III contain the mathematical formulation. The differential equations for the rigid-body motion and the elastic displacements as well as the boundary conditions for the latter are derived. The vector differential equations of motion of the missile shell and fluid elements are derived separately and then combined into a single vector equation of motion for a typical element. The equations for the shell and fluid elements are not independent, since the pressure of the fluid is transmitted to the case at both ends, as well as laterally along the missile. In addition, if friction is considered, a longitudinal force along the length of the missile is present. The vector differential equation of motion is transformed by a variational principle into five differential equations of motion in terms of the generalized coordinates. Whereas these equations are more amenable to solution than in vector form, they are coupled as well as highly nonlinear. In addition, the equations involve such fluid quantities, some of them in an implicit manner, as pressure and velocity distributions, mass flow rates, and internal viscous effects. No closed form solution of the general case of motion is possible. For certain special cases, however, such as the upright vertical flight, a reasonable solution can be obtained.

The internal fluid flow is investigated in Section IV and in particular the effect of using a noninertial control volume. Relative magnitudes of several fluid terms are considered in light of various assumptions concerning the physical system, and a correlation between the total thrust in terms of the mass flow rate and in terms of the integrated internal pressure is obtained.

A primary motion in the vertical direction is postulated with the remaining motions treated as perturbations which, together with the assumption of small, elastic motions, results in linearized equations. The solutions for the axial and lateral deformations are obtained in the form of series of the eigenfunctions of the corresponding constant-mass systems multiplied by time-dependent generalized coordinates. Whereas the set of equations for the axial motion may be uncoupled by assuming uniform mass, the equations for the lateral motion remain coupled due to the presence of the axial force; this coupling can be removed by limiting the number of terms in the series.

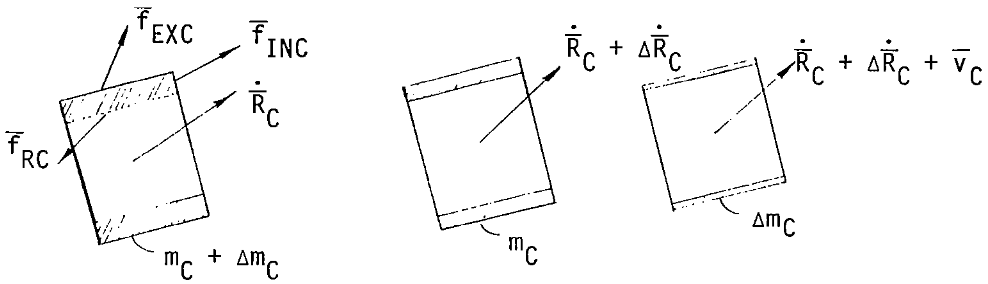
In Section VI, a numerical solution to the five coupled equations of motion is obtained. The equations are written in finite difference form and are programmed for the digital computer where an iterative procedure is utilized to cope with the nonlinearity of the equations. Similar assumptions concerning the fluid flow are retained, and results for a representative system are computed and compared with those obtained by the more restrictive analysis of Section IV. The agreement is found to be excellent.

II. MATHEMATICAL FORMULATION

2.1 The Differential Equation of Motion for the Case Element

An element of case is considered to consist of two parts: the case element and the fluid element. The element of case is treated as a "system," whereas a "control volume" approach is adopted for the fluid element. Two vector differential equations of motion, one for the system and one for the control volume, are derived separately and later combined into one vector equation.

A system may change shape, position and thermal condition but not the amount of matter. The differential equation of motion will be derived by using the system concept in conjunction with the force-linear momentum principle. To this end we consider the system at two instants,



(a). System at Time t　　(b). System at Time = t + Δt

Figure 2.1

t and t + Δt (Fig. 2.1). Next let us define

$\bar{f}_{EXC}$ - case external force/unit length of case

$\bar{f}_{INC}$ - case internal force/unit length of case

$\bar{f}_{RC}$ - force exerted by fluid on case/unit length of case

m_C - mass of case and unburned fuel/unit length of case

$\dot{\bar{R}}_C$ - absolute velocity of case element

$\overline{v}_C$ - velocity of burned mass relative to case

$\overline{p}_C$ - linear momentum/unit length of case

The linear momentum at time t is

$$\overline{p}_C = (m_C + \Delta m_C)\ \dot{\overline{R}}_C \tag{2.1}$$

and at time t + Δt is

$$\overline{p}_C + \Delta\overline{p}_C = m_C(\dot{\overline{R}}_C + \Delta\dot{\overline{R}}_C) + \Delta m_C(\dot{\overline{R}}_C + \Delta\dot{\overline{R}}_C + \overline{v}_C) \tag{2.2}$$

so that the change in momentum is

$$\Delta\overline{p}_C = m_C\Delta\dot{\overline{R}}_C + \Delta m_C(\Delta\dot{\overline{R}}_C + \overline{v}_C) \tag{2.3}$$

According to Newton's second law, the sum of the forces acting upon the element must be equal to the rate of change of momentum

$$\overline{f}_{EXC} + \overline{f}_{INC} + \overline{f}_{RC} = \lim_{\Delta t \to 0} \frac{\Delta\overline{p}_C}{\Delta t} = m_C\ddot{\overline{R}}_C + \overline{v}_C \frac{dm_C}{dt} \tag{2.4}$$

2.2 The Differential Equation of Motion for the Fluid Element. Reynold's Transport Theorem.[21]

In contrast with the system approach, the control volume approach envisions a definite volume in space which does not change. The boundary of that volume is called the control surface. The identity of the matter within the control volume may change but the shape must remain constant. Hence, we must assume that the elastic deformations of the case do not affect materially the dimensions of the control surface.

In Fig. 2.2, the boundary of the control volume is represented by the heavy line and the velocity field by $\overline{v}(x,y,z,t)$ so that at time t the system is identical with the fluid within the control volume. The

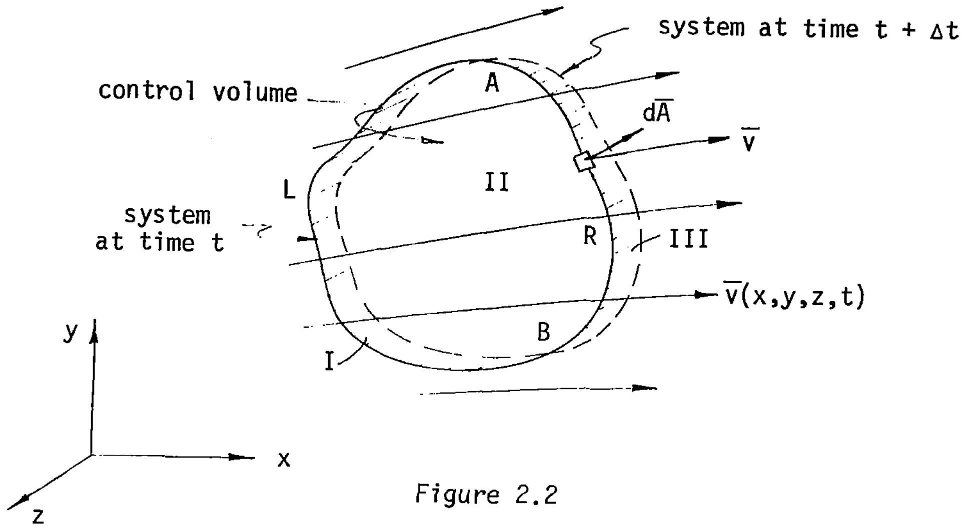


Figure 2.2

Control Volume

linear momentum associated with an element of fluid is $\overline{V}\rho d\nu$, where ρ is the mass per unit volume and $d\nu$ is the element of volume. The linear momentum, $\overline{p}_F$, of the fluid contained by the control volume at any instant t is therefore

$$\overline{p}_F = \int\int\int_{cv} \overline{V}\rho\, d\nu \qquad (2.5)$$

At time t the system occupies regions I and II, while at time t + Δt it occupies regions II and III. The time rate of change of the linear momentum is

$$\frac{d\overline{p}_F}{dt} = \lim_{\Delta t \to 0} \frac{\left(\iiint_{II} \overline{V}\rho d\nu + \iiint_{III} \overline{V}\rho d\nu\right)_{t+\Delta t} - \left(\iiint_{I} \overline{V}\rho d\nu + \iiint_{II} \overline{V}\rho d\nu\right)_{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left(\iiint_{II} \overline{V}\rho d\nu\right)_{t+\Delta t} - \left(\iiint_{II} \overline{V}\rho d\nu\right)_{t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\left(\iiint_{III} \overline{V}\rho d\nu\right)_{t+\Delta t}}{\Delta t} -$$

$$\lim_{\Delta t \to 0} \frac{\left(\iiint_{I} \overline{V}\rho d\nu\right)_{t}}{\Delta t} \qquad (2.6)$$

As $t \to 0$, the volume II tends to become the control volume so that

$$\lim_{\Delta t \to 0} \frac{\left(\iiint_{II} \bar{v}\rho d\nu\right)_{t+\Delta t} - \left(\iiint_{II} \bar{v}\rho d\nu\right)_{t}}{\Delta t} = \frac{\partial}{\partial t}\iiint_{CV} \bar{v}\rho d\nu \tag{2.7}$$

Furthermore, as $t \to 0$, the volume III approaches zero so that in the limit

$$\lim_{\Delta t \to 0} \frac{\left(\bar{v}\rho d\nu \Big|_{III}\right)_{t+\Delta t}}{\Delta t} = \bar{v}\,(\rho\bar{v} \cdot d\bar{A})_{III} \tag{2.8}$$

where $d\bar{A}$ is a vector representing an element of the control surface ARB and $\bar{v}$ is the velocity vector at that point. The right side of (2.8) represents the rate of efflux of $\bar{p}_F$ through the area $d\bar{A}$ of the control surface ARB. A similar treatment of the integral over region I results in the efflux of momentum through the surface ALB. But the sum of the integrals over ARB and ALB is the integral over the control surface so that the time rate of change of the momentum becomes

$$\frac{d\bar{p}_F}{dt} = \iint_{CS} \bar{v}(\rho\bar{v} \cdot d\bar{A}) + \frac{\partial}{\partial t} \iiint_{CV} \bar{v}\rho d\nu \tag{2.9}$$

Equation (2.9) is called the Reynold transport theorem for an inertial space and unifies the system and control volume concepts. Newton's second law of motion may once again be utilized to give

$$\bar{F}_{EXF} + \bar{F}_{INF} + \bar{F}_{RF} = \frac{d\bar{p}_F}{dt} = \iint_{CS} \bar{v}(\rho\bar{v} \cdot d\bar{A}) + \frac{\partial}{\partial t} \iiint_{CV} \bar{v}\rho d\nu \tag{2.10}$$

where $\bar{F}_{EXF}$, $\bar{F}_{INF}$, and $\bar{F}_{RF}$ are the total external, internal, and reactive forces, respectively, acting on the volume of fluid.

If the control volume is not fixed in space but accelerating with respect to an inertial system X_0, Y_0, Z_0 (Fig. 1.1), the equation of motion for an element of fluid is

$$d\bar{F}_{EXF} + d\bar{F}_{INF} + d\bar{F}_{RF} = \frac{d}{dt}\,(dM_F\dot{\bar{R}}_F) = dM_F\ddot{\bar{R}}_F \tag{2.11}$$

where dM_F is the mass of the fluid element and $\ddot{\bar{R}}_F$ is the absolute acceleration of the fluid.

If the position vector of the fluid, $\bar{R}_F$ is given in terms of a noninertial system of axes, the absolute acceleration, $\ddot{\bar{R}}_F$, has the form

$$\ddot{\bar{R}}_F = \ddot{\bar{R}} + \bar{a}_{rel} + 2\bar{\omega}x\bar{v}_{rel} + \dot{\bar{\omega}}x\bar{r} + \bar{\omega}x\ (\bar{\omega}x\bar{r}) \tag{2.12}$$

where

$\ddot{\bar{R}}$ = acceleration of the origin of the body axes x,y,z

$\bar{v}_{rel}$, $\bar{a}_{rel}$ = velocity and acceleration of fluid relative to x,y,z axes

$\bar{r}$ = position vector of fluid element relative to the x,y,z axes

$\bar{\omega}$, $\dot{\bar{\omega}}$ = angular velocity and acceleration of axes x,y,z

Using the notation $\dot{\bar{R}}_{Crel}$ to distinguish between the velocity of the case relative to the x,y,z system and the velocity, $\bar{v}$, of the fluid relative to the case, together with the corresponding relative accelerations $\ddot{\bar{R}}_{Crel}$ and $\dot{\bar{v}}$, respectively, Eq. (2.12) becomes

$$\ddot{\bar{R}}_F = \ddot{\bar{R}} + (\ddot{\bar{R}}_{Crel} + \dot{\bar{v}}) + 2\bar{\omega}x\ (\dot{\bar{R}}_{Crel} + \bar{v}) + \dot{\bar{\omega}}x\bar{r} + \bar{\omega}x\ (\bar{\omega}x\bar{r}) = \ddot{\bar{R}}_C + \dot{\bar{v}} + 2\bar{\omega}x\bar{v} \tag{2.13}$$

where

$$\ddot{\bar{R}}_C = \ddot{\bar{R}} + \ddot{\bar{R}}_{Crel} + 2\omega x\dot{\bar{R}}_{Crel} + \dot{\bar{\omega}}x\bar{r} + \bar{\omega}x\ (\bar{\omega}x\bar{r}) \tag{2.14}$$

is the absolute acceleration of the case element.

Introducing (2.13) and (2.14) into (2.11) and integrating we obtain the time rate of change of the momentum for a noninertial control volume

$$\bar{F}_{EXF} + \bar{F}_{INF} + \bar{F}_{RF} = \iiint_{cv} \ddot{\bar{R}}_F dM_F = \iiint_{cv} \ddot{\bar{R}}_F \rho d\nu =$$

$$\iiint_{cv} (\ddot{\bar{R}}_C + \dot{\bar{v}} + 2\bar{\omega}x\bar{v})\ \rho d\nu = \iiint_{cv} (\ddot{\bar{R}}_C + 2\bar{\omega}x\bar{v})\ \rho d\nu +$$

$$\iint_{cs} \bar{v}\ (\rho\bar{v} \cdot d\bar{A}) + \frac{\partial}{\partial t} \iiint_{cv} \bar{v}\ (\rho d\nu) \tag{2.15}$$

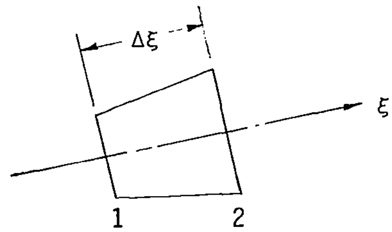


Figure 2.3

Fluid Element

In view of the nature of the physical problem under consideration, the flow may be restricted to the one-dimensional case, since the lateral deflection of the shell is small. Furthermore, if the density and velocity of the fluid are assumed uniform over a cross section of the missile, the change in any property of the fluid may be considered as a function of the ξ - coordinate only, where ξ is the axis passing through the centers of the missile cross-sectional areas.

For a small increment of fluid of length $\Delta\xi$, Eq. (2.15) reduces to

$$\Delta\overline{F}_{EXF} + \Delta\overline{F}_{INF} + \Delta\overline{F}_{RF} = (\ddot{\overline{R}}_C + 2\overline{\omega}x\overline{v})\ \Delta M_F + (m_F v)_2\ \overline{v}_2 - (m_F v)_1\ \overline{v}_1 + \frac{\partial}{\partial t}(\Delta M_F \overline{v}) \tag{2.16}$$

where m_F is the mass of the fluid per unit length. Dividing Eq. (2.16) by $\Delta\xi$ and letting $\Delta\xi \to 0$ we obtain

$$\lim_{\Delta\xi\to 0} \frac{\Delta\overline{F}_{EXF} + \Delta\overline{F}_{INF} + \Delta\overline{F}_{RF}}{\Delta\xi} = \lim_{\Delta\xi\to 0} (\ddot{\overline{R}}_C + 2\overline{\omega}x\overline{v})\ \frac{\Delta M_F}{\Delta\xi} +$$

$$\lim_{\Delta\xi\to 0} \frac{(m_F v)_2\ \overline{v}_2 - (m_F v)_1\ \overline{v}_1}{\Delta\xi} + \frac{\partial}{\partial t}\left(\lim_{\Delta\xi\to 0} \frac{\Delta M_F}{\Delta\xi}\ \overline{v}\right)$$

Finally, the vector differential equation of motion for the fluid element reduces to

$$\bar{f}_{EXF} + \bar{f}_{INF} + \bar{f}_{RF} = m_F\,(\ddot{\bar{R}}_C + 2\bar{\omega}\mathrm{x}\bar{v}) + \frac{\partial}{\partial\xi}\,(m_F v\bar{v}) + \frac{\partial}{\partial t}\,(m_F\bar{v}) + \sum_{i=1}^{k} \Delta\left[m_F v\,(\xi_i,t)\,\bar{v}\,(\xi_i,t)\right]\,\delta(\xi - \xi_i) \tag{2.17}$$

The terms denoted by the symbol Δ provide for an abrupt change in the time rate of change of momentum due to a change in the direction of flow at $\xi = \xi_i$ such as may result from a gimballing of the nozzle. The terms $\bar{f}_{RF}$, $\bar{f}_{INF}$, and $\bar{f}_{EXF}$ represent the reactive force per unit length on the fluid, the fluid pressure differential per unit length and gravitational force, and the external force per unit length, respectively. In this manner, $\bar{f}_{EXF}$ contains nonconservative forces only.

2.3 The Combined Differential Equation of Motion in Vector Form

For small lateral deformation, the spatial variable ξ of the last section approaches the spatial variable x as defined in Section I. Eqs. (2.4) and (2.17) may then be combined to give the vector differential equation of motion for the differential element of the missile

$$\bar{f}_{EXC} + \bar{f}_{INC} + \bar{f}_{RC} + \bar{f}_{EXF} + \bar{f}_{INF} + \bar{f}_{RF} = (m_C + m_F)\,\ddot{\bar{R}}_C + \dot{m}_C\bar{v}_C + m_F\,(2\bar{\omega}\mathrm{x}\bar{v}) + \frac{\partial}{\partial x}\,(m_F v\bar{v}) + \frac{\partial}{\partial t}\,(m_F\bar{v}) + \sum_{i=1}^{k} \Delta\left[m_F v\,(x_i,t)\,\bar{v}\,(x_i,t)\right]\,\delta(x - x_i) \tag{2.18}$$

However, by Newton's third law, the reactive force exerted on the case by the fluid is just the negative of the reactive force exerted on the fluid by the case. Also, the distributed internal force in the fluid is recognized as due to the pressure-area differential acting on the cross-sectional area as well as gravitational forces

$$\overline{f}_{INF} = - \frac{\partial}{\partial x} (pA_F)\ \overline{i} + m_F \overline{g} \tag{2.19}$$

where p is assumed uniform across the cross section. Furthermore, letting $m = m_C + m_F$ be the total mass per unit length, the combined differential equation for the missile element appears as

$$\overline{f}_{EX} + \overline{f}_{INC} - \frac{\partial}{\partial x} (pA_F)\ \overline{i} + m_F\overline{g} = m\ddot{\overline{R}}_C + \dot{m}_C\overline{v}_C + m_F\ (2\overline{\omega}x\overline{v}) + \frac{\partial}{\partial x}(m_F v\overline{v}) + \frac{\partial}{\partial t}(m_F\overline{v}) + \sum_{i=1}^{k} \Delta\left[m_F v\ (x_i,t)\ \overline{v}\ (x_i,t)\right] \delta(x - x_i) \tag{2.20}$$

where $\overline{f}_{EX} = \overline{f}_{EXC} + \overline{f}_{EXF}$ is the distributed external force which includes the drag. Only nonconservative forces are included in $\overline{f}_{EX}$, so that both the gravitational forces as well as the internal stresses in the vehicle case are represented by $\overline{f}_{INC}$.

III. THE DIFFERENTIAL EQUATIONS OF MOTION IN TERMS OF GENERALIZED COORDINATES

3.1 The Variational Principle

Whereas the vector differential equation of motion of an element of a solid-fuel missile provides a mathematical formulation of the physical system, it is not particularly amenable to mathematical solution. Equations of motion in terms of generalized coordinates are more suitable for that purpose. A variational principle similar to Hamilton's principle is adopted as the most direct means of obtaining the equations of motion in terms of generalized coordinates.

With planar motion assumed, the generalized coordinates are chosen as two rigid-body translations, $X(t)$ and $Y(t)$, one rigid-body rotation, $\theta(t)$, one axial elastic displacement, $u(x,t)$, and one lateral elastic displacement, $y(x,t)$, as shown in Fig. 3.1. Since the only spatial

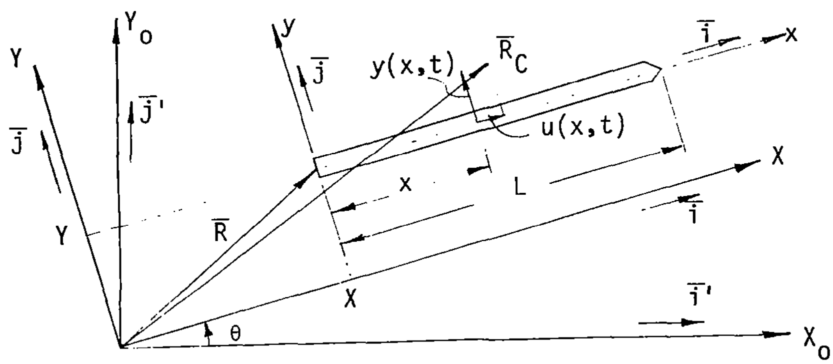


Figure 3.1
Coordinate System for Planar Motion

coordinate relative to the body is x, the position of a point on the missile case is

$$\overline{R}_C = \overline{R} + (x+u)\,\overline{i} + y\overline{j} = (X + x + u)\,\overline{i} + (Y + y)\,\overline{j} \tag{3.1}$$

where $\bar{i}$ and $\bar{j}$ are not constant, but rotating unit vectors. Also, since the motion is planar, the angular velocity of the body axes simplifies to

$$\bar{\omega} = \dot{\theta}\bar{k} \tag{3.2}$$

where $\bar{k}$ is a constant unit vector normal to the plane of motion. At this point it is convenient to disregard any abrupt changes in the time rate of change of the momentum of the flow other than that due to a gimbal angle of the nozzle. Therefore, the velocity of the fluid relative to the missile case at any point other than the nozzle end, x = 0, is

$$\bar{v}(x,t) = v(x,t)\,\bar{i} \tag{3.3}$$

With the above restrictions, the element of the missile described by Eq. (2.20) may be put in dynamic equilibrium by the use of D'Alembert's principle

$$\bar{E}(x,t) = \bar{f}_{EX} + \bar{f}_{INC} - \frac{\partial}{\partial x}(pA_F)\,\bar{i} + m_F\bar{g} - m\ddot{\bar{R}}_C - \left[\dot{m}_C v_C + \frac{\partial}{\partial x}(m_F v^2) + \frac{\partial}{\partial t}(m_F v)\right]\bar{i} - 2\dot{\theta}m_F v\bar{j} - \Delta\left[m_F v(0,t)\,\bar{v}\,(0,t)\right]\delta(x) = 0 \tag{3.4}$$

The virtual work associated with the combined element of the rocket is then

$$\delta\hat{W}(x,t) = \bar{E}(x,t)\cdot\delta\bar{R}_C(x,t) \tag{3.5}$$

where $\delta\bar{R}_C(x,t)$ is an arbitrary variation of $\bar{R}_C(x,t)$ such that $\delta\bar{R}_C(x,t)$ must vanish at $t = t_0$ and $t = t_1$. Since $\delta\bar{i} = \delta\theta\bar{j}$ and $\delta\bar{j} = -\delta\theta\bar{i}$, the virtual displacement may be expressed as

$$\delta\bar{R}_C(x,t) = \left[\delta X + \delta u - \delta\theta(Y + y)\right]\bar{i} + \left[\delta Y + \delta y + \delta\theta(X + x + u)\right]\bar{j} \tag{3.6}$$

The statement of the variational principle is

$$\int_{t_0}^{t_1}\int_0^L \delta\hat{W}(x,t)\,dx\,dt = 0 \tag{3.7}$$

3.2 The Equations of Motion. General Expressions

To derive the equations of motion in terms of generalized coordinates, we consider the various terms comprising $\delta\hat{W} = \bar{E} \cdot \delta\bar{R}_C$ separately. First we consider the term due to external forces

$$\bar{f}_{EX} \cdot \delta\bar{R}_C = \delta\hat{W}_{nc} \tag{3.8}$$

where $\delta\hat{W}_{nc}$ denotes the nonconservative work density. The virtual work resulting from the internal forces corresponds to the negative value of the variation in the potential energy density $\widehat{PE}$

$$\bar{f}_{INC} \cdot \delta\bar{R}_C = - \delta\widehat{PE} \tag{3.9}$$

The terms due to the fluid axial force and fluid gravity are

$$\left[- \frac{\partial}{\partial x} (pA_F)\ \bar{i} + m_F\bar{g}\right] \cdot \delta\bar{R}_C = \delta\hat{P} \tag{3.10}$$

Noting that

$$\frac{d}{dt} (\dot{\bar{R}}_C \cdot \delta\bar{R}_C) = \ddot{\bar{R}}_C \cdot \delta\bar{R}_C + \delta(\frac{1}{2}\dot{\bar{R}}_C \cdot \dot{\bar{R}}_C)$$

one can write

$$- m\ddot{\bar{R}}_C \cdot \delta\bar{R}_C = \delta\widehat{KE} + \delta\hat{T} \tag{3.11}$$

where

$$\widehat{KE} = \frac{1}{2} m\dot{\bar{R}}_C \cdot \dot{\bar{R}}_C \tag{3.12}$$

is the kinetic energy density and

$$\delta\hat{T} = -m \frac{d}{dt} (\dot{\bar{R}}_C \cdot \delta\bar{R}_C) = -\frac{d}{dt} (m\ \dot{\bar{R}}_C \cdot \delta\bar{R}_C) + \dot{m}\ \dot{\bar{R}}_C \cdot \delta\bar{R}_C \tag{3.13}$$

can be shown to represent a variation in the equivalent generalized distributed forces due to mass time rate of change. Furthermore, one can write

$$- \left\{\left[\dot{m}_C v_C + \frac{\partial}{\partial x} (m_F v^2) + \frac{\partial}{\partial t} (m_F v)\right] \bar{i} + 2\dot{\theta} m_F v\ \bar{j} + \Delta\left[m_F v(0,t)\ \bar{v}(0,t)\right] \delta(x)\right\} \cdot \delta\bar{R}_C = \delta\hat{V} \tag{3.14}$$

where $\delta\hat{V}$ is a variation in the equivalent generalized distributed forces due to fluid flow.

In view of the above definitions one can write

$$\int_0^L (\delta\hat{W}_{nc} + \delta\hat{P} + \delta\hat{T} + \delta\hat{V})\, dx = (F_X + P_X + T_X + V_X)\, \delta X +$$

$$(F_Y + P_Y + T_Y + V_Y)\, \delta Y + (F_\theta + P_\theta + T_\theta + V_\theta)\delta\theta \; +$$

$$\int_0^L (\hat{F}_u + \hat{P}_u + \hat{T}_u + \hat{V}_u)\, \delta u\, dx + \int_0^L (\hat{F}_y + \hat{P}_y + \hat{T}_y + \hat{V}_y)\, \delta y\, dx \qquad (3.15)$$

where the quantities in parantheses in the right side of Eq. (3.15) are recognized as corresponding generalized forces.

Next define the Lagrangian density, $\hat{L}$, in the form

$$\hat{L} = \widehat{KE} - \widehat{PE} \qquad (3.16)$$

and it suffices, for the moment, to state that the functional dependence of the Lagrangian density can be written as

$$\hat{L} = \hat{L}(X,\, Y,\, \theta,\, u,\, y,\, \dot{X},\, \dot{Y},\, \dot{\theta},\, \dot{u},\, \dot{y},\, u',\, y',\, y'') \qquad (3.17)$$

from which it follows that

$$\delta\hat{L} = \frac{\partial\hat{L}}{\partial X}\, \delta X + \frac{\partial\hat{L}}{\partial Y}\, \delta Y + \ldots + \frac{\partial\hat{L}}{\partial y'}\, \delta y' + \frac{\partial\hat{L}}{\partial y''}\, \delta y'' \qquad (3.18)$$

Note that primes indicate differentiations with respect to the spatial variable x. Introducing Eqs. (3.8) through (3.18) into Eq. (3.7), performing a number of integrations by parts and recalling the arbitrariness of the virtual displacements, we obtain the Lagrange equations of motion

$$- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) + \frac{\partial L}{\partial X} + F_X + P_X + T_X + V_X = 0$$

$$- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) + \frac{\partial L}{\partial Y} + F_Y + P_Y + T_Y + V_Y = 0$$

$$- \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \frac{\partial L}{\partial \theta} + F_\theta + P_\theta + T_\theta + V_\theta = 0 \tag{3.19}$$

$$- \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}}\right) + \frac{\partial \hat{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial u'}\right) + \hat{F}_u + \hat{P}_u + \hat{T}_u + \hat{V}_u = 0$$

$$- \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{y}}\right) + \frac{\partial \hat{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial y'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}}{\partial y''}\right) + \hat{F}_y + \hat{P}_y + \hat{T}_y + \hat{V}_y = 0$$

where the latter two equations are subject to the boundary equations

$$\frac{\partial \hat{L}}{\partial u'} = 0 \qquad \text{at } x = 0, L \tag{3.20}$$

and

$$\frac{\partial \hat{L}}{\partial y''} = 0 \qquad \text{at } x = 0, L \tag{3.21}$$

$$\frac{\partial L}{\partial y'} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y''}\right) = 0 \qquad \text{at } x = 0, L \tag{3.22}$$

3.3 The Equations of Motion. Specific Form

The specific form of the nonconservative forces F_X, F_Y and F_θ associated with the rigid motion and the nonconservative distributed forces $\hat{F}_u$ and $\hat{F}_y$ appears evident from Eqs. (3.6), (3.8) and (3.15). Furthermore, from Eqs. (3.6), (3.10) and (3.15) one obtains

$$P_X = - \int_0^L \frac{\partial}{\partial x}(pA_F)\, dx + \int_0^L m_F \bar{g} \cdot \bar{i}\, dx$$

$$P_Y = \int_0^L m_F \bar{g} \cdot \bar{j}\, dx \tag{3.23}$$

$$P_\theta = \int_0^L (Y + y) \frac{\partial}{\partial x} (pA_F)\, dx + \int_0^L m_F \Big[-(Y + y)\, \bar{g} \cdot \bar{i} + (X + x + u)\, \bar{g} \cdot \bar{j}\Big] dx$$

$$\hat{P}_u = - \frac{\partial}{\partial x} (pA_F) + m_F \bar{g} \cdot \bar{i}$$

$$\hat{P}_y = m_F \bar{g} \cdot \bar{j}$$

which are equivalent generalized forces resulting from the fluid internal pressure-area differential and the fluid gravity.

Noting that

$$\frac{d\bar{i}}{dt} = \dot{\theta}\bar{j} \qquad \text{and} \qquad \frac{d\bar{j}}{dt} = - \dot{\theta}\bar{i}$$

the absolute velocity of a point on the missile case in terms of the generalized coordinates is

$$\dot{\bar{R}}_C = \Big[\dot{X} + \dot{u} - (Y + y)\, \dot{\theta}\Big] \bar{i} + \Big[\dot{Y} + \dot{y} + (X + x + u)\, \dot{\theta}\Big] \bar{j} \tag{3.24}$$

With $\dot{M}$ denoting the time rate of change of the total mass of the system one can use Eqs. (3.6), (3.13), (3.15) and (3.24), perform the necessary integration and conclude that

$$T_X = \dot{M}\, (\dot{X} - Y\dot{\theta}) + \int_0^L m\ddot{u}\, dx - \dot{\theta} \int_0^L m\dot{y}\, dx$$

$$T_Y = \dot{M}\, (\dot{Y} + X\dot{\theta}) + \int_0^L m\ddot{y}\, dx + \dot{\theta} \int_0^L \dot{m}\, (x + u)\, dx$$

$$\begin{aligned} T_\theta = {} & \dot{M} \Big[X\dot{Y} - Y\dot{X} + \dot{\theta}(X^2 + Y^2)\Big] - \int_0^L (Y + y)\, \dot{m}\, \dot{u}\, dx \\ & + \int_0^L (X + x + u)\, \dot{m}\, \dot{y}\, dx - (\dot{X} - 2Y\dot{\theta}) \int_0^L \dot{m}\, y\, dx \\ & + (\dot{Y} + 2X\dot{\theta}) \int_0^L \dot{m}(x + u)\, dx + \dot{\theta} \int_0^L \dot{m} \Big[(x + u)^2 + y^2\Big] dx \end{aligned} \tag{3.25}$$

$$\hat{T}_u = \dot{m} \Big[\dot{X} + \dot{u} - \dot{\theta}(Y + y)\Big]$$

$$\hat{T}_y = \dot{m} \Big[\dot{Y} + \dot{y} + \dot{\theta}(X + x + u)\Big]$$

are the equivalent generalized forces due to the mass variation.

If the flow at the nozzle (x=0) is deflected such that it makes an angle γ with the x axis,

$$\bar{v}(0,t) = v(0,t)\ (\bar{i} \cos \gamma + \bar{j} \sin \gamma) \qquad (3.26)$$

and the change in the time rate of change of momentum of the fluid is

$$\Delta \left[m_F v(0,t)\ \bar{v}(0,t) \right] = m_F v^2(0,t) \left[(1 - \cos \gamma)\bar{i} - (\sin \gamma)\bar{j} \right] \qquad (3.27)$$

Equations (3.6), (3.14), (3.15) and (3.27) lead us to

$$V_X = - \int_0^L \left[\dot{m}_C v_C + \frac{\partial}{\partial x}(m_F v^2) + \frac{\partial}{\partial t}(m_F v) \right] dx - m_F v^2(0,t)(1 - \cos \gamma)$$

$$V_Y = - 2\dot{\theta} \int_0^L m_F v\ dx + m_F v^2(0,t) \sin \gamma$$

$$V_\theta = \int_0^L (Y + y) \left[\dot{m}_C v_C + \frac{\partial}{\partial x}(m_F v^2) + \frac{\partial}{\partial t}(m_F v) \right] dx$$

$$+ \left[Y + y(0,t) \right] m_F v^2(0,t)(1 - \cos \gamma) \qquad (3.28)$$

$$- 2\dot{\theta} \int_0^L (X + x + u)\ m_F v\ dx + \left[X + u(0,t) \right] m_F v^2(0,t) \sin \gamma$$

$$\hat{V}_u = - \left[\dot{m}_C v_C + \frac{\partial}{\partial x}(m_F v^2) + \frac{\partial}{\partial t}(m_F v) \right] - m_F v^2(0,t)(1 - \cos \gamma)\ \delta(x)$$

$$\hat{V}_y = - 2\dot{\theta}\ m_F v + m_F v^2(0,t) \quad \sin \gamma\, \delta(x)$$

Introducing Eq. (3.24) into (3.12) we obtain the kinetic energy density

$$\hat{KE} = \frac{1}{2} m \left\{ \left[\dot{X} + \dot{u} - \dot{\theta}(Y + y) \right]^2 + \left[\dot{Y} + \dot{y} + \dot{\theta}(X + x + u) \right]^2 \right\} \qquad (3.29)$$

and the potential energy density can be written in the form

$$\hat{PE} = \frac{1}{2} \left[EI(y'')^2 \right] + \frac{1}{2} \left[EA_C(u')^2 \right] + \frac{1}{2} \left[P(y')^2 \right] + m_C g\ \bar{R}_C \cdot \bar{j}' \qquad (3.30)$$

in which the following symbols have been used

E - modulus of elasticity of the case material

I - area moment of inertia of the case cross section about a transverse axis (neutral bending axis)

A_C - cross-sectional area of the case

P - axial force in the case (positive for tension)

$m_C g$ - gravitational force per unit length of case

$\bar{j}'$ - unit vector along the Y_0 axis

The Lagrangian density is obtained by introducing Eqs. (3.29) and (3.30) into (3.16). The Lagrangian L has the form

$$L = \int_0^L \hat{L}\, dx \tag{3.31}$$

Performing the indicated operations in the first of equations (3.19) yields

$$-M(\ddot{X} - \ddot{\theta}Y - 2\dot{\theta}\dot{Y} - \dot{\theta}^2 X) - \int_0^L m\, \ddot{u}\, dx + \ddot{\theta} \int_0^L m\, y\, dx +$$

$$\dot{\theta}^2 \int_0^L m(x + u)\, dx + 2\dot{\theta} \int_0^L m\, \dot{y}\, dx - \int_0^L \left[\frac{\partial}{\partial x} (pA_F) + \right.$$

$$\left. \frac{\partial}{\partial x} (m_F v^2) + \frac{\partial}{\partial t} (m_F v) \right] dx - \int_0^L \dot{m}_C v_C dx -$$

$$m_F v^2(o,t)\, (1 - \cos \gamma) - Mg \sin \theta + F_X = 0 \tag{3.32}$$

The scalar product of the unit vector $\bar{i}$ and Eq. (2.17) results in the expression

$$\bar{f}_{EXF} \cdot \bar{i} + \bar{f}_{RF} \cdot \bar{i} - \frac{\partial}{\partial x} (pA_F) - m_F g \sin \theta - m_F \ddot{\bar{R}}_C \cdot \bar{i} - \frac{\partial}{\partial x} (m_F v^2) -$$

$$\frac{\partial}{\partial t} (m_F v) - m_F v^2(0,t)\, (1 - \cos \gamma)\, \delta(x) = 0 \tag{3.33}$$

where Eq. (3.27) and the nature of $\bar{f}_{INC}$ are recalled. A combination of Eq. (3.32) with (3.33) results in a simplified expression for the motion of the case alone.

$$-M_C(\ddot{X} - \ddot{\theta}Y - 2\dot{\theta}\dot{Y} - \dot{\theta}^2X) - \int_0^L m_C \ddot{u} \, dx + \ddot{\theta} \int_0^L m_C y \, dx +$$

$$\dot{\theta}^2 \int_0^L m_C(x + u) \, dx + 2\dot{\theta} \int_0^L m_C \dot{y} \, dx - \int_0^L (\bar{f}_{RF} \cdot \bar{i} + \dot{m}_C v_C) \, dx -$$

$$M_C g \sin \theta + F_{XC} = 0 \qquad (3.34)$$

where M_C is the total mass of the missile case at any time, F_{XC} is the generalized force associated with the coordinate X due to external forces acting on the case only, and the terms due to the fluid pressure are represented in the integral

$$- \int_0^L (\bar{f}_{RF} \cdot \bar{i}) \, dx = \int_0^L (\bar{f}_{RC} \cdot \bar{i}) \, dx.$$

In a similar manner, the equation of motion corresponding to the Y coordinate is

$$-M_C(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X - \dot{\theta}^2Y) - \int_0^L m_C \ddot{y} \, dx - \ddot{\theta} \int_0^L m_C(x + u) \, dx$$

$$+\dot{\theta}^2 \int_0^L m_C \, y \, dx - 2\dot{\theta} \int_0^L m_C \dot{u} \, dx - M_C \, g \cos \theta$$

$$-\int_0^L (\bar{f}_{RF} \cdot \bar{j}) \, dx + F_{YC} = 0 \qquad (3.35)$$

and the equation for the θ coordinate is

$$- (\ddot{\theta}Y + \dot{\theta}^2X - \ddot{X} + 2\dot{\theta}\dot{Y}) \int_0^L m_C y \, dx + (-\ddot{\theta}X + \dot{\theta}^2Y - \ddot{Y} - 2\dot{\theta}\dot{X}) \cdot$$

$$\int_0^L m_C(x + u) \, dx + Y F_{XC} - XF_{YC} - \int_0^L m_C \left[\ddot{y}(x + u) - y \ddot{u}\right] dx$$

$$- \ddot{\theta}\int_0^L m_C \left[(x + u)^2 + y^2\right] dx - 2\dot{\theta} \int_0^L m_C \left[(x + u) \dot{u} + y\dot{y}\right] dx$$

$$- \int_0^L m_C g \left[(x + u) \cos \theta - y \sin \theta \right] dx + \int_0^L \left[(\dot{m}_C v_C + \overline{f}_{RF} \cdot \overline{i}) y - (\overline{f}_{RF} \cdot \overline{j}) (x + u) \right] dx + F_{\theta C} = 0 \tag{3.36}$$

The final two equations of motion corresponding to the coordinates u and y are obtained in an identical fashion.

$$\frac{\partial}{\partial x} (EA_C u') - m_C \left[\ddot{X} + \ddot{u} - \ddot{\theta}(Y + y) - 2\dot{\theta}(\dot{Y} + \dot{y}) - \dot{\theta}^2(X + x + u) \right] - \dot{m}_C v_C - \overline{f}_{RF} \cdot \overline{i} - m_C g \sin \theta + \hat{F}_{uC} = 0 \tag{3.37}$$

and

$$- \frac{\partial^2}{\partial x^2} (E\ I\ y'') + \frac{\partial}{\partial x} (Py') - m_C \left[\ddot{Y} + \ddot{y} + \ddot{\theta}(X + x + u) + 2\dot{\theta}(\dot{X} + \dot{u}) - \dot{\theta}^2(Y + y) \right] - \overline{f}_{RF} \cdot \overline{j} - m_C g \cos \theta + \hat{F}_{yC} = 0 \tag{3.38}$$

where the latter two equations are subject, respectively, to the boundary conditions

$$EA_C\ u' = 0 \qquad \text{at } x = 0, L \tag{3.39}$$

and

$$EI\ y'' = 0 \qquad \text{at } x = 0, L \tag{3.40}$$

$$-P\ y' + \frac{\partial}{\partial x} (EI\ y'') = 0 \qquad \text{at } x = 0, L \tag{3.41}$$

The five equations of motion are coupled and, in addition, highly nonlinear so that no solution in closed form exists. Furthermore, the evaluation of the reactive force due to the internal fluid flow is not yet completely defined since its determination requires additional information concerning the mass flow rate as a function of time, together with the pressure and velocity distributions throughout the missile. Also appearing in the equations of motion are quantities, some of them in an implicit manner, such as the physical shape, stiffness and density of the shell, nozzle configuration and flow characteristics, as well as drag and internal viscous forces.

IV. THE INTERNAL FLUID FLOW

4.1 Internal Pressure and Velocity Distribution

The formulation of an exact mathematical model treating the extremely complex physical conditions occurring within a solid-fuel missile is difficult in itself, and beyond the scope of this investigation. For instance, most of the double base solid propellants commonly in present use are susceptible to a variation in the burning rate with changes in both pressure and temperature as well as to the inertial stresses resulting from flight. Coupled with the variation of the burning rate of acoustic origin, these effects may become significant to such a degree as to cause unstable burning or "chuffing." Additional complexity arises in the consideration of the combustion process itself, such that a rigorous formulation would include the possibility of burning particles or other solids in the internal fluid.

In the present investigation, the missile is regarded as a cylindrical container of uniform cross section, closed at the end $x = L$, but open through a completely expanding nozzle at the end $x = 0$. Also, the length of the nozzle is considered relatively short compared to the total length of the rocket. The propellant is assumed to be bonded to the missile shell such that mass per unit length and the internal flow area vary with time. In view of the low elastic modulus (1 to $7 \cdot 10^3$ psi)[22] for most propellants, together with the complicating fact that the elastic modulus is nonlinear for even low stress levels, the stiffness of the missile does not include the stiffness due to the propellant and thus remains constant with time.

Since the forward end of the missile is closed, the products of combustion must flow in the negative x - direction such that at any point

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[22] R. N. Wimpress, p. 80.

x, the flow rate is equal to the total rate of change of mass between the point x and the closed end. This quantity is denoted as $\dot{M}_C(x,t)$, and from the continuity equation of fluids

$$\dot{M}_C(x,t) = \int_x^L \dot{m}_C(\xi,\ t)\ d\xi \qquad (4.1)$$

where ξ is a dummy variable of integration.

If the burning rate is assumed independent of temperature and pressure, the rate of change of mass per unit length of the missile is independent of the spatial coordinate x such that

$$m_C(x,t) = m_C(t) \qquad (4.2)$$

Furthermore, the missile will remain uniform in cross section and the center of mass of the system will not shift with respect to the x axis. In addition to the consideration of the burning rate as a function of the spatial coordinate, an assumption must be made concerning the mass flow rate as a function of time. This rate is largely determined by the grain configuration, together with the use and position of burning inhibitors as well as other less important conditions. The simple cylindrical grain shape treated in the present investigation results in approximately "neutral burning"[22] while various cruciform, multiple grain, or end burning forms may be designed to yield either "progressive" or "regressive" burning characteristics. Typical mass-flow rates amenable to analytical treatment may be approximated as shown in Fig. 4.1 by a rectangular function or slightly more accurately by a trapezoidal function.[3]

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[3] G. Leitmann, p. 142

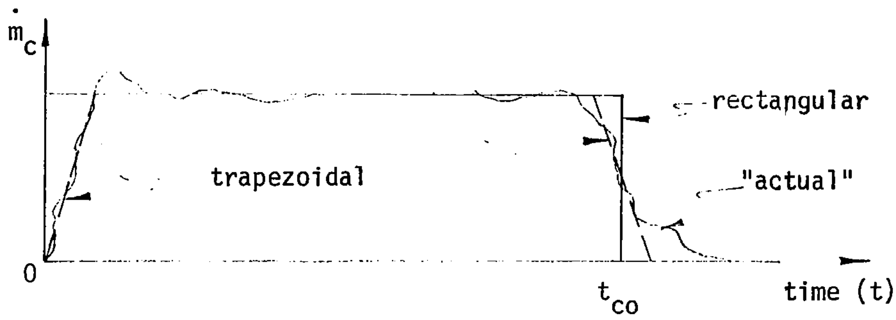

Figure 4.1

Actual and Approximated Mass Flow Rate

A common assumption is that the temperature and pressure in the combustion chamber are uniform[22] and the flow is frictionless, so that the density of the fluid ρ is constant. Since the flow area A_F is constant with respect to x (although not with respect to time), the fluid mass per unit length, $m_F = \rho A_F$, is constant, and a linear relative velocity distribution results from Equation (4.1). This assumption of uniform chamber pressure must be further justified for high accelerations, however, since it neglects the effects of a change of pressure due to flow acceleration and friction and due to the fact that the control volume is accelerating with respect to the inertial axes.

Equation (3.33), for frictionless, steady-state flow with constant area and zero engine gimballing angle, reduces to

$$- \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} (\rho v^2) + \rho(\ddot{\bar{R}}_C \cdot \bar{i} + g \sin \theta) \qquad (4.3)$$

With $\ddot{\bar{R}}_C$ assumed to be known, the above equation contains three unknowns: p, ρ, and v. However, additional equations are available in the form of

[22] Wimpress, p. 42

$$\dot{M}_C(x,t) = \rho A_F v(x,t) \tag{4.4}$$

together with the equation of state for a perfect gas

$$p = \rho RT \tag{4.5}$$

which, with the assumption that the gas constant R is known, introduces the absolute temperature T as a new unknown. Consistent with the assumption of a perfect gas are the relations

$$c_p - c_v = R \tag{4.6}$$

and

$$\gamma = c_p/c_v \tag{4.7}$$

where c_p and c_v are the specific heats at constant pressure and volume, respectively, and γ is the ratio of the two. Finally, these relations are supplemented by the heat-energy relations

$$h + \frac{1}{2} v^2 = h_L \tag{4.8}$$

and

$$h - h_r = c_p T \tag{4.9}$$

where h is the enthalpy at any point, h_L is the enthalpy at the forward end of the missile, and h_r is a reference enthalpy which may be assumed zero. Substitution of Eqs. (4.4) and (4.9) into Eq. (4.8) yields

$$v^2 + \frac{2pA_F c_p}{R\dot{M}_c(x,t)} - 2c_p T_L = 0 \tag{4.10}$$

from which a value for v may be obtained. If this value of v is in turn introduced into the integral form of Eq. (4.3), the result is an expression which, with the exception of ρ in the integral, is in terms of known quantities only

$$p_L - p = \pm \left[\left(\frac{p\gamma}{\gamma-1}\right)^2 + 2\,\frac{c_p T_L \dot{M}_C(x,t)}{A_F^2} \right]^{\frac{1}{2}} - \int_x^L \rho(\ddot{\bar{R}}_C \cdot \bar{i} + g \sin\theta)\, d\xi \quad (4.11)$$

A similar substitution for ρ in the above equation results in an equation that relates the pressure at any point in the chamber to the pressure at the forward end in terms of known quantities. However, the evaluation of the integral is impractical. If the integral is neglected for the moment, Eq. (4.11) simplifies to

$$\frac{p}{p_L} = \frac{1}{\gamma+1} \pm \frac{1}{\gamma+1} \left[1 + (\gamma - 1)(\gamma + 1)(1 - 2c_p T_L \dot{M}_C\, (x,t)/A_F^2\, p_L^2) \right]^{\frac{1}{2}} \quad (4.12)$$

which agress with Price[23] who has plotted p/p_L as a function of $\dot{M}_C(x,t)/\dot{M}_{C*}$ where $\dot{M}_{C*}$ is the mass rate of flow at sonic velocity in the combustion chamber. An examination of Price's results indicates that for a Mach number of 0.4 or less, such as exists in most physical systems, the maximum change in p/p_L is approximately 5 per cent due to the flow acceleration.

An indication of the order of magnitude of the integral term resulting from the acceleration of the control volume may now be obtained. Since p/p_L is nearly constant, ρ/ρ_L is likewise. Furthermore, the effect of gravity compared to $\ddot{\bar{R}}_C$ is small for a vehicle under high acceleration. In addition, if the motion is assumed to be simple translation, $\ddot{\theta} = \dot{\theta} = 0$, and $\ddot{\bar{R}}_C \cdot \bar{i} = \ddot{X} + \ddot{u}$. But $\ddot{u}$ is sign variable and may be ignored over the total length of the missile. Depending on the drag, burning rate, and nozzle configuration, a typical solid-fuel missile may exhibit a linear acceleration of the order of 5,000 ft sec^{-2}, so that the change in pressure resulting from the noninertial term is

$$\Delta p \simeq \int_0^L \rho\, \ddot{X}\, d\xi = 9 \text{ psi} \quad (4.13)$$

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[23] E. W. Price, p. 63

for each 100 inches of combustion-chamber length. For a missile 100 inches in length with a chamber pressure of 2,000 psi, this amounts to only 0.45 per cent of the pressure change due to flow acceleration and, hence, may be neglected. Also, this is a pressure increase rather than a pressure decrease in the negative x-direction, and as such, tends to render the chamber pressure more uniform.

The magnitude of the friction term $\overline{f}_{EXF}$ may be estimated by assuming a value for the friction factor, f = 0.005 (see Wimpress, p. 33), and again using the assumption of nearly constant internal fluid density. With z denoting the perimeter of the flow area, the pressure drop due to friction for a 10-inch-diameter missile with the above chamber pressure is

$$\Delta p = \int_0^L \frac{fv^2 \rho z}{2A_F} \, d\xi \simeq 3.3 \text{ psi} \tag{4.14}$$

for each 100 inches of chamber length, which is once again negligible.

Thus, even for a body under high acceleration, the assumption of a constant chamber pressure, together with the resulting linear velocity distribution, appears quite reasonable insofar as the dynamic characteristics of the missile are concerned, and this assumption is used for the remainder of the investigation.

4.2 Determination of the Forcing Function

In order to evaluate the forcing function resulting from the internal pressure and mass flow, it is necessary to determine the value of the expressions involving $\overline{f}_{RF}$ and $\dot{m}_C v_C$ in the equations of motion, Eqs. (3.34) through (3.38). If the propellant merely burns or ablates at the surface such that the mass has essentially no component of axial velocity relative to the case before it enters the control volume, the $\dot{m}_C v_C$ terms are negligible, and the thrust is determined from the evaluation of the integral

$$\int_0^L (\bar{f}_{RF} \cdot \bar{i})\, dx = \int_0^L p\, (\frac{\partial A_F}{\partial x})\, dx = \int_0^L p dA_F \qquad (4.15)$$

where the reactive distributed force is determined as indicated in Fig. 4.2.

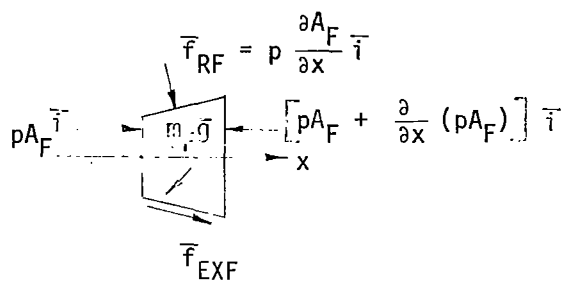

Figure 4.2

Forces on Fluid Element

With the exception of the nozzle and the forward end of the missile, the flow area is constant for vehicles of uniform cross section, so that $dA_F = 0$. If the length of the nozzle is small compared with the length of the missile, any net reactive force at the nozzle may be considered to act at x = 0. However, in order to evaluate the pdA_F term of Eq. (4.15) it is necessary to know the pressure as a function of the spatial coordinate, x, which will vary with nozzle configuration. For the purposes of this investigation, the assumption of a linear pressure drop through the nozzle with sonic flow at the throat is considered adequate, such that

$$\int_0^L p dA_F = p_L A_F \delta(L) - (\frac{p_N - p_T}{2})(A_N - A_T)\ \delta(0) \qquad (4.16)$$

where the subscripts correspond to the notation of Fig. 4.3.

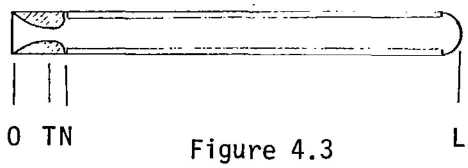

Figure 4.3

Axial Position Locations

Since the sonic pressure is known in terms of the chamber pressure, the reactive force resulting from the internal fluid is

$$\int_0^L (\bar{f}_{RF} \cdot \bar{i})\, dx \simeq p_L A_F\, \delta(L) - 0.236\, p_N A_F\, (1 - A_T/A_F)\, \delta(0) \qquad (4.17)$$

V. THE VERTICAL, UPRIGHT FLIGHT. CLOSED FORM SOLUTION

5.1 Rigid-Body Motion

Although the reactive force of the internal fluid is now tractable to calculation due to the assumptions of the previous section, the equations of motion remain highly nonlinear and coupled. To add to the complexity, the mass distribution is time-dependent so that a closed form solution of the equations in the present form is not possible.

Whereas an entirely numerical solution of the equations is obtainable, and is in fact accomplished in the subsequent section, such a solution often tends to obscure significant dynamic characteristics since it is valid only for a given set of physical parameters and initial conditions. Also, in the event that an instability is indicated, the question arises as to whether it is the physical system or merely the mathematical representation that is unstable. In any event, generalizations are difficult to ascertain, and since the purpose of this investigation is to study the dynamic characteristics of a solid-fuel missile, an analytical treatment is highly desirable, even under simplifying assumptions.

The flight of many sounding rockets consists of vertical, upright flight, and in order to study this case in more detail, the "primary motion" is defined as the vertical rigid-body flight of the vehicle. The deviations from the primary motion form "secondary motions" and are regarded as small perturbations from the primary motion. Since the missile is assumed initially uniform and since the mass is dissipated uniformly, the center of mass remains stationary relative to the vehicle shell at the mid-way point between the ends of the missile. In the subsequent material it is convenient to measure x from the center of the missile such that $X(t)$ and $Y(t)$ are the coordinates of the center of mass.

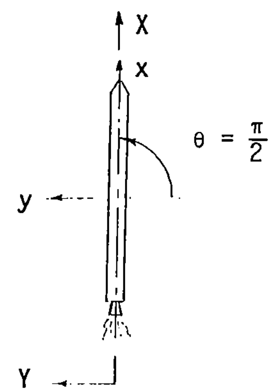

Figure 5.1

Vertical, Upright Flight

Equations (3.34) through (3.35) are considerably simplified by the assumption that $X >> Y$, $X >> \theta$ and $\theta \approx \frac{\pi}{2}$, so that second and higher order terms in Y and θ are ignored. Since the choice of the origin at the center of mass leads to $\int_{-L/2}^{L/2} m_C x \, dx = 0$, the above three equations reduce to

$$- M_C\ddot{X} - \int_{-L/2}^{L/2} m_C\ddot{u} \, dx - \int_{-L/2}^{L/2} (\bar{f}_{RF} \cdot \bar{i} + \dot{m}_C v_C) dx - M_C g + F_{XC} = 0 \qquad (5.1)$$

$$-M_C(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X) - \int_{-L/2}^{L/2} m_C\ddot{y} \, dx - \ddot{\theta}\int_{-L/2}^{L/2} m_C u \, dx$$

$$- 2\dot{\theta} \int_{-L/2}^{L/2} m_C\dot{u} \, dx - \int_{-L/2}^{L/2} \bar{f}_{RF} \cdot \bar{j} \, dx + F_{Y_C} = 0 \qquad (5.2)$$

$$\ddot{X} \int_{-L/2}^{L/2} m_C y \, dx - (\ddot{\theta}X + Y + 2\dot{\theta}\dot{X}) \int_{-L/2}^{L/2} m_C u \, dx - \int_{-L/2}^{L/2} m_C \left[\ddot{y}(x+u) - y\ddot{u} \right] dx$$

$$- \ddot{\theta} \int_{-L/2}^{L/2} m_C \left[(x + u)^2 + y^2 \right] dx - 2\dot{\theta} \int_{-L/2}^{L/2} m_C \left[(x + u) \dot{u} + y\dot{y} \right] dx$$

$$+ \int_{-L/2}^{L/2} \left[(\bar{f}_{RF} \cdot \bar{i} + \dot{m}_C v_C) y - \bar{f}_{RF} \cdot \bar{j} \, (x + u) \right] dx + \int_{-L/2}^{L/2} m_C g \, y \, dx$$

$$+ YF_{XC} - XF_{Y_C} + F_{\theta C} = 0 \qquad (5.3)$$

Equations (5.1) through (5.3) are further simplified by consideration of the fact that the average elastic motion, even when multiplied by a weighting function, is zero. This, combined with the assumption that the elastic displacements are sufficiently small that second-order terms are negligible, leads to

$$- M_C\ddot{X} - \int_{-L/2}^{L/2} (\overline{f}_{RF} \cdot \overline{i} + \dot{m}_C v_C)\, dx - M_C g + F_{XC} = 0$$

$$- M_C(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X) - \int_{-L/2}^{L/2} \overline{f}_{RF} \cdot \overline{j}\, dx + F_{YC} = 0 \qquad (5.4)$$

$$- \ddot{\theta} \int_{-L/2}^{L/2} m_C x^2\, dx - \int_{-L/2}^{L/2} (\overline{f}_{RF} \cdot \overline{j})\, x\, dx + YF_{XC} - XF_{YC} + F_{\theta C} = 0$$

which establishes the rigid-body motion of the missile in terms of the coordinates $X(t)$, $Y(t)$, and $\theta(t)$.

For no engine gimbal angle

$$\int_{-L/2}^{L/2} \overline{f}_{RF} \cdot \overline{j}\, dx = 0 \qquad (5.5)$$

which implies that there is no transverse component due to fluid flow. Furthermore, in order to concentrate on the equivalent of the free vibration case, the nonconservative external forces are ignored, or

$$F_{XC} = F_{YC} = F_{\theta C} = \hat{F}_{uC} = \hat{F}_{yC} = 0 \qquad (5.6)$$

so that the last two of Eqs. (5.4) are identically satisfied by

$$Y = \theta = 0 \qquad (5.7)$$

and the first one yields the axial acceleration

$$\ddot{X} = - g - \frac{1}{M_C} \int_{-L/2}^{L/2} (\overline{f}_{RF} \cdot \overline{i} + \dot{m}_C v_C)\, dx \qquad (5.8)$$

5.2 Elastic Motion

In view of the assumptions and results of the preceding section, the equations for the axial and transverse elastic motions become, respectively,

$$\frac{\partial}{\partial x}\left(EA_C \frac{\partial u}{\partial x}\right) + m_C\ddot{u} = -m_C\ddot{X} - \overline{f}_{RF} \cdot \overline{i} - \dot{m}_C v_C - m_C g = f(x,t) \tag{5.9}$$

and

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(P \frac{\partial y}{\partial x}\right) + m_C\ddot{y} = 0 \tag{5.10}$$

where the axial force, P, is given by

$$P = EA_C \frac{\partial u}{\partial x} \tag{5.11}$$

In addition, Eq. (5.9) is subject to the boundary conditions as expressed in Eq. (3.39) and Eq. (5.10) is subject to boundary conditions (3.40) and (3.41).

Whereas neither the differential equation nor the boundary conditions for the axial elastic displacement u(x,t) contains the transverse elastic displacement y(x,t), an examination of the differential equation and boundary conditions for y(x,t) indicates that both contain u(x,t) as is seen from Eq. (5.11). A solution for u(x,t) must, therefore, be obtained prior to the solution for y(x,t).

A solution of Eq. (5.9) is assumed in the form of a series

$$u(x,t) = \sum_{i=1}^{n} \phi_i(x)\, q_i(t) \tag{5.12}$$

where $q_i(t)$ are time-dependent generalized coordinates and $\phi_i(x)$ are the solutions of the eigenvalue problem consisting of the differential equation

$$-\frac{d}{dx}\left(EA_C \frac{d\phi}{dx}\right) = m_o\omega^2\phi \tag{5.13}$$

together with the boundary conditions

$$EA_C \frac{d\phi}{dx} = 0, \qquad x = -L/2,\ L/2 \qquad (5.14)$$

This eigenvalue problem corresponds to the axial vibration problem of a rod of uniform mass per unit length m_o. Furthermore, the functions ϕ_i are such that $\int_{-L/2}^{L/2} m_o \phi_i \phi_j \, dx = \delta_{ij}$ where δ_{ij} is the Kronecker delta, and in addition $\int_{-L/2}^{L/2} m_o \phi_i \, dx = 0$. The latter expression is consistent with the zero average elastic motion postulated in the preceding section. In this case, m_o is taken as the initial mass per unit length of the missile and, hence, independent of x. From the assumption of uniform burning introduced in Section IV, the distributed mass at any time is

$$m_C(t) = m_o - \dot{m}_C t = m_o (1 - \beta t) \qquad (5.15)$$

where $\beta = \dot{m}_C/m_o$.

If Eq. (5.12) and (5.15) are introduced into Eq. (5.9), the result is

$$\sum_{i=1}^{n} m_o \left[(1 - \beta t)\ \ddot{q}_i + \omega_i^2 q_i\right] \phi_i = f\ (x,t) \qquad (5.16)$$

where utilization has been made of Eq. (5.13). If the above expression is multiplied by ϕ_j and integrated over the length of the missile, a set of uncoupled ordinary differential equations is obtained in the form of

$$(1 - \beta t)\ \ddot{q}_i + \omega_i^2 q_i = U_i(t), \quad (i = 1, 2, ---, n) \qquad (5.17)$$

where

$$U_i(t) = \int_{-L/2}^{L/2} f(x,t)\ \phi_i(x)\ dx, \quad (i = 1, 2, ---, n) \qquad (5.18)$$

are the generalized forces and contain terms due to the internal fluid reactive forces. As a result of the relatively rapid transition to a steady-state situation of constant burning (see Fig. 4.1), the time dependency of $U_i(t)$ is assumed as a step function for the duration of powered flight.

A substitution of variables in the form of $1 - \beta t = \tau^2$ in the homogeneous part of Eq. (5.17) yields a Bessel equation of the type

$$\tau \frac{d^2 q_i}{d\tau^2} - \frac{dq_i}{d\tau} + \lambda_i^2 \tau q_i \qquad (5.19)$$

where

$$\lambda_i = \frac{2\omega_i}{\beta} \qquad (5.20)$$

Eq. (5.19) has the solution

$$q_i(t) = C_{1i}\ \tau\ J_1(\lambda_i \tau) + C_{2i}\ \tau\ Y_1(\lambda_i \tau) \qquad (5.21)$$

where J_1 and Y_1 are Bessel functions of the first order and first and second kind, respectively. A resubstitution of variables results in the solution of Eq. (5.17) in the form

$$q_i(t) = (1 - \beta t)^{\frac{1}{2}} \left[C_{1i}\ J_1\ (\lambda_i \sqrt{1-\beta t}) + C_{2i}\ Y_1\ (\lambda_i \sqrt{1-\beta t}) \right] + \frac{U_i}{\omega_i^2}\ ,$$

$$(i = 1, 2, ---, n) \qquad (5.22)$$

where U_i is the amplitude of $U_i(t)$ and C_{1i} and C_{2i} are constants of integration that are determined from the initial conditions. If the intial conditions are assumed as

$$u(x,t) \Big|_{t=o} = u_o(x),\ \frac{\partial u(x,t)}{\partial t} \Big|_{t=o} = 0 \qquad (5.23)$$

the solution for the axial elastic displacement reduces to

$$u(x,t) = \sum_{i=1}^{n} \left\{ \frac{(u_{oi} - U_i/\omega_i^2)(1-\beta t)^{\frac{1}{2}}}{J_1(\lambda_i)\ Y_o(\lambda_i) - J_o(\lambda_i)\ Y_1(\lambda_i)} \left[Y_o(\lambda_i)\ J_1(\lambda_i \sqrt{1-\beta t}) \right. \right.$$

$$\left. \left. - J_o(\lambda_i)\ Y_1(\lambda_i \sqrt{1-\beta t}) \right] + U_i/\omega_i^2 \right\} \phi_i(x) \qquad (5.24)$$

where

$$\int_{-L/2}^{L/2} m_o\, u_o(x)\, \phi_i\, (x)\, dx = u_{oi}, \quad (i=1,2,---,n) \tag{5.25}$$

Since $\lambda_i >> 1$, asymptotic expansions for Bessel functions of large argument are possible in the form of[24]

$$J_n\,(\lambda_j z) \sim \sqrt{2/\pi\lambda_j z}\, \left[\zeta_n(\lambda_j z)\cos\,\alpha - \xi_n(\lambda_j z)\sin\,\alpha\right]$$

and

$$Y_n(\lambda_j z) \sim \sqrt{2/\pi\lambda_j z}\, \left[\zeta_n(\lambda_j z)\,\sin\,\alpha - \xi_n(\lambda_j z)\,\cos\,\alpha\right]$$

where

$$\zeta_n(\lambda_j z) \sim 1 - \frac{(4n^2-1)(4n^2-3^2)}{2!\ \ (8\lambda_j z)^2} + \ldots$$

$$\xi_n(\lambda_j z) \sim \frac{4n^2-1}{1!\ (8\lambda_j z)} - \ldots$$

and

$$\alpha = \lambda_j z - \frac{\pi}{4} - \frac{n\pi}{2}$$

If the above expressions are introduced into Eq. (5.24) the result is

$$u(x,t) = \sum_{i=1}^{n} \left[(u_{oi} - \frac{U_i}{\omega_i^2})(1-\beta t)^{\frac{1}{4}} \cos\,\lambda_i(1-\sqrt{1-\beta t}) + \frac{U_i}{\omega_i^2}\right] \phi_i(x) \tag{5.26}$$

where the square bracket is identified as $q_i(t)$.

Now we are in the position to attempt a solution for the motion $y(x,t)$. As in the case of the axial motion, the transverse elastic displacement is assumed in the form of a series

$$y(x,t) = \sum_{j=1}^{n} \psi_j(x)\ \eta_j(t) \tag{5.27}$$

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[24] N. W. McLachlan, pp. 81-82

where the $\eta_j(t)$'s are once again time-dependent generalized coordinates, and the $\psi_j(x)$'s are eigenfunctions obtained from the solution of the eigenvalue problem defined by the differential equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2}{dx^2}\right) = m_o\,\Omega^2\psi \tag{5.28}$$

and the boundary conditions

$$EI\,\frac{d^2\psi}{dx^2} = 0 \text{ and } \frac{d}{dx}\left(EI\,\frac{d^2\psi}{dx^2}\right) = 0, \quad x = -L/2,\ L/2 \tag{5.29}$$

The eigenfunctions ψ_j are such that

$$\int_{-L/2}^{L/2} m_o\,\psi_i\,\psi_j\,dx = \delta_{ij}\,, \quad \int_{-L/2}^{L/2} m_o\,\psi_j\,dx = 0 \quad \text{and} \quad \int_{-L/2}^{L/2} m_o x\,\psi_j dx = 0,$$

where once again, the latter two expressions justify some of the simplifications obtained in the equations for the rigid-body motion in Section 5.1.

Equations (5.11), (5.12), and (5.27) are introduced into Eq.(5.10) which yields

$$\sum_{j=1}^{n} m_o\left[(1-\beta t)\,\ddot{\eta}_j + \Omega^2{}_j\eta_j\right]\psi_j \;-\sum_{i=1}^{n}\sum_{j=1}^{n} q_i\,\frac{d}{dx}\left(EA_C\,\frac{d\phi_i}{dx}\,\frac{d\psi_j}{dx}\right)\eta_j = 0 \tag{5.30}$$

where Eq. (5.28) has been used. If Eq. (5.30) is multiplied by ψ_r and integrated over the length of the missile we obtain a set of ordinary differential equations of the form

$$(1-\beta t)\,\ddot{\eta}_r + \Omega_r{}^2\,\eta_r - \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ijr}(t)\,\eta_j = 0 \tag{5.31}$$

where

$$P_{ijr}(t) = \int_{-L/2}^{L/2} q_i(t)\ \psi_r(x) \frac{d}{dx} \left(EA_C \frac{d\phi_i}{dx} \frac{d\psi_j}{dx}\right) dx \tag{5.32}$$

In view of the boundary conditions (5.14), Eq. (5.32) reduces to

$$P_{ijr}(t) = -q_i(t) \int_{-L/2}^{L/2} EA_C \frac{d\phi_i}{dx} \frac{d\psi_j}{dx} \frac{d\psi_r}{dx}\, dx \tag{5.33}$$

Unfortunately, unlike the set of equations for the axial motion, the set of equations (5.31), for the transverse motion, are coupled due to the presence of the axial force, P. However, by retaining a limited number of terms in the expansions (5.12) and (5.27), Eqs. (5.31) become uncoupled. It is observed, for instance, that when the integrand in Eq. (5.33) is an odd function, the resultant integral is zero. But ϕ_i is an odd function if i is odd and ψ_r is an even function if r is even and vice versa. In addition, ϕ_i and ψ_r are such that if they are even functions their derivatives are odd functions, and vice versa, so that

$$\begin{aligned}
P_{111}(t) &= -q_1(t) \int_{-L/2}^{L/2} EA_C \frac{d\phi_1}{dx} \left[\frac{d\psi_1}{dx}\right]^2 dx = -Q_{111}\ q_1(t) \\
P_{112}(t) &= P_{121}(t) = 0 \\
P_{122}(t) &= -q_1(t) \int_{-L/2}^{L/2} EA_C \frac{d\phi_1}{dx} \left[\frac{d\psi_2}{dx}\right]^2 dx = -Q_{122}\ q_1(t)
\end{aligned} \tag{5.34}$$

If only the first term in the series (5.12), together with the first two terms in the series (5.27) are retained, Eqs. (5.31) reduce to the uncoupled equations

$$\begin{aligned}
(1 - \beta t)\ \ddot{\eta}_1 + \Omega_1^2 \eta_1 - Q_{111}\ q_1(t)\ \eta_1 &= 0 \\
(1 - \beta t)\ \ddot{\eta}_2 + \Omega_2^2 \eta_2 - Q_{122}\ q_1(t)\ \eta_2 &= 0
\end{aligned} \tag{5.35}$$

A solution to the typical equation

$$(1-\beta t)\,\ddot{\eta} + \Omega^2\eta - \left(\sum_{k=0}^{m} Q_k t^k\right)\eta = 0 \tag{5.36}$$

is obtained by the method of Frobenius and has the form

$$\eta = \sum_{n=0}^{\infty} B_n t^{n+s} \tag{5.37}$$

so that

$$\sum_{n=0}^{\infty} B_n(n + s)(n + s - 1)t^{n + s - 2} - \beta \sum_{n=0}^{\infty} B_n(n + s)(n + s - 1)t^{n + s - 1}$$

$$+ (\Omega^2 - Q_0) \sum_{n=0}^{\infty} B_n t^{n + s + 1} - Q_2 \sum_{n=0}^{\infty} B_n t^{n + s + 2} --- = 0$$

which can be written as

$$\sum_{n=0}^{\infty} B_n(n + s)(n + s - 1)\, t^{n + s - 2} - \beta \sum_{n=1}^{\infty} \beta_{n-1}(n + s - 1)(n + s - 2)^{n+s-2}$$

$$+ (\Omega^2 - Q_0) \sum_{n=2}^{\infty} B_{n-2} t^{n + s - 2} - Q_1 \sum_{n=3}^{\infty} B_{n-3} t^{n + s - 2}$$

$$- Q_2 \sum_{n=4}^{\infty} B_{n-4} t^{n + s - 2} - --- = 0$$

The above equation is satisfied if

$$B_0\, s(s-1) = 0$$

$$B_1\,(s + 2)\,s - \beta B_0\, s(s - 1) = 0$$

$$B_2\,(s + 2)(s + 1) - \beta B_1(s + 1)\,s + B_0(\Omega^2 - Q_0) = 0 \tag{5.38}$$

$$B_3\,(s + 3)(s + 2) - \beta B_2(s + 2)(s + 1) + B_1(\Omega^2 - Q_0) - B_0 Q_1 = 0$$

$$B_4(s + 4)(s + 3) - \beta B_3(s + 3)(s + 2) + B_2(\Omega^2 - Q_0) - B_1 Q_1 - B_0 Q_2 = 0$$

etc.

A solution is obtained by setting $s = 0$, where B_0 and B_1 remain arbitrary, so that from Eq. (5.38) are obtained all the coefficients B_n, $n \geq 2$, in terms of B_0 and B_1, where the latter are the constants of integration. For s = 0, Eqs. (5.38) assume the form of the recurrence formula

$$B_n = \beta \frac{n-2}{n} B_{n-1} - \frac{\Omega^2 - Q_0}{n(n-1)} + \frac{1}{n(n-1)} \sum_{k=1}^{\infty} Q_k B_{n-2-k}, \quad n \geq 2 \tag{5.39}$$

and $B_n = 0$ for n negative.

Equation (5.39) yields

$$B_2 = -\tfrac{1}{2}(\Omega^2-Q_0)B_0$$

$$B_3 = \frac{1}{6}\left[-\beta(\Omega^2-Q_0) + Q_1\right]B_0 - \frac{1}{6}(\Omega^2-Q_0)B_1$$

$$B_4 = \frac{1}{12}\left[-\beta^2(\Omega^2-Q_0) + \tfrac{1}{2}(\Omega^2-Q_0)^2 + \beta Q_1 + Q_2\right]B_0$$
$$+ \frac{1}{12}\left[-\beta(\Omega^2-Q_0) + Q_1\right]B_1 \tag{5.40}$$
$$B_5 = \frac{1}{20}\left[-(\beta^3+Q_1)(\Omega^2-Q_0) + \frac{2}{3}(\Omega^2-Q_0)^2 + \beta^2 Q_1 + \beta Q_2 + Q_3\right]B_0$$
$$+ \frac{1}{20}\left[-\beta^2(\Omega^2-Q_0) + \frac{1}{6}(\Omega^2-Q_0)^2 + \beta Q_1 + Q_2\right]B_1$$

so that the introduction of the above coefficients into Eq. (5.37) results in

$$n = \sum_{n=0}^{\infty} B_n t^n = B_0 + B_1 t - \frac{1}{2}(\Omega^2-Q_0)B_0 t^2 + \{\frac{1}{6}\left[-\beta(\Omega^2-Q_0) + Q_1\right]B_0$$
$$- \frac{1}{6}(\Omega^2-Q_0)B_1\}t^3 + \{\frac{1}{12}\left[-\beta^2(\Omega^2-Q_0) + \frac{1}{2}(\Omega^2-Q_0)^2 + \beta Q_1 + Q_2\right]B_0$$
$$+ \frac{1}{12}\left[-\beta(\Omega^2-Q_0) + Q_1\right]B_1\}t^4 + \frac{1}{20}\left[-(\beta^3+Q_1)(\Omega^2-Q_0) + \frac{2}{3}(\Omega^2-Q_0)^2\right.$$
$$\left.+ \beta^2 Q_1 + \beta Q_2 + Q_3\right]B_0 + \frac{1}{20}\left[-\beta(\Omega^2-Q_0) + \frac{1}{6}(\Omega^2-Q_0) + \beta Q_1 \pm Q_2\right]B_1\}t^5 + \text{----}$$
$$= B_0\{1 - \frac{1}{2}(\Omega^2-Q_0)t^2 + \frac{1}{6}\left[Q_1 - \beta(\Omega^2-Q_0)\right]t^3 + \frac{1}{12}\left[Q_2 + \beta Q_1\right. \tag{5.41}$$
$$\left.+ \frac{1}{2}(\Omega^2-Q_0) - \beta^2(\Omega^2-Q_0)\right]t^4 + \frac{1}{20}\left[Q_3 + \beta Q_2 + \beta^2 Q_1 + \frac{2}{3}(\Omega^2-Q_0)^2\right.$$
$$\left.- (\beta^3+Q_1)(\Omega^2-Q_0)\right]t^5 + \text{---------}\}$$
$$+ B_1\{t - \frac{1}{6}(\Omega^2-Q_0)t^3 + \frac{1}{12}\left[Q_1 - \beta(\Omega^2-Q_0)\right]t^4 + \frac{1}{20}\left[Q_2 + \beta Q_1\right.$$
$$\left.+ \frac{1}{6}(\Omega^2-Q_0)^2 - \beta^2(\Omega^2-Q_0)\right]t^5 + \text{---------}\}$$

Hence, the solution of Eqs. (5.35) assumes the simplified form

$$\eta_j(t) = B_{0j}\, \eta_{0j}(t) + B_{1j}\, \eta_{1j}(t), \qquad (j = 1,2) \tag{5.42}$$

where

$$\begin{aligned} \eta_{0j}(t) &= 1 - \frac{1}{2}(\Omega_j^2 - Q_0)t^2 + \frac{1}{6}\left[Q_1 - \beta(\Omega_j^2 - Q_0)\right]t^3 + --- \\ \eta_{1j}(t) &= t - \frac{1}{6}(\Omega_j^2 - Q_0)t^3 + \frac{1}{12}\left[Q_1 - \beta(\Omega_j^2 - Q_0)\right]t^4 + --- \end{aligned} \tag{5.43}$$

and B_{0j} and B_{1j}, $(j = 1,2)$, are constants of integration which are determined from the initial conditions. Finally, introduction of Eq. (5.42) into Eq. (5.27) yields the expression for the transverse elastic motion

$$y(x,t) = \sum_{j=1}^{2} \psi_j(x)\left[B_{0j}\, \eta_{0j}(t) + B_{1j}\, \eta_{1j}(t)\right] \tag{5.44}$$

If $y(x,o)$ and $\dot{y}(x,o)$ are the initial transverse displacement and velocity, respectively, the constants of integration are

$$\begin{aligned} B_{0j} &= \int_{-L/2}^{L/2} m_0\, \psi_j(x)\, y(x,0)\, dx \\ B_{1j} &= \int_{-L/2}^{L/2} m_0\, \psi_j(x)\, \dot{y}(x,0)\, dx \end{aligned} \tag{5.45}$$

It is apparent that, for no external forces, $y(x,t)$ is zero if the initial conditions are zero or if they are proportional to either $\psi_j(x)$ or $x\psi_j(x)$ for $j > 2$.

5. 3 Results

The equation for the axial elastic motion

$$u(x,t) = \sum_{j=1}^{n} \phi_j(x)\left[(1-\beta t)^{1/4}\left(u_{0j} - \frac{U_j}{\omega_j^2}\right)\cos\frac{2\omega_j}{\beta}\left(1-\sqrt{1-\beta t}\right) + \frac{U_j}{\omega_j^2}\right] \tag{5.46}$$

is not particularly amenable to numerical calculation because of the extremely large argument of the cosine term. However, for small time t, the radical in the above expression may be expanded by means of the binomial expansion formula to give

$$\cos \frac{2\omega_j}{\beta}\left[1-\left(1-\frac{\beta t}{2}+\frac{(\beta t)^2}{8}-\ldots\right)\right] = \cos \frac{2\omega_j}{\beta}\left(\frac{\beta t}{2}-\frac{(\beta t)^2}{8}+\ldots\right) \tag{5.47}$$

For $t << 1$, the $(\beta t)^2$ term as well as all higher order terms in t may be neglected. In addition

$$(1 - \beta t)^{1/4} \approx 1 \tag{5.48}$$

so that a simplified expression for u(x,t) appears as

$$u(x,t) \approx \sum_{j=1}^{n} \phi_j(x)\left[\frac{U_j}{\omega_j^2} + \left(u_{0j} - \frac{U_j}{\omega_j^2}\right)\cos \omega_j t\right] \tag{5.49}$$

Although this form has definite utility for the calculation of the axial displacement for very small t, it neglects the change in the frequency and amplitude of vibration as a function of time due to the change in the mass. However, since the axial displacement u(x,t) is necessary for the calculation of y(x,t), its evaluation is required for the entire period of powered flight.

For this reason, Eq. (5.47) has been programed for solution by means of a digital computer. Input to the program requires values of Young's modulus, length and area of the missile case, area of the fluid, nozzle throat area, and internal fluid pressure, together with the mass per unit length and the time rate of change of the mass per unit length of the missile. The number of terms in the series is also read in as input in order to be able to evaluate the rate of convergence of the series. The output consists of values of the axial elastic displacement at discrete intervals of x for discrete values of time from time = 0 to burnout. The time increment is determined from the relation

$$\tau = \frac{h}{c} \tag{5.50}$$

where h is the spatial increment and

$$c = \sqrt{EA_C/m_C} \tag{5.51}$$

so that a sufficiently large number of deflected shapes are obtained over a cycle to ensure representative coverage. Results for 1, 2, 10, and 20 term series have been obtained for a typical solid-fuel missile with the following physical properties:

$E = 30 \cdot 10^6$ lb per sq in.

$A_C = 7.53$ sq in.

$A_F = 36.4$ sq in.

$A_T = 18.2$ sq in.

$m_0 = 0.011$ lb-sec^2 per sq in.

$\dot{m}_C = 0.004$ lb-sec per sq in.

$L = 100$ in.

$p_{L/2} = 2{,}000$ lb per sq in.

For nine increments of length, a time increment of approximately 0.0000775 second results for the above set of parameters. As an indication of the rate of convergence of the series, a comparison of the deflected shape for different numbers of terms at a typical time appears in Fig. 5.2. Fig. 5.3 shows the missile at several selected times throughout the first cycle, while deformations at equivalent times throughout the 40th cycle are depicted in Fig. 5.4. The change in the period of vibration as well as in the amplitudes at various times during the cycles is evidenced by a comparison of the latter two figures.

It should be stressed that, while the eigenfunctions of a constant mass system are used, this is merely a mathematical convenience and does not imply normal mode vibration. On the contrary, the system does not possess any natural frequencies in the ordinary sense, and the amplitudes are not constant at a given period in different cycles.

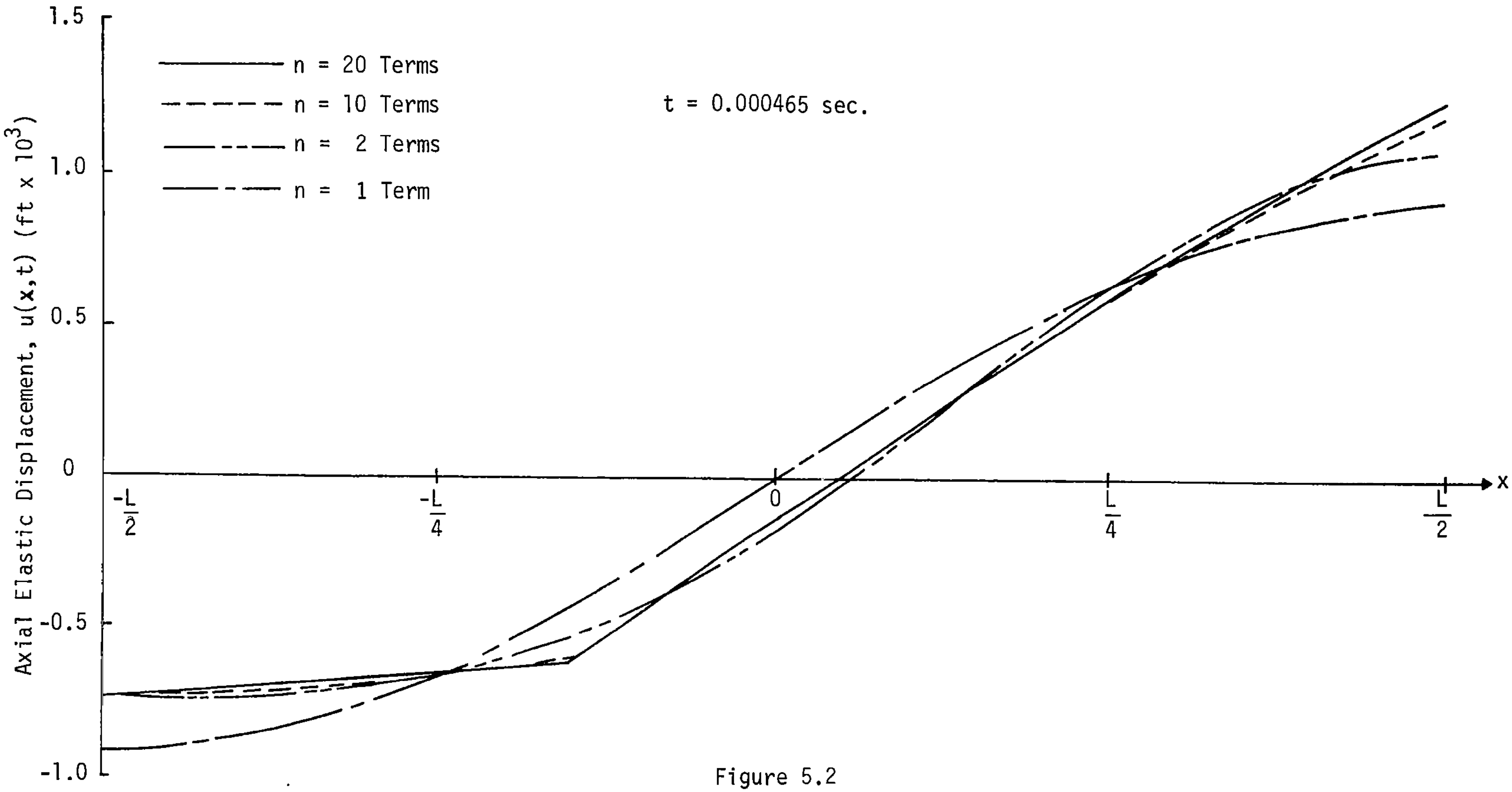

Figure 5.2

Comparison of Deflected Shapes

For Different Numbers of Terms

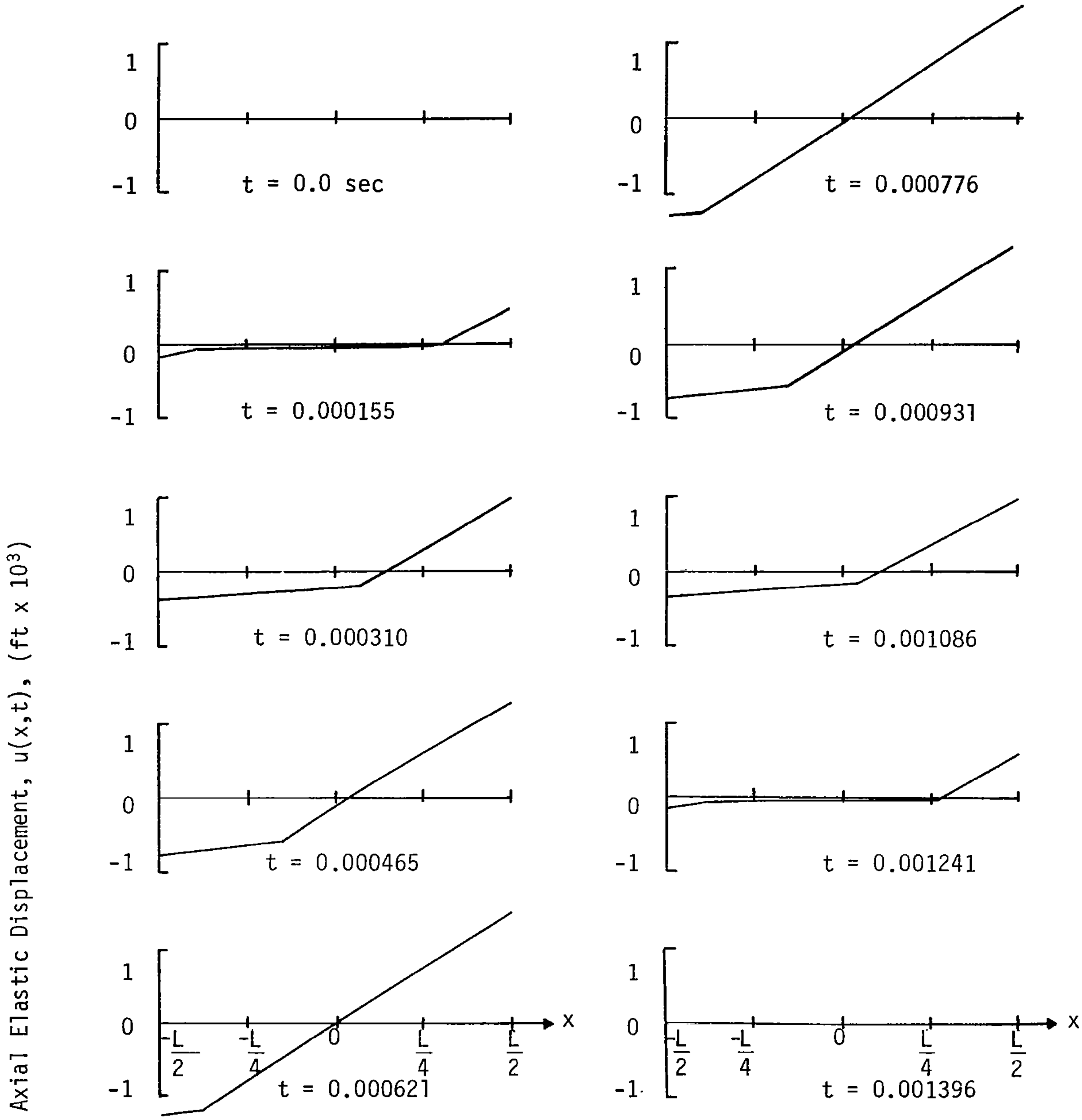

Figure 5.3

Axial Deflection Throughout

The First Cycle

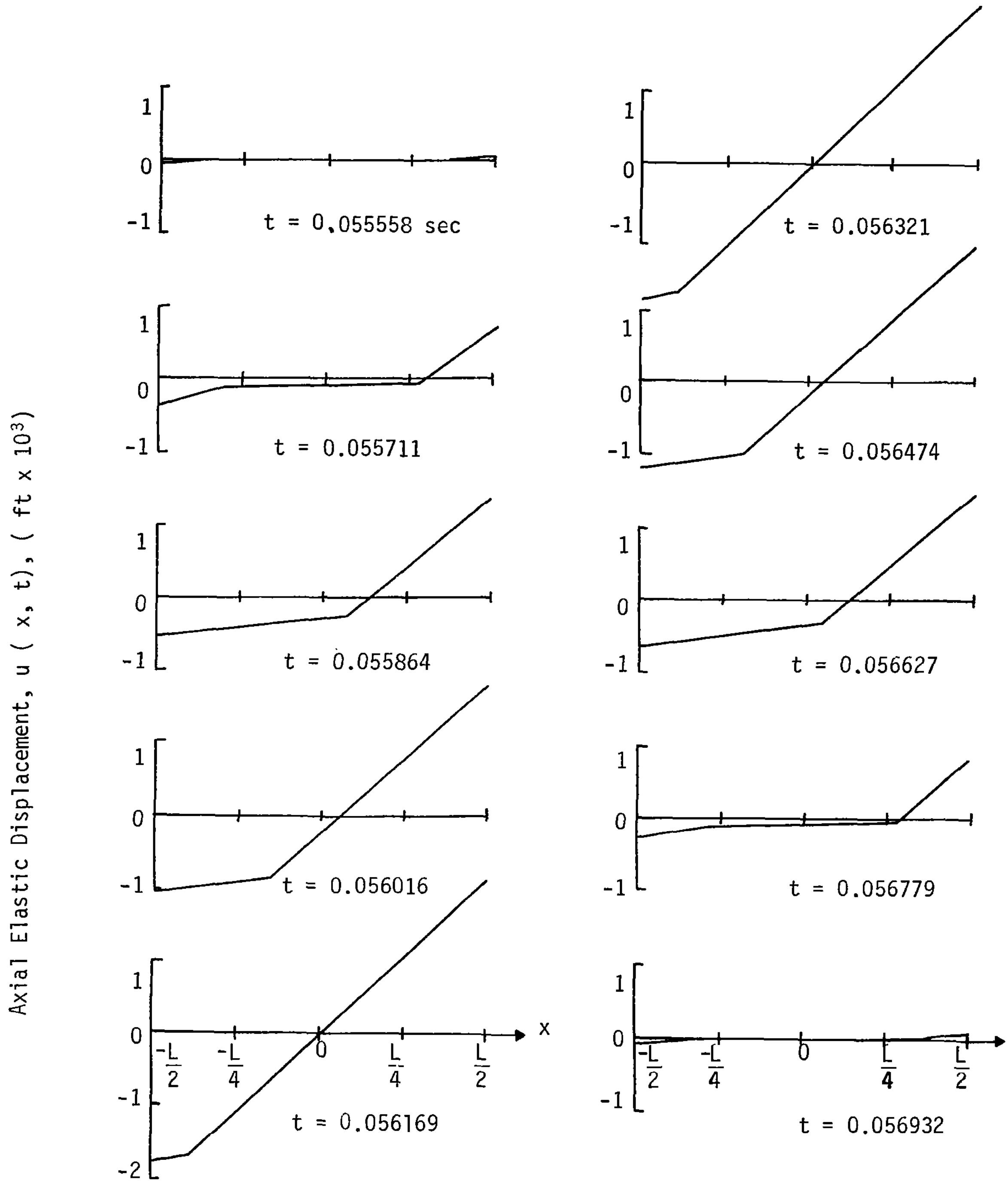

Figure 5.4

Axial Deflection Throughout

The Fortieth Cycle

VI. NUMERICAL SOLUTION FOR PLANAR MOTION

6.1 Equations of Motion in Difference Form

To obtain a thorough understanding of the dynamic characteristics of a slender body of time-dependent mass under high accelerations, it is desirable to secure an analytical solution, even under certain limiting conditions. As indicated in the previous section, these conditions require assumptions concerning the relative magnitudes of the generalized coordinates.

To justify these assumptions, it is necessary to investigate the complete set of five coupled nonlinear equations of motion by means of a numerical method together with a high-speed digital computer. Once again, the missile is envisioned as a uniform, circular cylinder of time-dependent mass, closed at one end, but open through a nozzle at the other. The mass variation along the centroidal axis is assumed to be constant and of constant magnitude from ignition until burnout. As in the previous section, the internal fluid is considered to be nonviscous and of constant pressure within the combustion chamber such that a linear velocity distribution results from the closed end to the entrance of the nozzle. External pressure, as well as nonconservative external forces, is assumed negligible, and gravity is once more assumed constant such that with the exception of the limitation concerning the relative magnitudes of the coordinates, all conditions are identical with those of Section V.

The equations of motion are written in finite difference form, and boundary and starting values for all variables are calculated from the corresponding boundary and initial conditions of the differential equations of motion. Central difference expressions are utilized throughout the analysis with the exception of certain terms involving $\dot{\theta}^2$, where a backward (left-slanted) difference is used to avoid a quadratic recurrence relation for θ. Numerical integration is used to evaluate the integrals occurring in Eqs. (3.34) through (3.36).

An iterative sequence is necessary to cope with the nonlinearity of the equations. Since the elastic displacement in the axial direction is needed for the calculation of $y(x,t)$, and since for most physical systems, the value of $u(x,t)$ will be the most rapidly varying parameter, it is first calculated at the new time, $t + \Delta t$, by assuming that the values of the other variables are unchanged from time t. Values of the elastic displacement in the transverse direction are then computed by using the values of $u(x, t + \Delta t)$, but once again with the use of the values of the rigid-body terms at time t. In a similar manner, values of X, Y, and θ are calculated for time $t + \Delta t$. Refined values of the axial elastic displacement are then obtained with use of the new values of the remaining variables just computed, and the process is repeated until the desired accuracy is obtained. Since the magnitude of the time increment is quite small (approximately 10^{-4} seconds) for most physical systems, the rigid-body terms do not vary significantly over this increment, and a relatively low number of iterations are required.

Also, in addition to the assumptions already stated, the following initial conditions are assumed:

$$\begin{aligned} X(0) = Y(0) = \dot{X}(0) = \dot{Y}(0) = \dot{\theta}(0) = 0 \\ u(x,0) = \dot{u}(x,0) = y(x,0) = \dot{y}(x,0) = 0 \end{aligned} \tag{6.1}$$

and $\theta(0)$ = any value between 0 and 2π.

Equation (3.34) is written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{1}{c^2}\Big[\ddot{X} - \ddot{\theta}(Y+y) - 2\dot{\theta}(\dot{Y}+\dot{y})$$

$$- \dot{\theta}^2(X + x + u)\Big] + \frac{1}{c^2} g \sin\theta + \frac{1}{EA_C}(\bar{f}_{RF} \cdot \bar{i} + \dot{m}_C v_C - \hat{F}_{uC}) \qquad (6.2)$$

where

$$c^2 = \frac{EA_C}{m_C}$$

If any viscous or drag forces are once again neglected, Eq. (6.2) for a typical element other than the end elements is written in finite difference form by utilizing such simple central difference expressions as

$$\frac{1}{h^2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j}) = \frac{1}{c^2\tau^2}(u_{i,j+1} - 2u_{ij} + u_{i,j-1})$$

$$+ \frac{1}{c^2\tau^2}\Big[X_{j+1} - 2X_j + X_{j-1} - (\theta_{j+1} - 2\theta_j + \theta_{j-1})(Y_j + y_{ij})$$

$$- \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(Y_{j+1} - Y_{j-1} + y_{i,j+1} - y_{i,j-1}) - \frac{1}{4}(\theta_{j+1} - \theta_{j-1})^2 \cdot$$

$$(X_j + ih + u_{ij})\Big] + \frac{1}{c^2} g \sin\theta_j \qquad (6.3)$$

where h is the spatial increment and $\tau = \Delta t$ is the time increment. A recurrence relation for u is immediately apparent for time $t + \tau$ from the above expression.

$$u_{i,j+1} = -u_{i,j-1} + 2(1 - \frac{c^2\tau^2}{h^2})\, u_{ij} + \frac{c^2\tau^2}{h^2}(u_{i+1,j} + u_{i-1,j})$$

$$- \Big[X_{j+1} - 2X_j + X_{j-1} - (\theta_{j+1} - 2\theta_j + \theta_{j-1})(Y_j + y_{ij})$$

$$- \frac{1}{2} (\theta_{j+1} - \theta_{j-1})(Y_{j+1} - Y_{j-1} + y_{i,j+1} - y_{i,j-1}) - \frac{1}{4} (\theta_{j+1} - \theta_{j-1})^2 \cdot$$

$$(X_j + ih + u_{ij}) \Big] - \tau^2 g \sin \theta_j \qquad (6.4)$$

From the initial condition equation (6.1), $\frac{\partial u(x,o)}{\partial t} = 0$, which in difference form is

$$\frac{1}{2\tau} (u_{i,1} - u_{i,-1}) = 0 \qquad (6.5)$$

so that the starting values for u are simply

$$u_{i,1} = u_{i,-1} \qquad (6.6)$$

Since the internal pressure exerts an axial force only on the two ends, it may be treated as a nonhomogeneous boundary condition rather than a spatial Dirac delta function,

$$\frac{\partial u(L/2,t)}{\partial x} = \frac{p_{L/2}A_F}{EA_C}$$

or, in difference form

$$u_{n+1,j} - u_{n-1,j} = 2 \frac{hp_{L/2}A_F}{EA_C} \qquad (6.7)$$

At the nozzle end, a linear drop in pressure is assumed so that

$$\frac{\partial u(-L/2,t)}{\partial x} \cong - 0.236 \frac{p_{L/2}A_F(1-A_T/A_F)}{EA_C}$$

or

$$u_{1,j} - u_{-1,j} = -0.472 \frac{hp_{L/2}A_F(1-A_T/A_F)}{EA_C} \qquad (6.8)$$

In the recurrence relation for $u_{i,j+1}$, Eq. (6.4), many values such as X_{j+1}, Y_{j+1}, θ_{j+1}, and $y_{i,j+1}$ are required for time $t + \tau$ which are not immediately available at this stage of the calculation. Consequently, it is necessary to adopt an iterative procedure by using for such values as $\ddot{X}$, $\dot{\theta}$, etc., which are not available at $t + \tau$, the value existing for time t.

The equation for the elastic transverse motion, Eq. (3.38), appears in difference form as

$$\frac{1}{h^4}(y_{i+2,j} - 4y_{i+1,j} + 6y_{ij} - 4y_{i-1,j} + y_{i-2,j}) + \frac{1}{\tau^2\beta^4}(y_{i,j+1}$$

$$- 2y_{ij} + y_{i,j-1}) = \frac{A_C}{2h^3 I}\Big[(u_{i+1,j} - 2u_{ij} + u_{i-1,j})(y_{i+1,j} - y_{i-1,j})$$

$$+ (u_{i+1,j} - u_{i-1,j})(y_{i+1,j} - 2y_{ij} + y_{i-1,j})\Big] - \frac{1}{\tau^2\beta^4}\Big[Y_{j+1} - 2Y_j + Y_{j-1}$$

$$+ (\theta_{j+1} - 2\theta_j + \theta_{j-1})(X_j + ih + u_{ij}) + \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(X_{j+1} - X_{j-1}$$

$$+ u_{i,j+1} - u_{i,j-1}) - \frac{1}{4}(\theta_{j+1} - \theta_{j-1})^2 (Y_j + y_{ij})\Big] - \frac{1}{\beta^4} g \cos\theta_j$$

$$- \frac{\rho A_F}{gEI\tau}(\theta_{j+1} - \theta_{j-1}) v_{ij} \tag{6.9}$$

where $\beta^4 = EI/m_C$. The recurrence relation for $y(x,t+\tau)$ is then

$$y_{i,j+1} = \frac{\tau^2\beta^4}{h^4}(-y_{i+2,j} + 4y_{i+1,j} + 4y_{i-1,j} - y_{i-2,j}) + 2(1 - \frac{3\tau^2\beta^4}{h^4}) y_{ij}$$

$$- y_{i,j-1} + \frac{\tau^2\beta^4 A_C}{2h^3 I}\Big[(u_{i+1,j} - 2u_{ij} - u_{i-1,j})(y_{i+1,j} - y_{i-1,j})$$

$$+ (u_{i+1,j} - u_{i-1,j})(y_{i+1,j} - 2y_{ij} + y_{i-1,j})\Big] + \Big[Y_{j+1} - 2Y_j + Y_{j-1}$$

$$+ (\theta_{j+1} - 2\theta_j + \theta_{j+1})(X_j + ih + u_{ij}) + \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(X_{j+1}$$

$$- X_{j-1} + u_{i,j+1} - u_{i,j-1}) - \frac{1}{4}(\theta_{j+1} - \theta_{j-1})^2 (Y_j + y_{ij})\Big]$$

$$+ \tau^2 g \cos\theta_j + \frac{\tau\rho A_F}{g m_C}(\theta_{j+1} - \theta_{j-1}) v_{ij} \tag{6.10}$$

Once again, from the initial condition for the transverse elastic motion,

$$\frac{\partial y(x,0)}{\partial t} = \frac{1}{2\tau}(y_{i,1} - y_{i,-1}) = 0$$

the simple starting difference relation results

$$y_{i,1} = y_{i,-1} \tag{6.11}$$

In addition to the initial condition described above, Eq. (6.10) is subject to the boundary conditions, Eqs. (3.40) and (3.41). From Eq. (3.40) at $x = -L/2$

$$EI \frac{\partial^2 y(-L/2,t)}{\partial x^2} = \frac{EI}{h^2} (y_{1,j} - 2y_{0j} + y_{-1,j}) = 0$$

or

$$y_{-1,j} = 2y_{0j} - y_{1,j} \tag{6.12}$$

Also, since $P = EA_C \frac{\partial u}{\partial x}$, Equation (3.41) for $x = -L/2$ yields

$$-\frac{EA_C}{4h^2} (u_{1,j} - u_{-1,j})(y_{1,j} - y_{-1,j}) + \frac{EI}{2h^3} (-y_{2,j} + 2y_{1,j} - 2y_{-1,j} + y_{-2,j})$$
$$= 0$$

or

$$y_{-2,j} = y_{2,j} - 2y_{1,j} - 2y_{-1,j} + \frac{hA_C}{2I} (u_{1,j} - u_{-1,j})(y_{1,j} - y_{-1,j}) \tag{6.13}$$

Similarly, at the closed end, $x = L/2$, Eq. (3.40) results in

$$EI \frac{\partial^2 y(L/2,t)}{\partial x^2} = \frac{EI}{h^2} (y_{n+1,j} - 2y_{nj} + y_{n-1,j}) = 0$$

or

$$y_{n+1,j} = 2y_{nj} - y_{n-1,j} \tag{6.14}$$

whereas Eq. (3.41) yields

$$-\frac{EA_C}{4h^2} (u_{n+1,j} - u_{n-1,j})(y_{n+1,j} - y_{n-1,j}) + \frac{EI}{2h^3} (-y_{n+2,j} + 2y_{n+1,j} -$$
$$2y_{n-1,j} + y_{n-2,j}) = 0$$

or

$$y_{n+2,j} = y_{n-2,j} - 2y_{n+1,j} + 2y_{n+1,j} - \frac{hA_C}{2I} (u_{n+1,j} - u_{n-1,j})(y_{n+1,j} - y_{n-1,j}) \tag{6.15}$$

Once again, whereas the values of the elastic displacement in the axial direction are know, the rigid-body quantities X_{j+1}, Y_{j+1}, and θ_{j+1} appearing in Eq. (6.10) are still unknown. Consequently, values of $\ddot{X}$, $\dot{\theta}$, etc. must be assumed to remain unchanged from time t to time t + τ in the initial calculation of the lateral elastic displacement.

Before proceeding to the difference equations for the rigid-body terms, it is necessary to devote some attention to the size of the time and space increments, τ and h, in order to insure numerical stability of the difference equations corresponding to the partial differential equations (3.37) and (3.38). The value of h is determined from

$$h = L/n \tag{6.16}$$

where n is the number of stations required to yield adequate information as to the deflected shape in both the axial and the lateral direction. For stability of the homogeneous wave equation, the time increment is then

$$\tau^2 \leq h^2/c^2 \tag{6.17}$$

Stability of the homogeneous beam equation in difference form, on the other hand, requires that

$$\tau^2 \leq h^4/4\beta^4 \tag{6.18}$$

Obviously, with the exception of certain isolated combinations of physical parameters, the values of τ determined from Eqs. (6.17) and (6.18) are not equal, although for most systems of interest, they are of the same order of magnitude. Therefore, in order to insure mathematical stability, it is necessary to compute two values of τ and utilize whichever is smaller. Also, since the mass per unit length of the missile varies as a function of time, the time mesh size varies slightly from one time to the next.

To determine the difference relations for the rigid-body equation of motion, it is necessary to adopt a means of numerical integration, since integrals as well as differentials are involved. Simple trapezoidal integration, rather than one of Simpson's methods or even more complex expressions, is used so as to preclude the necessity of deciding at the outset to use either an odd or an even number of space increments.

Equation (3.34) appears in difference form as

$$\frac{m_C nh}{\tau^2}\Big[(X_{j+1} - 2X_j + X_{j-1}) + \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(Y_{j+1} - Y_{j-1})$$

$$-Y_j(\theta_{j+1} - 2\theta_j + \theta_{j-1}) - \frac{X_j}{4}(\theta_{j+1} - \theta_{j-1})^2\Big] + \sum_{i=0}^{n-1}\frac{m_C h}{2\tau^2}(u_{i,j+1}$$

$$- 2u_{ij} + u_{i,j-1} + u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}) - \frac{1}{\tau^2}(\theta_{j+1} - 2\theta_j + \theta_{j-1}) \cdot$$

$$\sum_{i=0}^{n-1}\frac{m_C h}{2}(y_{ij} + y_{i+1,j}) - \frac{1}{4\tau^2}(\theta_{j+1} - \theta_{j-1})^2\sum_{i=0}^{n-1}\frac{m_C h}{4}\Big[ih + u_{ij}$$

$$+ (i+1)h + u_{i+1,j}\Big] - \frac{1}{\tau^2}(\theta_{j+1} - \theta_{j-1})\sum_{i=0}^{n-1}\frac{m_C h}{4}(y_{i,j+1} - y_{i,j-1}$$

$$+ y_{i+1,j+1} - y_{i+1,j-1}) + g\, m_C nh \sin\theta_j - p_L A_F\Big[1 - 0.236(1 - A_T/A_F)\Big] \quad (6.19)$$

which yields the recurrence relation for the rigid-body translation in the X-direction.

$$X_{j+1} = 2X_j - X_{j-1} - \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(Y_{j+1} - Y_{j-1}) + Y_j(\theta_{j+1} - 2\theta_j + \theta_{j-1})$$

$$+ \frac{X_j}{4}(\theta_{j+1} - \theta_{j-1})^2 \frac{1}{2n}\sum_{i=0}^{n-1}(u_{i,j+1} - 2u_{ij} + u_{i,j-1} + u_{i+1,j+1} - 2u_{i+1,j}$$

$$+ u_{i+1,j-1}) + \frac{1}{2n}(\theta_{j+1} - 2\theta_j + \theta_{j-1})\sum_{i=0}^{n-1}(y_{ij} + y_{i+1,j})$$

$$+ \frac{1}{8n}(\theta_{j+1} - \theta_{j-1})^2\sum_{i=0}^{n-1}\Big[(2i+1)h + u_{ij} + u_{i+1,j}\Big]$$

$$+ \frac{1}{4n}(\theta_{j+1} - \theta_{j-1})\sum_{i=0}^{n-1}(y_{i,j+1} - y_{i,j-1} + y_{i+1,j+1} - y_{i+1,j-1})$$

$$- \tau^2 g \sin \theta_j + \frac{\tau^2 p_L A_F}{m_C n h}\left[1. - 0.236 (1 - A_T/A_F)\right] \tag{6.20}$$

From the initial condition

$$\dot{X}(0) = \frac{1}{2\tau}(X_{j+1} - X_{j-1})$$

results the starting difference relation

$$X_1 = X_{-1} \tag{6.21}$$

Similarly, Eq. (3.35) appears in difference form as

$$\frac{m_C nh}{\tau 2}\left[(Y_{j+1} - 2Y_j + Y_{j-1}) + \frac{1}{2}(\theta_{j+1} - \theta_{J-1})(X_{j+1} - X_{j-1})\right.$$

$$\left. + X_j(\theta_{j+1} - 2\theta_j + \theta_{j-1}) - \frac{Y_j}{4}(\theta_{j+1} - \theta_{j-1})^2\right] + \sum_{i=0}^{n-1} \frac{m_C h}{2\tau^2}(y_{i,j+1}$$

$$- 2y_{ij} + y_{i,j-1} + y_{i+1,j+1} - 2y_{i+1,j} + y_{i+1,j-1}) + \frac{1}{\tau^2}(\theta_{j+1} - 2\theta_j$$

$$+ \theta_{j-1}) \sum_{i=0}^{n-1} \frac{m_C h}{2}\left[ih + u_{ij} + (i+1)h + u_{i+1,j}\right] - \frac{1}{4\tau^2}(\theta_{j+1} - \theta_{j-1})^2 \cdot$$

$$\sum_{i=0}^{n-1} \frac{m_C h}{2}(y_{ij} + y_{i+1,j}) + \frac{1}{\tau^2}(\theta_{j+1} - \theta_{j-1}) \sum_{i=0}^{n-1} \frac{m_C h}{4}(u_{i,j+1} - u_{i,j-1}$$

$$+ u_{i+1,j+1} - u_{i+1,j-1}) + g\, m_C nh \cos \theta_j - (\theta_{j+1} - \theta_{j-1}) \frac{\rho A_F h}{2g\tau} \cdot$$

$$\sum_{i=0}^{n-1} (v_{ij} + v_{i+1,j}) = 0 \tag{6.22}$$

which yields the recurrence relation for the Y direction

$$Y_{j+1} = 2Y_j - Y_{j-1} - \frac{1}{2}(\theta_{j+1} - \theta_{j-1})(X_{j+1} - X_{j-1}) - X_j(\theta_{j+1} - 2\theta_j + \theta_{j-1})$$

$$+ \frac{Y_j}{4}(\theta_{j+1} - \theta_{j-1})^2 - \frac{1}{2n}\sum_{i=0}^{n-1}(y_{i,j+1} - 2y_{ij} + y_{i,j-1} + y_{i+1,j+1} - 2y_{i+1,j}$$

$$+ y_{i+1,j-1}) - \frac{1}{2n}(\theta_{j+1} - 2\theta_j + \theta_{j-1})\sum_{i=0}^{n-1}\left[(2i+1)h + u_{ij} + u_{i+1,j}\right.$$

$$+ \frac{1}{8n} (\theta_{j+1} - \theta_{j-1})^2 \sum_{i=0}^{n-1} (y_{ij} + y_{i+1,j}) - \frac{1}{4n} (\theta_{j+1} - \theta_{j-1}) \cdot$$

$$\sum_{i=0}^{n-1} (u_{i,j+1} - u_{i,j-1} + u_{i+1,j+1} - u_{i+1,j-1}) - \tau^2 g \cos \theta_j + \frac{\tau}{2n} \cdot$$

$$(\theta_{j+1} - \theta_{j-1}) \frac{\rho A_F}{m_C g} \sum_{i=0}^{n-1} (v_{ij} + v_{i+1,j}) \qquad (6.23)$$

and once again, from the initial condition

$$\dot{Y}(0) = \frac{1}{2\tau}(Y_{j+1} - Y_{j-1}) = 0$$

the starting difference relation is obtained

$$Y_1 = Y_{-1} \qquad (6.24)$$

Finally, Eq. (3.36) is written in difference form as

$$\Big[Y_j(\theta_{j+1} - 2\theta_j + \theta_{j-1}) + X_j (\theta_j - \theta_{j-1})^2 - (X_{j+1} - 2X_j + X_{j-1})$$

$$+ \frac{1}{2} (\theta_{j+1} - \theta_{j-1})(Y_{j+1} - Y_{j-1})\Big] \sum_{i=0}^{n-1} \frac{m_C h}{2} (y_{ij} + y_{i+1,j}) + \Big[X_j (\theta_{j+1}$$

$$- 2\theta_j + \theta_{j-1}) - Y_j(\theta_j - \theta_{j-1})^2 + (Y_{j+1} - 2Y_j + Y_{j-1}) + \frac{1}{2} (\theta_{j+1} - \theta_{j-1}) \cdot$$

$$(X_{j+1} - X_{j-1})\Big] \sum_{i=0}^{n-1} \frac{m_C h}{2} \Big[(2i+1) h + u_{ij} + u_{i+1,j}\Big] + \sum_{i=0}^{n-1} \frac{m_C h}{2} \Big\{(y_{i,j+1}$$

$$- 2y_{ij} + y_{i,j-1})(ih + u_{ij}) + (y_{i+1,j+1} - 2y_{i+1,j} + y_{i+1,j-1}) \cdot$$

$$\Big[(i+1) h + u_{i+1,j}\Big]\Big\} - \sum_{i=0}^{n-1} \frac{m_C h}{2} \Big[(u_{i,j+1} - 2u_{ij} + u_{i,j-1}) y_{ij} + (u_{i+1,j+1}$$

$$- 2u_{i+1,j} + u_{i+1,j-1}) y_{i+1,j}\Big] + (\theta_{j+1} - 2\theta_j + \theta_{j-1}) \sum_{i=0}^{n-1} \frac{m_C h}{2} \Big\{(ih + u_{ij})^2 +$$

$$\left[(i+1)\ h + u_{i+1,j}\right]^2 + y_{ij}^2 + y_{i+1,j}^2\Big\} + (\theta_{j+1} - \theta_{j-1}) \sum_{i=0}^{n-1} \frac{m_c h}{4} \Big\{(ih +$$

$$u_{ij})(u_{i,j+1} - u_{i,j-1}) + \left[(i+1)\ h + u_{i+1,j}\right](u_{i+1,j+1} - u_{i+1,j-1}) +$$

$$y_{ij}\ (y_{i,j+1} - y_{i,j-1}) + y_{i+1,j}\ (y_{i+1,j+1} - y_{i+1,j-1})\Big\} + \sum_{i=0}^{n-1} \frac{m_c h}{2}\ \tau^2 g \Big\{\Big[ih +$$

$$u_{ij} + (i+1)\ h + u_{i+1,j}\Big] \cos\theta_j - (y_{ij} + y_{i+1,j}) \sin\theta_j\Big\} + \tau^2 p_L A_F \Big[1 -$$

$$0.236\ (1 - A_T/A_F)\Big] \sum_{i=0}^{n-1} \frac{h}{2} (y_{ij} + y_{i+1,j}) + \frac{\tau\rho A_F}{g}\ (\theta_{j+1} - \theta_{j-1}) \sum_{i=0}^{n-1} \frac{h}{2} (v_{ij} +$$

$$v_{i+1,j}) = 0 \qquad (6.25)$$

which results in the somewhat involved recurrence relation

$$\theta_{j+1} = \Big\{- \Big[Y_j\ (-2\theta_j + \theta_{j-1}) + X_j\ (\theta_j - \theta_{j-1})^2 - (X_{j+1} - 2X_j + X_{j-1}) -$$

$$\frac{1}{2}\ \theta_{j-1}\ (Y_{j+1} - Y_{j-1})\Big] \sum_{i=0}^{n-1} (y_{ij} + y_{i+1,j}) - \Big[X_j\ (-2\theta_j + \theta_{j-1}) - Y_j(\theta_j -$$

$$\theta_{j-1})^2 + (Y_{j+1} - 2Y_j + Y_{j-1}) - \frac{1}{2}\ \theta_{j-1}\ (X_{j+1} - X_{j-1})\Big] \sum_{i=0}^{n-1} \Big[(2i+1)\ h +$$

$$u_{ij} + u_{i+1,j}\Big] - \sum_{i=0}^{n-1} \Big[(y_{i,j+1} - 2y_{ij} + y_{i,j-1})(ih + u_{ij}) + (y_{i+1,j+1} +$$

$$2y_{i+1,j} + y_{i+1,j-1})(u_{i+1,j} - (i+1)\ h)\Big] + \sum_{i=0}^{n-1} \Big[(u_{i,j+1} - 2u_{ij} +$$

$$u_{i,j-1})\ y_{ij} + (u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1})\ y_{i+1,j}\Big] -$$

$$(-2\theta_j + \theta_{j-1}) \sum_{i=0}^{n-1} \Big\{(ih + u_{ij})^2 + \Big[u_{i+1,j} + (i+1)\ h\Big]^2 + y_{ij}^2$$

$$+ y_{i+1,j}^{2}\Big\} + \frac{1}{2}\theta_{j-1}\sum_{i=0}^{n-1}\Big[(ih + u_{ij})(u_{i,j+1} - u_{i,j-1}) + u_{i+1,j}$$

$$+ (i+1)\,h)(u_{i+1,j+1} - u_{i+1,j-1}) + y_{ij}(y_{i,j+1} - y_{i,j-1}) + y_{i+1,j}(y_{i+1,j+1}$$

$$- y_{i+1,j-1})\Big] - \tau^2 g \sum_{i=0}^{n-1}\Big[(u_{ij} + u_{i+1,j} + (2i+1)\,h)\cos\theta_j - (y_{ij}$$

$$+ y_{i+1,j})\sin\theta_j\Big] - \frac{\tau^2 \rho_L A_F}{m_C g}\Big[1 - 0.236\,(1 - A_T/A_F)\Big]\sum_{i=0}^{n-1}(y_{ij}$$

$$+ y_{i+1,j}) - \theta_{j-1}\frac{\tau\,\rho_F A_F}{m_C g}\sum_{i=0}^{n-1}(v_{ij} + v_{i+1,j})\Big\} \Big/ \Big\{\frac{1}{2}(Y_{j+1} + 2Y_j - Y_{j-1})\,\cdot$$

$$\sum_{i=0}^{n-1}(y_{ij} + y_{i+1,j}) + \frac{1}{2}(X_{j+1} + 2X_j - X_{j-1})\sum_{i=0}^{n-1}\Big[(2i+1)\,h + u_{ij}$$

$$+ u_{i+1,j}\Big] + \sum_{i=0}^{n-1}\Big[(ih + u_{ij})^2 + \Big[u_{i+1,j} + (i+1)\,h\Big]^2 + y_{ij}^{2} + y_{i+1,j}^{2}\Big]$$

$$+ \frac{1}{2}\sum_{i=0}^{n-1}\Big[(ih + u_{ij})(u_{i,j+1} - u_{i,j-1}) + \Big[u_{i+1,j} + (i+1)\,h\Big](u_{i+1,j}$$

$$- u_{i+1,j+1}) + y_{ij}(y_{i,j+1} - y_{i,j-1}) + y_{i+1,j}(y_{i+1,j+1} - y_{i+1,j-1})\Big]$$

$$+ \frac{\tau\,\rho\,A_F}{m_C g}\sum_{i=0}^{n-1}(v_{ij} + v_{i+1,j})\Big\} \qquad (6.26)$$

As with the other rigid-body terms, the initial condition

$$\dot{\theta}(0) = \frac{1}{2\tau}(\theta_{j+1} - \theta_{j-1}) = 0$$

results in the starting relation for θ

$$\theta_1 = \theta_{-1} \qquad (6.27)$$

In view of the length and complexity of the above equations of motion, a computer solution is obviously required, especially when the necessity of a number of iterations for each time increment is considered. Therefore, the recurrence relations, together with the necessary starting relations, have been programed for the Control Data 3400 digital computer. At a typical time, the value of the elastic axial displacement is first calculated by using values for the rigid-body coordinates, velocities, and accelerations. Next, the lateral elastic displacement is computed by using the new value of the axial elastic displacement just obtained, but once again using existing values for the rigid-body terms. Subsequently, new values for X, Y, and θ at $t + \tau$ are calculated, in that order, by using new values of all available variables, but existing values of any terms not yet computed. A new value of the elastic axial displacement is then calculated by utilizing the new values for the remaining variables, and the process is repeated until the desired accuracy is obtained. Output consisting of the time, t, together with the rigid-body terms X, Y, and θ, and the elastic displacement u(x) and y(x) is then printed. New values of the fluid flow area, case mass per unit length, and time increment are then computed and the calculation is repeated continuously until burnout.

6.2 Results

Although no stipulation as to the relative magnitudes of the values of the generalized coordinates was made for the numerical solution, the only rigid-body motion of consequence noted was the linear translation in the X-direction. Only a very slight drift in the value of Y (less than 10^{-7} times the value of X) was observed for vertical flight. Also, the value of θ remained constant to the number of decimal places printed in the output for all cases run. Thus, the assumptions concerning the rigid-body motion used in the perturbation solution appear justified. The linear rigid-body displacement for one case is plotted as a function of time in Fig. 6.1, where, for comparative purposes, the displacement of a constant-mass missile of constant thrust is also shown.

A comparison of the axial elastic displacements obtained by the finite-difference method and the series solution with 20 terms is shown in Fig. 6.2 for a typical time during the first cycle of oscillation. Excellent correlation is observed, and the slight variation could undoubtedly be further decreased by the use of additional terms in the series. Deviation of the two solutions increases somewhat with time, however, and a comparison of the displacements obtained at approximately the same time during the 40th cycle is shown in Fig. 6.3. The actual time at which the indicated displacement occurs is slightly (0.275 per cent) different for the two methods, which is attributed to error accumulation in the finite-difference solution over the 720 time increments. Additional axial deformations during the 40th cycle are presented in Fig. 6.4.

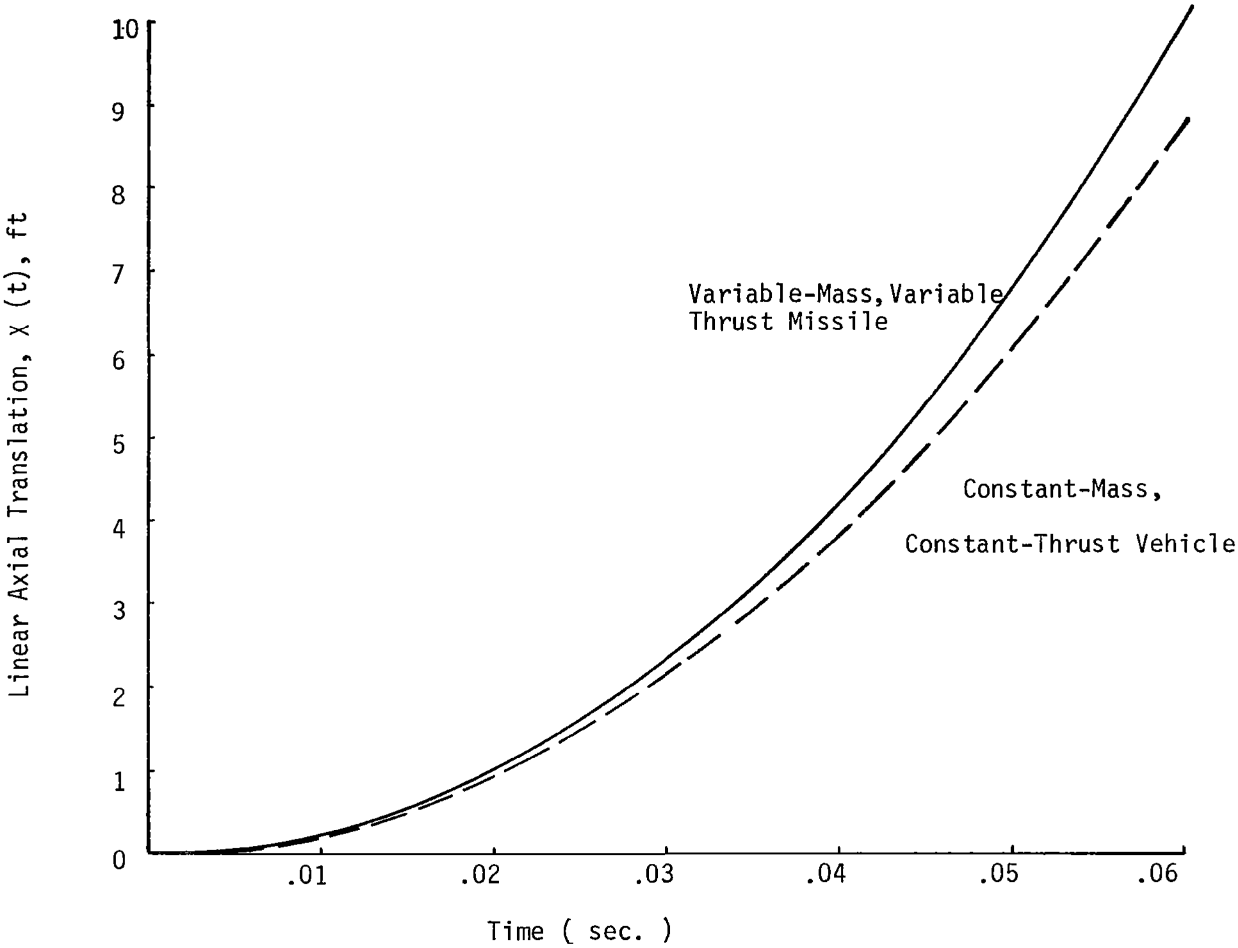

Figure 6.1

Linear Axial Translation

Further substantiation of the analytical solution was indicated by the lack of any lateral deformation. Test cases included those where the ratio of the initial periods of lateral and axial vibration was unity, as well as 2:1, 3:1, 4:1, and 0.5:1.

In order to investigate the effect of a variable axial strain on the lateral vibration characteristics of a solid-fuel missile, an example with an initial lateral deformation was obtained. These results appear in Fig. 6.5 and 6.6. Although, for the internal pressure used, the effect on the lateral vibration is not especially pronounced, the trends agree with Seide[10] with the exception that the axial force is variable. The stiffening due to the internal pressure also produced a decrease in the intial period of vibration of 5.8% compared to the period of a force-free beam of constant mass.

Of final interest, several test cases were run with nonconstant chamber pressure. For a pressure oscillation frequency close to that of the axial vibration of the case, a rapid increase in amplitude was experienced which soon exceeded the limitation of small deflection theory. Unfortunately, the results were complicated by the appearance of numerical instability such that accurate generalizations proved impossible.

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[10] P. Seide, p. 20

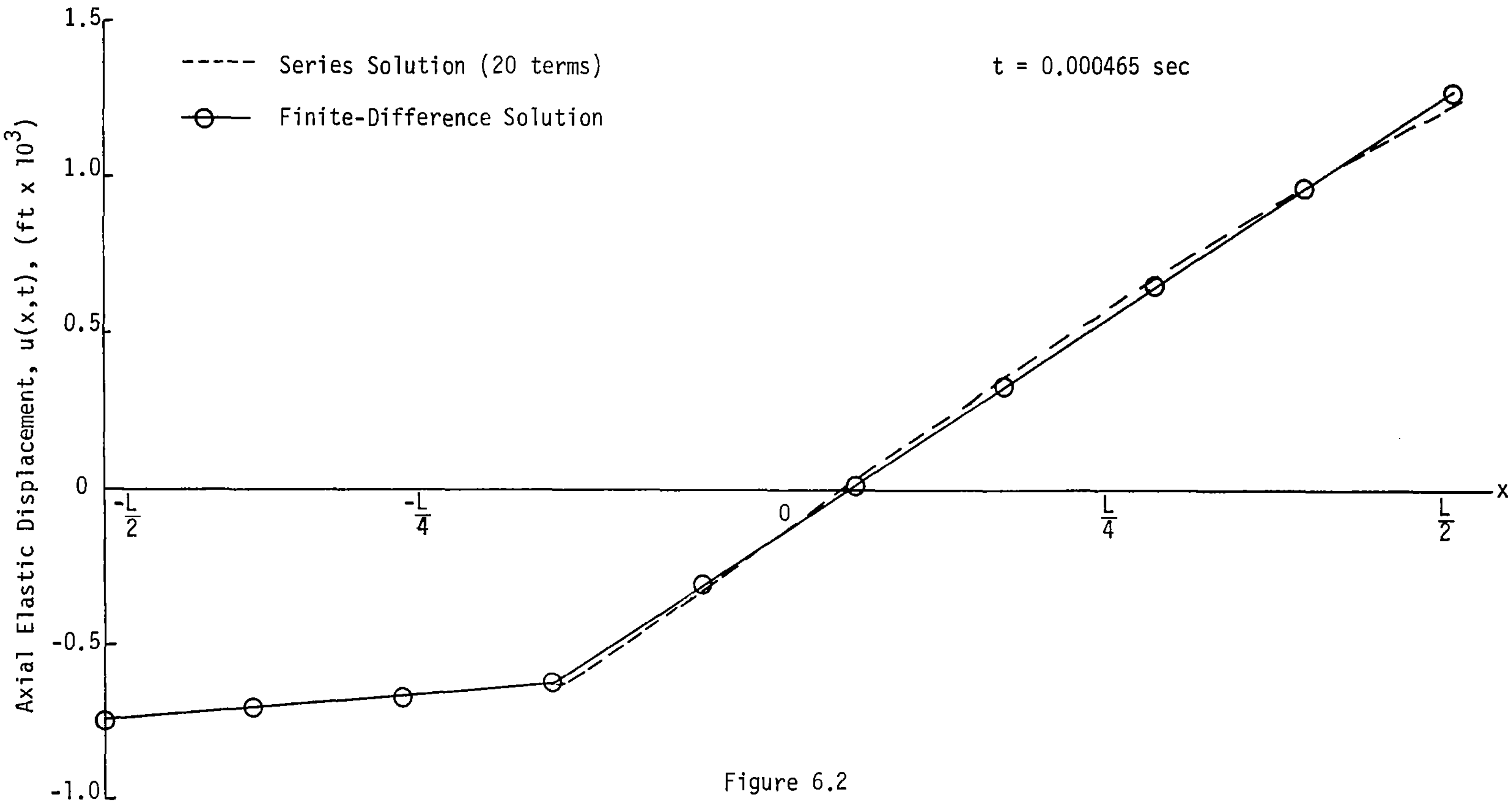

Figure 6.2

Comparison of Series and Finite-Difference

Solutions During the First Cycle

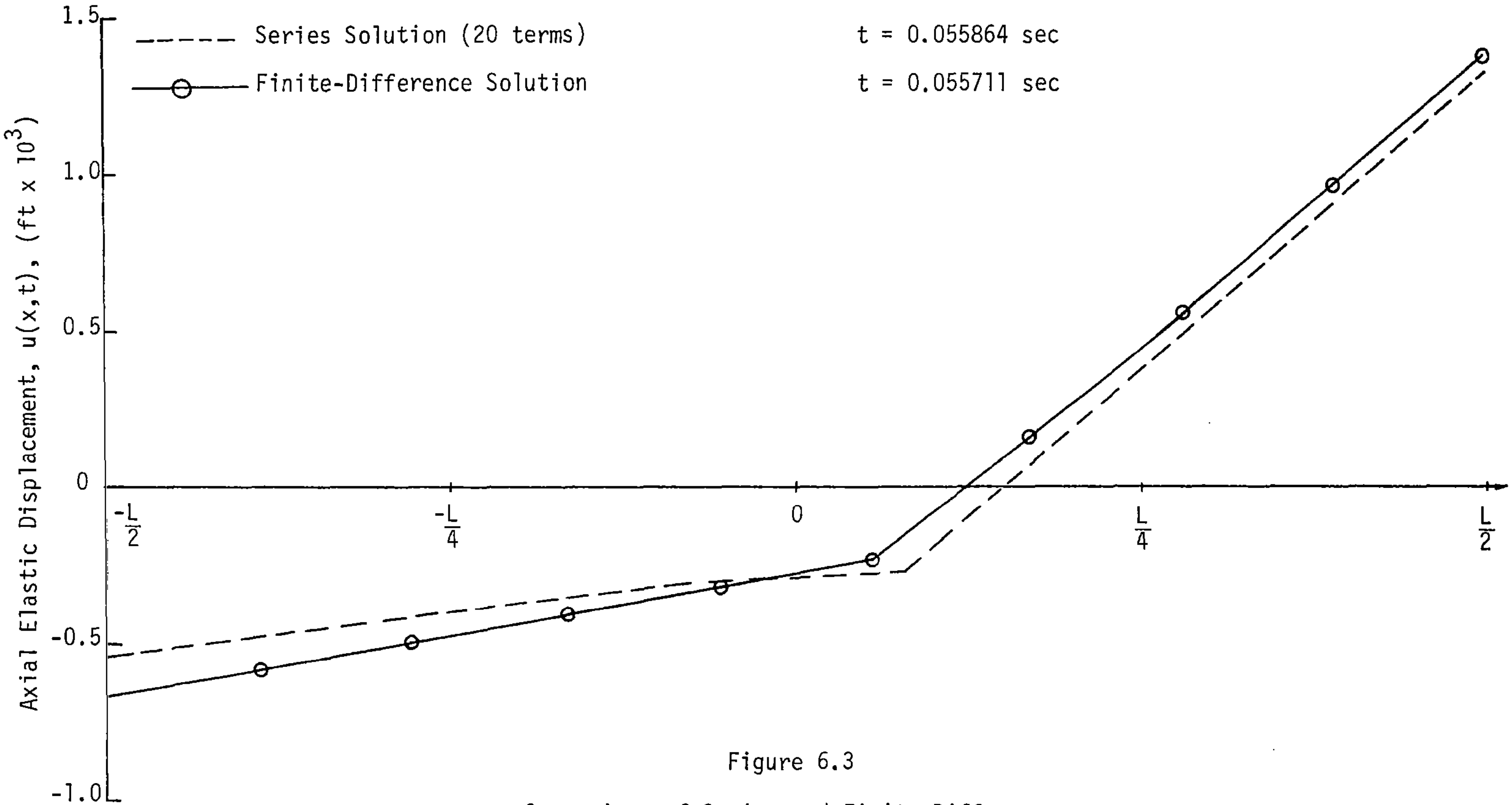

Figure 6.3

Comparison of Series and Finite-Difference

Solutions During the Fortieth Cycle

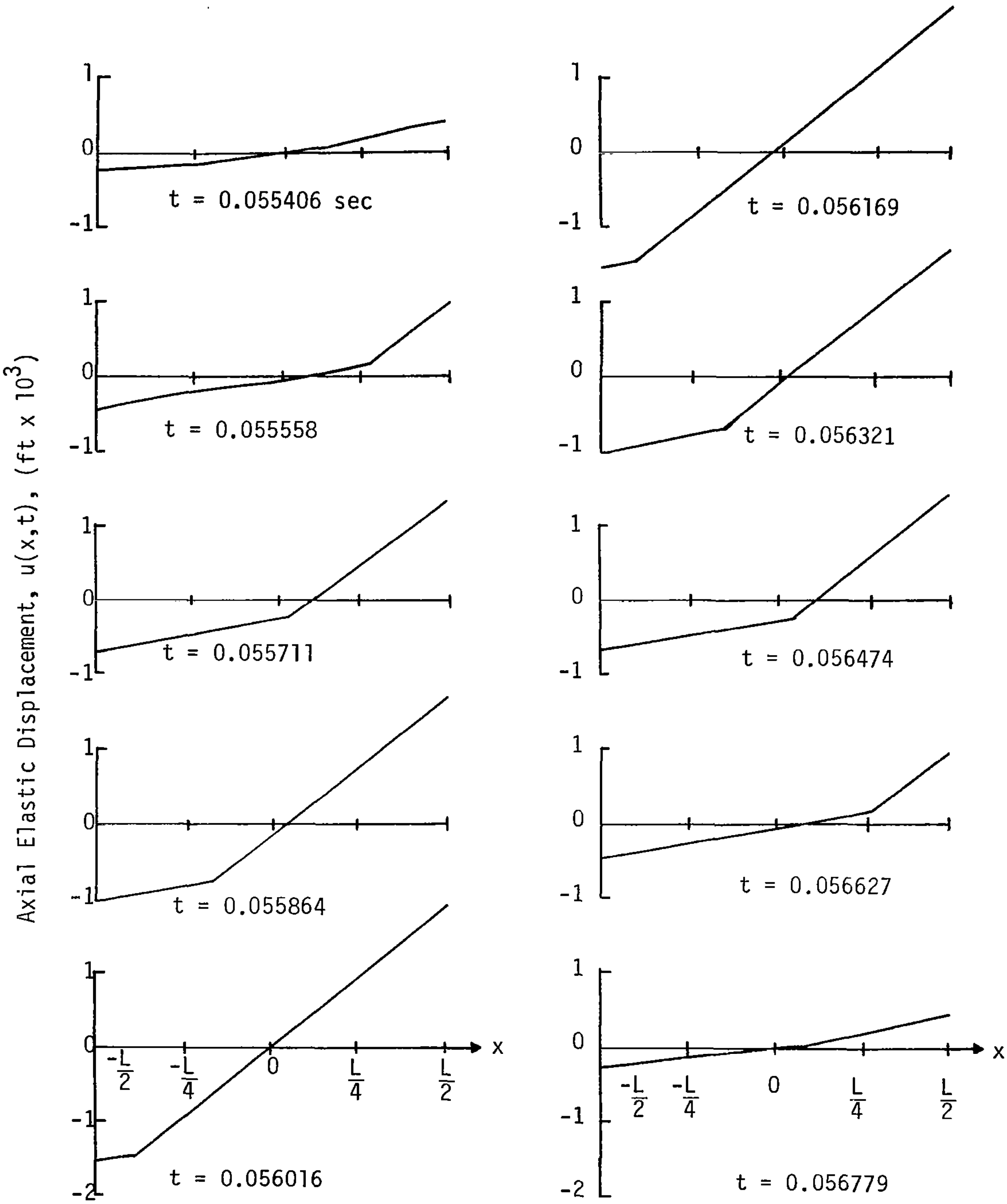

Figure 6.4

Axial Displacement Throughout

The Fortieth Cycle

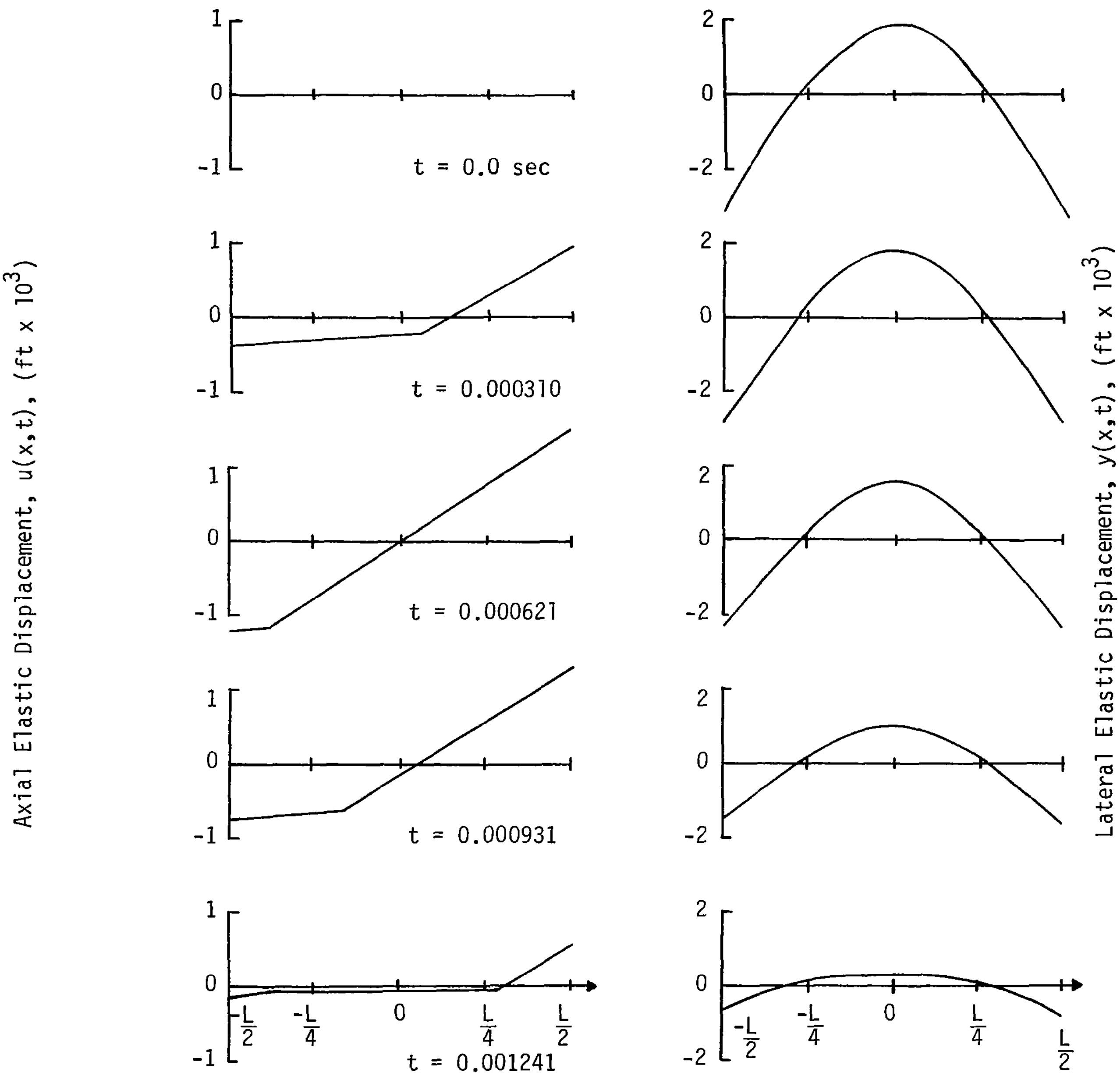

Figure 6.5

Effect of Axial Displacement

On Lateral Displacement

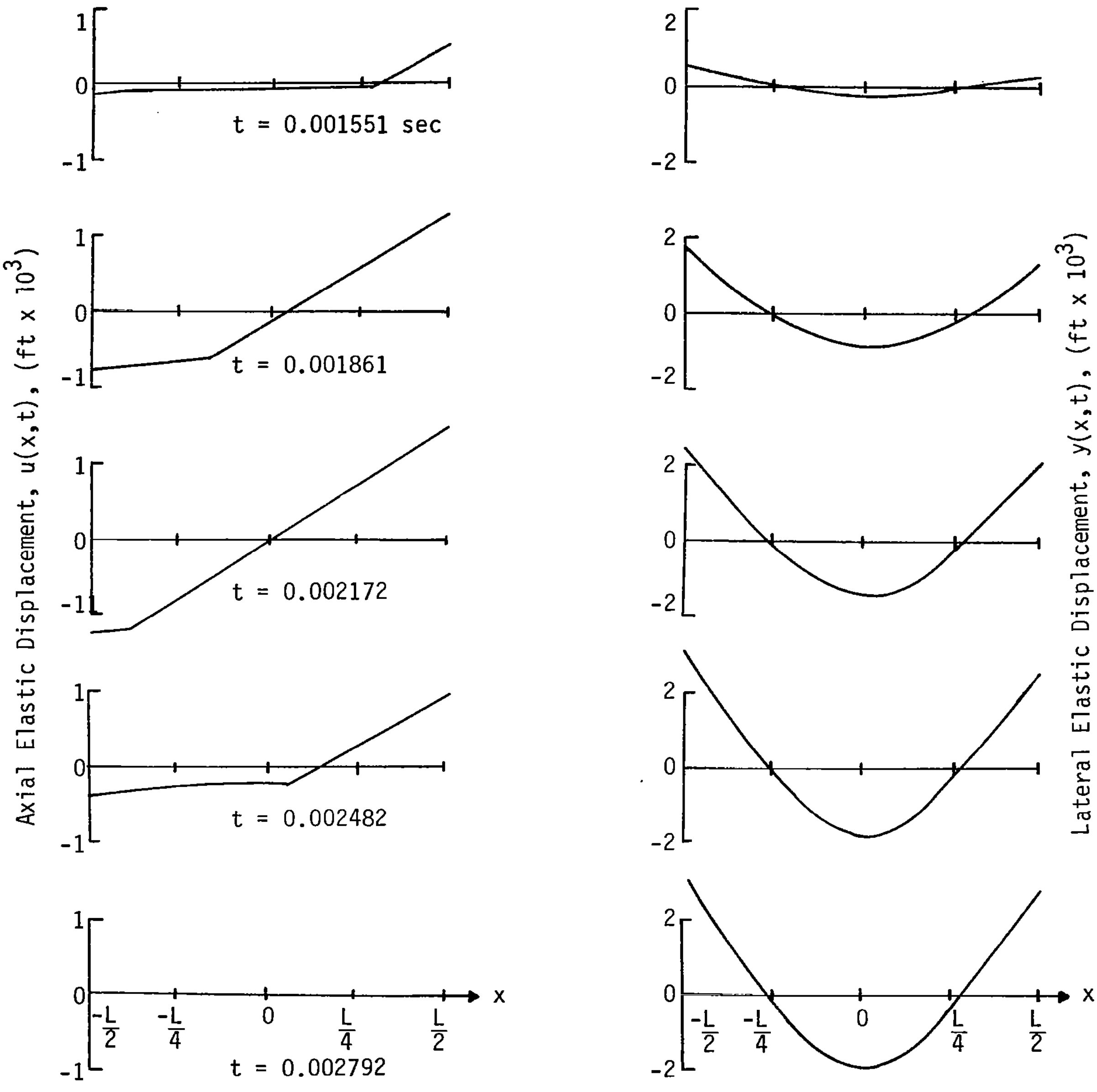

Figure 6.6

Effect of Axial Displacement

On Lateral Displacement

VII. SUMMARY AND CONCLUSIONS

The dynamic characteristics of a slender, elastic body of variable mass were investigated. The analysis is applicable to a solid-fuel missile which was envisioned as a slender, cylindrical body capable of both rigid-body motion as well as axial and transverse elastic deformation. During the powered flight of the vehicle, the mass was considered as a function of time with the products of combustion flowing relative to the missile structure and finally exhausting through a nozzle to the atmosphere.

The equations of motion of an element of the case and a fluid element were derived in vector form and then unified into a single equation of motion, which was then transformed by means of a variational principle into coupled equations of motion in terms of generalized coordinates.

To determine the effect of the fluid flow on the dynamic characteristics of the missile, the internal pressure and velocity distributions were investigated for a body under high accelerations, with consideration of the contribution of terms resulting from fluid flow, friction, and a non-inertial control volume. It was shown that the assumption of constant chamber pressure, as well as a linear velocity distribution through the combustion chamber, was justified.

A primary motion in the rectilinear direction was then postulated, with the remaining rigid-body motions and the elastic deformations treated as perturbations. Assuming a uniform mass, although depending on time, a solution for the axial vibration was obtained in terms of a series of the eigenfunctions of the corresponding constant-mass system multiplied by time-dependent generalized coordinates. A set of uncoupled ordinary differential equations for the generalized coordinates resulted, the solutions of which were determined by the initial conditions and the forcing functions resulting from fluid flow. It is stressed that, although the eigenfunctions of a constant-mass missile were used, it did not imply normal mode vibration. On the contrary, a variable-mass system does not possess any natural frequencies in the ordinary sense, and the amplitudes are not constant at times corresponding to the same fraction of the changing periods.

The same series approach to the lateral vibration resulted in coupled equations due to the presence of the axial force in the equation for $y(x,t)$. In order to uncouple this set of equations, it was necessary to assume that the deformation in the axial direction could be approximated by the first term of the series for the axial vibration and the first two terms of the series for the lateral motion. Of final consequence, it was shown that no lateral elastic motion resulted under the assumed initial conditions $y(x,0) = \dot{y}(x,0) = 0$.

To verify the solution for the axial deformations obtained by means of the perturbation technique, as well as to determine the effect of relaxing the requirement of linear rigid-body translation, a numerical solution to the coupled equations of motion was obtained. Excellent correlation between the two solutions was observed, especially during the early portion of the powered flight. Although no restrictions were imposed on $Y(t)$ and $\theta(t)$ only slight variation was noted in these parameters with time, and it was concluded that the jet damping availble from fluid flow was adequate to prevent unstable tumbling, even for an elastic body. Also in agreement with the perturbation solution, no lateral elastic vibration was excited unless an initial lateral deformation or velocity was stipulated, even for cases in which the periods of vibration for the lateral and axial motion were the same -- 2:1, 3:1, 4:1, and 0.5:1.

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