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HEAT-TRANSFER ANALYSIS FOR LIQUID-METAL FLOW IN RECTANGULAR CHANNELS WITH HEAT SOURCES IN THE FLUID

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SUMMARY

For an understanding of turbulent liquid-metal heat transfer in rectangular channels with heat generation in the fluid stream, an analysis was performed for forced-convection heat transfer to slug flow in rectangular channels with aspect ratios from 1 to ∞ and with uniform heat sources in the fluid.

The analysis was carried out for both uniform wall heat flux and uniform wall temperature. The results obtained apply in the thermal entrance region as well as in the fully developed region. For the wall-heat-flux situation, the four channel walls were considered to be uniformly heated - the heat flux on the short sides was considered an arbitrary fraction or multiple of the heat flux on the long sides. For the uniform temperature case, the wall temperature was assumed to be constant and equal on the four walls of the channel.

The analysis was based on the additional assumptions that (1) heat transport by eddy conduction is negligible compared with that by molecular conduction, (2) internal heat generation is spatially uniform, and (3) fluid properties are invariant with temperature. The temperature distributions were determined by utilizing the method of superposition.

The effects on temperature distributions of (1) the ratios that determine the role of internal heat generation relative to that of wall heat transfer or temperature driving force, (2) specified aspect ratio, and (3) specified wall heat fluxes or wall temperatures around the channel periphery were investigated. Numerical results for wall temperatures, heat-transfer rates, and bulk mean fluid temperatures are presented graphically as functions of significant parameters. The solutions identify the locations of maximum temperature or wall heat-transfer rates.

The results are useful for estimating local heat-transfer characteristics in turbulent heat-generating liquid-metal flow in rectangular channels when the Péclet number is approximately 100.

INTRODUCTION

Technological developments in space power generation have stimulated interest in the problem of forced-convection heat transfer to liquid-metal flow in passages with internal heat generation in the fluid stream. This system applies, for example, in the design of liquid-metal-fuel reactors, electromagnetic pumps and flowmeters, and liquid-metal magnetohydrodynamic (MHD) generators. The fluid in these devices will be heated by radioactive fission products or by an electric current flowing through the fluid. A schematic diagram of a simple MHD generator, which consists of a rectangular channel with sidewall electrodes, is shown in figure 1. The liquid metal flows in the channel with velocity U in the x-direction. A uniform magnetic field H is applied in the y-direction. Electric current flows from the right electrode through an external load to the left electrode, and then through the liquid metal back to the right electrode to complete the circuit. In many cases, the flow in such devices will be accompanied by heat, both that dissipated internally through viscous or Joule heating and that occurring through wall or electrode heating.

For the proper operation of such devices, a satisfactory temperature distribution along the passage walls must be maintained. The designer, therefore, must be able to compute the temperature distribution along the walls and to know how much heat must be removed to cool the walls and thus prevent temperatures from exceeding design limits. The problem involves studying liquid-metal channel flow with combined internal heat sources and wall heat transfer. Turbulent flows, which are very often encountered in practice, are studied in this analysis.

This investigation is concerned with hydrodynamically developed turbulent flow of a
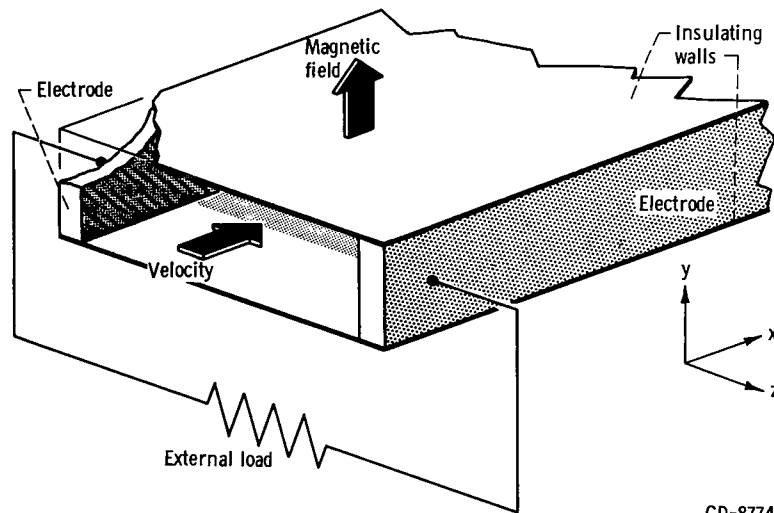

Figure 1. - Schematic diagram of magnetohydrodynamic channel.

liquid metal in rectangular channels with aspect ratios from 1 to ∞ and with uniform internal heat generation. An analysis including both uniform wall heat flux and uniform wall temperature is carried out for both the thermal entrance region and the fully developed region. For the case of uniform wall heat flux, the four channel walls are considered to be uniformly heated, but the heat flux on the short sides of the channel is an arbitrary fraction or multiple of the heat flux on the long sides. The walls of an MHD generator (fig. 1), for example, must insulate electrically, and if the wall temperature is less than that of the liquid metal, the wall must conduct heat from the fluid into an adjacent coolant. In addition, heat produced by electric currents in the electrodes will be transferred to the liquid metal. These effects may lead to unequal heat addition on adjacent sides. Unequal heat addition may also be the result of unwanted heat leakage or addition through insulation. For the uniform temperature case, the wall temperature is assumed constant and equal on all four walls of the channel.

This report is concerned with the determination of the axial and peripheral temperature distribution and heat-transfer characteristics in rectangular channels with aspect ratios from 1 to ∞. Attention is focused on the rectangular channel because of its increasing use in the applications mentioned. Within the knowledge of the author, the experimental and analytical studies of turbulent liquid-metal channel flows with internal heat generation have been confined to elementary geometries such as the circular tube (refs. 1 to 4) and the parallel-plate channel (refs. 4 to 6). In contrast to this moderate amount of information, turbulent-flow heat transfer to a heat-generating liquid metal in a rectangular channel has apparently received little analytical and no experimental work.

A few studies related to the problem considered in the present investigation are noted. In the absence of internal heat generation in the fluid, solutions have been developed which approximate situations that might occur with liquid metals in turbulent flow through rectangular channels. Fully developed slug-flow Nusselt numbers and wall-temperature distributions are presented in references 7 to 9. In these references, the problem was solved by assuming, in addition to a uniform velocity throughout the channel, that turbulent eddying does not contribute to conduction of heat within the fluid. The results pertain to systems characterized by low Reynolds and Prandtl numbers and to the portion of the rectangular channel beyond the thermal entrance region. In the discussion section of reference 10, Hoagland presents work done on the thermal entrance region for laminar slug flow in rectangular channels and gives some numerical results. Slug-flow wall-temperature distributions in the entrance region of channels having a variety of noncircular shapes, including a square channel, are presented in reference 11. The data were obtained by an analog technique. In reference 12, forced-convection heat transfer to laminar slug flow in a rectangular channel is examined for the boundary condition of a channel wall temperature both peripherally and axially uniform. This analysis was carried out without consideration of internal heat sources.

Magnetohydrodynamic heat transfer in a rectangular channel with specified heat-flux boundary conditions is considered in reference 13. The flow is assumed to be hydromagnetically and thermally fully developed, and heat conduction is by molecular conduction only. The effects of viscous dissipation and Joule heating are included in a general manner.

The present, very limited knowledge of turbulent liquid-metal flow and eddy diffusivity variation in noncircular passages makes theoretical progress very unlikely without appeal to simplified models. It does not seem likely, moveover, that any single model, for which a mathematical analysis is feasible, will prove adequate for all Reynolds and Prandtl numbers. Therefore, in order to gain some understanding of the complex problem of turbulent liquid-metal flow in rectangular channels with wall heat transfer and internal heat sources, consideration is given to a simplified but representative model. This specific model not only retains many of the physical characteristics of turbulent liquid-metal channel flow, but also leads to a tractable mathematical problem. This model should, therefore, provide information on the temperature distribution and heat-transfer characteristics for such flows in rectangular passages.

The idealized system assumed to approximate the forced-convection system under consideration is based on the following postulates:

(1) The established turbulent velocity profile is represented by a uniform distribution.

(2) The thermal eddy diffusivity is small compared to the thermal molecular diffusivity and is neglected.

(3) Longitudinal heat conduction is small compared to longitudinal convection and transverse conduction and is neglected.

(4) The internal heat generation is spatially uniform.

It is pointed out (e.g., refs. 7 and 14) that the blunt-nosed turbulent velocity distribution for a liquid-metal system can be represented satisfactorily by a uniform distribution. (References 15 and 16 remark that this assumption is even better satisfied for turbulent flow in MHD devices.) The second postulate implies (ref. 17) that the thermal solution pertains to systems characterized by Péclet numbers equal to, or less than, approximately 100. The third postulate (ref. 18) introduces a negligible error for Péclet moduli that are equal to, or greater than, approximately 100. Turbulent-flow heat transfer to liquid metals can be estimated, at least in the absence of internal heat generation, by the use of the slug-flow solutions for molecular conduction (refs. 7 to 9).

In this analysis, numerical results are provided for the case of internal heat sources which are uniform across the channel cross section and along the channel length. The results can be extended, however, to include sources which vary in the transverse and longitudinal directions (refs. 19 and 20).

This investigation is divided into two major sections dealing, respectively, with internal heat generation in the presence of prescribed wall-heat fluxes and with internal

heat generation in the presence of an axially and peripherally uniform wall temperature. Beginning the analysis with the case of internal heat generation and prescribed wall heat fluxes will facilitate the development for the case of internal heat generation with uniform equal channel surface temperatures.

## INTERNAL HEAT GENERATION WITH PRESCRIBED WALL HEAT FLUXES

The rectangular channel and its coordinate system for the present case are shown in figure 2. The turbulent velocity is postulated to be fully established at $x = 0$. The fluid temperature at the channel entrance is uniform across the section at a value t_e. Within the channel a heating process takes place that includes a uniform heat generation within the liquid metal and a uniform heat transfer at the channel walls. The fluid is assumed to have constant physical properties, and only steady-state heat transfer is investigated.

The established turbulent velocity profile is represented by

$$u = U = \text{constant} \tag{1}$$

(All symbols are defined in appendix A.) The differential equation describing convective heat transfer for the idealized system takes the form

$$U \frac{\partial t}{\partial x} = \frac{\kappa}{\rho c_p} \left(\frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) + \frac{Q}{\rho c_p} \tag{2}$$
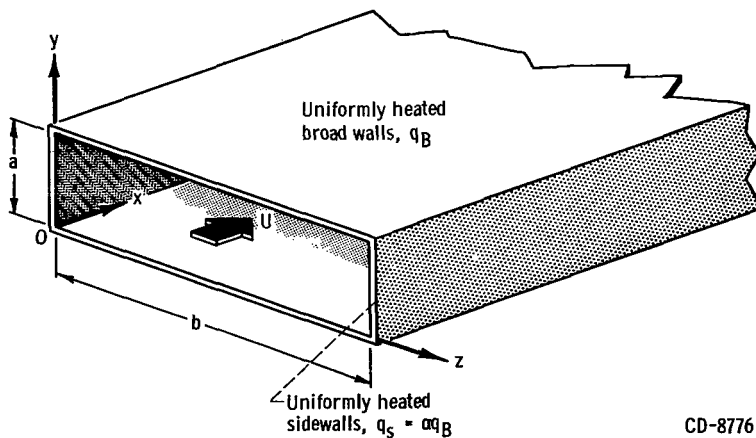
Figure 2. – Coordinate system for rectangular channel with different uniform heating on each pair of opposite walls.

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and the boundary conditions are

$$\frac{\partial t}{\partial y} = -\frac{q_B}{\kappa} \qquad \text{at } y = 0 \text{ (specified wall heat flux)} \qquad (3a)$$

$$\frac{\partial t}{\partial y} = \frac{q_B}{\kappa} \qquad \text{at } y = a \text{ (specified wall heat flux)} \qquad (3b)$$

$$\frac{\partial t}{\partial z} = -\frac{q_S}{\kappa} \qquad \text{at } z = 0 \text{ (specified wall heat flux)} \qquad (3c)$$

$$\frac{\partial t}{\partial z} = \frac{q_S}{\kappa} \qquad \text{at } z = b \text{ (specified wall heat flux)} \qquad (3d)$$

$$t(0, y, z) = t_e \qquad \text{(entrance condition)} \qquad (3e)$$

It can be shown (appendix B) that the following equation satisfies the differential equation (2) and the previous conditions, and is thus the solution to the problem where internal heat generation and wall heat transfer occur simultaneously:

$$\frac{t - t_e}{\dfrac{q_B a}{\kappa}} = \left[2\left(1 + \frac{\alpha}{\sigma}\right) + R\right]x' + y'^2 - y' + \frac{\alpha}{\sigma}z'^2 - \alpha z' + \frac{1}{6}(1 + \sigma\alpha)$$

$$- \sum_{k=2,4,\ldots}^{\infty} \frac{4}{k^2\pi^2}\cos(k\pi y')e^{-k^2\pi^2 x'} - \alpha\sigma \sum_{l=2,4,\ldots}^{\infty} \frac{4}{l^2\pi^2}\cos\left(l\pi\,\frac{z'}{\sigma}\right)e^{-l^2\pi^2 x'/\sigma^2} \qquad (4)$$

where

$$R = \frac{Qa}{q_B} \qquad (5)$$

There may arise practical applications where it is desirable to express the heat flux on the broad walls in terms of sidewall heat transfer as $q_B = \overline{\alpha}q_S$. The temperature distribution can also be expressed in terms of sidewall heat transfer by substituting $1/\overline{\alpha}$

for α, $\overline{\alpha}q_s$ for q_B, and replacing the parameter R by the parameter S, which is defined as

$$S = \frac{Qa}{q_s} \qquad (6)$$

The parameters R and S are the ratios of internal heat evolution to the heat transferred at the channel walls, and they give a measure of the importance, in connection with temperature development, of internal heat generation relative to wall heat transfer.

Heat-Transfer Results

From the temperature distribution given by equation (4), various quantities of engineering interest can be determined. In the developments to follow, the analytical results are expressed in terms of the broad-wall heat transfer q_B.

Of practical application are the wall-temperature variations corresponding to prescribed wall heat transfer and internal heat generation. The local temperatures $t_w(x', z') \equiv t_{w,B}$ along the broad walls are found from equation (4) by evaluating it at $y' = 0$ or at $y' = 1$:

$$\frac{t_{w,B} - t_e}{\dfrac{q_B a}{\kappa}} = \left[2\left(1 + \frac{\alpha}{\sigma}\right) + R\right]x' + \frac{\alpha}{\sigma}z'^2 - \alpha z' + \frac{1}{6}(1 + \sigma\alpha)$$

$$- \sum_{k=2,4,\ldots}^{\infty} \frac{4}{k^2\pi^2} e^{-k^2\pi^2 x'} - \alpha\sigma \sum_{l=2,4,\ldots}^{\infty} \frac{4}{l^2\pi^2} \cos\left(l\pi \frac{z'}{\sigma}\right) e^{-l^2\pi^2 x'/\sigma^2} \qquad (7)$$

The local temperatures $t_w(x', y') \equiv t_{w,s}$ along the sidewalls are found in a similar manner from equation (4) by evaluating it at $z' = 0$ or at $z' = \sigma$:

$$\frac{t_{w,s} - t_e}{\dfrac{q_B a}{\kappa}} = \left[2\left(1 + \frac{\alpha}{\sigma}\right) + R\right]x' + y'^2 - y' + \frac{1}{6}(1 + \alpha\sigma)$$

$$- \sum_{k=2,4,\ldots}^{\infty} \frac{4}{k^2\pi^2}\cos(k\pi y')e^{-k^2\pi^2 x'} - \alpha\sigma \sum_{l=2,4,\ldots}^{\infty} \frac{4}{l^2\pi^2}e^{-l^2\pi^2 x'/\sigma^2} \qquad (8)$$

Another more convenient form of these equations is obtained by introducing the bulk mean temperature t_b. For a uniform heat source and a uniform wall heat transfer, the bulk temperature is given by

$$\frac{t_b - t_e}{\dfrac{q_B a}{\kappa}} = \left[2\left(1 + \frac{\alpha}{\sigma}\right) + R\right]x' \qquad (9)$$

Then the local wall- to bulk-temperature difference along the broad walls is given by

$$\frac{t_{w,B} - t_b}{\dfrac{q_B a}{\kappa}} = \frac{\alpha}{\sigma}z'^2 - \alpha z' + \frac{1}{6}(1 + \alpha\sigma) - \sum_{k=2,4,\ldots}^{\infty} \frac{4}{k^2\pi^2}e^{-k^2\pi^2 x'}$$

$$- \alpha\sigma \sum_{l=2,4,\ldots}^{\infty} \frac{4}{l^2\pi^2}\cos\left(l\pi\,\frac{z'}{\sigma}\right)e^{-l^2\pi^2 x'/\sigma^2} \qquad (10)$$

while the temperature difference along the sidewalls is

$$\frac{t_{w,s} - t_b}{\frac{q_B a}{\kappa}} = y'^2 - y' + \frac{1}{6}(1 + \alpha\sigma) - \sum_{k=2,4,\dots}^{\infty} \frac{4}{k^2\pi^2} \cos(k\pi y') e^{-k^2\pi^2 x'}$$

$$- \alpha\sigma \sum_{l=2,4,\dots}^{\infty} \frac{4}{l^2\pi^2} e^{-l^2\pi^2 x'/\sigma^2} \tag{11}$$

Equations (10) and (11) are independent of the internal heat generation rate and, therefore, are valid for all heat generation rates for the proposed model. The differences between the fully developed wall and bulk temperatures are

$$\frac{(t_{w,B} - t_b)_d}{\frac{q_B a}{\kappa}} = \frac{\alpha}{\sigma} z'^2 - \alpha z' + \frac{1}{6}(1 + \alpha\sigma) \tag{12a}$$

$$\frac{(t_{w,s} - t_b)_d}{\frac{q_B a}{\kappa}} = y'^2 - y' + \frac{1}{6}(1 + \alpha\sigma) \tag{12b}$$

The ratios of local to fully developed temperature differences along the broad walls or sidewalls at any location in the channel are found from equations (10) to (12) as

$$\frac{(t_{w,B} - t_b)}{(t_{w,B} - t_b)_d} = 1 - \frac{\displaystyle\sum_{k=2,4,\dots}^{\infty} \frac{4}{k^2\pi^2} e^{-k^2\pi^2 x'} + \alpha\sigma \sum_{l=2,4,\dots}^{\infty} \frac{4}{l^2\pi^2} \cos\left(l\pi \frac{z'}{\sigma}\right) e^{-l^2\pi^2 x'/\sigma^2}}{\frac{\alpha}{\sigma} z'^2 - \alpha z' + \frac{1}{6}(1 + \alpha\sigma)}$$

$$\tag{13a}$$

$$\frac{(t_{w,s} - t_b)}{(t_{w,s} - t_b)_d} = 1 - \frac{\displaystyle\sum_{k=2,4,\ldots}^{\infty} \frac{4}{k^2\pi^2}\cos(k\pi y')e^{-k^2\pi^2 x'} + \alpha\sigma \sum_{l=2,4,\ldots}^{\infty} \frac{4}{l^2\pi^2} e^{-l^2\pi^2 x'/\sigma^2}}{y'^2 - y' + \frac{1}{6}(1 + \alpha\sigma)}$$

(13b)

It can be observed from the foregoing equations that a prescribed heat flow from the channel surfaces to the fluid produces, at a given axial location, a local wall temperature or a wall- to bulk-temperature difference which varies around the periphery of the channel and which also varies with channel aspect ratio.

Results and Discussion

To illustrate the effects of internal heat generation and prescribed wall heat transfer on wall-temperature distributions, local bulk mean temperature, and thermal entry lengths for liquid-metal flow in rectangular passages, a number of solutions have been obtained for various combinations of aspect ratio σ, wall heat flux ratio α or $\overline{\alpha}$, and heat flux ratio R or S. For brevity, only some of these results are included in this report. In particular, values of the parameters α and $\overline{\alpha}$ chosen for the computations correspond to the following cases:

(1) For $\alpha = -1$, the broad walls are heated and the sidewalls are cooled.

(2) For $\alpha = 0$, the sidewalls are insulated.

(3) For $\alpha = 1$, uniform heating (or cooling) takes place all around the channel periphery.

(4) For $\overline{\alpha} = 0$, the broad walls are insulated.

Designers of channels for space-power-generation systems are interested in the temperatures reached by the walls, and in particular the peripheral location where the wall temperature will assume its highest value for a known wall heat input and a known internal heat generation rate. From an examination of the analytical results, it is expected that the peak temperatures will occur either in the corner of a channel or at the centerline of the broad wall or short wall, depending on the wall heat flux ratio α. Therefore, knowledge of temperature conditions in such regions is essential. The wall temperatures given here are relative to the bulk temperature, since the heat flux ratio R or S is then eliminated as a parameter.

The longitudinal variations of the dimensionless wall temperatures were evaluated from the analytical solution, and are presented in figure 3 for channel aspect ratios of
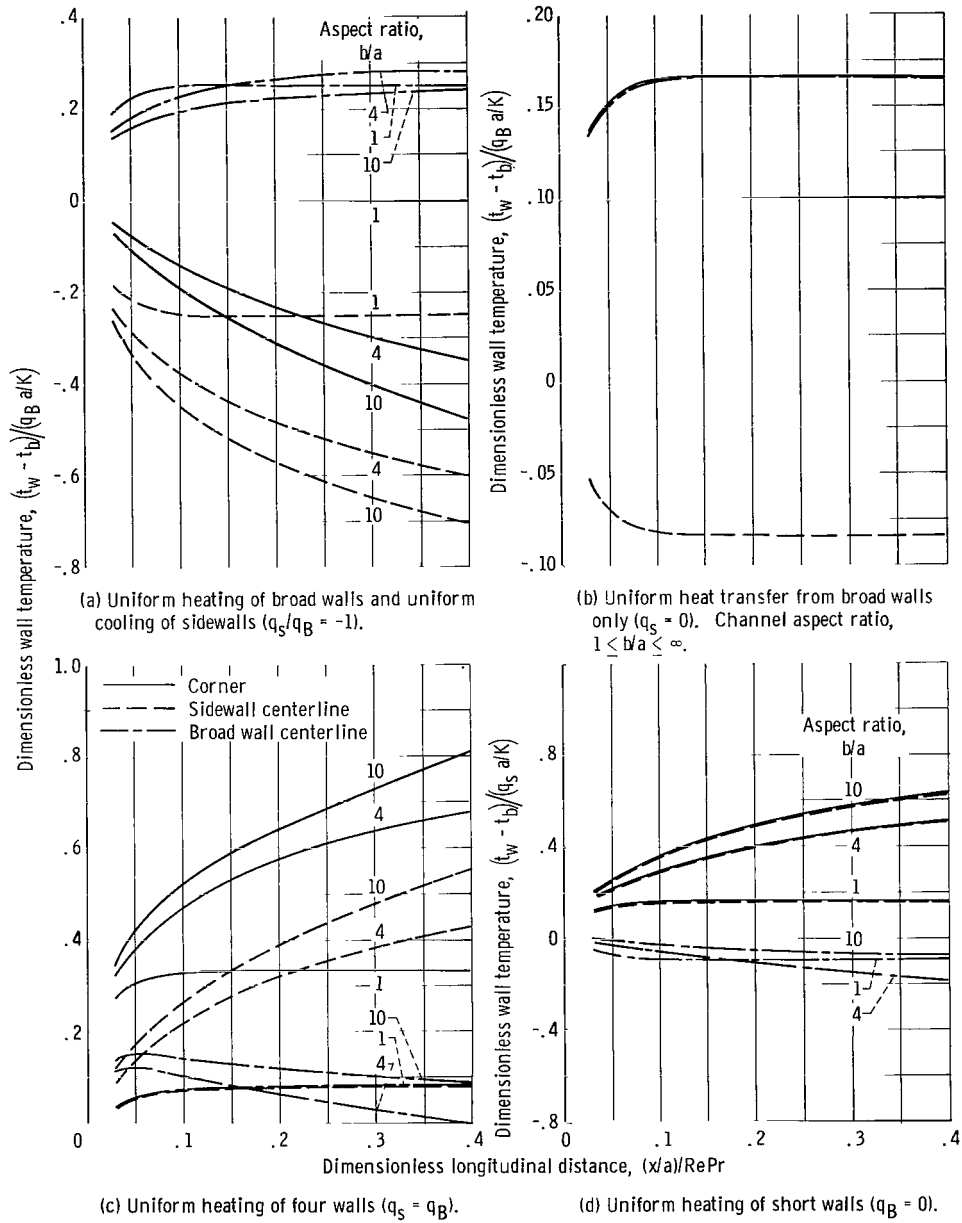
Figure 3. – Wall-temperature development in thermal entrance region of rectangular channels.

(a) Uniform heating of broad walls and uniform cooling of sidewalls (q_S/q_B = -1).

(b) Uniform heat transfer from broad walls only (q_S = 0). Channel aspect ratio, $1 \leq b/a \leq \infty$.

(c) Uniform heating of four walls (q_S = q_B).

(d) Uniform heating of short walls (q_B = 0).

1, 4, and 10. For the special case of insulated sidewalls ($\alpha = 0$), the solution is independent of the channel aspect ratio; therefore, the result applies for all aspect ratios. The result corresponds to that result for heat-generating slug flow between parallel plates with uniform wall heat flux. At some places along a wall, the wall- to bulk-temperature difference may be negative. This is understandable if it is recalled that t_w is a local value along the wall, while t_b is an average value over the entire cross section. When the broad walls are heated and the sidewalls are cooled ($\alpha = -1$), the peak temperatures occur along the centerline of the broad walls at all axial positions for the aspect ratios shown (fig. 3(a)). When only the broad walls are heated ($\alpha = 0$) (fig. 3(b)), these walls assume a uniform temperature, for all aspect ratios, which is higher than the temperatures along the insulated sidewalls. For a rectangular channel with uniform heat flux around the entire periphery (fig. 3(c)), the peak temperatures for all σ occur at the channel corners. Finally, when heat is transferred from only the short walls (fig. 3(d)), these walls attain a uniform temperature that is higher than the temperatures along the adiabatic broad walls.

Another quantity with practical application is the bulk-mean-temperature variation along the length of the channel. The bulk temperature is given here relative to the temperature of the fluid at the entrance to the channel. The dimensionless bulk temperatures are presented in figure 4 for channel aspect ratios of 1, 20, and ∞ (parallel-plate channel) with the heat flux ratio appearing as a family parameter.

Positive and negative values of the heat flux ratios R and S are considered in figure 4. It is assumed that Q is positive (a heat source). A positive value of R, therefore, implies that q_B is positive - that is, that heat is being transferred from the broad walls to the fluid. A negative value of R, on the other hand, implies that q_B is negative, or that heat is being transferred from the fluid to the broad walls. For positive R, therefore, internal heat generation and broad-wall heat transfer reinforce one another to produce a bulk temperature larger than that obtained in the absence of internal heat generation. Conversely, for negative R, heat transfer to the broad wall opposes internal heat generation in the bulk-temperature development. Similar arguments apply in connection with sidewall heat transfer q_s and heat flux ratio S.

For very small values of $|R|$ or $|S|$, wall heat transfer dominates the bulk-temperature development, while for large values of $|R|$ or $|S|$, the effects of internal heat generation dominate. This explains the variety of trends that are evident in each of the figures. The bulk-temperature development is, in general, affected slightly by channel aspect ratios σ to about 20 and insignificantly thereafter. For a channel with insulated sidewalls, however, the bulk-temperature development is independent of the channel aspect ratio, and corresponds to that for heat-generating slug flow between parallel plates with uniform wall heat flux.

The foregoing presentation of results has been concerned with wall temperatures in
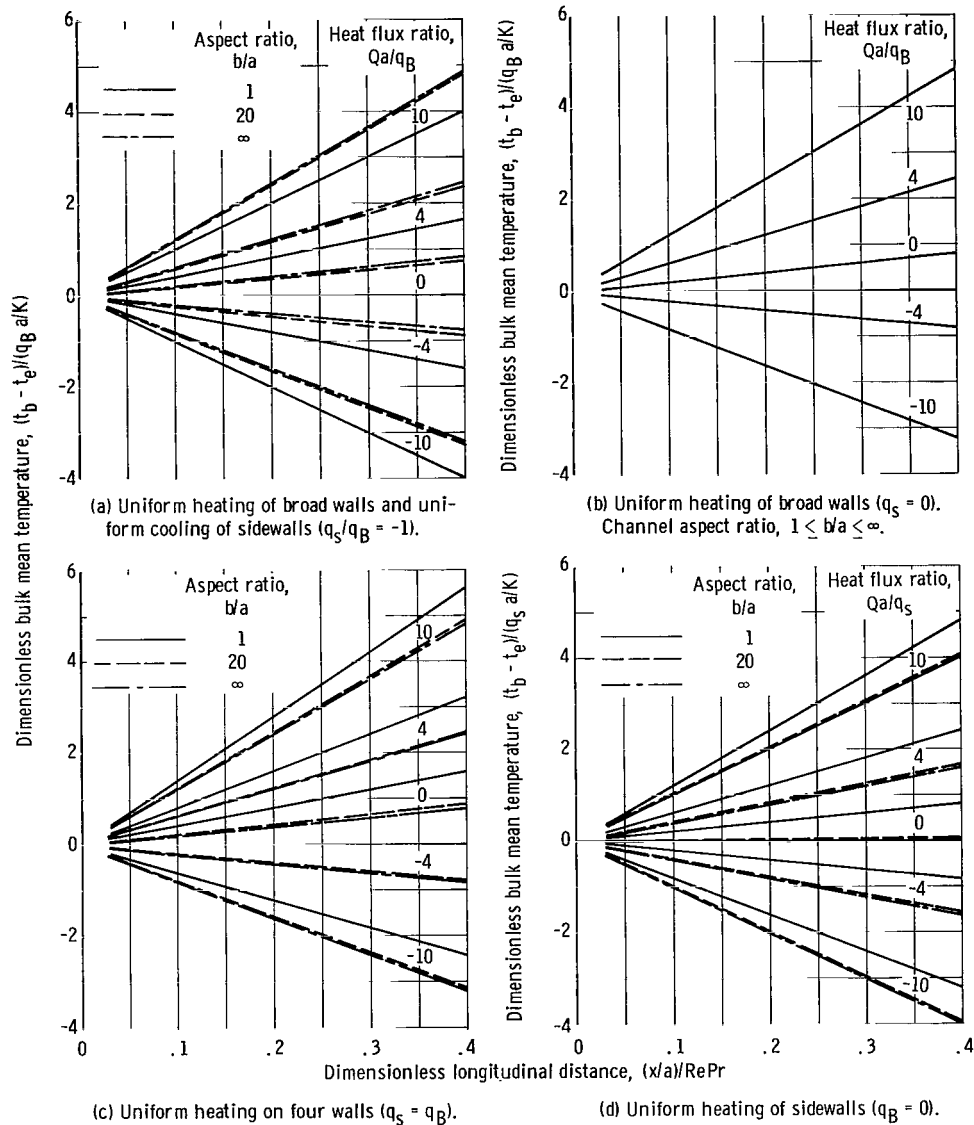
(a) Uniform heating of broad walls and uniform cooling of sidewalls (q_s/q_B = -1).

(b) Uniform heating of broad walls (q_s = 0). Channel aspect ratio, $1 \leq b/a \leq \infty$.

(c) Uniform heating on four walls (q_s = q_B).

(d) Uniform heating of sidewalls (q_B = 0).

Figure 4. - Longitudinal variation of bulk mean temperature for various values of heat flux ratio Qa/q_B or Qa/q_S.

the thermal entrance region. As a matter of general interest, the wall-temperature variation around the periphery of the channel in the fully developed region is considered. The wall temperatures are again given relative to the bulk temperature, since in the fully developed region the temperature difference $(t_w - t_b)_d$ is independent of x. In addition, as noted earlier, the temperature difference $t_w - t_b$ is independent of the heat flux ratio R at all axial positions.

Fully developed wall temperatures are presented in figure 5 for the values of α and $\overline{\alpha}$ considered earlier and for various aspect ratios. It is worthwhile to recall that t_w

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(a) Uniform heating of broad walls and uniform cooling of sidewalls ($q_s/q_B = -1$).

(b) Uniform heating of broad walls only ($q_s = 0$). Channel aspect ratio, $1 \leq b/a \leq \infty$.

(c) Uniform heat flux on four walls ($q_s = q_B$).
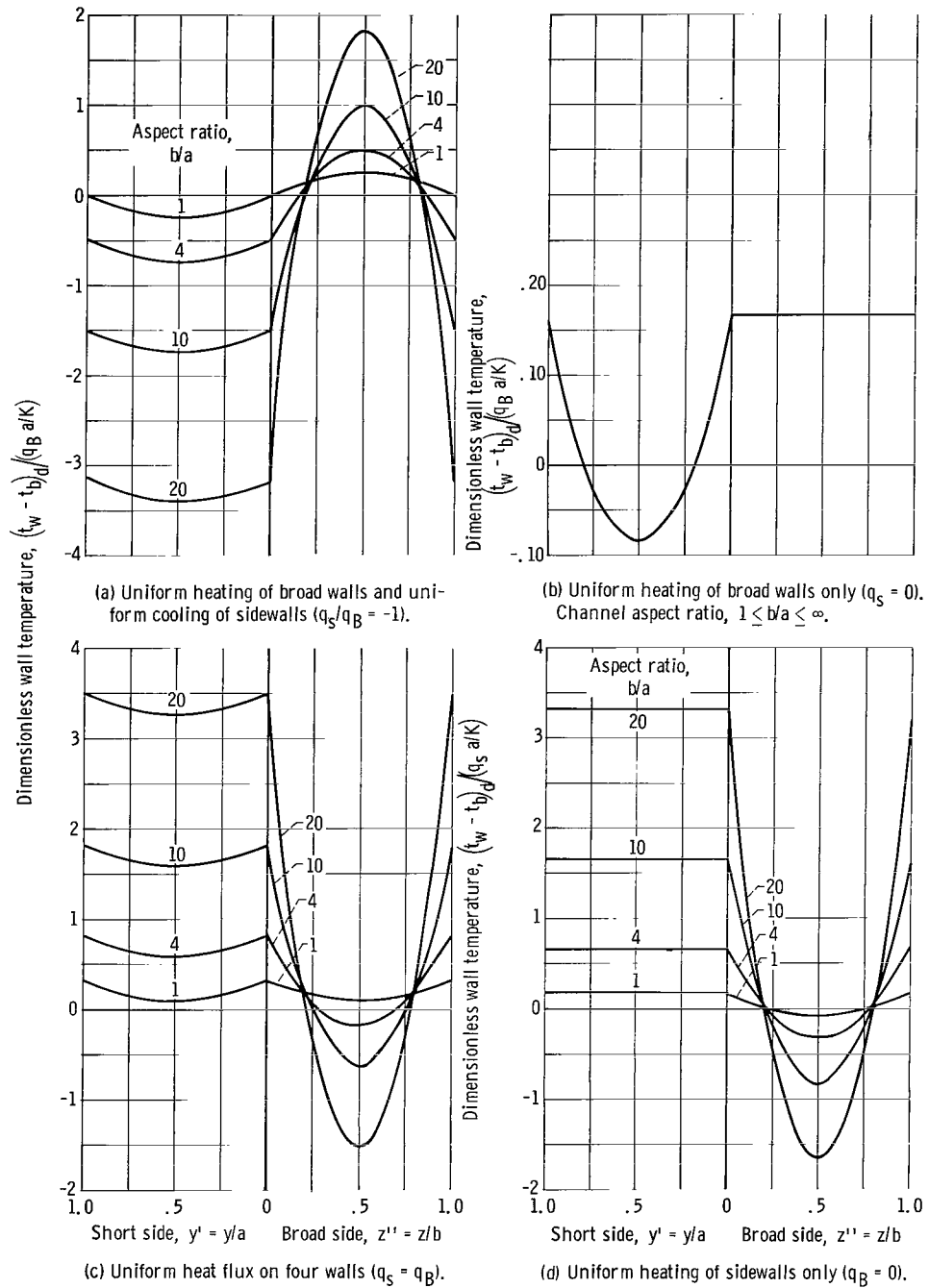
(d) Uniform heating of sidewalls only ($q_B = 0$).

Figure 5. – Fully developed wall temperatures of rectangular channels.

is a local value along the wall, while t_b is an average value over the entire cross section. Therefore, at some places along the wall, the temperature difference $(t_w - t_b)_d$ is negative, which means that t_b is larger than t_w. The hot spots are strikingly displayed in these figures, appearing in the corners, or at the broad-wall or short-wall midpoints. It is also apparent that the aspect ratio has a significant effect on the temperature distribution.

Of considerable practical importance to the designer is a knowledge of the conditions under which entrance effects must be accounted for in heat-transfer calculations. In theory, the approach of a local- to bulk-temperature difference to the fully developed value is asymptotic. Consequently, it is difficult to identify a specific length of channel as a thermal entrance length. It is practice to define a thermal entrance length in terms of the downstream distance $x/a\text{RePr}$ at which the temperature difference comes within 5 percent of the fully developed value. The variation with dimensionless axial distance of local to fully developed wall temperatures at the channel corner and at the broad-wall and short-wall centerlines was evaluated from the analytical expressions. The results thus obtained are plotted in figure 6 for values of $\alpha = -1$ and $\overline{\alpha} = 0$ and for parametric aspect ratio values. The corner temperature ratio is not shown for the square channel when $\alpha = -1$, because for this situation the corner temperature difference is zero at all axial positions. Lines delineating the condition $(t_w - t_b)/(t_w - t_b)_d = 0.95$ have been drawn in figure 6 to graphically illustrate the approach to fully developed conditions.

From an inspection of the graphs, it is seen that, for the wall heating conditions represented, the wall-temperature profiles approach fully developed conditions at widely different distances from the entrance to the heating section, depending on the particular wall location chosen for consideration. It is also evident that the sidewalls of the channels strongly influence the temperature developments, with the thermal entrance lengths decreased with decreased aspect ratio.

It is the practice in reporting heat transfer connected with flow through noncircular passages to present average heat-transfer coefficients or Nusselt numbers based on a heat flow averaged around the channel periphery and on an average wall temperature. This practice might be useful when the wall temperature remains constant everywhere, or at least is constant around the periphery of a channel at a given axial position. In the present situation where the wall boundary condition is one of peripherally and axially uniform heat input, it is apparent that the knowledge of the resulting local wall-temperature distribution is more important to the designer than Nusselt numbers or even average thermal-entry lengths. Therefore, no attempt has been made to determine the latter quantities. It is felt that the results are presented in a form more convenient for engineering calculations.
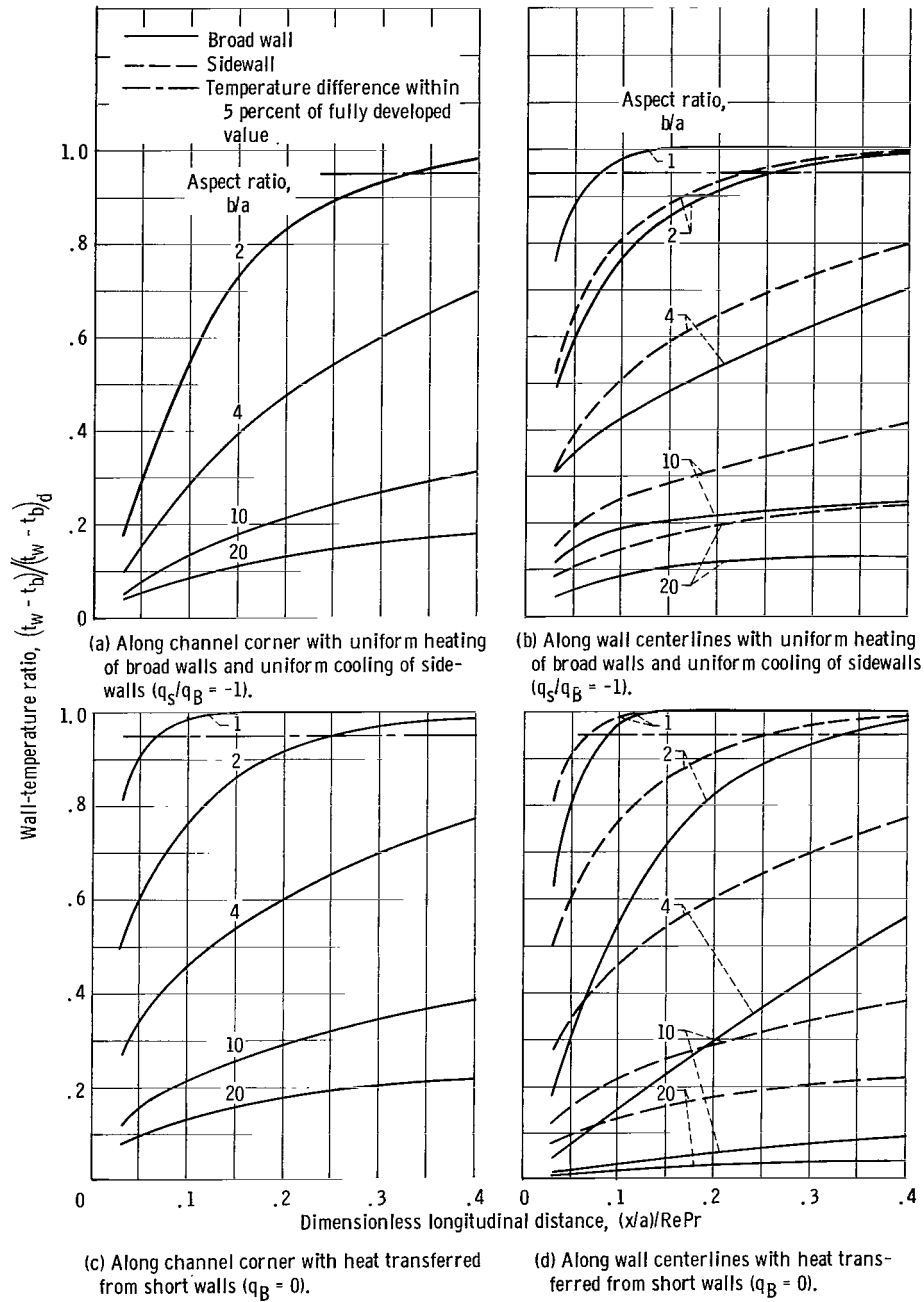
(a) Along channel corner with uniform heating of broad walls and uniform cooling of sidewalls ($q_s/q_B = -1$).

(b) Along wall centerlines with uniform heating of broad walls and uniform cooling of sidewalls ($q_s/q_B = -1$).

(c) Along channel corner with heat transferred from short walls ($q_B = 0$).

(d) Along wall centerlines with heat transferred from short walls ($q_B = 0$).

Figure 6. – Wall-temperature ratios in thermal entrance region.

INTERNAL HEAT GENERATION WITH UNIFORM WALL TEMPERATURE

Consideration is now given to the problem where a liquid metal with a uniform temperature t_e enters a rectangular channel whose four walls are maintained at uniform equal temperatures t_w, which are different from the entrance value (fig. 7). In addition,
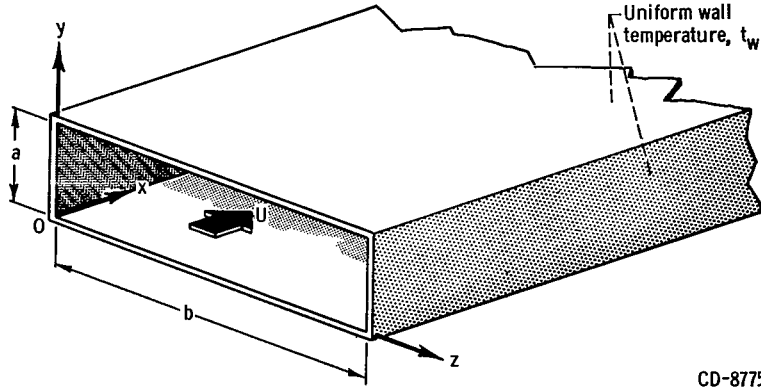
Figure 7. - Coordinate system for rectangular channel with uniform and equal wall temperatures.

a uniform heat generation begins within the fluid at $x = 0$. Although many wall-temperature combinations are possible, the boundary conditions of axially and peripherically uniform wall temperature are of some technical interest and will serve to demonstrate the effects of internal heat generation on heat-transfer characteristics in the presence of uniform wall temperatures.

The velocity profile is again represented by equation (1), and the energy transfer processes in the liquid metal are assumed to be still governed by equation (2). The boundary conditions are now

$$t = t_w \qquad \text{at } y = 0 \text{ and } y = a \text{ (specified wall temperatures)} \qquad (14a)$$

$$t = t_w \qquad \text{at } z = 0 \text{ and } z = b \text{ (specified wall temperature)} \qquad (14b)$$

$$t(0, y, z) = t_e \qquad \text{(entrance condition)} \qquad (14c)$$

A solution to equation (2) that satisfies the boundary conditions of equations (14) is given by

$$\frac{t - t_w}{t_e - t_w} = T = \frac{1}{N} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{mn\pi^2 \Gamma_{mn}^2}\left(1 - e^{-\Gamma_{mn}^2 x'}\right)\sin(m\pi y')\sin\left(n\pi \frac{z'}{\sigma}\right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{mn\pi^2} \sin(m\pi y')\sin\left(n\pi \frac{z'}{\sigma}\right)e^{-\Gamma_{mn}^2 x'} \tag{15}$$

where

$$N = \frac{\dfrac{\kappa}{a^2}(t_e - t_w)}{Q} \tag{16}$$

and $m = 1, 3, 5, \ldots$ and $n = 1, 3, 5, \ldots$ The parameter N is essentially the ratio of temperature difference $t_e - t_w$ to internal heat generation. The derivation and discussion of equation (15) is presented in appendix C.

Heat-Transfer Results

From the temperature distribution given by equation (15), quantities of engineering interest can be determined. The local bulk temperature T_b for the situation of combined internal heat generation and uniform equal wall temperatures, for example, is obtained from (appendix C)

$$\frac{t_b - t_w}{t_e - t_w} \equiv T_b = \frac{1}{N} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{64}{m^2 n^2 \pi^4 \Gamma_{mn}^2}\left(1 - e^{-\Gamma_{mn}^2 x'}\right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{64}{m^2 n^2 \pi^4} e^{-\Gamma_{mn}^2 x'} \tag{17}$$

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When the channel wall temperatures are specified, the quantity of greatest practical interest is the longitudinal variation of the wall heat fluxes that is required to maintain the wall temperatures constant. If, for example, the wall temperatures are less than that of the liquid metal entering the channel, and in addition internal heat sources are present, the walls must conduct heat from the liquid metal into a surrounding coolant. An expression for the local peripheral heat flux along the sidewall $z' = 0$ is obtained by applying Fourier's law

$$q_z(0) = -\kappa\left(\frac{\partial t}{\partial z}\right)_{z=0} = -\frac{\kappa}{a}(t_e - t_w)\left(\frac{\partial T}{\partial z'}\right)_{z'=0}$$

to the temperature solution (eq. (15)) to obtain

$$-\frac{q_z(0)a}{\kappa(t_e - t_w)} = \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{16}{m\pi\sigma}\sin(m\pi y')e^{-\Gamma_{mn}^2 x'}$$

$$+ \frac{1}{N}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{16}{m\pi\sigma\Gamma_{mn}^2}\left(1 - e^{-\Gamma_{mn}^2 x'}\right)\sin(m\pi y') \qquad (18)$$

The local heat flux $q_z(\sigma)$ along the opposite sidewall $z' = \sigma$ is obtained as the negative of that along the wall $z' = 0$; that is, $q_z(\sigma) = -q_z(0)$. The local peripheral heat flux along the broad channel wall $y' = 0$ is found in a similar manner with the result

$$-\frac{q_y(0)a}{\kappa(t_e - t_w)} = \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{16}{n\pi}\sin\left(n\pi\frac{z'}{\sigma}\right)e^{-\Gamma_{mn}^2 x'}$$

$$+ \frac{1}{N}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{16}{n\pi\Gamma_{mn}^2}\left(1 - e^{-\Gamma_{mn}^2 x'}\right)\sin\left(n\pi\frac{z'}{\sigma}\right) \qquad (19)$$

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The local dimensionless heat flux $q_y(1)$ at the opposite broad wall $y' = 1$ is given by $q_y(1) = -q_y(0)$.

It should be noted from equations (18) and (19) that the boundary condition of a constant peripheral and axial channel wall temperature requires (at a given axial position) the local heat flow from the channel surfaces to the fluid, or from the fluid to the channel surfaces, to vary around the periphery. The local heat flow also varies with channel aspect ratio.

(a) Channel aspect ratio, 1 (square channel).

(b) Channel aspect ratio, 4.

(c) Channel aspect ratio, 10.

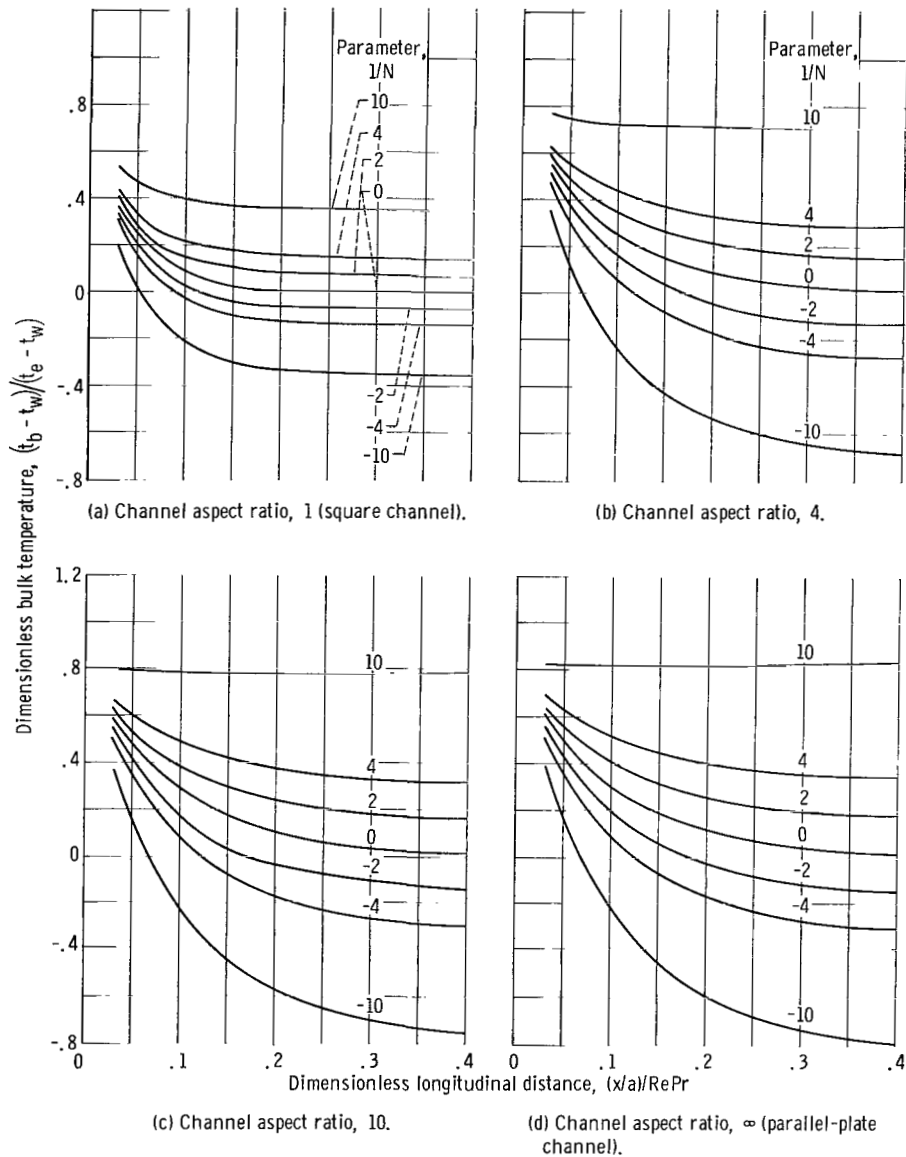(d) Channel aspect ratio, ∞ (parallel-plate channel).

Figure 8. - Bulk or mixed-mean temperature variation for internal heat generation with equal wall temperatures.

Results and Discussion

To illustrate the results, local bulk mean temperatures and wall heat fluxes were computed for several combinations of aspect ratio σ and the heat generation parameter N. Bulk temperature results for internal heat generation with peripherally and axially uniform wall temperatures are presented in figure 8 for channel aspect ratios of 1, 4, 10, and ∞ and for parametric values of the heat generation ratio. It is again assumed that Q is positive, but clearly $T_e - T_w$ may be either positive or negative, depending on whether the walls are cooler or warmer than the entering liquid metal. The heat generation ratio or parameter N, therefore, is also either positive or negative. Consequently, both positive and negative values are considered in the figure. The curves of $(t_b - t_w)/(t_e - t_w)$ change continuously from an initial value of one to an asymptote (for $X \rightarrow \infty$) of $(1/N) \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} 64/m^2 n^2 \pi^4 \Gamma_{mn}^2$

as t_b varies from t_e to

$t_w + \left[(t_e - t_w)/N\right] \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} 64/m^2 n^2 \pi^4 \Gamma_{mn}^2$. In the absence of internal heat sources $(N \rightarrow \infty)$, the bulk temperature asymptotically approaches the temperature of the walls.

To illustrate more clearly the influence of channel aspect ratio, the dimensionless bulk temperature is plotted against the dimensionless longitudinal distance in figure 9 for values of $1/N$ equal to -10, 0, and 10, with the channel aspect ratio as the family parameter. The bulk temperature development for a given value of $1/N$ is affected by channel aspect ratio σ to about 10 and insignificantly affected thereafter.

Figure 9. – Bulk temperature dependence on aspect ratio for various values of heat generation ratio.

A plot of the longitudinal variation of the dimensionless sidewall heat flux at the wall locations $y' = 0.25$ and 0.50 was prepared and is presented in figure 10. Channel aspect ratios of 1, 4, 10, and 20 are considered, and the heat generation ratio is the parameter. The heat fluxes commence with large negative values near the channel entrance and change continuously with increasing downstream distance. For a channel of given aspect ratio, the magnitude and sign of the parameter N have a profound influence on the wall heat fluxes. In the absence of internal heat generation the wall heat fluxes decrease to zero at large downstream distances. For negative values of N, $q_z(0)$ can change sign at some position along the wall length, which indicates
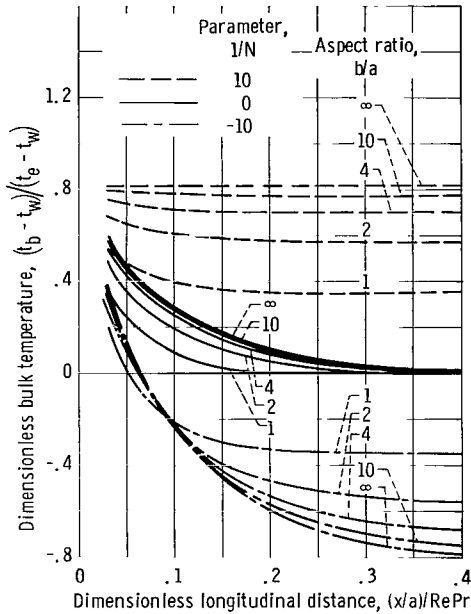
Position along
sidewall,
y/a
— — — —   0.25
— · — · —   .50

Parameter,
1/N

Parameter,
1/N

-10

-4

0

4

10

Dimensionless sidewall heat flux, $q_z(0)a/K(t_e - t_w)$

-10

-4

0

4

10

(a) Channel aspect ratio, 1 (square channel).

(b) Channel aspect ratio, 4.

-10

-4

0

4

10

-10

-4

0

4

10

Dimensionless longitudinal distance, $(x/a)/RePr$

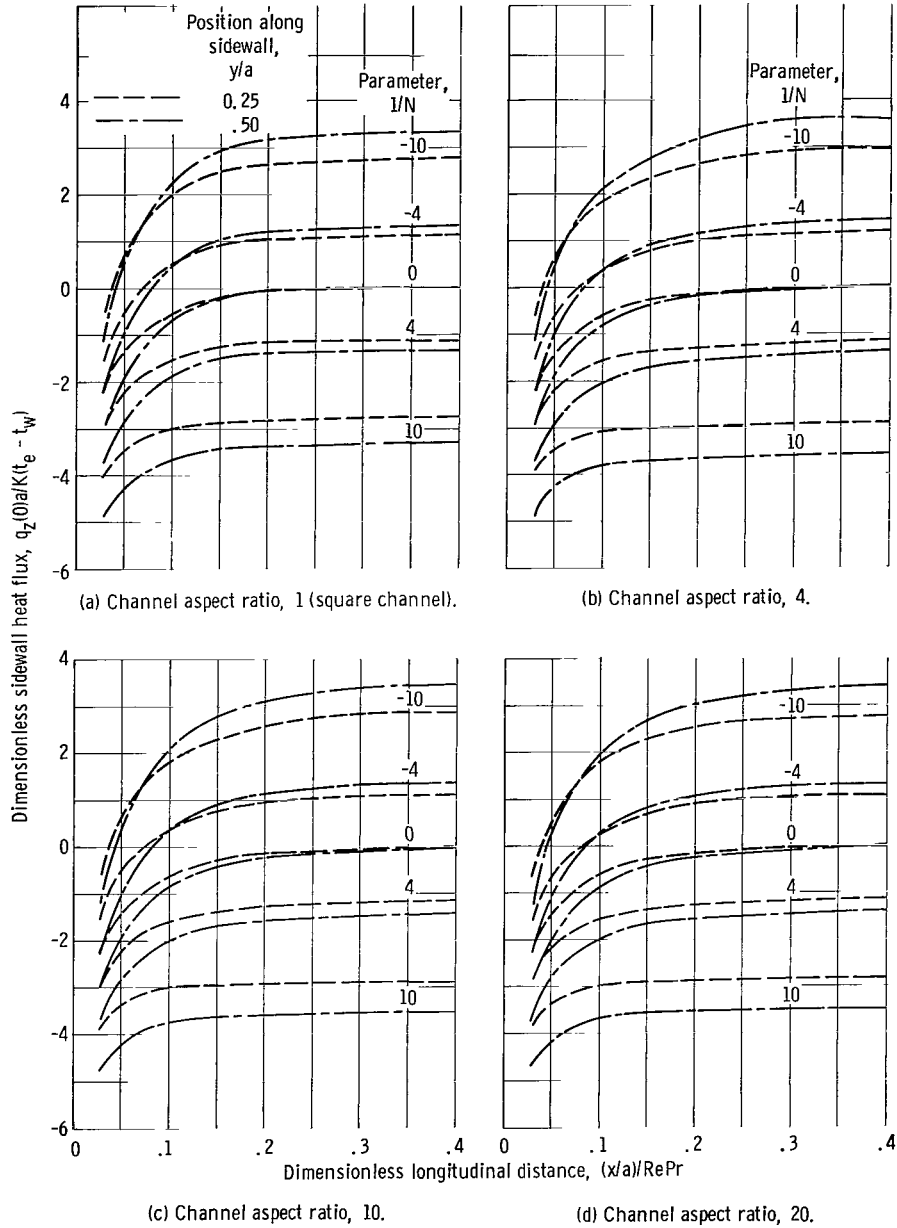(c) Channel aspect ratio, 10.

(d) Channel aspect ratio, 20.

Figure 10. - Local sidewall heat-flux results for internal heat generation with equal wall temperatures.

a change in direction of the heat transfer at that location along the wall. For a given value of $1/N$, the influence of channel aspect ratio on wall heat fluxes is slight for the range of aspect ratios chosen for illustration. When $\sigma \to \infty$ (infinite parallel plates), however, the sidewall heat fluxes diminish to zero at all positions along the sidewalls (see eq. (18)).

Longitudinal variations of the dimensionless broad-wall heat fluxes at the wall locations $z''(= z'/\sigma)$ of 0.25 and 0.50 were evaluated and are presented in figure 11 for channels with aspect ratios of 1, 4, 10, and 20. The heat generation ratio N appears again as the parameter. The figure shows the influence of aspect ratio on wall heat fluxes. As σ is increased, for a specified value of $1/N$, the peripheral wall heat flux distribution becomes more uniform. The broad-wall heat fluxes, at a given longitudinal position and for a selected value of N, remain almost constant for all aspect ratios σ beyond approximately 10. The heat generation ratio has a significant influence on the broad-wall heat fluxes for a channel of given aspect ratio. For $N \to \infty$, the wall heat fluxes decrease to zero at large downstream distances. For negative values of N, it is observed that $q_y(0)$ also may change sign at some position along the broad-wall length.

It is the practice in considering heat transfer in noncircular passages to define a heat-transfer coefficient as the average heat flux at the wall (averaged around the channel periphery) divided by the difference between the fluid bulk temperature and the wall temperature. In the foregoing figures, however, for negative values of the parameter N, the wall- to bulk-temperature difference became zero when the bulk temperature of the fluid was equal to the wall temperature and negative when the wall temperature was less than the bulk temperature. In addition, for negative values of N, there occurred changes in direction of the wall heat transfer at certain locations. Infinities and zeros, therefore, occurred in the heat-transfer coefficient, and this variation opens to question the utility of a heat-transfer coefficient under the condition of internal heat generation in the fluid. Provision of adequate, adjacent cooling channels for effective heat removal to avoid over-heating the fluid and to maintain the required wall temperatures is of practical concern to the designer. Also, the local wall-heat-flux conditions are necessarily more important from the standpoint of effective heat removal than consideration of the average heat-transfer conditions. It is conjectured that independent consideration of local wall- to bulk-temperature differences and wall heat fluxes will prove more useful for preliminary heat-transfer design purposes. For these reasons, information on the average heat-transfer coefficients for rectangular channels containing flowing, heat-generating liquid metals are not reported.

(a) Channel aspect ratio, 1 (square channel).

(b) Channel aspect ratio, 4.

(c) Channel aspect ratio, 10. Results along broad wall coincide.

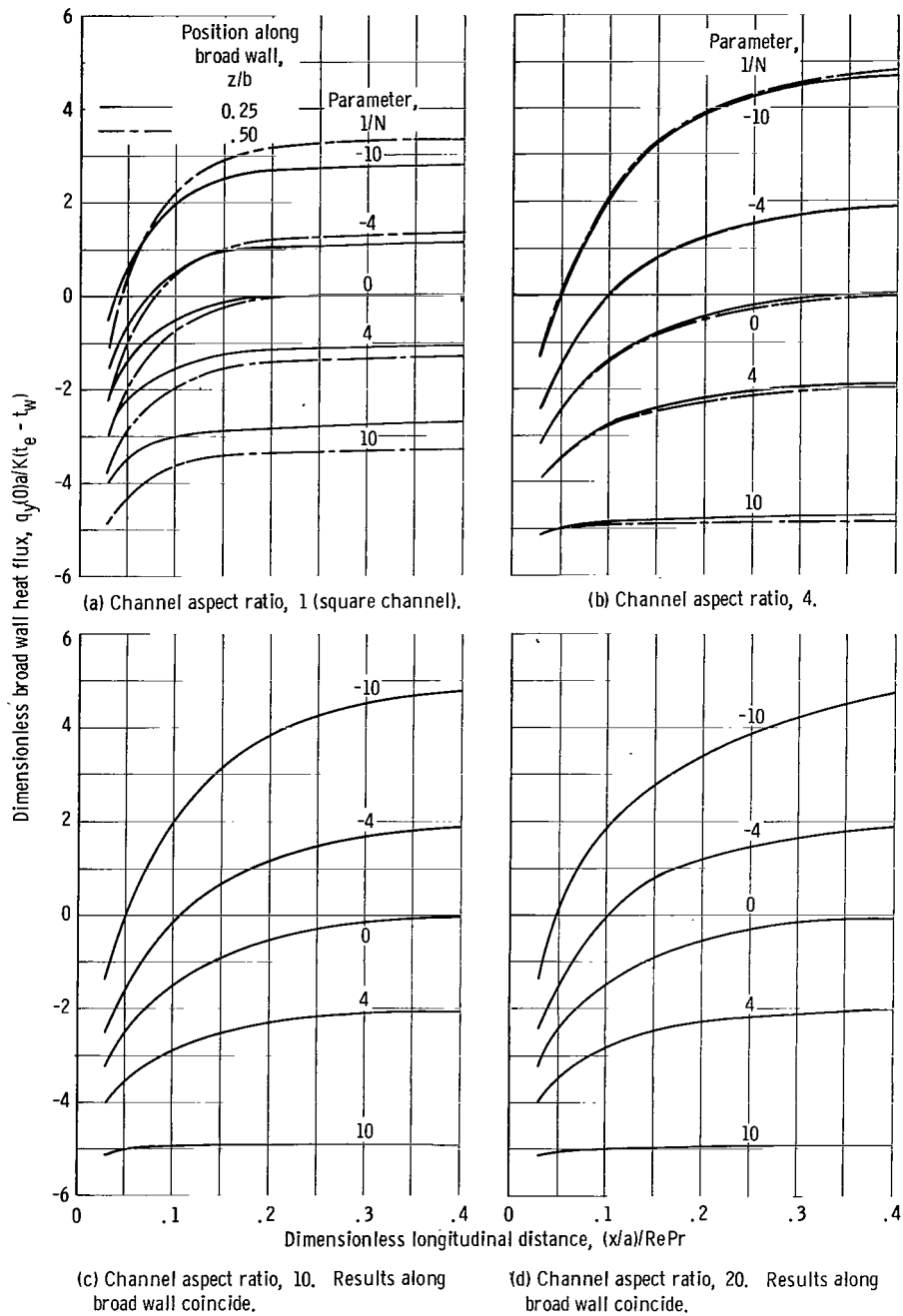(d) Channel aspect ratio, 20. Results along broad wall coincide.

Figure 11. – Local broad-wall heat-flux results for internal heat generation with equal wall temperatures.

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CONCLUDING REMARKS

Theoretical results have been presented for heat transfer to liquid metals flowing in rectangular channels with prescribed wall heat fluxes or wall temperatures and heat sources uniformly distributed in the fluid. The various boundary and internal heating effects considered herein have yielded a qualitative understanding of the axial and peripheral temperature or heat-flux distributions as well as general heat-transfer characteristics.

It should be emphasized that the present results are strictly valid only for the slug-flow velocity distribution with heat transfer by molecular conduction. The simplification provided by these assumptions, however, has made it possible to obtain exact mathematical solutions to the governing energy equation. A fact of greater importance is that these temperature distributions approximate conditions expected for turbulent liquid-metal flow in rectangular channels for relatively low Prandtl and Reynolds numbers.

Some of the characteristics of turbulent, heat-generating liquid-metal heat transfer are the following:

1. Wall temperature distributions for specified wall heat fluxes are, in general, sensitive to the channel aspect ratio.

2. For the slug-flow velocity distribution, wall- to bulk-temperature differences for specified wall heat fluxes are independent of the internal heat generation rate. It is conjectured, therefore, that with nonuniformity of the velocity profile the influence of an internal source on wall- to bulk-temperature differences will be slight.

3. The thermal entrance length is quite sensitive to channel aspect ratio and wall heating conditions. For the conditions investigated, the thermal entrance length is least for a square channel.

4. Bulk temperature development in channels with uniform temperature along all walls is influenced by channel aspect ratio to an aspect ratio of about 10. Beyond this value, the influence of aspect ratio is slight.

5. The wall-heat-flux variations required to maintain the wall temperatures constant are dependent on the channel aspect ratio, internal heat source strength, and magnitude of the entrance temperature in relation to the value of the wall temperature. In particular, for wall temperatures cooler than the fluid entrance temperature, a change in direction of the heat transfer may occur at some position along a given wall. An engineering design of a wall heating or cooling system would necessarily have to take this into consideration. The results, in addition, point out the locations of maximum wall temperatures or maximum wall heat-transfer rates.

More refined analyses would undoubtedly take into account velocity-profile and eddy-diffusivity variations to provide a more realistic description. Turbulent velocity profiles for circular-pipe and parallel-plate channel systems have been conveniently and satisfac-

torily represented by power-law expressions. Reference 21 presents and applies a variational method for determining velocity distributions for flow of a power-law fluid in rectangular channels. The results might prove useful for heat-transfer studies. Within the knowledge of the author, eddy-diffusivity distributions for turbulent flow in rectangular channels have received little theoretical consideration. In reference 6, the writer derived analytical heat-transfer solutions for turbulent flow of a heat-generating liquid metal in a parallel-plate channel for high Reynolds numbers where the effect of radial eddy-diffusivity variations must be included. Uniform wall heat flux and uniform wall-temperature cases were considered. A linear approximation of the complicated eddy-diffusivity function was used as in references 22 and 23, and good agreement of the heat-transfer results with existing numerical solutions was demonstrated. A linear diffusivity function could be applied to the problem of turbulent heat transfer in rectangular passages. However, within the author's knowledge, no work has thus far been done along these lines.

For fully developed turbulent flow in straight channels of rectangular cross section, there exists a transverse mean flow, commonly known as secondary flow, superimposed upon the axial mean flow (refs. 24 to 27). This secondary flow interacts with the axial mean flow and the turbulence structure in a complex manner. Qualitative observations indicate that secondary flow tends to equalize the velocities and temperatures within any cross section of a rectangular channel. However, because little quantitative knowledge exists on the influence of the secondary flow on temperature distributions within a fluid, particularly for fully developed turbulent liquid-metal flow, an exact theoretical calculation of the temperature field in rectangular channels is not possible at the present time. Consequently, the wall-temperature and wall-heat-flux distributions determined in the present analyses may be taken as limiting cases.

The present analytical treatments are adequate for preliminary design purposes and should give the designer a reasonable estimate of the locations and magnitudes of maximum temperatures or wall heat fluxes that could be expected for a given channel configuration in the presence of internal heat generation.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 1, 1966,
129-01-09-07-22.

APPENDIX A

SYMBOLS

A_n coefficients in series for temperature distribution with internal heat generation and zero wall temperatures

a length of short side

b length of broad side

C_{mn} coefficients in double series for temperature distribution with internal heat generation and zero temperatures

c_p specific heat of fluid at constant pressure

D_{rs} coefficients in double series for temperature distribution with internal heat generation and zero wall temperatures

H uniform magnetic field

N heat generation ratio, $\dfrac{\dfrac{K}{a^2}(t_e - t_w)}{Q}$

Pr Prandtl number, $\mu c_p / \kappa$

Q rate of internal heat generation per unit volume

q heat flux per unit wall area

$q_y(0)$ peripheral heat flux along broad wall $y' = 0$

$q_z(0)$ peripheral heat flux along side-wall $z' = 0$

R heat flux ratio, Qa/q_B

Re Reynolds number, $\rho U a / \mu$

S heat flux ratio, Qa/q_s

T dimensionless temperature, $(t - t_w)/(t_e - t_w)$

t temperature

U velocity, uniform over channel cross section

u local fluid velocity

W_n function of y', defined by eq. (C14)

x coordinate measured along the axial direction

x' dimensionless axial coordinate, $x/aRePr$

y coordinate measured along short side

y' dimensionless coordinate, y/a

z coordinate measured along long side

z' dimensionless coordinate, z/a

z'' dimensionless coordinate, $z/b = z'/\sigma$

α wall heat flux ratio, q_s/q_B

$\bar{\alpha}$ wall heat flux ratio, $q_B/q_s = 1/\alpha$

β_n eigenvalue, $n\pi\sqrt{1 + (1/\sigma^2)}$

Γ^2_{mn} function of m, n, and σ, $\pi^2\left[m^2 + (n^2/\sigma^2)\right]$

Γ^2_{rs} function of r, s, and σ, $\pi^2\left[r^2 + (s^2/\sigma^2)\right]$

κ thermal conductivity of fluid

μ fluid viscosity

ρ fluid density

σ channel aspect ratio

Subscripts:

B   broad wall

b   bulk or mixed-mean value

d   fully developed

e   entrance value

Q   insulated wall or zero wall temperature, internal heat generation

q   wall heat flux, no internal generation

s   sidewall

T   uniform wall temperature, no internal heat generation

w   value at wall

Superscript:

*   entrance region

APPENDIX B

TEMPERATURE DISTRIBUTION FOR INTERNAL HEAT GENERATION

WITH WALL HEAT TRANSFER

The steady-state temperature distribution for slug flow in a rectangular channel with internal heat generation and wall heat transfer is the solution to equation (2) subject to the boundary conditions expressed by equations (3).

The problem has been examined from the standpoint of superposition of basic solutions in an earlier paper (ref. 28). This approach is a standard in heat-transfer analysis and need not be repeated here. The basic solutions may be found in reference 28; the results have been used in the present report.

This same problem can be treated by using the fact that the steady-state slug-flow convection problem is the same as the problem of unsteady-state heat conduction in a solid with the same cross-sectional shape and the same boundary conditions. Therefore, the results presented in reference 28, for the situation where a nongenerating liquid metal entering at a temperature t_e flows through a channel with heat transfer q_B and q_s at the walls, might also have been derived from appropriate superposition of the solution of the problem of one-dimensional heat flow in a slab with uniform heat flux at the surface (ref. 29, p. 113). (This fact was called to the author's attention by Mr. Aaron J. Friedland in a written discussion of reference 28 that was submitted to the 1966 Heat Transfer and Fluid Mechanics Institute.)

The temperature t_q (ref. 28) may be expressed as the sum $t_e + t_{q,B} + t_{q,s}$, where $t_{q,B}$ corresponds to the case of no internal heat generation, zero inlet temperature, heat flux q_B on the broad walls, and insulated sidewalls. The temperature $t_{q,s}$ corresponds to the case of no internal heat generation, zero inlet temperature, heat flux q_s on the sidewalls, and insulated broad walls. The problem is then reduced to the case of one-dimensional heat flow in a slab with uniform heat flux at the surface, and the general solution is obtained by adding the results for the two directions. The details are omitted, since the results present nothing new.

Therefore, the solutions previously obtained in reference 28 for steady-state slug-flow convection could be found from consideration of the conduction of heat in the unsteady state. The advantage of the former approach to the problem, however, lies in the fact that it is possible, in principle, to apply it to more difficult cases (e.g., more realistic turbulent velocity profile; inclusion of turbulent diffusion as an additional mechanism for heat transfer) which are not easily dealt with by analogy to the unsteady-state heat-conduction problem.

APPENDIX C

TEMPERATURE DISTRIBUTION FOR INTERNAL HEAT GENERATION

WITH UNIFORM, EQUAL WALL TEMPERATURES

The temperature t may be expressed as a sum $t_Q + t_T$, or in dimensionless form,

$$\frac{t(x,y,z) - t_w}{t_e - t_w} = \frac{t_Q}{t_e - t_w} + \frac{t_T - t_w}{t_e - t_w} = T_Q + T_T \tag{C1}$$

The governing equations and boundary conditions for T_Q and T_T become

$$\frac{\partial T_Q}{\partial x'} = \frac{\partial^2 T_Q}{\partial y'^2} + \frac{\partial^2 T_Q}{\partial z'^2} + \frac{Qa^2/\kappa}{(t_e - t_w)} \tag{C2}$$

$$
\left.
\begin{aligned}
T_Q = 0 \qquad &\text{at } y' = 0 \text{ and } y' = 1 \\[10pt]
T_Q = 0 \qquad &\text{at } z' = 0 \text{ and } z' = \sigma \\[10pt]
T_Q = 0 \qquad &\text{at } x' = 0
\end{aligned}
\right\} \tag{C3}
$$

$$\frac{\partial T_T}{\partial x'} = \frac{\partial^2 T_T}{\partial y'^2} + \frac{\partial^2 T_T}{\partial z'^2} \tag{C4}$$

$$
\left.
\begin{aligned}
T_T = 0 \qquad &\text{at } y' = 0 \text{ and } y' = 1 \\[10pt]
T_T = 0 \qquad &\text{at } z' = 0 \text{ and } z' = \sigma \\[10pt]
T_T = 1 \qquad &\text{at } x' = 0
\end{aligned}
\right\} \tag{C5}
$$

The problems involving the temperatures T_Q and T_T have a definite physical meaning. The former is the case of a heat-generating liquid metal at $t = 0$ that enters a channel whose wall temperatures are also $t = 0$. The latter is the case of a nongener-

ating liquid metal at t_e that enters a channel whose wall temperatures are t_w. The two problems are treated separately, and the results are combined to yield information for the general situation.

Internal Heat Generation with Wall Temperatures $t_w = 0$

In the fully developed region the fluid flows isothermally $(\partial T_{Q,d}/\partial x' = 0)$ and therefore from equation (C2) $T_{Q,d}$ satisfies

$$\frac{\partial^2 T_{Q,d}}{\partial y'^2} + \frac{\partial^2 T_{Q,d}}{\partial z'^2} = -\frac{Qa^2/\kappa}{(t_e - t_w)} \qquad (C6)$$

with the boundary conditions

$$T_{Q,d} = 0 \qquad \text{at } y' = 0 \text{ and } y' = 1 \qquad (C7)$$

$$T_{Q,d} = 0 \qquad \text{at } z' = 0 \text{ and } z' = \sigma \qquad (C8)$$

The boundary value problem posed by equations (C6) to (C8) is analogous to the problem of torsion in rectangular bars (ref. 30). The analogy has been used also to treat the problem of laminar flow in a rectangular conduit (ref. 31). Thus, from the results given in reference 30, the temperature distribution in a heat-generating slug flow in a rectangular channel with wall temperatures $t_w = 0$ is given by

$$T_{Q,d}(y',z') = \frac{4\sigma^2}{\pi^3} \frac{Qa^2}{t_e - t_w} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3}\left[1 - \frac{\cosh\dfrac{n\pi}{\sigma}\left(y' - \dfrac{1}{2}\right)}{\cosh\dfrac{n\pi}{2\sigma}}\right]\sin\frac{n\pi z'}{\sigma} \qquad (C9)$$

The solution as given by equation (C9) has the advantage of avoiding the use of a double Fourier series for the temperature distribution. In principle, an extension of the technique employed in reference 30 to obtain a single series solution of equation (C6) for solving the analogous equation for torsion in rectangular beams makes possible a single Fourier series solution for the entrance region temperature $T_Q^*(x',y',z')$. The analytical

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difficulties associated with satisfying the thermal entrance condition, however, lead to the mathematically intractable problem now presented.

To determine the solution in the thermal entrance region, an entrance temperature $T_Q^*(x', y', z')$ is introduced such that

$$T_Q = T_{Q,d} + T_Q^* \tag{C10}$$

After equation (C10) is introduced into equation (C2) and it is taken into account that $T_{Q,d}$ satisfies equation (C6), the function T_Q^* must satisfy the differential equation

$$\frac{\partial T_Q^*}{\partial x'} = \frac{\partial^2 T_Q^*}{\partial y'^2} + \frac{\partial^2 T_Q^*}{\partial z'^2} \tag{C11}$$

with the boundary conditions

$$\left. \begin{array}{ll} T_Q^* = 0 & \text{at } y' = 0 \text{ and } y' = 1 \\[2mm] T_Q^* = 0 & \text{at } z' = 0 \text{ and } z' = \sigma \end{array} \right\} \tag{C12}$$

At the channel entrance $(x' = 0)$, the condition is

$$T_Q(0, y', z') = 0 = T_{Q,d}(y', z') + T_Q^*(0, y', z')$$

or, by rearranging,

$$T_Q^*(0, y', z') = -\frac{4\sigma^2}{\pi^3} \frac{\dfrac{Qa^2}{\kappa}}{t_e - t_w} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \left[1 - \frac{\cosh \dfrac{n\pi}{\sigma} \left(y' - \dfrac{1}{2}\right)}{\cosh \dfrac{n\pi}{2\sigma}} \right] \sin \frac{n\pi z'}{\sigma} \tag{C13}$$

Equation (C13) suggests taking $T_Q^*(x', y', z')$ in the form of a series:

$$T_Q^* = \frac{\frac{Qa^2}{\kappa}}{t_e - t_w} \sum_{n=1,3,\ldots}^{\infty} A_n \sin\frac{n\pi z'}{\sigma} W_n e^{-\beta_n^2 x'} \tag{C14}$$

where A_1, A_3, . . . are constant coefficients and W_1, W_3, . . . are functions of y' only. Substituting equation (C14) into equation (C11) and taking into account the boundary conditions (C12) yield the general expression for the entrance temperature:

$$T_Q^*(x', y', z') = \frac{\frac{Qa^2}{\kappa}}{t_e - t_w} \sum_{n=1,3,\ldots}^{\infty} A_n \sin n\pi y' \sin\frac{n\pi z'}{\sigma} e^{-\beta_n^2 x'} \tag{C15}$$

where $\beta_n^2 = n^2\pi^2(1 + 1/\sigma^2)$. The coefficients A_n have yet to be determined. Applying the boundary condition at the channel entrance as given by equation (C13) results in

$$\sum_{n=1,3,\ldots}^{\infty} A_n \sin n\pi y' \sin\frac{n\pi z'}{\sigma} = -\frac{4\sigma^2}{\pi^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3}\left[1 - \frac{\cosh\frac{n\pi}{\sigma}\left(y' - \frac{1}{2}\right)}{\cosh\frac{n\pi}{2\sigma}}\right]\sin\left(\frac{n\pi z'}{\sigma}\right) \tag{C16}$$

The solution of equation (C16) for the constant coefficients A_n does not appear possible, and it is concluded that the formulation of the temperature distribution T_Q in terms of a single Fourier series does not appear to offer any significant advantages. The temperature distribution, on the other hand, is susceptible to a straightforward analysis in terms of a double Fourier series. Therefore, the problem is now treated from the standpoint of double Fourier series. The solution of equation (C6) can be expressed as a double sine series:

$$T_{Q,d}(y', z') = \frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi y')\sin n\pi\frac{z'}{\sigma} \tag{C17}$$

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The individual terms and the entire series vanish on the boundaries. Hence, the boundary conditions (eqs. (C7) and (C8)) are satisfied.

The next task is to determine the Fourier expansion coefficients C_{mn} so that equation (C17) will satisfy the governing equation (C6). Substituting equation (C17) into equation (C6), multiplying by a typical product of sines, and finally integrating over the cross section of the channel give

$$C_{mn} = \frac{16}{mn\pi^2 \Gamma_{mn}^2} \tag{C18}$$

where $\Gamma_{mn}^2 = \pi^2 \left[m^2 + (n^2/\sigma^2) \right]$ and where m and n are odd integers. If either m or n or both are even numbers, $C_{mn} = 0$.

The entrance temperature T_Q^* must now be determined, and it must satisfy equations (C11) and (C12). The boundary condition corresponding to the requirement that the fluid enter the channel with a temperature value $t = 0$ is now

$$T_Q^*(0, y', z') = -T_{Q,d}(y', z') = -\frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi y') \sin\left(n\pi \frac{z'}{\sigma}\right) \tag{C19}$$

Equation (C11) and the associated boundary conditions can be solved by the separation-of-variables technique. Let

$$T_Q^* = \varphi_1(x') \varphi_2(y') \varphi_3(z') \tag{C20}$$

where $\varphi_1(x')$, $\varphi_2(y')$, and $\varphi_3(z')$ are functions of x', y', and z', respectively. Substitution of equation (C20) into the differential equation (C11) and consideration of the boundary conditions (eq. (C12)) give the solution for T_Q^* in the form of a double sine series:

$$T_Q^* = \frac{Qa^2}{\kappa(t_e - t_w)} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} D_{rs} \sin(r\pi y') \sin\left(s\pi \frac{z'}{\sigma}\right) e^{-\Gamma_{rs}^2 x'} \tag{C21}$$

where $\Gamma_{rs}^2 = \pi^2 \left[r^2 + (s^2/\sigma^2) \right]$. The coefficients D_{rs} remain to be determined. Applying the boundary condition at the channel entrance as given by equation (C19) yields

$$-\frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin(m\pi y')\sin\left(n\pi \frac{z'}{\sigma}\right)$$

$$= \frac{Qa^2}{\kappa(t_e - t_w)} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} D_{rs} \sin(r\pi y')\sin\left(s\pi \frac{z'}{\sigma}\right) \qquad (C22)$$

The equality of the two doubly infinite series in equation (C22) is interpreted to be equivalent to the results $r = m$, $s = n$, and $D_{rs} = -C_m$. Hence,

$$T_Q^* = -\frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} C_{mn} \sin(m\pi y')\sin\left(n\pi \frac{z'}{\sigma}\right) e^{-\Gamma_{mn}^2 x'} \qquad (C23)$$

Combining the results of the preceding sections in accordance with equation (C10) gives an expression for the temperature distribution that applies in both the entrance and fully developed regions:

$$T_Q(x',y',z') = \frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16}{mn\pi^2 \Gamma_{mn}^2}\left(1 - e^{-\Gamma_{mn}^2 x'}\right)\sin(m\pi y')\sin n\pi \frac{z'}{\sigma}$$

$$(C24)$$

The local bulk temperature $T_{Q,b}$ of the liquid metal is given by

$$\frac{t_{Q,b}}{t_e - t_w} = T_{Q,b} = \frac{\int_0^\sigma \int_0^1 U T_Q \, dy' \, dz'}{\int_0^\sigma \int_0^1 U \, dy' \, dz'} = \frac{1}{\sigma}\int_0^\sigma \int_0^1 T_Q \, dy' \, dz' \qquad (C25)$$

Introducing equation (C24) into equation (C25), integrating and then rearranging yield

$$T_{Q,b} = \frac{Qa^2}{\kappa(t_e - t_w)} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{64}{m^2 n^2 \pi^4 \Gamma_{mn}^2} \left(1 - e^{-\Gamma_{mn}^2 x'}\right) \tag{C26}$$

Wall Temperatures t_w Without Internal Heat Generation

The dimensionless temperature $T_T(x', y', z')$ is the solution to equations (C4) and (C5). (The problem is treated in ref. 12 from the standpoint of a Fourier analysis.) The temperature field is described by the double infinite Fourier series (ref. 12)

$$T_T(x', y', z') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{mn\pi^2} \sin(m\pi y') \sin\left(n\pi \frac{z'}{\sigma}\right) e^{-\Gamma_{mn}^2 x'} \tag{C27}$$

where $m = 1, 3, 5, \ldots$ and $n = 1, 3, 5, \ldots$. The local bulk temperature $T_{T,b}$ may be found as follows:

$$\frac{t_{T,b} - t_w}{t_e - t_w} = T_{T,b} = \frac{1}{\sigma} \int_0^{\sigma} \int_0^1 T_T \, dy' \, dz' \tag{C28}$$

Substituting the temperature distribution from equation (C27) into equation (28) and integrating result in

$$T_{T,b} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{64}{m^2 n^2 \pi^4} e^{-\Gamma_{mn}^2 x'} \tag{C29}$$

Combining the solutions for T_Q and T_T yields the general result

$$\frac{t - t_w}{t_e - t_w} = T = \frac{1}{N} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{mn\pi^2 \Gamma_{mn}^2} \left(1 - e^{-\Gamma_{mn}^2 x'}\right) \sin(m\pi y') \sin\left(n\pi \frac{z'}{\sigma}\right)$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16}{mn\pi^2} \sin(m\pi y') \sin\left(n\pi \frac{z'}{\sigma}\right) e^{-\Gamma_{mn}^2 x'} \qquad \text{(C30)}$$

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