CONCERNING METRIZATION AND SEPARATION IN NORMAL, SEPARABLE MOORE SPACES

D. Reginald Traylor

Recently, [3] E. E. Grace and R. W. Heath raised a question which is stated below as Conjecture A.

Conjecture A: Suppose that S is a connected, normal Moore space such that S contains no cut points and it is true that if each of P and Q is a point of S and R is a region containing P then some separable, closed, connected subset N of R separates P from Q in S. Then S is separable.

The purpose of this note is to answer Conjecture A in the negative, provided there exists a normal, separable, nonmetrizable Moore space. It follows that, should Conjecture A be found true, it thus would remove the condition of the continuum hypothesis from Jones' result ([7], Theorem 5), that each normal, separable Moore space is metrizable, provided $2^{\aleph_0} < 2^{\aleph_1}$.

For definitions and results related to the question of metrization of normal Moore spaces, refer to ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10]).

The following lemmas prove helpful in describing the construction of a space which denies Conjecture A. There is much reliance on the methods which were employed in ([2], Theorem 1), ([9], Theorem 3 and Theorem 7), and ([10], Theorem 4). No proof of Lemma 1 is included here, as it only states formally a property of E^3.

Lemma 1. There exist, in E^3, a countably infinite discrete point set K

[redacted]

and a collection G of mutually exclusive arcs such that

i) if each of x and y is a point of K some arc in G has x as one end point and y as the other,

ii) each arc in G has its end points in K, and

iii) if g is an arc in G, then g contains no limit point of G^*-g.

Lemma 2. If there exists a normal, separable, nonmetrizable Moore space (S,Ω) then there exists one, say (S',Ω'), such that S' is a subset of E^3 and (S',Ω') is locally compact.

Proof. Denote by (S,Ω) a normal, separable, nonmetrizable Moore space. There exists [7, Lemma C] an uncountable subset N of S with no limit point and a countable dense subset L of $S - N$. If $S^o = L + N$, let (S^o,Ω^o) denote the subspace of (S,Ω) induced by the relative topology.

If x is a point of N, denote by $P_{x,1}, P_{x,2}, \ldots$ a sequence of points of L which converges, in the Ω^o sense, sequentially to x. In [2, Theorem 2] it is established that there exists a space (S_1,Ω_1) with the following properties:

i) $S_1 = S^o$,

ii) Ω_1 is the topology induced by the following definition of region: The point set R is a region if and only if either

(a) for some point P of L, R is the degenerate set whose only point is P, or

(b) for some point x of N and some integer K, R is the set to which p belongs if and only if $P = x$ or $P = P_{x,j}$ for some $j \geq k$, and

iii) (S_1,Ω_1) is normal, separable, locally compact, nonmetrizable, and no region has boundary.

If G_n^1 denotes the collection to which the region R belongs if and only if R is a degenerate region, or, for some point x of N and some positive integer $i \geq n$, $R = x + \sum_{j=i}^{\infty} P_{x,j}$ then $\{G_n^1\}_{n=1}^{\infty}$ gives a development for (S_1,Ω_1).

Denote by K the subset of E^3 and by G the collection of arcs described in Lemma 1. There exists a reversible transformation T from K onto L. Let G' denote the subcollection of G to which the arc [a,b] belongs if and only if there exist a point x of N, points y and z of K, and a positive integer i such that $T(y) = P_{x,i}$, $T(z) = P_{x,i+1}$, and a = y, b = z, or a = z, b = y. Denote by M an uncountable subset of E^3 such that $\bar{\bar{M}} = \bar{\bar{N}}$, and M is a subset of $E^3 - (K + \overline{G'^*})$. It is no restriction to assume that T has been extended such that T is a reversible transformation from M + K to N + L with T(M) = N and T(K) = L.

Let S' = M + K and consider the space (S',Ω') where Ω' is the topology induced by the following definition of region: The statement that the point set R is a region of G_n' means that there exists a region g of G_n^1 such that T(g) = R. Clearly, (S',Ω') is topologically equivalent to (S_1,Ω_1) and thus satisfies the lemma.

Now let $S_2 = S' + G'^*$ and consider the space (S_2,Ω_2) where Ω_2 is the topology induced by the following definition of region: The statement that

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the point set R is a region of G_n^2 means that either

i) there exists a region g of G_n' such that P belongs to R if and only if either

(a) P is a point of g, or

(b) there exists an arc [a,b] of G which has both end points in g and P is a point of [a,b], or

(c) there exists an arc [a,b] of G such that a is in g, b is not in g and P is some point of that component of [a,b] which contains a and (in E^3) each of whose points is less than 1/n from a, or

(d) there exists an arc [a,b] of G such that g contains b but not a and P is some point of that component of [a,b] which contains b and (in E^3) each of whose points is less than 1/n from b, or

ii) there exists an arc [a,b] of G which contains a subsegment g whose length (in E^3) is less than 1/n and R = g.

It follows, as in [9, Theorem 3], that (S_2,Ω_2) with the development $\{G_n^2\}_{n=1}^{\infty}$ is a normal, separable, arcwise connected, locally connected, nonmetrizable space. The following lemma is thus established.

Lemma 3. If there exists a normal, separable, nonmetrizable Moore space then there exists one, say (S_2,Ω_2), such that S_2 is a subset of E^3 and (S_2,Ω_2) is normal, separable, arcwise connected, locally connected and nonmetrizable.

Lemma 4. If there exists a normal, separable, nonmetrizable Moore space

(S,Ω) and N is a discrete uncountable subset of S then there exists a normal, separable, arcwise connected, locally connected, nonmetrizable Moore space (S_2,Ω_2) which is embedded in a normal, arcwise connected, locally connected, nonmetrizable Moore space (S_3,Ω_3) which contains a collection H of mutually exclusive domains such that $\bar{\bar{H}} = \bar{\bar{N}}$.

Proof. Consider (S_2,Ω_2) of Lemma 3. There exists a subset M of S_2 which is discrete and uncountable. Denote by Q a point of E^3 and by H a collection of mutually exclusive horizontal line segments of E^3 such that $\overline{(H^* + Q)}$ does not intersect S_2 in E^3 and $\bar{\bar{H}} = \bar{\bar{M}}$. There exists a reversible transformation T from H onto M.

Let $S_3 = S_2 + H^* + Q$ and consider the space (S_3,Ω_3) where Ω_3 is the topology induced by the following definition of region: The statement that the point set R is a region of G_n^3 means that either

i) there is a region g of G_n^2 such that g does not intersect M and $R = g$, or

ii) there is a region g of G_n^2 which contains a point x of M such that the point P belongs to R if and only if P is a point of g or, if (a,b) is the element of H such that $T[(a,b)] = x$, then P is a point of (a,b) less than $1/n$ (in E^3) from a, or

iii) there exists a segment (a,b) of H and a subsegment (c,d) of (a,b) such that the length of (c,d), in E^3, is less than $1/n$ and $R = (c,d)$, or

iv) R is the set to which P belongs if and only if $P = Q$ or there exists a segment (a,b) of H such that P is a point of (a,b) which is less than $1/n$ from b (in E^3).

Clearly, (S_3,Ω_3), with the development $\{G_n^3\}_{n=1}^{\infty}$, satisfies the lemma.

Lemma 5. If there exists a Moore space (S,Ω) satisfying the hypothesis of Lemma 4 then there exists a Moore space (S_3,Ω_3) satisfying the conclusion of Lemma 4 and, in addition, is embedded in a normal, connected, locally connected, arcwise connected Moore space (S_4,Ω_4) such that if each of P and Q is a point of S_3 and R is a region in (S_4,Ω_4) then there is a closed, connected, separable subset N of R which separates P from Q in (S_4,Ω_4).

Proof. Consider the space (S_3,Ω_3) of Lemma 4. If W is a set such that $\bar{\bar{W}} = \bar{\bar{S}}_3$ and W does not intersect S_3 and for each positive integer n, C_n denotes a circle with radius $1/n$ such that no C_n intersects S_3 or W, then for each element w of W, let $C_{w,n} = w \times C_n$. There is a reversible transformation T from W onto S_3. If $T(w) = P$, then with each point P of S_3 there is associated an infinite sequence of circles $C_{w,1}, C_{w,2}, \cdots$. For each i and each point P of S_3, let $C_{w,i} = C_i^P$.

Remark: In the space (S_2,Ω_2) each point of K is an end point of some arc of G'. The set K is embedded in (S_3,Ω_3). Suppose that each of x and y is a point of K and [x,y] is that arc of G' having end points x and y. There exist, in [x,y],,two subsets: $A = \sum A_{x,y,i}$ and $B = \sum B_{x,y,i}$ where $A_{x,y,1}, A_{x,y,2}, \cdots$ converges sequentially and monotonically to x and $B_{x,y,1}, B_{x,y,2}, \cdots$ converges sequentially and monotonically to y. If C_i^x is a circle, associated under T with x, and K_x is that subset of K consist of those points each of which is an end point of an arc having the other end point x, there is a homeomorphic image of C_i^x, in E^3, which contains $A_{x,y,i}$ in its boundary, for each y in K_x. For simplicity and notational purposes, it is assumed here that C_i^x has that property itself. Thus, in the following treatment, if x is in S_2, each C_i^x contains points of S_3 as described above.

Let Ω_4 denote the topology induced by the following definition of region: The statement that the point set R is a region of G_n^4 means that either

i) there is a point P of S_3 and a positive integer i such that $i \geq n$ and P belongs to a connected open (in the subspace C_i^P of E^3) subset of $(C_i^P - S_3 \cdot C_i^P)$ which has length (in E^3) less than $1/i$, or

ii) there exist points x and y of K, an arc [x,y] of G having x and y as end points, a positive integer i and a point $A_{x,y,i}$ such that P belongs to R if and only if either

(a) $P = A_{x,y,i}$, or

(b) P is a point of an open connected subset of [x,y] which contains $A_{x,y,i}$ and is of length less than $1/n$, or

(c) P is a point of an open connected subset of C_i^x which contains $A_{x,y,i}$ and is of length less than $1/n$, or

(d) there exists a point $A_{x,y,j}$ or Bx,y,j which belongs to the open connecetd set satisfying (b) such that P is a point of an open connected subset of some C_j^y which contains $A_{x,y,j}$ or Bx,y,j and is of length less than $1/n$, or

(e) replace $A_{x,y,i}$ by $B_{x,y,i}$ in ii), or

iii) there exists a region g of G_n^3 such that P belongs to R if and only if either

(a) P is a point of g, or

(b) there exist a point x of g and a positive integer $1 \geq n$ such that P is a point of C_i^x, or

(c) there is a point x of S_3 such that for some j, C_j^x intersects g at only one point, say y, and P is a point of an open connected subset of C_j^x which contains y and has length less than 1/n.

It follows that (S_4,Ω_4) is a Moore space with development $\{G_n^4\}_{n=1}^{\infty}$. That it has the properties described in the lemma follows as in [2, Theorem 1] and from the property that if P is a point of S_3 and R is a region of (S_4,Ω_4) then there exists a closed, connected, separable subset N of R (in particular, some C_i^p) such that $S_4 - N = H + U$ where H and U are mutually separated, H is a subset of R and $S_4 - R$ is a subset of U.

Lemma 6. Suppose that (S_4,Ω_4) is a Moore space satisfying Lemma 5. Then for each positive integer $n \geq 4$, there exists a normal, arcwise connected, locally connected, nonmetrizable Moore space (S_{n+1},Ω_{n+1}) such that (S_n,Ω_n) is embedded in (S_{n+1},Ω_{n+1}), no point of S_{n+1} is a limit point of S_n in (S_{n+1},Ω_{n+1}), and it is true that if each of P and x is a point of S_n and R is a region in (S_{n+1},Ω_{n+1}) containing P then there exists a closed, connected, separable subset N of R which separates P from x in (S_{n+1},Ω_{n+1}).

Proof. The construction only need by indicated. Consider (S_4,Ω_4) of Lemma

5. Each point of $S_4 - S_3$ is a point of some C_j^x for some point x of S_3 and some positive integer j. Indeed, no point of $S_4 - S_3$ is a limit point of any subset of S_3 in (S_4, Ω_4). Using the constructive device of Lemma 5. there may be associated with each point P of $S_4 - S_3$ a sequence $C_1^P, C_2^P, \ldots$ of homeomorphic images of circles such that C_i^P intersects a connected subset of $C_j^x \cdot (S_4 - S_3)$ in two and only two points.

Definition of (S_5, Ω_5): The statement that P is a point of S_5 means that P is a point of S_4 or P is a point of some C_i^y for some point y in $S_4 - S_3$ and some positive integer i. The statement that the point set R is a region in G_n^5 means that there exists a region g in G_n^4 such that the point z belongs to R if and only if either

(a) there exist a point x of $S_4 - S_3$ and a positive integer j and a connected subset C of $C_j^x - C_j^x \cdot S_4$ which has length less than $1/n$ and z is a point of C, or

(b) i) z is a point of g, or

ii) there exists a point x of $(S_4 - S_3) \cdot g$ and a positive integer $i > n$ such that z is a point of C_i^x, or

iii) there exists a point x of $S_4 - S_3$ which is not in g but such that, for some positive integer j, C_j^x intersects g (this intersection consists of only one point) and z is a point of a connected subset of C_j^x which contains $C_j^x \cdot g$ and has length less than $1/n$.

Using an argument similar to that of the preceding lemma, it follows that (S_5, Ω_5) meets the conditions of the lemma.

Indeed, it is readily seen that (S_5,Ω_5) may be embedded in a space (S_6,Ω_6) in a similar fashion, meeting the conditions of the lemma. The lemma follows from a formal induction which only repeats the above described construction.

Theorem. If Conjecture A is true then each normal, separable Moore space is metrizable.

Proof. Assume there exists a normal, separable, nonmetrizable Moore space and consider the sequence (S_1,Ω_1), (S_2,Ω_2),... given by the preceding lemmas. Let $S = \sum_{i=1}^{\infty} S_i$ and consider the space (S,Ω) where Ω is the topology induced by the following definition of region: The statement that the point set R of G_n is a region means there exist a positive integer k and a sequence $R_k, R_{k+1}, R_{k+2},\ldots$ such that:

i) for each i, R_{k+1} is a region of G_n^{k+i} in (S_{k+1},Ω_{k+1}),

ii) $R_{k+i+1}\cdot S_{k+i} = R_{k+i}$ for each i,

iii) R_{k+i} does not intersect S_{k+i-1}, and

iv) $\sum_{i=k}^{\infty} R_i = R$.

Using an argument quite similar to that employed in [2, Theorem 1] or [10, Theorem 4], it follows that (S,Ω) is a normal, nonmetrizable, connected, arcwise connected Moore space. That (S,Ω) is not separable follows from the construction of (S_3,Ω_3). Indeed, each (S_n,Ω_n) contains uncountably many mutually exclusive domains if $n \geq 3$. The construction of the space (S,Ω) was such that if each of P and x is a point and R is a region containing P then there exists a closed, separable, connected set (a topological copy of some circle in the construction) which separates P from x. This would deny the conjecture and the theorem is proved.

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University of Houston

Houston, Texas

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