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BINARY DIFFUSION IN AN EXPONENTIAL MEDIUM

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Greenbelt, Md.

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ABSTRACT

The equation of diffusion of a minor, light gaseous component through an exponential medium is examined in this detailed report. The equation's one-dimensional version may be expressed in dimensionless units in the form

$$\frac{\partial n}{\partial t} = e^{z} \left[\frac{\partial^2 n}{\partial z^2} + (1+\mu) \frac{\partial n}{\partial z} + \mu n \right] ,$$

with μ the mass ratio of the component particles. With a large variety of initial and boundary conditions, the solutions can be expressed in the general form

$$n(z, t) = ae^{-z} + be^{-\mu z} + e^{-1+\mu/2\, z} \left[\sum_{\ell=1}^{\infty} C_\ell (z)\, e^{-\gamma_{\nu,\ell}^2 t/4} + \sum_{-\infty}^{\infty} d_k (z)\, e^{ik\omega t} \right] .$$

$\gamma_{\nu,\ell}$ is the ℓth zero of the Bessel function $J_\nu (x)$ ($\nu = 1-\mu$), and ω is the fundamental frequency of periodic boundary conditions, if present.

For $t \approx 4\gamma_{\nu,1}^{-2} \approx 1$, $n(z, t)$ tends, therefore, to approach a steady state regime, which may be of periodic character. The former time is also the characteristic time scale for transport of the minor component from a source at $z = 0$ to high altitudes. For both hydrogen and helium in the terrestrial atmosphere, this is in the order of one day.

The simple equation is, of course, a most idealized description of diffusion in the atmosphere. Nonetheless, the coincidence of the time scales for vertical transport to the escape region and the periodic variation found there indicate that a realistic evaluation of the density profile of hydrogen or helium should at least, take into account the diurnal variation of conditions in the ambient atmosphere. It casts some doubt on models of diurnal variation constructed as a superposition of aperiodic stationary density profiles.

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INTRODUCTION

Binary diffusion of different gaseous species is common in many physical processes and is a typical manifestation of the tendency toward equalization of gross material properties of heterogeneous gaseous mixtures. In this particular case, the properties in question are the relative concentrations of the individual components, their partial pressures, the total pressure of the mixture, its temperature, etc.

Consequently, the gases are set in motion relative to each other by a great variety of generalized forces. Under laboratory conditions of moderate scale with respect to external forces, such as gravity, the composition of gaseous mixtures will soon tend toward homogeneity (except near the boundaries of containers, where a variety of processes may take place). Diffusion is well described by Fick's second law of diffusion,

$$\frac{\partial n_{10}}{\partial n} = \nabla \cdot \left(D_1 \nabla n_{10} \right) . \tag{1}$$

Here $n_{10} = n_1/N$, the ratio of the number density (per cm^3) of particles of species 1 to the total number density $N = \left(n_1 + n_2\right)/cm^3$ for the case of a binary mixture. $D_1 \left(cm^2\ t^{-1}\right)$ is the diffusion coefficient of species 1. In view of the approximate homogeneity soon attained under ordinary conditions, Equation 1 simplifies to

$$\frac{\partial n_{10}}{\partial t} = D \nabla^2 n . \tag{2}$$

This equation, of the same form as the standard equation of heat conduction, is extensively covered in References 1 and 2, and admits a great variety of solutions, depending on geometry, boundary conditions, and initial conditions.

Under special conditions in the laboratory and on the grand scale of planetary atmospheres and space, Equations 1 and 2 are no longer satisfactory descriptions of the diffusion process. A far more complicated equation, taking into account all of the previously mentioned factors, begins with the expression for the velocity of species 1 relative to the other component (Reference 3),

$$\underline{w}_1 - \underline{w}_2 = -\frac{N^2}{n_1 n_2} D_{12} \left[\nabla n_{10} + \frac{n_1 n_2 (m_2 - m_1)}{N\rho} \log p - \frac{\rho_1 \rho_2}{p\rho} (\underline{F}_1 - \underline{F}_2) + k_T \nabla \log T\right] . \tag{3}$$

Here $\underline{w}_1$ and $\underline{w}_2$ (cm/sec) are the velocities of species 1 and 2, m_1 and m_2 (gm) are their respective masses/particle, p (gm/cm t^2) is the total specific pressure, $\rho_1 = n_1 m_1$, $\rho_2 = n_2 m_2$, $\rho = \rho_1 + \rho_2$, and $\underline{F}_1$ and $\underline{F}_2$ (cm/sec^2) are the accelerations acting on the particles of the two species. $K_T = D_T/D_{12}$ is the thermal diffusion factor, and T(°K) is the temperature of the gas.

The time dependent diffusion equation may then be obtained by taking the divergence of the flux

$$\frac{\partial n_1}{\partial t} = -\nabla \cdot \left[n_1 (\underline{w}_1 - \underline{w}_2)\right] . \tag{4}$$

This equation is not completely rigorous; a more rigorous description is given by the "telegrapher's equation," including a second derivative with respect to time (Reference 4). For most practical purposes, however, the solutions of the two equations are indistinguishable.

Since the scope of interest lies in the binary diffusion of neutral particles in the atmosphere, and gravitational acceleration is independent of mass, the acceleration term in Equation 3 may be dropped. The following concerns the diffusion of a minor constituent through an effectively stationary major constituent, such that $\underline{w}_2 \approx 0$, $n_2 \approx N$. The rigid sphere approximation for the diffusion coefficient is

$$D_{12} = \frac{3}{8N\sigma}\left[\frac{\pi k T}{2m_1}(1+\mu)\right]^{1/2} = \alpha_1 \frac{T^{1/2}}{N} , \tag{5}$$

where k is Boltzmann's constant, σ is the initial collision cross section, and $\mu = m_1/m_2$.

To simplify matters further, the one-dimensional diffusion equation will be exclusively considered. In view of the above considerations, using Equation 3, and the ideal gas law, (and dropping the subscript 1 for n_1), Equation 4 may be written

$$\frac{\partial n}{\partial T} = -\frac{\partial}{\partial Z}\left\{-D(z,t)\left[\frac{\partial n}{\partial Z} - \left(\frac{\mu}{N}\frac{\partial N}{\partial z} - \frac{(1-\mu+\alpha_T)}{T(z,t)}\frac{\partial T(z,t)}{\partial z}\right)n\right]\right\} , \tag{6}$$

where we assumed that μ and $\alpha_T = K_T/n_{10}n_{20}$ are approximately constant. Denoting $\alpha_2 = -\mu\alpha_1$, and $\alpha_3 = (1-\mu+\alpha_T)\alpha_1$, Equation 6 may be obtained in explicit form as

$$\frac{\partial n}{\partial t} = \alpha_1 \frac{T^{1/2}}{N}\frac{\partial^2 n}{\partial z^2} + \left(\frac{(\alpha_2-\alpha_1)T^{1/2}}{N^2}\frac{\partial N}{\partial z} + \frac{(\alpha_1+2\alpha_3)}{2NT^{1/2}}\frac{\partial T}{\partial z}\right)\frac{\partial n}{\partial z} + \left[\frac{(\alpha_2-2\alpha_3)}{2N^2T^{1/2}}\left(\frac{\partial N}{\partial z}\right)\left(\frac{\partial T}{\partial z}\right)\right.$$

$$\left. - \frac{2\alpha_2 T^{1/2}}{N^3}\left(\frac{\partial N}{\partial z}\right)^2 - \frac{\alpha_3}{2NT^{3/2}}\left(\frac{\partial T}{\partial z}\right)^2 + \frac{\alpha_2 T^{1/2}}{N^2}\left(\frac{\partial^2 N}{\partial z^2}\right) + \frac{\alpha_3}{NT^{1/2}}\frac{\partial^2 T}{\partial z^2}\right] n . \tag{7}$$

Changing to $X = 1/N$ offers a slight simplification

$$\frac{\partial n}{\partial t} = \alpha_1 T^{1/2} X \frac{\partial^2 n}{\partial z^2} + \left[\left(\alpha_1 - \alpha_2\right) T^{1/2} \frac{\partial X}{\partial z} + \frac{\left(\alpha_1 + 2\alpha_3\right)}{2} T^{-1/2} X \frac{\partial T}{\partial z}\right] \frac{\partial n}{\partial z} + \left[\frac{\left(2\alpha_3 - \alpha_2\right)}{2} T^{-1/2} \frac{\partial T}{\partial z} \frac{\partial X}{\partial z}\right.$$

$$\left. - \alpha_2 T^{1/2} \frac{\partial^2 X}{\partial z^2} - \frac{\alpha_3}{2} T^{-3/2} X \left(\frac{\partial T}{\partial z}\right)^2 + \alpha_3 T^{-1/2} X \frac{\partial^2 T}{\partial z^2}\right] n . \tag{8}$$

In general, both T and X (and in the worst cases also $\alpha_1, \alpha_2, \alpha_3$) are functions of both z and t; other than numerical solutions of Equation 8 cannot be expected. Though numerical solutions are useful and necessary in many situations, they do not provide the desired general insight into the behavior of the process investigated. It may, therefore, be profitable to study the properties of an analytic solution of a simplified equation. Fortunately the density profile of the main atmosphere is roughly exponential, and the temperature, which appears in Equations 7 and 8 in essentially logarithmic terms, does not vary over orders of magnitudes. Under the assumption that

$$T = T_0, \text{ a constant} , \tag{9a}$$

$$X = X_0 \exp(z/H) , \tag{9b}$$

and where

$$H = bT_0 , \tag{10}$$

Equation 8 simplifies to

$$\frac{\partial n}{\partial t} = T_0^{1/2} X_0 e^{z/H} \left(\alpha_1 \frac{\partial^2 n}{\partial z^2} + \frac{\left(\alpha_1 - \alpha_2\right)}{bT_0} \frac{\partial n}{\partial z} - \frac{\alpha_2 n}{b^2 T^2}\right) . \tag{11}$$

It is convenient to transform Equation 11 into dimensionless form by changing to dimensionless variables

$$z = \frac{z}{bT_0} \tag{12}$$

and

$$t^1 = \left(\frac{X_0 \alpha_1}{b^2 T_0^{3/2}}\right) t ; \tag{13}$$

Equation 11 becomes

$$\frac{\partial n}{\partial t} = e^z \left[\frac{\partial^2 n}{\partial z^2} + (1+\mu)\frac{\partial n}{\partial z} + \mu n\right], \tag{14}$$

where the superscripts on z and t are omitted. It is this equation, and its solution in the region from $z = 0$ to $z = \infty$, which is discussed in the remaining sections.

PREVIOUS SOLUTIONS OF THE DIFFUSION EQUATION WITH EXPONENTIAL DIFFUSION COEFFICIENT

Equation 14 has been treated previously by a number of authors (References 5, 6, and 7), during investigation of the descent of a pulse of small particles released instantaneously at some altitude z_0 in a stationary exponential atmosphere. The solution given in Reference 7 is very useful for the study of this special problem. But it, and results directly deducible from it, can hardly be applied to a realistic description of binary diffusion processes continuing in the upper atmosphere. Such processes are, for instance, the diffusion of hydrogen or helium through an ambient medium, composed predominantly of atomic oxygen and molecular nitrogen.

In changing $y = e^{-z/2}$ to a variable, the region $(0, \infty)$ in z is transformed to a region (0, 1) in y. To avoid complicated boundary condition at $y = 1$, the authors in Reference 7 now extend the region of y to ∞ (which is equivalent to extending z to $-\infty$), and impose homogeneous boundary conditions on n for $y = 0$ and $y \to \infty$. The initial condition corresponds to an instantaneous particle source at time $t = 0$ and $z = z_0$, i.e., $n(y, 0) \propto \delta(y - y_0)$.

The resulting solution is

$$n(y, t) = \frac{n_0}{Ht} y_0^{1-\mu} y^{1+\mu} \exp\left(\frac{-y^2 + y_0}{t}\right) I_{|\mu-1|}\left(\frac{2yy_0}{t}\right), \tag{15}$$

where $I_{|\mu-1|}$ is the modified Bessel function of order $|\mu - 1|$. Equation 15 predicts the development of a constant density profile and its descent with a velocity which is approximately proportional to e^z. Now, this is a reliable description of the particular transient process studied, but fails to represent the structure of a component which is supplied by a permanent (though not necessarily constant) source located at z_0. Such a source, for instance, supplies neutral atomic hydrogen in the vicinity of the turbopause. This becomes most evident for a light component.

By Equation 3 the net flux of the minor component becomes under the assumptions underlying Equation 15, i.e., Equations 9a and 9b

$$F = -e^z\left(\frac{\partial n}{\partial z} + \mu n\right)$$

$$= \frac{1}{2y}\frac{\partial n}{\partial y} - \frac{\mu}{y^2} n . \tag{16}$$

Choosing for the sake of simplicity the location of the source y_0 at $y_0 = 1$ (which is permissible since the region of y to $y \to \infty$ has been extended) and writing $s = 2yy_0/t$, the instantaneous net flux at $y = 1$ is obtained as

$$F(1, s) = \frac{s}{2} \exp(-s) I_\beta(s) \left(1 - \frac{s}{2} - \mu + \frac{s}{2} \frac{I_{\beta+1}(s)}{I_\beta(s)}\right), \tag{17}$$

where $\beta = (\mu - 1)$. The next step is to investigate the asymptotic behavior of F for $s \to \infty$ $(t \to 0)$, and $s \to 0$ $(t \to \infty)$. Since the factor multiplying the expression in the fourth set of parentheses in Equation 17 is non-negative for all s, it is sufficient to study the asymptotic behavior of the expression within the parentheses.

For $t \to 0$, or $s \to \infty$

$$I_\beta(s) \approx \frac{e^s}{(2\pi s)^{1/2}} \left[1 - \frac{4\beta^2 - 1}{1!8s} + 0\left(\frac{1}{s^2}\right)\right] \tag{18}$$

so that

$$\frac{s}{2} \frac{I_{\beta+1}(s)}{I_\beta(s)} \to \frac{s}{2} - \frac{3}{4} + \frac{\mu}{2} \tag{19}$$

and the expression within the fourth set of parentheses (Equation 17) tends to

$$\left(1 - \mu + \frac{\mu}{2} - \frac{s}{2} + \frac{s}{2} - \frac{3}{4}\right) = \frac{1}{4} - \frac{\mu}{2}. \tag{20}$$

Thus for small t the flux will be positive, given $\mu < 1/2$. For $t \to \infty$, $s \to 0$

$$\frac{I_{\beta+1}(s)}{I_\beta} \to \frac{s}{2(\beta + 1)} \tag{21}$$

and the expression becomes

$$\left[1 - \mu + 0(s)\right] \to 1 - \mu. \tag{22}$$

For large t the net flux will, therefore, be positive whenever $\mu < 1$, a well known result of kinetic theory. The somewhat paradoxical situation is reached that the particle density profile (i.e., the bulk of the particles released) moves in one direction, while the net flux is toward the opposite direction. Physically this means that the overwhelming majority of particles descends with relatively small velocities, while a fast minority soars to high altitudes.

But Equation 15 is not suitable for application to the more continuous natural diffusion processes for another reason. Since the assumed sources of particles operate over long times, it is presumed that some quasi-stationary state is reached. Therefore, a set of boundary conditions and initial conditions which provides a solution converging for large t to the solution of the stationary equation is a desirable choice (Equation 4). Moreover, the conditions should not display any singularities, except possibly at the location of the source.

In order to use the Green's function, Equation 15, in obtaining the particle distribution emanating from a continuous source, it is necessary to substitute $t - t_0$ for t, multiply $n(y,\ t - t_0)$ by the specific source yield, $s(y_0,\ t_0)$, and perform the integration

$$n(y,\ t) = \int_0^t s(t_0)\, n\, (y, t - t_0)\, dt_0 \quad . \tag{23}$$

In view of the complex and singular character of $n(y,\ t - t_0)$, the integral of its product with $s(t_0)$ may diverge for some values of the parameter μ.

This divergence, resulting from the superposition of "instantaneous" plane sources, has an analog in the theory of heat conduction.

The superposition of constant point-sources $\phi(t_0) = q$ yields a temperature

$$T(r,\ t) = \frac{q}{4\pi\kappa r}\, \mathrm{erfc}\left(\frac{r}{\sqrt{4\kappa t}}\right) ; \tag{24}$$

reducing for $t \to \infty$ to

$$T(r) = \frac{q}{4\pi\kappa r} , \tag{25}$$

where κ is the thermal conductivity and $\mathrm{erfc}\,(x)$, is the complimentary error function

$$\mathrm{erfc}\,(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \quad .$$

$T(r)$ as given by Equation 25 is well behaved except for the singularity at the origin.

The corresponding solution for a constant plane source in the plane $z = 0$ yields

$$T(z,\ t) = q\left(\frac{t}{\pi\kappa}\right)^{1/2} e^{-z^2/4\kappa t}\ \frac{q|z|}{2\kappa}\ \mathrm{erfc}\left(\frac{|z|}{2\sqrt{\kappa t}}\right) , \tag{26}$$

which obviously diverges for large t for all finite z. The difference in behavior between Equations 25 and 26 is described to the difference between a singular source of infinitesimal extent and one of infinite extent; this explanation appears to carry over to the case of particle diffusion. At any rate, a solution of the type of Equation 26 is not useful since it evidently does not describe the physical situation in a realistic manner.

WELL-BEHAVED SOLUTIONS TO THE BINARY DIFFUSION EQUATION

For this purpose well-behaved solutions of the diffusion equation solutions are defined as non-singular over all $t > 0$ (except possibly at $t = 0$), and over all $z > 0$ (or equivalently $0 < y < 1$), except possibly at $z = 0$. Moreover, a solution is desirable which for $t \to \infty$ converge to the solution of the time independent equation

$$\frac{\partial n}{\partial t} = 0 \tag{27}$$

or to a solution of an equation with periodic boundary conditions such as

$$n(0, t) = \sum_{-\infty}^{\infty} C_k e^{ik\omega t} . \tag{28}$$

The reason for this choice is an interest in continuous atmospheric diffusion phenomena which are likely to display a stationary or periodic behavior.

Let us then first examine the solution of Equation 27 and Equation 14 under the simple boundary condition (Equation 28) for large t. In the first case we have to solve the equation of continuity

$$e^z \left(\frac{\partial n}{\partial z} + \mu n \right) = - B . \tag{29}$$

Here B is the (constant) flux. Its solution is

$$n = (1 - \mu)^{-1} \left\{ \left[n_0 (1 - \mu) - B \right] e^{-\mu z} + B e^{-z} \right\} \tag{30}$$

where $n_0 = n(0)$. In particular, if $B = n_0 (1 - \mu)$ then

$$n = n_0 e^{-z} . \tag{31}$$

Solution of Equation 14 with the boundary conditions Equation 28 is very similar to the simplified treatment of the earth's temperature (Reference 9). First we change to a variable $y = e^{-z/2}$, which transforms Equation 14 to

$$y^2 \frac{\partial^2 n}{\partial y^2} - (2\mu + 1) y \frac{\partial n}{\partial y} + 4\mu n = y^2 \frac{\partial n}{\partial t} , \tag{32}$$

while the boundary condition becomes

$$n(1, t) = \sum_{-\infty}^{\infty} C_k \, e^{ik\omega t} \quad . \tag{33}$$

Assuming that for $t \to \infty$ the periodicity of the boundary conditions persists at all altitudes, we look for solutions of the form

$$n(y, t) = \sum_{n=-\infty}^{\infty} C_k \, u_k \, (y) \, e^{ik\omega t} \quad . \tag{34}$$

We thus obtain for the kth term $u_k \, (y)$ the equation

$$y^2 \frac{d^2 u_k}{dy^2} - (2\mu + 1) \, y \frac{du_k}{dy} + \left(4\mu - in\omega y^2\right) u_k = 0 \tag{35}$$

with the boundary condition

$$u_k \, (1) = 1 \quad . \tag{36}$$

The general solution of Equation 35 is

$$u_k \, (y) = y^{1+\mu} \left[a_k \, J_{|1-\mu|} \left(2i \sqrt{iK\omega y}\right) + a_k^1 \, J_{-|1-\mu|} \left(2i \sqrt{iK\omega y}\right)\right] \cdot \tag{37}$$

For a well-behaved solution $J_{-|1-\mu|}$ has to be discarded since it diverges at $y = 0$. Writing $\ell = |k|$ and $\nu = 1 - \mu$ one obtains therefore

$$u_{\pm\ell} = a_{\pm\ell} \, y^{1+\mu} \, J_\nu \left(2i \sqrt{\pm i\ell\omega y}\right) \tag{38}$$

or, in terms of the Kelvin functions, $\mathrm{ber} \, (z)$ and $\mathrm{bei} \, (z)$

$$u_{+\ell} \, (y) = a_{+\ell} \, y^{1+\mu} \left[\mathrm{ber}_\nu \left(2 \sqrt{\ell\omega} \; y\right) + i \, \mathrm{bei}_\nu \left(2 \sqrt{\ell \omega} \; y\right)\right] \tag{39a}$$

$$u_{-\ell} \, (y) = a_{-\ell} \, y^{1+\mu} \, e^{i\pi\nu} \left[\mathrm{ber}_\nu \left(2\sqrt{\ell\omega} \; y\right) - i \, \mathrm{bei} \left(2 \sqrt{\ell\omega} \; y\right)\right] \cdot \tag{39b}$$

By virtue of the boundary conditions (Equation 36)

$$u_{+\ell}(y) + u_{-\ell}(y) = 2y^{1+\mu} p_\nu^{-1}\left(2\sqrt{\ell\omega}\right)\left[\mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\right)\mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\, y\right) + \mathrm{bei}_\nu(2\ell\omega)\,\mathrm{bei}_\nu\left(2\sqrt{\ell\omega}\, y\right)\right]. \quad (40)$$

Here $p_\nu(z) = \mathrm{ber}_\nu^2(z) + \mathrm{bei}_\nu^2(z)$. Writing $C_{\pm\ell} = 1/2\,(A \mp iB)$, the steady state periodic solution of Equation 14 is obtained as

$$\begin{aligned} n(z, t) = C_0 e^{-z} + \sum_{\ell=1}^{\infty} \Bigg(& \exp\left[-\frac{(1+\mu)z}{2}\right] p_\nu^{-1}\left(2\sqrt{\ell\omega}\right) \\ & \Bigg\{\left[\mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\right)\mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\, e^{-z/2}\right) + \mathrm{bei}_\nu\left(2\sqrt{\ell\omega}\right)\mathrm{bei}_\nu\left(2\sqrt{\ell\omega}\, e^{-z/2}\right)\right] \\ & \left[A_\ell \cos \ell\omega t + B_\ell \sin \ell\omega t\right] + \\ & \left[\mathrm{bei}_\nu\left(2\sqrt{\ell\omega}\right)\mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\, e^{-z/2}\right) - \mathrm{ber}_\nu\left(2\sqrt{\ell\omega}\right)\mathrm{bei}_\nu\left(2\sqrt{\ell\omega}\, e^{-z/2}\right)\right] \\ & \left[A_\ell \sin \ell\omega t - B_\ell \cos \ell\omega t\right]\Bigg\}\Bigg), \end{aligned} \quad (41)$$

a result valid for $0 \le z < \infty$.

The most convenient manner of looking for time dependent solutions of Equation 14 is with the use of Laplace transforms

$$f(p) = \mathfrak{L}\left[F(t)\right] = \int_0^\infty e^{-pt} F(t)\, dt. \quad (42)$$

Changing to $y = e^{-z/2}$ and applying the Laplace transformation to Equation 14, the equation

$$y^2 \frac{d^2 N}{dy^2} - (2\mu + 1)\, y \frac{dN}{dy} + 4\left(\mu - py^2\right) N = -\, n(y, +0)\, y^2 \quad (43)$$

is obtained for the transform of the density, $N(y, p) = \mathfrak{L}\left[n(y, t)\right]$. We have not, as yet, specified boundary conditions for $n(y, t)$. For well-behaved solutions it is natural to assume that $n \to 0$ as $y \to 0$. As for $n(1, t)$, it will be assumed only that it is some form $n_1(t)$ with transform $N_1(p)$. The initial condition $n(y, 0+)$ will not be considered at first because it is seen from the Green's function, Equation 15, that the effect of the initial condition will soon die out, even if it is of highly singular character, as long as a finite region of space is considered. Moreover, the continuous and more permanent processes usually have a gradual onset, represented as a gradual increase of the density from vanishing values at the start. Also, the discussion of boundary conditions is greatly facilitated by dealing with the homogeneous part of Equation 43.

The general solution of this equation is

$$N(y, p) = y^{1+\mu} \left[aJ_{\nu} \left(2ip^{1/2} y \right) + bJ_{-\nu} \left(2ip^{1/2} y \right) \right] \tag{44}$$

(for integral ν replace $J_{-\nu}$ by $Y_{-\nu}$). But for regular solutions in $0 \le y \le 1$, b must vanish. Therefore, upon considering the boundary condition $N(1, p) = N_1(p)$, $N(y, p)$ is always of the form

$$N(y, p) = y^{1+\mu} N_1(p) \frac{J_{\nu}\left(2ip^{1/2} y\right)}{J_{\nu}\left(2ip^{1/2}\right)} . \tag{45}$$

The character, and the mere possibility of evaluation of $n(y, t)$ is therefore crucially dependent on the analytic properties of $R_{\nu} = y^{1+\mu} J_{\nu}\left(2ip^{1/2} y\right) / J_{\nu}\left(2ip^{1/2}\right)$. Except for the trivial case when $\mu = 1$, both numerator and denominator have a branch point at $p = 0$ in the complex p plane, as well as infinite real simple zeros, for $\nu > -1$. In the fraction, however, the branch point is removed, and it becomes single valued for all p. It may be written as

$$R_{\nu}(p) = \frac{F_{\nu}\left(2ip^{1/2} y\right)}{F_{\nu}\left(2ip^{1/2}\right)} y^{1+\mu} , \tag{46}$$

where $F_{\nu}(z)$ are the entire functions of Reference 11.

When $p = \rho e^{i\theta}$ is written, it is easily seen that the argument $2\rho^{1/2} e^{i\pi+\theta/2}$ will be real only for $\theta = \pm\pi$, i.e., all (simple) poles of $R_{\nu}(p)$ will be located on the negative real axis in the p-plane, and $R_{\nu}(p)$ is a meromorphic complex function with a half-plane of analyticity. This, as will be shown in Appendix A, will bear on the existence and properties of the inverse Laplace transform.

It will, in general, not be possible to invert $N(y, p)$ directly, i.e., to evaluate directly the integral

$$n(y, t) = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} e^{pt} N(y, p)\, dp , \tag{47}$$

and, in most cases, it will be necessary to resort to contour integration. In this context it is also necessary to examine the behavior of $R_\nu(p)$ for $|p| = \rho \to \infty$. Writing $J_\nu(iz) = (i)^\nu I\nu(z)$, the asymptotic expansion of the latter function may be employed to obtain

$$
\begin{aligned}
\lim_{p\to\infty} R_\nu(p) &= \lim_{p\to\infty}\left[y^{1+\mu}\frac{e^{2yp^{1/2}y}}{\sqrt{2\pi p^{1/2}y}}\Bigg/\frac{e^{2p^{1/2}y}}{\sqrt{2\pi p^{1/2}}}\right], \\
&= y^{\mu+1/2}\lim_{\rho\to\infty}\left[e^{-2\rho^{1/2}}(1-y)(\cos\theta/2+i\sin\theta/2)\right], \\
&\leq y^{\mu+1/2}\lim_{\rho\to\infty}\left[e^{-2\rho^{1/2}}(1-y)\cos\theta/2\right], \\
&\leq y^{\mu+1/2}, \qquad (48)
\end{aligned}
$$

where $|\theta| \leq \pi$, $y \leq 1$. Thus, if $N_1(p)$ is such that for $|p| \to \infty$ it tends to zero, another indispensable condition for contour integration is met (Appendix).

It is necessary to examine the behavior of $N(y, p)$ in the half plane of convergence, in order to assure that

$$N(y, p) = \mathfrak{L}\left[\mathfrak{L}^{-1}(N)\right] = N . \qquad (49)$$

Or in other words, in Laplace transformation of the inverse transform $n(y_1 t) = \mathfrak{L}^{-1}[N(y, p)]$ the original transform $N(y, p)$ is recovered, and the resulting inverse transform does not depend on the particular abscissa to the right of the abscissa of convergence chosen as the path of integration. As outlined in the Appendix, a practically useful sufficient condition is found in Reference 14.

Assuming $N(p)$ to be analytic in $R_e(p) > p_{1r} \geq 0$, and assuming it is possible to represent it in the form

$$N(p) = \frac{C}{p^\alpha} + \frac{G(p)}{p^{1+\epsilon}} , \qquad (50)$$

where $(0 < \alpha \leq 1,\ \epsilon > 0)$ and where $G(p)$ is bounded in $Re(p) \geq p_{1r} + \delta > p_{1r}$ $(\delta > 0)$. Then $N(p)$ is the Laplace transform of

$$n(t) = \int_{p_{r-i\infty}}^{p_{r+\infty}} e^{pt} N(p)\, dp .$$

As indicated by Equation 45, $N(p) = N_1(p) R_\nu(p)$. If then $N_1(p)$ is such that the above condition is satisfied, the resulting inverse transform $n(t)$ is proper, regardless of the legitimate manner in which it is evaluated. In particular, since $R_\nu(p)$ is holomorphic in the half plane to the right of the first zero of $J_\nu\left(2ip^{1/2}\right)$, $N_1(p)$ must be such that the product $N(p)$ fulfills the condition of Equation 50. By resorting to contour integration

$$n(y, t) = \sum r_\lambda(y, t) \tag{51}$$

is obtained where r_λ is the λth residue of $e^{pt} N(y, p)$. This criterion is applied to a small variety of simple, but physically meaningful and instructive, boundary conditions, $n_1(t)$:

constant $n(1, t)$

$$n(1, t) = n_0 , \tag{52a}$$

saturation type

$$n(1, t) = n_0 \left(1 - e^{-nt}\right) , \tag{52b}$$

periodic

$$n(1, t) = \sum_{k \neq 0}^{\infty} C_K e^{ik\omega t} , \tag{52c}$$

increasing in t

$$n(1, t) = n_0 t^{\alpha} ,$$

$$\text{where} \quad \alpha > 0 , \tag{52d}$$

and

decreasing in t

$$n(1, t) = n_0 t^{-\alpha}$$

$$\text{where} \quad 0 < \alpha < 1 . \tag{52e}$$

Case A.

$$N(y, p) = \frac{n_0 R_\nu}{p} \tag{53}$$

The abscissa of convergence is $\mathrm{Re}\,(p) = 0$. But in view of Equation 50, the representability of the inverse $n(y, t)$ as a Laplace transform is not assured, for denoting $R_\nu(y, p) = g(p)$, $\epsilon = 0$ is obtained. Since, however, Equation 50 is merely a sufficient condition, a contour integration along a suitable path to obtain

$$n(y, t) = \sum_{k=0}^{\infty} r_k^\nu$$

with r_k^ν the kth residue of $p^{-1} R_\nu e^{pt}$ can be performed. Now, $N(y, p)$ has the poles r_0^ν at the origin and r_ℓ^ν $(\ell \neq 0)$ at the zeros of $J_\nu(2ip^{1/2})$. These, as was shown, may be written as $p_\nu^\ell = -\gamma_{\nu,\ell}^2/4$, where $\pm\gamma_{\nu,\ell}$ are the simple zeros of $y_\nu(x)$. Consequently,

$$r_0^\nu = n_0 y^2 , \tag{54}$$

since $\lim_{p\to 0} R_\nu(y, p) = y^{1+\mu} y^\nu = y^2$. Denoting by differentiation with respect to p, also

$$r_\ell^\nu = \left. \frac{n_0 y^{1+\mu} J_\nu\left(2ip^{1/2} y\right) e^{pt}}{p J_\nu\left(2ip^{1/2}\right)} \right| p = \gamma_{\nu,\ell}^2/4$$

$$r_\ell^\nu = \frac{-2n_0 y^{1+\mu} J_\nu\left(\gamma_{\nu,\ell} y\right)}{\gamma_{\nu,\ell} J_{\nu+1}\left(\gamma_{\nu,\ell}\right)} e^{-\gamma_{\nu,\ell}^2 t/4} . \tag{55}$$

Thus as the zeros of Bessel functions of consecutive orders interlace, one obtains the alternating series

$$n(y, t) = n_0\left[y^2 - 2y^{1+\mu} \sum \frac{J_\nu\left(\gamma_{\nu,\ell}\right)}{\gamma_{\nu,e} J_{\nu+1}\left(\gamma_{\nu,\ell}\right)} e^{-\gamma_{\nu,\ell}^2 t/4}\right], \tag{56}$$

which for $t \to \infty$ converges to the steady state, solution $n_0 y^2$, identical with Equation 31. Equation 56 also obviously satisfies the boundary condition Equation 52a, at all times, i.e., for all $t \geq 0$

$$\lim_{y\to 1} n(y, t) = n_0 . \tag{57}$$

However, for the convergence at all fixed y of $\lim_{t\to 0} n(y, t)$ to vanishing initial value, the $t \to 0$ theorem based on the conditions of Equation 50 is not applicable since these conditions are not satisfied, as indicated above. This criterion may be applied, however, when dealing with boundary conditions of type b.

Case B.

$$N(y,\, p) \;=\; \frac{n_0\, \kappa\, y^{1+\mu} J_\nu \left(2ip^{1/2}\, y\right)}{p(p+\kappa)\, J_\nu \left(2ip^{1/2}\right)} \;. \tag{58}$$

Equation 58 may be written

$$N(y,\, p) \;=\; \frac{n_0\, \kappa R_\nu\,(p)}{p^2 \left(1 + \frac{\kappa}{p}\right)} \;. \tag{59}$$

Again the abscissa of convergence is the imaginary axis. Then for any $\delta > 0$, where $\delta = \mathrm{Re}\,(p)$, $R_\nu/(1+\kappa/p) = G(p)$ is bounded, and the conditions of Equation 50 are satisfied with $\mathcal{E} = 1 > 0$. An additional term in the series expansion for $n(y,\, t)$ is obtained, resulting from the pole at $p = -\kappa$. Since κ is quite arbitrarily chosen, $\kappa \neq \gamma_{\nu,\ell}^2/4$ is assumed, to obviate the evaluation of the residue at a second order pole at $p = -\kappa$ is necessary. With this reservation, our series expansion yields

$$n(y,\, t) \;=\; n_0 \left[y^2 - y^{1+\mu} \left(\frac{J_\nu \left(2\kappa^{1/2} y\right) e^{-\kappa t}}{J_\nu \left(2\kappa^{1/2}\right)} + 8 \sum_{\ell=1}^{\infty} \frac{\kappa J_\nu \left(\gamma_{\nu,\ell}\, y\right)}{\gamma_{\nu,\ell} \left(4\kappa - \gamma_{\nu,\ell}^2\right) J_{\nu+1} \left(\gamma_{\nu,\ell}\right)}\, e^{-\gamma_{\nu,\ell}^2 t/4} \right) \right] \tag{60}$$

Again $\lim\limits_{t\to\infty}(y,\, t) = n_0\, y^2$. But on the strength of the considerations in the Appendix the convergence of $n(y,\, t)$ to zero at $t = 0$ is assured, since $N(y,\, p) = 0\left(1/p^2\right)$ for $p \to \infty$, and therefore $n(y,\, t) = 0(t)$ for $t \to 0$. For finite t Case A may be approached arbitrarily close by letting $\kappa \to \infty$.

Case C.

$$N(y_1\, p) \;=\; R_\nu\,(y_1\, p) \sum_{-\infty}^{\infty} \left(\frac{p + ik\omega}{p^2 + k^2 \omega^2}\, C_k \right) \tag{61}$$

again converges for positive real p. The poles are distributed along the negative real axis, and along the imaginary axis at $p = \pm i\ell\omega$, ℓ spanning all non-negative integers. Using contour integration,

$$n(y,\, t) \;=\; C_0\, y^2 + y^{1+\mu} \left[2 \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \frac{\left(4m\omega\, \gamma_{\nu,\ell}\, B_m - \gamma_{\nu,\ell}^3\, A_m\right) J_\nu \left(\gamma_{\nu,\ell}\, y\right)}{\left(\gamma_{\nu,\ell}^4 + 16m^2 \omega^2\right) J_\nu \left(\gamma_{\nu,\ell}\right)}\, e^{-\gamma_{\nu,\ell}^2 t/4} + P(y,\, t) \right] , \tag{62}$$

where $P(y,\, t)$ is the periodic term in the right member of the steady state solution, Equation 41, to which $n(y,\, t)$ converges for $t \to \infty$. In view of the convergence criterion Equation 50, convergence is also assured for $t \to 0$, if $\Sigma C_k\, e^{ik\omega t}$ is a pure sine series.

Case D.

$$N(y, p) = n_0 \Gamma(\alpha + 1) R_\nu (p) p^{-(\alpha+1)} \tag{63}$$

It is shown in the Appendix that $n(y, t) = 0(t^\alpha)$ for $t \to \infty$, and, in view of finite solutions for all t, this Case can be dismissed.

Case E.

$$N(y, p) = n_0 \Gamma(1 - \alpha) R_\nu (p) p^{1-\alpha} , \tag{64}$$

is shown to vanish as $n(y, t) \approx 0(t^{\alpha-2})$.

THE EFFECT OF INITIAL CONDITIONS

Having dealt at length with the dependence of the solution on the boundary conditions, without modification of the problem by extension of the investigation to negative values of z (i.e. $y > 1$), an estimate of the effect of initial conditions can be given. The first step is to examine the appropriate Green's function $G(y, y_0; p)$ obtained from the solution of the equation

$$L\left[G(y, y_0; p)\right] = \delta(y - y_0) \tag{65}$$

where L is the operator on the left side of Equation 43. Homogeneous boundary conditions at $y = 0$, and $y = 1$ are chosen. The complete solution $n(y, t)$ will be a sum of the solution to the equation with zero initial conditions and the solution with zero boundary conditions. It is easily shown that

$$G(y, y_0; p) = \frac{\pi y^{1+\mu}}{2 y_0^{2+\mu}} \left[- \frac{J_\nu (\alpha y_0) Y_\nu (\alpha)}{J_\nu (\alpha)} J_\nu (\alpha y) + \theta(y_0 - y) Y_\nu (\alpha y_0) J_\nu (\alpha y) + \theta(y - y_0) J_\nu (\alpha y_0) Y_\nu (\alpha y)\right] \tag{66}$$

with $\alpha = 2ip^{1/2}$. Thus

$$N(y, p) = N_0 (y, p) + N_i^* (y, p) . \tag{67}$$

Here $N_0 (y, p)$ denotes the solutions of the previous section,

$$N^* (y, p) = - 4 \int_0^1 G(y, y_0; p) n_0^* (y_0) y_0^2 \, dy_0 . \tag{68}$$

Two typical distributions may be of interest.

(A) an initial pulse, as in Reference 7, where $n(z, 0) = n_0 \delta(z - z_1)$, or

$$n(y, 0) = \frac{n_0^* y_1 \delta(y - y_1)}{2} . \tag{69}$$

Then from Equations 66 and 68

$$N^*(y, p) = \pi n_0^* y_1^{1-\mu} y^{1+\mu} \left[\frac{J_\nu(\alpha y_1) Y_\nu(\alpha) J_\nu(\alpha y)}{J_\nu(\alpha)} - J_\nu(\alpha y) Y_\nu(\alpha y_1)\right]$$

where $y_1 < y$, and

$$= \pi n_0^* y_1^{1-\mu} y^{1+\mu} \left[\frac{Y_\nu(\alpha) J_\nu(\alpha y_1) J_\nu(\alpha y)}{J_\nu(\alpha)} - J_\nu(\alpha y_1) Y_\nu(\alpha y)\right] \tag{70}$$

where $y > y_1$. The second term on the right side of Equation 69 is identical with the solution in Reference 7 while the first term accounts for the adjustment of the initial density to homogeneous boundary conditions.

(B) an initial density corresponding to a previously established steady state $n(z, 0) = n_0^* e^{-\sigma z}$ $(\sigma > 0)$, or

$$n(y, 0) = n_0^* y^{2\sigma} \tag{71}$$

Though Equation 71 does not agree with the homogeneous condition stipulated at $y = 1$, the relation of Equation 71 may be assumed to be valid up to $y = 1 - \epsilon$, while beyond that point, it may be assumed that $n(y, 0) = n_0^* y^{2\sigma}(y - 1)$, and then passes to the limit $\epsilon \to 0$. Then

$$N^*(y, p) = 4n_0^* \int G(y, y_0; p)\, y_0^{2\sigma+2}\, dy_0 . \tag{72}$$

The integral in Equation 71 is evaluated for two selected values of σ: $\sigma_1 = 1$, $\sigma_2 = \mu$, such that $N^*(y, p)$ may be expressed in terms of a simple combination of Bessel functions. For general σ, Lommel functions $S_{2\sigma, \nu}(z)$ and $S_{2\sigma-1, \nu-1}(z)$ would be used.

(1) $\sigma = 1$ corresponds to an initial density equal to the steady state density $n(z) \propto e^{-z}$. Equation 72 yields

$$\frac{N^*(y, p)}{4n_0^*} = \frac{\pi y^{1+\mu}}{2}\left[\frac{Y_\nu(\alpha) J_\nu(\alpha y)}{J_\nu(\alpha)} \int_0^1 y_0^{\nu+1} J_\nu(\alpha y_0)\, dy_0 - J_\nu(\alpha y) \int_y^1 y_0^{\nu+1} Y_\nu(\alpha y_0)\, dy_0 - Y_\nu(\alpha y) \int_0^y y_0^{\nu+1} J_\nu(\alpha y_0)\, dy_0\right]$$

$$= \frac{\pi y^{1+\mu}}{2\alpha}\left\{\left[Y_\nu(\alpha) J_{\nu+1}(\alpha) - J_\nu(\alpha) Y_{\nu+1}(\alpha)\right] \frac{J_\nu(\alpha y)}{J_\nu(\alpha)} + \left[J_\nu(\alpha y) Y_{\nu+1}(\alpha y) - Y_\nu(\alpha y) J_{\nu+1}(\alpha y)\right] y^{\nu+1}\right\}$$

$$= \frac{y^{1+\mu}}{\alpha^2} \frac{J_\nu(\alpha y)}{J_\nu(\alpha)} - \frac{y^2}{\alpha^2} \tag{73}$$

(2) $\sigma = \mu$ corresponds to diffusive equilibrium, $n(t) \alpha e^{-\mu z}$, and, in a similar manner, Equation 72 yields

$$\frac{N^*(y, p)}{4n_0^*} = \frac{\pi y^{1+\mu}}{2\alpha} \left[J_\nu(\alpha) Y_{\nu-1}(\alpha) - Y_\nu(\alpha) J_{\nu-1}(\alpha)\right] \frac{J_\nu(\alpha y)}{J_\nu(\alpha)} - \frac{y^{2\mu}}{\alpha^2}$$

$$= \frac{y^{1+\mu}}{\alpha^2} \frac{J_\nu(\alpha y)}{J_\nu(\alpha)} - \frac{y^{2\mu}}{\alpha^2} \quad . \tag{74}$$

Substitution $y = 0$, and $y = 1$ shows $N^*(y, p)$ to satisfy the boundary conditions, and it is easy to verify that Equations 73 and 74 satisfy the inhomogeneous Equation 43.

THE CHARACTERISTIC STRUCTURE OF SOLUTIONS, NUMERICAL RESULTS, AND POSSIBLE APPLICATIONS

Having obtained a variety of solutions, we now proceed to discuss their salient features. The solutions of greater physical interest are Equations 60 and 62. For the former it can be readily verified that $n(y, 0) = 0$. For applying the representation Equation A10 in the Appendix, substitution $z = 2n^{1/2}$ yields

$$\frac{J_\nu\left(2n^{1/2} y\right)}{J_\nu\left(2n^{y} 2\right)} \equiv y^\nu - 8 \sum_{\ell=1}^{\infty} \frac{n J_\nu\left(\gamma_{\nu,\ell}\, y\right)}{\gamma_{\nu,\ell}\left(4n - \gamma_{\nu,\ell}^2\right) J_{\nu+1}\left(\gamma_{\nu,\ell}\right)} \quad , \tag{75}$$

thereby proving the assertion on the convergence of $n(y, t)$ for $t \to 0$. Moreover, by passing to the limit $n \to \infty$

$$y^\nu - 2 \sum_{\ell=1}^{\infty} \frac{J_\nu\left(\gamma_{\nu,\ell}\, y\right)}{\gamma_{\nu,\ell}\, J_{\nu+1}\left(\gamma_{\nu,\ell}\right)} \to y^{-1/2} e^{-2(1-y)n^{1/2}} \to 0 \quad , \tag{76}$$

is obtained for $y > 1$, such that zero initial conditions are also satisfied by Equation 56 for $y < 1$.

From our series expansion of $n(y, t)$ it is also evident that the solutions are, indeed, of relaxation type for later times, whatever the initial and boundary conditions. This is in accordance with the established aims with respect to the desired solutions. It is also clear that for large t (in the order of $\gamma_{\nu,1}^{-2}$) the dominant non-periodic, time-dependent term is the first term in the series, on account of the factor $e^{-\gamma_{\nu,\ell}^2 t/4}$ (assuming $4n > \gamma_{\nu,1}^2$). This term is always of negative sign, since $J_\nu(\gamma_{\nu,1} y)/J_{\nu+1}(\gamma_{\nu,1}) < 0$ for $y < 1$. By comparing it with the stationary term, y^2, a characteristic time, $\tau_\mu(y)$, for approach to steady state conditions can be deduced. For $n \gg \gamma_{\nu,1}$, this becomes

$$\tau_\mu(y) = \frac{4}{\gamma_{\nu,1}^2}\left\{1 + \log\left[\frac{2y^{\mu-1} J_\nu(\gamma_{\nu,1} y)}{\gamma_{\nu,1} J_{\nu+1}(\gamma_{\nu,1})}\right]\right\} \tag{77}$$

Typical values of τ_μ are presented in Table 1.

Table 1

τ_μ (y), *Characteristic Dimensionless Time for Approach to Steady State Conditions.*

Exponential Depth y	Mass Ratio (μ) 0	1/2	1
0.75	0.160	0.180	0.270
0.5	0.385	0.505	0.730
0.25	0.485	0.645	0.935
0.05	0.520	0.685	1.000

Table 1 already bears evidence to the easier penetration of lighter particles into a heavier medium. The relatively mild dependence of τ on μ appearing in Table 1 is, however, somewhat misleading. By considering dimensional time (using Equations 5 and 13) the dependence is considerably accentuated. Applying the values in the table to the diffusion of H, O, N_2 respectively, into a medium of O_2, the respective τ_μ (y) have to be divided in the ratio $\sqrt{77}$: $\sqrt{7}$: $\sqrt{5}$. It is assumed that the cross-section σ appearing in Equation 5 does not change appreciably with μ, as well as that $\tau_{1/32} \approx \tau_0$, and $\tau_{7/8} \approx \tau_1$.

On the other hand, the dependence of τ on y, appearing in Table 1, explains qualitatively the steep slope portion of the density profile progressing in time towards smaller y, i.e., higher altitudes z. τ(y) is, therefore, a rough but quite reliable measure of the diffusion velocity, which increases noticeably with mounting altitude.

These results are, of course, not altered substantially if one includes the effect of initial conditions. As one can see from Equation 73 and 74 the decay of the initial density is governed by the same time dependent series as Equation 55, i.e.,

$$\sum_{\ell=1}^{\infty} \frac{J_\nu(\gamma_{\nu,\ell}\, y)}{\gamma_{\nu,\ell} J_{\nu+1}(\gamma_{\nu,\ell})} e^{-\gamma_{\nu,\ell}^2 t/4} .$$

As an illustration of results, Figures 1 through 6 represent the evolution of the density profile for μ = 0, 0.5, 1 respectively according to Equations 59 and 60; in the latter, a value of 4_n about

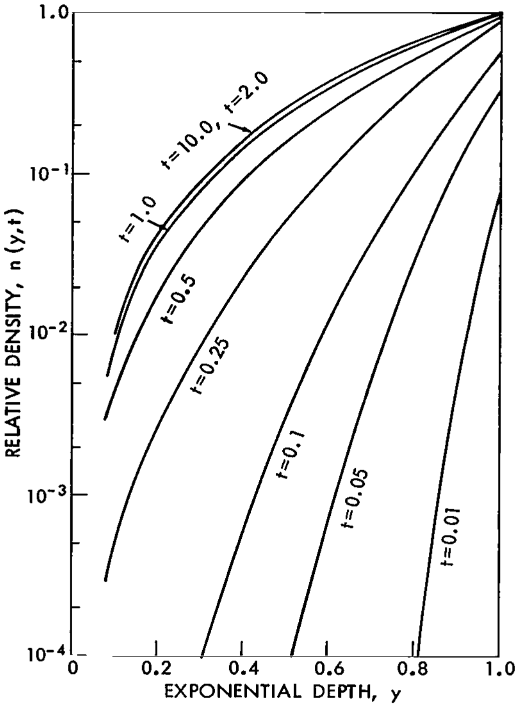

Figure 1—Density, n(y, t), versus depth, y; $\mu = 0$, $\kappa = 8.0$.

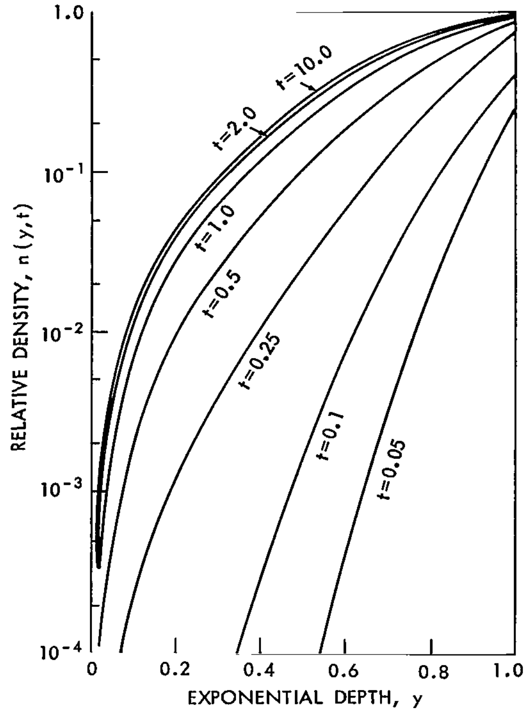

Figure 2—Density, n(y, t), versus depth, y; $\mu = 1/2$, $\kappa = 5.5$.

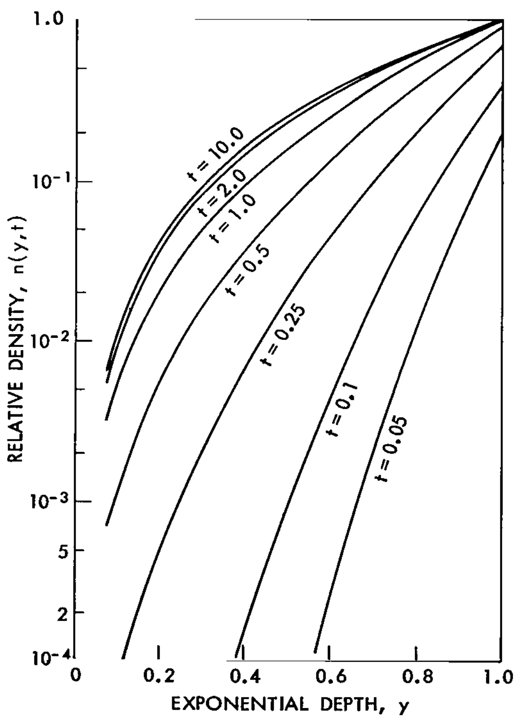

Figure 3—Density, n(y, t), versus depth, y; $\mu = 1$, $\kappa = 4.5$.

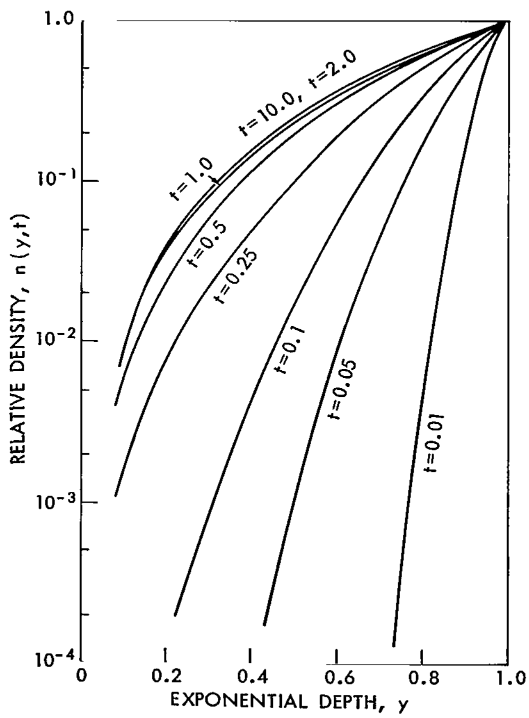

Figure 4—Density, n(y, t), versus depth, y; $\mu = 0$, $\kappa = \infty$.

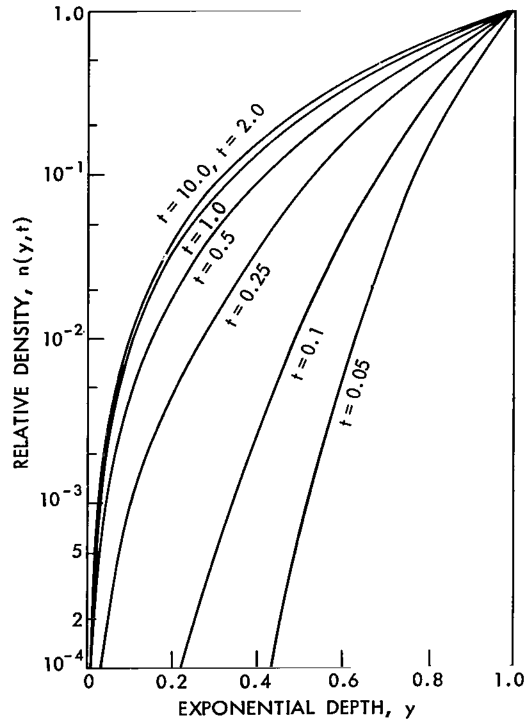

Figure 5—Density, n(y, t), versus depth, y; μ = 1/2, $\kappa = \infty$.

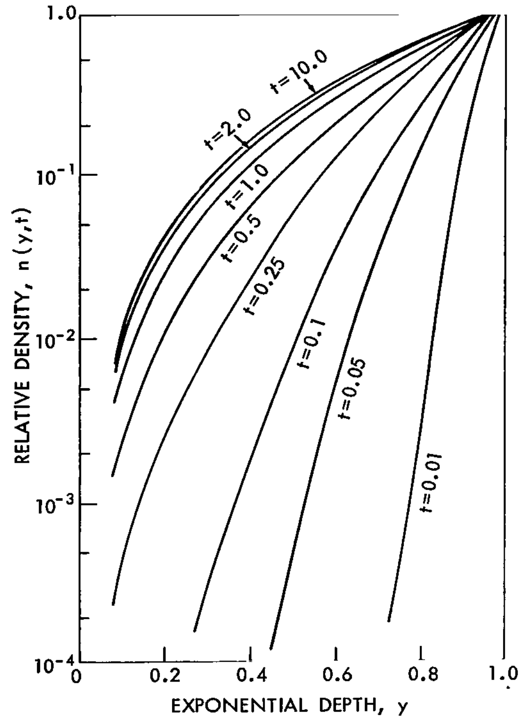

Figure 6—Density, n(y, t), versus depth, y; μ = 1, $\kappa = \infty$.

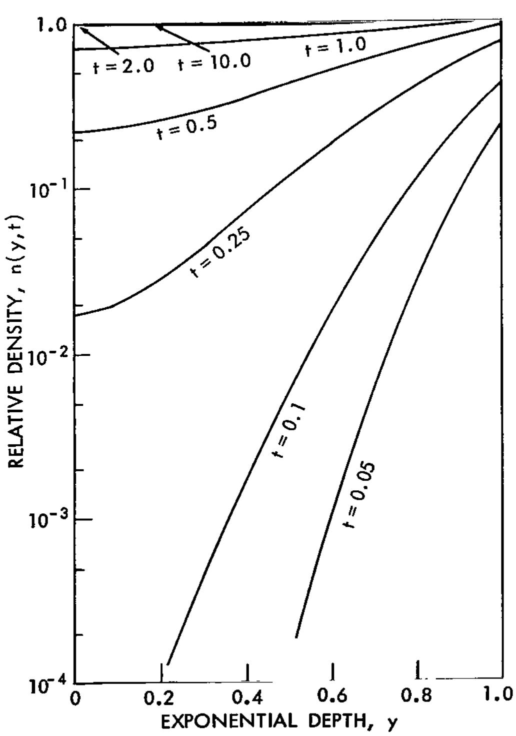

Figure 7—Relaxation of density to steady state; n(y, t)/n(y, ∞) versus y for various times, t.

halfway between $\gamma_{\nu,1}^2$ and $\gamma_{\nu,2}^2$ has been selected. The respective curves show that, even in this case, τ as computed under the assumption $\kappa \to \infty$ is in the right order of magnitude.

Figure 7 shows the convergence of the density profile to the steady state for μ = 1/2. It extends the small sample of τ_μ values presented in Table 1.

Though it appears that the results of this study apply to a very idealized model of the diffusion processes going on in the earth's atmosphere, they constitute a necessary step in the investigation of more complex phenomena. These results were primarily intended to serve as zero order solutions of the far more complex Equation 8, and provide a convenient starting point for a perturbation treatment of this equation.

Nonetheless some meaningful physical information may be gleaned by exercising due caution. Of importance in this respect are the characteristic terms τ_μ

which effectively measure the time of transport for minor components from the source region to high altitudes. With the aid of Equations 5 and 13, these are found to be $\approx 10^5$ sec. for the diffusion of atomic hydrogen starting at $\approx$120km and $\approx 3 \times 10^5$ for helium. These light components are subject to escape, depending sensitively on the temperature at $\approx$500 km, which displays a strong diurnal variation.

In the first approximation the diurnal variation of escape may be regarded as a periodic boundary condition, whose period is about the same as the time scale of supply from the source. In view of these results (Equation 41), it appears therefore questionable to deduce the density profile of minor constituents from the equation of continuity, as has been the usual practice (References 15, 16 and 17). This will be even more so in the case of hydrogen, where the time of depletion, or "life time" with respect to escape (Reference 19) is on the same scale.

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APPENDIX A

Some Useful Results from the Theory of Laplace Transforms and Complex Function Theory

Many of the results in this discussion are based on theorems and formulas from the theory of Laplace transforms and complex function theory. In fact, the solutions of form $F(y,p)$, are obtained directly from solutions of the Laplace transformed equation $\mathfrak{L}[Lf(y, t)]$, where $\mathfrak{L}(y, t, \partial/\partial y, \partial/\partial t \cdots)$ is the differential operator applied to $f(y,t)$.

The immediate question arises whether application of the inverse operator $\mathfrak{L}^{-1}$ to $F(y, p)$ yields $f(y, t)$ such that $\mathfrak{L}[f(y, t)] = F(y, p)$, and for which $F(y, p)$ this can, indeed, be shown to hold. Unfortunately only sufficient conditions are known so far (Reference 14). A rather simple, but readily applicable condition is expressed by Theorem 1.

Let $F(p)$ be analytic in the right half plane $\mathrm{Re}\,(p) > p_{1r} \geq 0$, where $p = p_r + ip_i$. Let it be possible to represent $F(p)$ in this half plane as

$$F(p) = \frac{C}{p^{\alpha}} + \frac{G(p)}{p^{1+\epsilon}} \tag{A1}$$

where $(0 < \alpha \leq 1,\ \epsilon > 0)$, with $G(p)$ bounded in $\mathrm{Re}\,(p) \geq p_{1r} + \delta > p_{1r}\,(\delta > 0)$, then $F(p)$ is the Laplace transform of the function $(t > 0)$

$$f(t) = P(2\pi i)^{-1} \int_{p_r - i\infty}^{p_r + i\infty} e^{pt}\, F(p)\, dp \tag{A2}$$

where $p_r > p_{1r}$. Frequent use has been made of this theorem in the examination of "admissible" boundary conditions.

The next question arises with respect to convergence of $f(y, t)$ at $t = 0$. In many cases an answer may be found in Theorem 2 (Reference 18). Let $F(p)$ be analytic in $\mathrm{Re}\,(p) \geq p_{0r}$, and let

$$F(p) = 0\left(\frac{1}{|p|^{\mu}}\right) \text{ as } p \to \infty ,$$

with $\mu > 1$, independent of the path. Then $\mathfrak{L}^{-1}\ F(p)$ converges for $t \geq 0$, and represent a function $f(t)$ for which

$$f(t) = 0\left(t^{\mu-1}\right) \text{ as } t \to 0 . \tag{A3}$$

Similar Abelian theorems (Reference 18) enable the examination of the asymptotic behavior of $f(t)$ for $t \to \infty$. Theorem 3, below, is a strong theorem of rather wide applicability. Let $F(p)$ be analytic in the sector $\arg(p - p_0) \le \psi$ $(\pi/2 < \psi < \pi)$, except at p_0. Let $F(p) \to A(p - p_0)^\lambda$ (λ real), as $p \to p_0$ in this sector. On the rays $\arg(p - p_0) = \pm\psi$ let $F(p) = 0(e^{n|p|})$ as $|p| \to \infty$. Then

$$F(t) \approx Ae^{p_0 t} \frac{t^{-\lambda-1}}{\Gamma(-\lambda)} \quad \text{as} \quad t \to \infty . \tag{A4}$$

Following the problem of existence of the inverse Laplace transform and its asymptotic behavior for $t \to 0$, and $t \to \infty$ the question of its evaluation is discussed briefly. Since it is well known that this can often be done by simple contour integration, it is sufficient to state two of the necessary conditions on $F(p)$ (Reference 18),

(a) $F(p)$ is analytic in the complex p plane except at the poles $p_0, p_1 \cdots$, all left of $R_e(p) \le p_{0r}$.

(b) $|F(p)| \to 0$ on the limiting contour C_n, i.e., as $|p| \to \infty$.

It turns out that many of the residues appearing in these results incorporate a factor of the form

$$Q_\nu(w, z) = J_\nu(w)/J_\nu(z) , \tag{A6}$$

where w and z are complex variables, connected by the linear relation $w = yz$. Q_ν is then a meromorphic function of z, and one may invoke the Mittag-Leffler theorem to find a convenient representation of $Q_\nu(y, z)$. The branch point of $J_\nu(z)$ at the origin is removed in the quotient, and apart from it, $J_\nu(z)$ has simple zeros, yielding the desired representation in the form

$$Q_\nu(y, z) = Q_\nu(y, 0) + \sum_k^1 r_k^\nu \left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) \tag{A7}$$

where r_k^ν is the residue of $Q_\nu(y, z)$ at $z = z_k$. For $\nu > -1$ all zeros of $J_\nu(z)$ are real and symmetric about the imaginary axis, i.e., $z_{\pm k} = \pm\gamma_{\nu,k}$. Also,

$$r_{+k}^\nu = -\frac{J_\nu(\gamma_{\nu,k})}{J_{\nu+1}(\gamma_{\nu,k})} \tag{A8}$$

while

$$r_{-k}^\nu = \frac{J_\nu(\gamma_{\nu,k})}{J_{\nu+1}(\gamma_{\nu,k})} \tag{A9}$$

so that

$$Q_\nu(y, z) = y^\nu - \sum_{-\infty}^{\infty}\left(\frac{1}{z - z_k} + \frac{1}{z_k}\right)\frac{J_\nu(z_k y)}{J_{\nu+1}(z_k)}$$

$$= y^\nu - \sum_{k=1}^{\infty}\left(\frac{1}{z - \gamma_{\nu,k}} - \frac{1}{z + \gamma_{\nu,k}} + \frac{2}{\gamma_{\nu,k}}\right)\frac{J_\nu(\gamma_{\nu,k} y)}{J_{\nu+1}(\gamma_{\nu,k})}$$

$$= y^\nu - 2\sum \frac{z^2 J_\nu(\gamma_{\nu,k} y)}{\gamma_{\nu,k}\left(z^2 - \gamma_{\nu,k}^2\right) J_{\nu+1}(\gamma_{\nu,k})} . \tag{A10}$$

This useful result could have been applied in the inversion of $N(p)$. This would have entailed, however, a complicated examination of the convergence properties of the infinite series in (A10), which could be obviated by the use of contour integration.

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