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## RESEARCH ON THE DESIGN OF ADAPTIVE CONTROL SYSTEMS

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Prepared for:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION  
ELECTRONICS RESEARCH CENTER  
CAMBRIDGE, MASSACHUSETTS

CONTRACT NAS 12-59

STANFORD RESEARCH INSTITUTE

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## INTRODUCTION

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This second volume of the final report contains seven technical memoranda in their original form as generated during the course of the contract.

In Memoranda 1 and 2, the problem of minimizing tracking errors of a large radio antenna for tracking space vehicle motion is formulated as a combined optimum estimation and control problem. The purpose of this work is to illustrate the applicability of combined optimization theory to high-precision pointing problems.

In Memoranda 3 and 4, reliability is considered from a systems point of view. A mathematical formulation for maximizing the service provided by a system comprising unreliable components is developed and shown to be solvable by standard optimization techniques.

In Memorandum 5, a simplified design technique for model-referenced adaptive systems, which are a subclass of performance feedback adaptive systems, is discussed. In Memorandum 6, analysis-synthesis adaptive systems are treated, and the adaptive control problem is formulated in terms of combined estimation and control theory (combined optimization theory).

In Memorandum 7, the problem of high-precision tracking of space vehicles is solved by application of the theory of optimum estimation and control. A computer program to implement the linearized estimator and controller has been developed and tested. This program is intended as a study tool to determine the relation between tracking performance and various design parameters, such as receiver noise, readout noise, trajectory model inaccuracies, wind disturbance, mechanical resonance, etc.



MEMORANDUM 1

APPLICATION OF ADAPTIVE FILTERING TO  
SATELLITE TRACKING AND ATTITUDE CONTROL



## MEMORANDUM 1

### APPLICATION OF ADAPTIVE FILTERING TO SATELLITE TRACKING AND ATTITUDE CONTROL

#### I INTRODUCTION

The purpose of this memorandum is to illustrate the applicability of modern adaptive filtering techniques to two specific problems: 1) the tracking of an earth satellite from an earth-bound tracking site, and 2) the attitude control of a satellite or space vehicle via information obtained from an on-board horizon sensor or star tracker. It will be shown that these two problems, although they appear quite different, can be treated by exactly the same methods, yielding similar system configurations. These configurations, while representing the application of advanced system theory to important present-day problems, also have the property that they can be implemented with presently available systems hardware. Thus closed loop angle tracking of deep space vehicles, which is presently impossible due to the very low signal to noise ratio experienced, would be brought within the realm of practicability.

It is shown in Sec. III that the system equations for the two problems of interest are either basically linear or can be appropriately linearized. As a result of this fact, these problems may be considered as a special case of the general problem of combined optimal estimation and control. The unique feature of this special case is that the functions of estimation and control separate; this fact being reflected in the system configurations shown in Sec. III.

The use of the term "adaptive" in describing the techniques discussed in this memorandum is based on two properties of these techniques. The first is that the filters vary their own dynamic behavior as a function of the existing noise environment (open-loop adaptation). Secondly, they can be arranged to estimate not only the system state but also the values of system parameters. Hence these filters may be considered as part of an analysis-synthesis adaptive system as well.

The novel feature of the systems discussed for the problems of

satellite tracking and attitude control is the use of a state estimating filter in a system which would more commonly be designed as a conventional feedback servo system. The advantages of this approach, which are discussed in detail in Sec. IV, include: improved accuracy due to enhanced immunity to noise and load disturbances, ability to operate in signal to noise conditions where conventional designs fail completely, improved reliability, and optimum utilization of a priori information.

Several remaining research questions, both applied and theoretical, are discussed in Sec. V.

## II THE SATELLITE TRACKING AND ATTITUDE CONTROL PROBLEMS

In approaching the satellite tracking and attitude control problems, four points must be considered: 1) the input signal generating mechanism, 2) the response mechanism, 3) the data collection system, and 4) the system performance criterion and constraints that limit the performance achievable. In what follows, each of these points is discussed in the context of satellite tracking and attitude control, however the development is of a sufficiently general nature that the applicability of the method to a wide variety of dynamic systems will be clear.

### A. Input Signal Generating Mechanism

Since both problems to be considered involve an earth satellite, it is convenient to employ a nonrotating rectangular earth centered coordinate system. It will be assumed (although this assumption is not necessary) that the earth satellite problem is a two-body problem and that the motion of the earth within the solar system may be ignored.\* Furthermore, the earth will be assumed homogeneous, spherical, and centered at the origin of the coordinate system.

The input signal generating mechanism for the satellite angle tracking problem\*\* is just the relative motion of the satellite and the tracking

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\* For a space vehicle this latter assumption is not valid and a space fixed coordinate system must be used. This, however, does not invalidate the method.

\*\* An analogous development exists for doppler and other tracking techniques.



site. This motion arises as the result of two dynamical systems, the first governing the motion of the satellite around the earth and the second the rotation of the earth on its axis. For the three position components and three velocity components of the satellite and tracking site given respectively by  $x_1 - x_6$  and  $y_1 - y_6$ , these dynamical systems may be represented by vector differential equations of the form

$$\dot{\underline{x}} = f_1[\underline{x}, \underline{g}(\underline{x}), \underline{w}(t)] \quad (1)$$

$$\dot{\underline{y}} = f_2(\underline{y})$$

where  $f_1$  and  $f_2$  are, in general, nonlinear functions of their arguments. The term  $\underline{g}(\underline{x})$  is included to account for the inhomogeneous gravitational field of the earth, and  $\underline{w}(t)$  represents the unpredictable drag effects experienced by the satellite.<sup>1</sup>

Since in the tracking context it is the motion of the satellite relative to the tracking site that is of interest, the vector

$$\underline{z} = \underline{x} - \underline{y}$$

representing the line joining the satellite and the tracking site will be considered. Furthermore, by a straightforward change of variables,  $\underline{z}$  may be expressed in the so-called "radar parameters" for the particular tracking site. These parameters are the azimuth  $\theta$ , elevation  $\varphi$  and range  $\rho$ , and their respective time derivatives. Together, these six parameters comprise the components of the state vector  $\underline{s}$  of the satellite expressed in a polar coordinate system centered at the tracking site. The input signal generating mechanism for the satellite tracking case, which produces the time profiles  $\theta(t)$ ,  $\varphi(t)$  and  $\rho(t)$ , may therefore be expressed as

$$\dot{\underline{s}} = f_3[\underline{s}, \underline{g}(\underline{x}), \underline{w}(t)] \quad (3)$$

where the form of  $f_3$  is derived from  $f_1$  and  $f_2$  together with the change of variables employed.

For the attitude control case two variants, which require somewhat different approaches, are considered: the use of a star tracker and the use of horizon sensors. It should be pointed out that although complete three axis control can be achieved with the use of a single star tracker, a single horizon sensor can yield control in only one axis. At most, the pitch and roll axes of an earth satellite can be controlled by the use of two horizon sensors, the yaw axis must be controlled by other means.

In the case of a star tracker, the quantities of interest are the angles between the line connecting the star and the satellite and each of two reference lines on the satellite. If the satellite is to be stabilized with respect to space, then it is required that these angles be kept constant. However, if the satellite is earth oriented, the desired values of these angles varies as a function of the satellite state  $\underline{x}$ . Since this requires an on-board knowledge of the satellite state at all times, the use of a star tracker for earth orientation is probably not as attractive as other techniques.

The input signal generating mechanism for a star tracking attitude control system is particularly simple. The angular orientation of the reference line to the star remains approximately constant with respect to a non-rotating earth centered coordinate system for any vehicle trajectory within the solar system. This is true since the distance to even the nearest star is much larger than any dimension of the solar system. For example, taking the nearest star to be 63,000 a.u. distant, and the diameter of the earth's orbit about the sun to be 2 a.u., the maximum variation in the orientation of the reference line for any position within the earth's orbit is approximately 31.8 microradians - a negligible quantity. As a result, if  $\underline{y}$  is taken in this case to represent the angular orientation of the reference line in space, the input signal generating mechanism may be characterized by

$$\dot{\underline{y}} = 0 \quad (4)$$

A horizon sensing system should be considered only in the context of an earth oriented satellite. The use of a horizon sensor to accomplish any other orientation leads to undue difficulties. Subject to this restriction, the input signal to a horizon sensing system is also approximately a constant if a spherical earth and circular orbit are assumed. (Relaxing these assumptions does not lead to any increase in complexity as will be shown in Sec. III). This may be illustrated simply by means of Fig. 1. It may be seen from simple geometry that for any given orbit, the angle  $\alpha$  between the local vertical and the horizon, is independent of the satellite's position along the orbit. That is  $\alpha$  is independent of the value  $\beta$ . Hence if  $\underline{y}$  is taken to represent the input signals in the pitch and roll coordinates, the input signal generating mechanism for a horizon sensing system may likewise be characterized by  $\dot{\underline{y}} = 0$ .

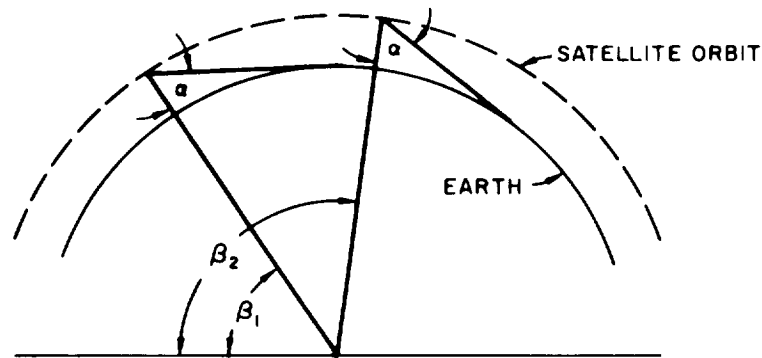


FIG. 1 SYSTEM GEOMETRY

#### B. Response Mechanism

The response mechanism for the satellite tracking case consists of the tracking antenna and its associated drive system. For the most common tracking configuration, the azimuth-elevation mount, this drive system is divided into two independent channels, one for azimuth and one for elevation. These channels are usually of the velocity servo type in

that a constant input signal gives rise to a steady state output velocity. The drive systems are for the most part linear, with some possible non-linear coupling arising from the change in effective moment of inertia as the antenna changes position. The predominant nonlinearity in a tracking antenna system is a strong effective saturation on the drive system input signal. This arises from power limitations, and is necessary to prevent high accelerations from causing structural damage to the antenna. Letting  $\underline{s}'$  be the state vector of the tracking antennas,\* in the same radar parameter coordinate system described above, the behavior of the tracking antenna drive system may be described by:

$$\dot{\underline{s}}' = A \underline{s}' + B \underline{u} \quad (6)$$

where  $A$  is the system matrix that describes the system dynamics,  $B$  is the distribution matrix indicating the manner in which the drive system input  $\underline{u}$  is applied. A restriction of the form

$$a_i \leq u_i \leq b_i \quad (7)$$

must be placed on the components of  $\underline{u}$  to account for the saturation constraints.<sup>2</sup>

The satellite itself, together with its torqueing system, comprises the response mechanism in the attitude control case. The system dynamics take on a particularly simple form in this instance since the satellite may be represented as a pure inertia in free space. Two methods of torqueing a satellite commonly employed in stabilization systems of this type are the use of reaction jets and the use of reaction flywheels. Reaction jets have the property that the torque they produce is quantized to specific values, usually two, one positive and one negative. As a result, systems employing reaction jet torquers behave as contactor control systems. However, if techniques such as pulse duration modulation are employed, a reaction jet system can be considered linear for analysis purposes. Reaction flywheel torquers, on the other hand, are continuous and in general linear, producing an output torque proportional to the

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\* Note that  $\underline{s}'$  does not contain components relating to range, as does  $\underline{s}$ .

input signal. For either torquing system there is a saturation constraint on the maximum torque that can be produced. In addition, the reaction jet system has a limitation on the total impulse that can be generated set by the amount of fuel on board. A similar limitation for the reaction flywheel system can be avoided if power is obtained, for example, from solar cells.

Since there is no coupling between the axes, the equations for the satellite motion can be partitioned and only one axis need be considered. Taking  $\underline{x}'$  to be the state vector whose components are, for example, pitch angle and pitch angle rate, the satellite response mechanism can be described by

$$\dot{\underline{x}}' = A \underline{x}' + B \underline{u} \quad (7)$$

where in this case the matrix A takes on a particularly simple form

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (8)$$

In those instances where a star tracker is employed to produce other than space oriented attitude control, there will be a second response mechanism within the satellite to position the star tracker axis relative to the satellite. This system, however, may be described and analyzed in a manner completely analogous to that of the satellite tracking system described above.

### C. Data Collection Systems

The data collection system for a satellite tracking station is made up of a conical scan or monopulse receiving antenna structure and the associated electronic demodulating equipment. The output information obtained from this system is two signals which are proportional to the azimuth and elevation components of the antenna pointing error. This pointing error is defined as the angle between the boresight of the antenna and the line from the tracking site to the satellite. This data collection system is non-dynamic (i.e.: it operates much faster than anything else in the tracking system) and may be closely approximated by a linear relationship for small values of the error. Hence the data collection system may be represented by

$$\epsilon_{\theta} = k_{\theta}(\theta - \theta') \quad \epsilon_{\varphi} = k_{\varphi}(\varphi - \varphi') \quad (9)$$

where  $\theta$  and  $\varphi$  are elements of  $\underline{s}$  and  $\theta'$  and  $\varphi'$  are elements of  $\underline{s}'$ . Errors in the form of noise are introduced into these measurements as the result of atmospheric distortion of the radio line of sight, r.f. sky noise, and receiver front end noise.<sup>3</sup>

The data collection systems on-board a satellite for attitude control produce essentially the same information as does that for satellite tracking. A star tracker produces two signals proportional to the components of the pointing error between the axis of the tracker telescope and the line to the star. A horizon sensor produces an output proportional to the angle between its axis and a line from the satellite to the horizon. The principle source of error in a star tracker is distraction by light from stars near the one being tracked, or by illuminated space dust coming into the field of view. Horizon sensors are subject to errors arising from atmospheric refraction and the presence in the atmosphere of clouds, dust, and other effects which tend to obscure the horizon. Both systems are subject to noise generated within the electronic signal processing equipment, but this tends to be of lesser significance.

#### D. Performance Criterion

In both the satellite tracking and satellite attitude control problems, the objective is to maintain the system error as small as possible. In this sense the performance criterion for these systems is instantaneous in that the performance at the present moment is given by the present value of the error. The strict application of such an instantaneous performance criterion, however, can lead to undesirable overall behavior since attempts to immediately null the present error can produce large subsequent overshoots. A more realistic performance criterion is some average measure of system error such as the mean square error. Such a measure provides a stationary criterion that is, in general, dependent only on the configuration of the system and not on the nature of the system signals at any given instant. Such an average measure must be tempered by the constraint that the instantaneous error not be allowed to become so large that input signal is lost due to passing outside the field of view.

### III SYSTEM CONFIGURATION

In the previous section, the basic factors involved in the problems of satellite tracking and attitude control have been discussed. In the present section, system configurations are developed, employing modern filtering, which are capable of alleviating some of the shortcomings inherent in the present methods used to solve these problems.

#### A. Satellite Tracking

The configuration of a satellite tracking system employing modern filtering is shown in Fig. 2.

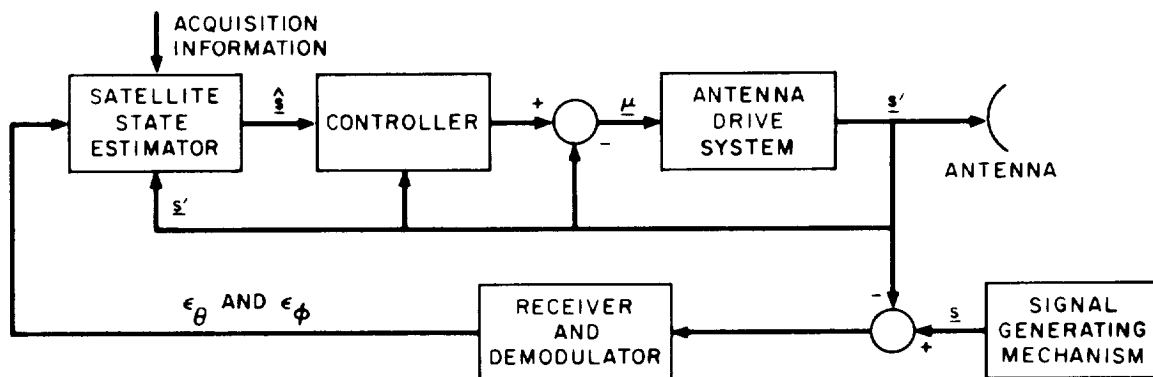


FIG. 2 SATELLITE TRACKING

The operation of this system is as follows: The satellite state estimator (for example a Kalman filter) generates an optimum estimate  $\hat{s}$  of the state of the satellite to be tracked.\* This estimate is based on precalculated acquisition information as well as any radar return data collected. The estimation is done by making use of the model of the signal generating mechanism as given by Eq. 3 and a priori information regarding the parameters to be used in the system model. This estimate is adjusted by the controller to account for such antenna errors as sag, known warpage, encoder errors, etc. In addition, adjustments may be made based on the velocity components of  $\hat{s}$  which will effectively eliminate the dynamic following error of the antenna drive

\*The range components of  $\hat{s}$  are not used in the operation of the tracking system.

system. This latter compensation is similar to the lead employed in shooting at a moving target. The resulting adjusted estimate constitutes a position input signal to the antenna drive system. By feeding back the antenna position around the velocity servo, thus converting it to a position servo, maximum performance of the antenna positioning system can be achieved without affecting the properties of the tracking loop. Hence the position loop can be made very "stiff" to reduce the effects of disturbances such as wind gusts.<sup>4</sup> This contrasts with normal tracking practice where the tracking error signals  $\epsilon_\theta$  and  $\epsilon_\varphi$  are used directly as rate input signals to the respective antenna drive system channels. Hence, if low bandwidth filters are required to combat a low signal-to-noise ratio, this low bandwidth is impressed on the antenna drive system thus deteriorating its immunity to load disturbances.

Tracking error data, obtained via the data collection system described earlier, is fed back to the state estimator, together with the measured antenna state  $\underline{s}'$ . These two pieces of information combine to provide measurements of actual satellite position. After these measurements are corrected to compensate for known system errors (in a manner exactly analogous to that in which  $\hat{\underline{s}}$  is adjusted), they are used to improve the accuracy of the estimation. The manner in which this improvement is accomplished is discussed in Sec. III-C.

In addition to the generation of optimum estimates of the satellite state  $\hat{\underline{s}}$ , several other properties of the estimator can be exploited to enhance system performance. For example, the measured data can be used to improve the parameter values used in the model, to the end that the model more faithfully represents the actual input signal generating mechanism. Another example is the fact that in addition to the estimate of the satellite state, the estimator also provides the covariance associated with this estimate. Such information is useful in assessing system operation and improving system performance. A third example applies in those cases where maneuvering commands are sent to the space vehicle being tracked. These same commands can be applied to the model contained within the estimator with the result that the estimated state reflects anticipation of the space vehicle maneuver. This eliminates the errors caused by waiting to sense the vehicle maneuver via the feedback data.



## B. Satellite Attitude Control

For the case of satellite attitude control the configuration shown in Fig. 2 must be modified to that shown in Fig. 3.

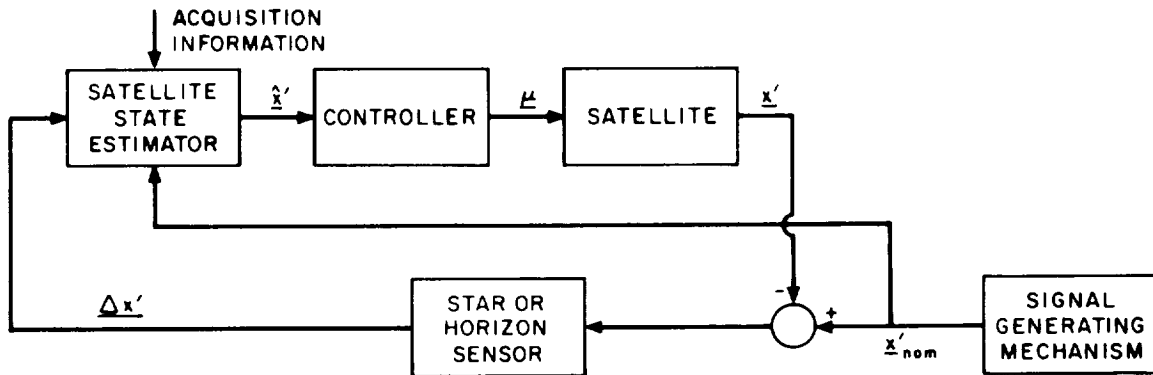


FIG. 3 ALTITUDE CONTROL

In this figure the vector  $\underline{x}'$  represents the state of the satellite (3 angular positions and 3 angular rates) and  $\underline{x}'_{nom}$  represents the desired or nominal value of this state. The difference between these two vectors, as measured by the star tracker (a horizon sensor can measure the difference in only one angle and one angle rate) is used as the measured input data to the satellite state estimator. When this difference vector  $\Delta \underline{x}'$  is added to the known angular orientation of the star tracker axis within the satellite, the result is a measured value for  $\underline{x}'$ . Based on these measurements, a model of the satellite dynamics as given in Eqs. 7 and 8, and initial acquisition information obtained during the lock-on phase, the satellite state estimator generates an optimum estimate of the satellite state  $\hat{\underline{x}}'$ . This is compared in the controller with the nominal state, and torquer control inputs generated to eliminate any deviation from the prescribed attitude.

The optimal controller must be computational in nature since it must calculate, for a pure inertia plant, that time profile of torque about each axis that will just reduce the indicated state error to zero. This is necessary here, unlike the tracking case, since no direct accurate feedback of the satellite state  $\underline{x}'$  is available. (Accurate information about  $\underline{x}'$  could be obtained from, for example a gyro stabilized platform,

but this would involve considerable cost and increase the system complexity and, in large part, would obviate the need for the star or horizon sensing system.) Hence the control of the satellite position cannot be accomplished by a straightforward position loop as was previously possible.

In Sec. II several assumptions were made which resulted in the input signal  $\underline{x}'_{\text{nom}}$  being a constant. If these assumptions are relaxed to take into consideration a non-spherical, non-homogeneous earth and a non-circular orbit, then it is clear that  $\underline{x}'_{\text{nom}}$  will no longer be a constant but will depend on the satellite's position  $\underline{x}$  in its orbit. It is evident that this does not add any complexity to the attitude control system per se, but considerable effort would be involved in obtaining the  $\underline{x}'_{\text{nom}}$  signal.

The observations made for the tracking case regarding the estimator's capability for improving the estimate of the state as well as the model also apply to this case.

### C. Optimum Estimation and the Kalman-Bucy Estimator

The general problem of optimum estimation is formulated with reference to Fig. 4 and in discrete time as follows:

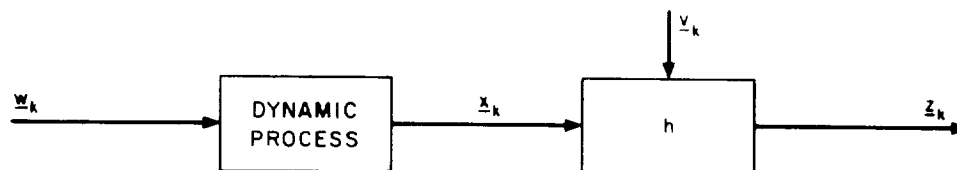


FIG. 4 OPTIMUM ESTIMATION

- $\underline{x}_k$ : state of process at  $k^{\text{th}}$  sampling time
- $w_k$ : random perturbation at time  $k$
- $v_k$ : measurement noise at time  $k$
- $z_k$ : measurement at time  $k$

Given:

- (1) The known difference equation

$$\underline{x}_{k+1} = f(\underline{x}_k, \underline{u}_k, \underline{w}_k, k) \quad (10)$$

- (2) The initial probability density  $p(\underline{x}_0)$

- (3) The statistics of  $\underline{w}$ ,  $p(\underline{w}_i)$ ,  $i = 0, 1, \dots, k$

- (4) The known relation

$$\underline{z}_k = h(\underline{x}_k, \underline{v}_k) \quad (11)$$

- (5) The statistics of  $\underline{v}$ ,  $p(\underline{v}_i)$ ,  $i = 0, 1, \dots, k$

- (6) The measurements  $\underline{z}_i$ ,  $i = 0, 1, \dots, k$

Find: The most likely estimate  $\hat{\underline{x}}_{k/k}^*$  of the system state  $\underline{x}_k$  where most likely is defined as that estimate for which

$$p(\underline{x}_k / \underline{z}_0, \dots, \underline{z}_k)$$

is maximized.

A completely rigorous solution to this problem has been obtained by Kalman and Bucy,<sup>4</sup> for the special case where Eqs. 10 and 11 take the form

$$\underline{x}_{k+1} = \Phi(k) \underline{x}_k + \Gamma(k) \underline{w}_k \quad (12)$$

$$\underline{z}_k = H(k) \underline{x}_k + \underline{v}_k \quad (13)$$

respectively, and in addition the statistics  $p(\underline{x}_0)$ ,  $p(\underline{w}_i)$ ,  $p(\underline{v}_i)$  are uncorrelated Gaussian probability density functions. For this case optimum estimator is given by the equation

$$\hat{\underline{x}}_{k/k} = \Phi \underline{x}_{k-1/k-1} + P_{k/k} H^T R^{-1} [z_k - H \Phi \hat{\underline{x}}_{k-1/k-1}] \quad (14)$$

where  $R$  is the covariance matrix of the noise process  $\underline{v}$ . The matrix  $P_{k/k}$  is defined by

$$P_{k/k} = \langle (\hat{\underline{x}}_{k/k} - \underline{x}_k) (\hat{\underline{x}}_{k/k} - \underline{x}_k)^T \rangle \quad (15)$$

and is the covariance of the estimate  $\hat{\underline{x}}_{k/k}$ . The variations of  $P_{k/k}$  with

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\* The notation  $\hat{\underline{x}}_{k/k}$  denotes the estimate of the value of the state vector  $\underline{x}_k$ , the estimate being made at time  $k$ .

$k$  is given by the equation

$$P_{k/k} = [\phi P_{k-1/k-1} \phi^T + \Gamma Q \Gamma^T + H^T R^{-1} H]^{-1} \quad (16)$$

where  $Q$  is the covariance matrix of the disturbance process  $w$ . A possible implementation of the Kalman - Bucy estimator is given in Fig. 5.

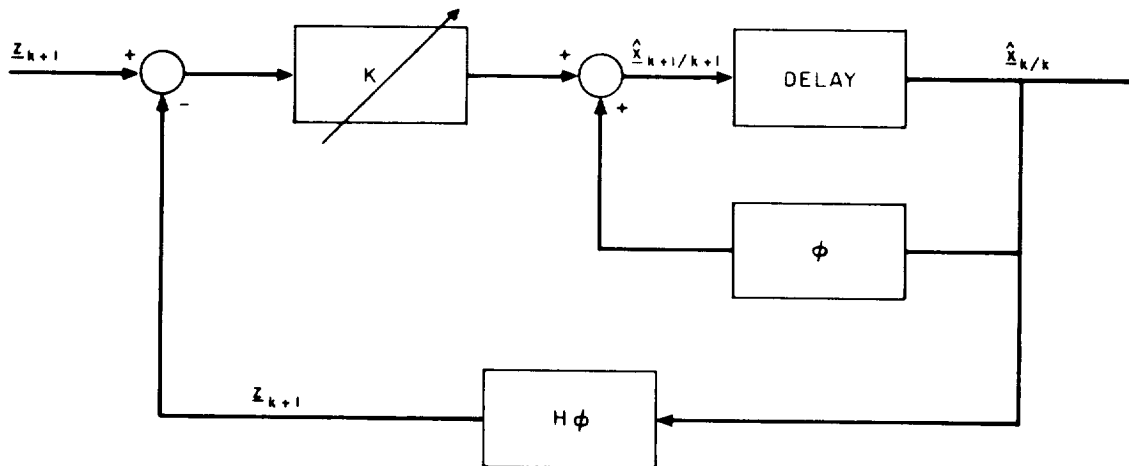


FIG. 5 KALMAN-BUCY ESTIMATOR

$$K = P_{k/k} H^T R^{-1}$$

The development given above is valid only for linear systems in which all stochastic variables are generated by uncorrelated (white) Gaussian processes. The requirement that the noise processes be uncorrelated can be relaxed to permit correlated Gaussian noise, and still retain a rigorous result, by adding "coloring" prefilters to the plant which appropriately shape the spectrum of an input white noise. The problem is thus returned to the original formulation, at the cost of some increase in the complexity of the plant model.

Several suggestions have been made for methods of applying the estimator configuration given by Eqs. 14 and 16, to the case where the plant and/or the observation system are non-linear. These techniques employ linearization of the plant and/or observation system about the present state values or about a nominal set of values (nominal state

trajectory). The problem is then treated as being linear in the small. A theoretical development for this case has been given based on the assumption that the statistics of all signals within the system may be adequately approximated by Gaussian multivariate density functions.<sup>6,7</sup> Furthermore, the concept of local linearization has been tested by at least two simulations, the results indicating satisfactory performance.<sup>7,8</sup>

The linear, or linearized, estimation problem described here, together with the control of an essentially linear plant, constitute a special case of the general problem of combined optimal estimation and control.<sup>9</sup> For this case, and with the further assumption that all probability densities may be approximated by Gaussian density functions, it has been shown<sup>10</sup> that the functions of estimation and control separate and may be performed in cascade. (That this is not generally true is discussed in Sec. V). As a result, the configurations shown in Figs. 2 and 3, where the estimator is followed by a controller, are truly optimal for the linear, or linearized, Gaussian case.

In the event that the values of one or more of the elements in the matrices  $\Phi$  or  $H$  of eqs. 12 and 13 are imperfectly known, these elements can be added as additional state variables to form an augmented state vector. In the process of estimating the state vector, estimates of the values of these parameters will be obtained in addition to an estimate of the plant state. Augmenting the state vector, however, immediately leads to a nonlinear plant equation, and the techniques discussed above for treating nonlinear plants must be employed.<sup>11</sup>

Several other rigorous results have been obtained to the general problem of estimation. The formulation due to Cox<sup>12</sup> arrives at essentially the same results described here, but via the technique of dynamic programming. A somewhat more general class of problems may be handled by this method, however a search over a set of values for the minimum is entailed at each iteration. The recursive application of Bayes Rule for estimation as suggested by Lee<sup>13</sup> and Meier<sup>6</sup>, is completely general and capable of handling any systems and statistical descriptions. However,

they lead to infinite dimensional expressions, thus rendering the method, in a strict sense, uncomputable. The need for suitable approximations to reduce this method to practice is discussed in Sec. V.

#### IV POTENTIAL ADVANTAGES

In this section several potential advantages to be derived from the use of modern adaptive filtering techniques in the context of satellite tracking and attitude control are discussed.

A primary advantage is the enhanced accuracy that can be achieved. This comes about principally as the result of two properties that are inherent in a system employing an optimum estimator and controller. These properties are: immunity to observation noise and immunity to load disturbances. Immunity to observation noise is the consequence of the fact that the estimator produces a least variance estimate of the state from the noisy data received. Hence it is capable of better noise rejection than any other filtering technique. The immunity to load disturbances results from the separation of the functions of filtering and plant output control by an inner feedback loop. (This does not apply in some cases of satellite attitude control.) The system for controlling the plant output can be optimized to reject load disturbances, without degrading the filtering operation of the estimator. Conversely the estimator can optimally reject input noise without degrading the plant performance.

The use of a model of the system dynamics in the structure of the estimator provides the unique capability of continuing to produce state estimates in the absence of feedback data. Whereas in a conventional system design the loss of feedback information constitutes open loop operation, and almost certain unacceptable system behavior, this is not the case for the Kalman estimators described. Loss of feedback data to a modeling estimator simply implies that further estimates will be made on the basis of the system model and data up to that point, with no further corrections being made. In one sense, this operation might be termed operation on the basis of predicted rather than measured values. The direct result of this property is that such systems will be able to successfully cope with temporary loss of feedback data due to such phenomena as fading or scintillation. The reliability of a system is therefore enhanced by the use of modern filtering techniques.

Along these same lines, the use of a modelling estimator permits system operation under feedback signal to noise ratios that would preclude operation of a conventional system. For example, in the case of a satellite tracking system with very low signal to noise ratio (say -20 db), a conventional system design would require a very low pass filter to be included in the signal processing path to reduce the debilitating effects of the noise. Since this filter is in the primary loop, it sets the effective bandwidth of the entire tracking system. In many cases this bandwidth is so low that the system cannot keep up with the satellite and hence tracking is impossible under these circumstances. On the other hand, in the system shown in Fig. 2, the bandwidth of the antenna positioning system is independent of the noise in the feedback channel. Furthermore, since it is driven by the estimator output, it can follow the satellite during high noise conditions--or even in the complete absence of feedback signal as pointed out above. The bandwidth of the feedback signal processing done by the estimator can be made as low as necessary and is governed by the variable gain  $K$  as described in Sec. III.

A priori information regarding the nature of the system dynamics and signal properties is utilized in an efficient (if not optimum) manner in the design and operation of an estimator. The details of the model chosen reflect knowledge of the system dynamics, whereas signal and noise properties are included in the values chosen for the covariance matrices which determine the variable gains of the estimator. In a sense these modern filtering techniques may be thought of as generalizations and extensions of the "matched filter" idea.

It might appear that modern filters such as the Kalman estimator would entail considerable complexity in implementation. This is not necessarily true. To be sure a complete and general realization would most likely require a digital computer. Even in this case, however, for low order systems even the smallest of digital computers will suffice. In those instances where, for example, the variable gain can be pre-programmed, or approximated by a constant value, or where the model takes a particularly simple form (as in the satellite attitude control case), the estimator might well be realizable by simple analog elements.

The Kalman Filter is adaptive in two senses of the word and hence is capable of achieving performance superior to that of a fixed filter. In the first sense, the effective gain  $K$  of the correction portion of the filter is dependent on the quality of the feedback information. If the variance of the measurement is low and that of the present estimate is high, then the measured data is weighted heavily. The converse is also true. Thus, the Kalman filter adapts its behavior to the estimated quality of its own output and to the quality of the measurement data it is receiving. This form of adaptation has been referred to as open-loop adaptation since adaptive action is taken as a direct function of measured quantities (the covariances in this case) and no feedback is involved. Secondly, when the state vector is augmented, so that the estimator is also estimating the values of system parameters, then it is adaptive in an analysis-synthesis sense as well. The estimation of the system parameter values constitutes the analysis step, whereas the synthesis step consists of using these estimated values to improve the estimator model and hence improve its overall performance.

Since, under equal operating conditions, modern filtering techniques promise performance superior to conventional techniques, it is reasonable to expect that their performance will still be acceptable under deteriorated conditions when conventional filters can no longer perform properly. Advantage can be taken of this feature, for example, by reducing the required transmitter power for reliable tracking, thus reducing the on-board weight required for this function and making room for other scientific equipment or reducing the boost energy required. In the satellite attitude control context it presents the possibility of using less sensitive detectors at a possible savings in system weight and complexity.

The emerging field of laser communication will place very stringent requirements on future earth-based and space vehicle based tracking, pointing, and attitude control systems. The control methods discussed in this memorandum constitute one possible solution to this demanding problem.

The development presented in Secs. II and III has been kept purposely general in nature. This generality can rightfully be cited as an advantage of this approach to systems control. Upon reflection, it becomes



evident that the approach indicated in these previous sections is quite general, and could easily be applied to a variety of system problems differing widely in context and detail from the two examples quoted throughout this memo.

#### V RESEARCH PROBLEMS

The principles discussed in this memorandum have been developed to the point where they can be applied in the immediate future to a satellite tracking or stabilization problem with almost complete assurance of success. Furthermore, the expected advantages to be gained from such application have been pointed out. The next logical step in the development of this method would be its implementation in an actual tracking or attitude control situation and the determination from operating experience of the benefits realized and the practical application problems that are yet to be resolved.

There remain several interesting questions of a theoretical nature in the application of modern filtering techniques to the problems of satellite tracking and attitude control. The most specific question concerns the mathematical models used to describe the signal generating and response mechanisms. In some cases, such as satellite rotational dynamics, the model is straightforward and simple in form. However, the model for satellite orbit mechanics is quite complex. The extent to which such complex models can be simplified for practical implementation, and yet retain essentially optimum estimator performance has not yet been determined. In addition, appropriate descriptions for the noise processes and possible simplifications thereof, have not in many cases been formulated.

The results of combined optimization theory<sup>9</sup> indicate that in a combined estimation and control problem such as we have here, the optimum control decisions must in general be made on the basis of the probability density over the state space, rather than on the basis of any specific estimate of the state vector. The only known exception to this statement is the linear Gaussian case where it has been shown<sup>10</sup> that the information contained in the distribution is completely summarized in the

estimate. For other situations, the extent of system performance degradation suffered by using only a single estimate rather than the entire density function should be determined. Since the determination and evaluation of a complete probability density function involves considerable computational complexity, it would be desirable to approximate the complete density function by perhaps its first several moments. The incremental gain in performance for each additional term included in the approximation should be evaluated.

The properties of the Kalman-Bucy estimator have been rigorously determined only for the case of a linear plant and Gaussian noise and disturbances. The several techniques available for extending the application of these filtering methods to nonlinear and nonGaussian situations should be rigorously examined to determine the significance and properties of the resulting estimates.

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MEMORANDUM 2

EVALUATION OF MODERN ESTIMATION AND CONTROL TECHNIQUES  
AS APPLIED TO SATELLITE TRACKING AND ATTITUDE STABILIZATION



## MEMORANDUM 2

### EVALUATION OF MODERN ESTIMATION AND CONTROL TECHNIQUES AS APPLIED TO SATELLITE TRACKING AND ATTITUDE STABILIZATION

#### I INTRODUCTION

This memorandum is a sequel to Memorandum 1.<sup>1\*</sup> Its purpose is to specify those tasks necessary to evaluate the extent of system performance improvement resulting from the application of modern estimation and control techniques to satellite tracking or attitude control. In the interest of brevity, only the satellite tracking problem<sup>\*\*</sup> will be discussed in this memo, although a directly analogous discussion applies to the attitude control problem.

The evaluation of system performance, when conventional or modern techniques are used, may be divided into three distinct tasks. The first involves the development of a complete mathematical description of the system and its inputs. Second, this description is converted to a program suitable for use on any of a variety of digital computers. Finally, the system performance is evaluated for several sets of conditions, using first conventional techniques and then modern methods. The completion of these tasks will result in a flexible general-purpose program that can be used to test a variety of systems. In particular, by testing systems of both modern and conventional design, a quantitative measure of their relative performance is obtained. In addition, since system optimality is only in terms of the system description employed, evaluation of system

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\* Reserences are listed at the end of this Memorandum.

\*\* In this context, the word satellite does not necessarily refer to a near-earth vehicle only.

performance via this program will indicate those areas where the system description must be expanded or revised.

These three tasks are discussed in greater detail in the following sections.

## II SYSTEM DESCRIPTION

For the purpose of a mathematical description, a satellite tracking system can be considered to consist of five major parts:

- (1) Signal generating process
- (2) Measurement system including measurement noise
- (3) State estimator
- (4) Controller
- (5) Drive system dynamics including load disturbances.

These divisions are illustrated in Fig. 1.

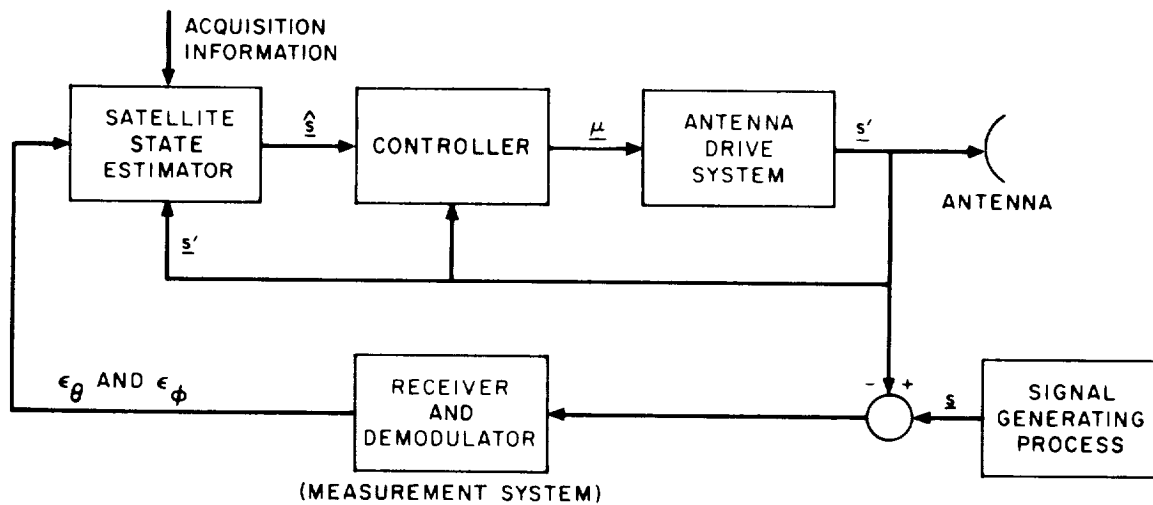


FIG. 1 SYSTEM CONFIGURATION

The equations describing the behavior of each of these components must be stated explicitly in order to carry out a system simulation.



#### A. Signal Generating Process

For an initial simulation of the satellite orbit signal generating process, it will be sufficient to consider the orbit to be the solution of the classical two-body problem, ignoring higher order terms which account for the effects of other celestial bodies and the non-uniformity of the earth's gravitational field. Making this approximation does not limit the generality of the approach since the omitted terms can easily be added to the equations of motion at a future time. One advantage gained by using the simplified orbit equations is the fact that the solution is known to be the Keplerian ellipse, which can be described by six orbital elements. Employing this representation will greatly simplify the computation required for an initial simulation.

The motion of the earth, and hence of the tracking site, can be easily described by a set of kinematic relationships describing the earth's rotation about its axis. (Since the origin of the coordinate system lies at the earth's center, and the earth's axis remains fixed in this coordinate system, the earth's rotation on its axis is the only motion of interest here.) The vector difference between the position of the satellite in its orbit and the position of the tracking site constitutes the input signal for the tracking system.

#### B. Measurement System

The measurement system of a satellite tracking system includes the receiving antenna structure together with the receivers and demodulators. To model the operation of the antenna structure it is necessary to convert

the input signal into "radar coordinates", i.e., azimuth, elevation, and range. This can be easily accomplished by means of a well known coordinate transformation matrix. The antenna structure is then represented by two non-linear functions (one for azimuth and one for elevation) which relate the received signals to the antenna angular pointing error in azimuth and elevation respectively. Noise signals must be added to each of these signals. The principal source of this noise in an actual system is the thermal noise introduced by the receivers themselves. Experience has shown that this noise can be accurately described as narrow band Gaussian and can be easily simulated by means of a digital noise generation routine.

The bandwidth of the receiver and demodulator electronics is usually high enough that their dynamic effects are entirely negligible. The measurement system is therefore completely described by the coordinate transformation matrix, the non-linear functions and the noise statistics.

### C. State Estimator

The purpose of the estimator is to generate from the available data a best estimate of the present state of the satellite with respect to the tracking site. This estimate is used as the input to the antenna control system.

The equations of a Kalman-Bucy recursive estimator are described in detail in Section III C of Memorandum 1.<sup>1</sup> These equations are already stated in a form to facilitate direct implementation by digital computer. Much of the programming required for this implementation is

already available in an automatic synthesis program developed by Kalman and Englar.<sup>2</sup>

In the event that the tracking site is operating in a radar mode, there is a two-way transport lag between the transmission of the radar pulse and the receipt of the echo. The equations for the Kalman-Bucy estimator given in Ref. 1 can be easily modified to accommodate this situation.

#### D. Controller

Several of the intended functions of the controller would be superfluous in an initial program. These include preprogrammed compensation for calibratable errors such as sag, warpage, and encoder errors. Implementation of these compensations would, in effect, amount to adding in a known error, only to subtract it out again. Hence very little further information on the system's performance would be gained.

On the other hand, the very important function of compensating the drive system for dynamic lags must be included. It is well known that for a linear system the optimal controller (in a least mean square error sense) consists of a linear function of the state and input variables, and is constant for steady state operation. Since some of the system state variables may not be directly measurable, as discussed in the next section, estimates for their values can be obtained from the state estimator by enlarging the model used.

#### E. Drive System Dynamics

During a tracking operation, when the incremental input commands are not likely to be large, the dynamics of the antenna drive system can

be considered to be completely linear\*. A quite accurate representation is obtained when the system dynamics are represented by a differential equation of third order in each axis (azimuth and elevation). In addition to the above dynamics, there exist resonances of the reflector and feed structures which affect the pointing direction of the antenna but are not measurable by the antenna position or velocity pickups. These dynamics give rise to the unmeasurable state variables referred to above. Their effects, however, are included in the monopulse error signals and hence their values may be estimated by the estimator.

The predominant load disturbance experienced by a tracking antenna is that of disturbance torques resulting from wind gusts. Newton, Gould and Kaiser<sup>3</sup> have shown that these torques may be described as a random variable with power density spectrum given by

$$\Phi(s) = \frac{\beta^2 \nu}{\pi(-s^2 + \nu^2)}$$

$\beta, \nu$  : measurable parameters

A random variable generated digitally according to this specification can easily be included in the above description of plant dynamics to include these load disturbances in the overall simulation.

### III PROGRAMMING

The system description obtained in task 1 in the form of a set of equations and definition of variables, can be translated into a digital

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\* It is assumed that anti-backlash devices are used, as is now common practice.

computer program. A universal programming language such as FORTRAN IV should be used so that the simulation can be performed on any convenient computer of appropriate size \*.

It is intended to organize the program in such a manner that individual sections can be conveniently modified or replaced. For example, the state estimator and optimal controller portion of the program would be replaced by a simulated conventional controller to obtain a comparison of system performance under both conventional and modern control.

#### IV SYSTEM PERFORMANCE EVALUATION

Included in this final task is the actual operation of the program. Both conventional and modern techniques will be used in a variety of environmental situations ranging from minimal noise and disturbance difficulties to input signal to noise conditions which preclude successful operation of the conventional methods.

Analysis of the results obtained produce two valuable pieces of information. First, a quantitative measure of system performance will be obtained. The performance of the system employing modern estimation and control techniques may be used as a "yardstick" against which to compare the performance of systems using other techniques. Second, an optimal system controller is only optimal for the system description for which it is derived. If this description is not appropriate, optimal performance

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\* All programs developed at SRI would be made available to ERC, together with program descriptions, to enable ERC personnel to perform concurrent system simulations.

will not be achieved. By testing several modifications of the system description, those equations or expressions that must be expanded or revised will become evident.

Ultimately the program could be used with actual recorded tracking data to test a particular system design's performance under actual operating conditions.

#### V LEVEL OF EFFORT

The estimated level of effort required to complete the tasks included in the preliminary evaluation of the performance of modern filtering techniques applied to satellite tracking is as follows:

- (1) Obtain explicit development of a mathematical system description, including both modern and conventional control, in a form suitable for computer programming 1 man-month
- (2) Program of the above equations in a programming language such as FORTRAN IV, and debug the program 1 man-month
- (3) Carry out several comparisons and perform data analysis and evaluation 2 man-months

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MEMORANDUM 3

RELIABILITY CONSIDERATIONS IN SYSTEM DESIGN



### MEMORANDUM 3

#### RELIABILITY CONSIDERATIONS IN SYSTEM DESIGN

In the present technical memorandum, the problem of designing a system which strikes a best balance between performance and reliability subject to equality and inequality constraints is formulated. A discussion concerned with trade-offs between reliability parameters and performance parameters has not been found in the reliability literature.

In a forthcoming technical memorandum, the problem of optimally allocating (in terms of satisfying the mission objectives) the remaining equipment and other resources after a failure has occurred is formulated and the implementation of a monitoring and control system capable of accomplishing this function is discussed. The first formulation given in the present memorandum does not contain any adaptive concepts; the second as discussed in Memorandum 4 does in the sense that the system makes best use of the remaining resources, thus adapting itself to the changed conditions resulting from equipment failure.

#### I DEFINITIONS AND TERMINOLOGY

The system under discussion contains  $N$  interconnected subsystems of capability  $x_i$  and failure probability  $\mu_i$ ,  $i=1, \dots, N$ . The general term capacity encompasses all those variables associated with a subsystem which contribute to the performance of the overall system. This performance will be measured by  $Q(\underline{x})$ .

The function  $Q_i(\underline{x})$  is defined as the performance after the  $i$ 'th subsystem or equipment has failed. This definition of  $Q_i$  allows for the

rearrangement of the subsystems as well as for redundancy. The performance  $Q_i$  occurs with probability  $\mu_i$ .

The function  $V(\underline{x}, \underline{\mu})$  is defined as the value or expected system performance assuming single, and complete failures; thus

$$V(\underline{x}, \underline{\mu}) = Q(\underline{x})(1 - \sum_i \mu_i) + \sum_i Q_i(\underline{x}) \mu_i \quad (1)$$

The cost or weight  $c_i$  of the  $i$ 'th subsystem depends on the capacity  $x_i$  and the failure probability  $\mu_i$ ; thus

$$c_i = c_i(x_i, \mu_i) \quad (2)$$

The total system cost (or weight) is thus

$$C = \sum_i c_i(x_i, \mu_i) \quad (3)$$

The selection of the system parameters  $x_i$  and  $\mu_i$  is not completely arbitrary, but must satisfy certain constraints; thus

$$\begin{aligned} x_i &\in X_i \\ \mu_i &\in U_i \end{aligned} \quad (4)$$

In particular

$$0 \leq \mu_i \leq 1 \quad ; \quad x_i \geq 0 \quad (5)$$

## II PROBLEM STATEMENT

Given a quantitative statement of system performance in terms of the  $x_i$  and an upper bound on the cost (or weight) of the system, it is desired to maximize the value  $V(\underline{x}, \underline{\mu})$  subject to the constraints  $x_i \in X_i$ ,  $\mu_i \in U_i$ . Alternatively, one may wish to minimize  $C$  subject to a minimum bound on the value.

As stated, this is a nonlinear static optimization problem which,

as a rule, cannot be handled by standard calculus because more often than not the solution is located on a constraint. Nonlinear programming and dynamic programming both appear capable of handling problems of this kind.

### III THE KUHN AND TUCKER THEOREM<sup>1</sup>

This theorem is of major importance in nonlinear programming. Since control engineers are usually not familiar with this theorem, it is restated here.

Given: (1) A cost  $F(\underline{x})$  of the vector variable  $\underline{x}$

(2) Inequality constraints of the form

$$\underline{G}(\underline{x}) \leq 0 \quad (6)$$

(3) Equality constraints

$$\underline{H}(\underline{x}) = 0 \quad (7)$$

and assuming convexity for  $F$ ,  $G$ , and  $H$ , the conditions under which  $F(\underline{x})$  is minimum is  $dL = 0$  with

$$\mathcal{L} = F(\underline{x}) + \underline{\alpha}^T \underline{G}(\underline{x}) + \underline{\beta}^T \underline{H}(\underline{x}) \quad (8)$$

$$\alpha_i \geq 0, \beta_i \text{ arbitrary} \quad (9)$$

and

$$\alpha_i G_i(\underline{x}) = 0 \quad (10)$$

The  $\alpha_i$  and  $\beta_i$  are the dual variables associated with the inequality constraints (6) and the equality constraints (7), respectively. Condition (10) implies that  $\alpha_i = 0$  if constraint  $G_i(\underline{x})$  is not reached and  $\alpha_i > 0$  in the alternative.

If the function  $F(\underline{x})$  is a profit function to be maximized, the  $\alpha_i$  must satisfy the condition

$$\alpha_i \leq 0 \quad (11)$$

The main shortcomings of the Kuhn and Tucker theorem are:

1. The satisfaction of the convexity assumption which is difficult to prove.
2. The requirement for continuous variation of the variables in the permissible range.
3. The frequent difficulty in solving the optimization equations.

The dual variables measure the sensitivity of performance with respect to the constraint values imposed.<sup>2</sup> This is major advantage in many applications.

#### IV ILLUSTRATIVE EXAMPLES

To illustrate the method, the following contrived two examples will be treated.

Assume that a space vehicle has the mission of making a measurement and of telemetering this measurement back to the earth. The measurement instrument is characterized by its standard deviation  $\sigma$  (the inverse of which is taken as its capacity) and its failure probability  $\mu_1$ . The transmitter is characterized by its failure probability  $\mu_2$ .

The performance of the system is measured in terms of  $Q$  by

$$Q = \frac{1}{\sigma} \quad (12)$$

with

$$\sigma \geq \frac{1}{2} \quad (13)$$

for example 1 and no lower bound on  $\sigma$  for example 2.

There is an upper limit on the combined weight of the instrument and the transmitter

$$C \leq 200 \quad (14)$$

The individual weights of the instrument and the transmitter are

$$c_1 = \frac{1}{\sigma \mu_1} \quad (15)$$

$$c_2 = \frac{1}{\mu_2} \quad (16)$$

with

$$0.01 \leq \mu_1 \leq 0.1 \quad (17)$$

$$0.005 \leq \mu_2 \leq 0.1$$

The value associated with the data received is thus

$$v = \frac{1}{\sigma} (1 - \mu_1 - \mu_2) + 0 \cdot \mu_1 + 0 \cdot \mu_2 \quad (18)$$

The exclusion equations (10) of the Kuhn and Tucker theorem are

$$\varphi_1 \left( \frac{1}{2} - \sigma \right) = 0$$

$$\varphi_2 (\mu_1 - 0.1) = 0$$

$$\varphi_3 (0.01 - \mu_1) = 0$$

$$\varphi_4 (\mu_2 - 0.1) = 0$$

$$\varphi_5 (0.005 - \mu_2) = 0 \quad (19)$$

The function  $\mathcal{L}$  is

$$\begin{aligned} \mathcal{L}(\sigma, \mu_1, \mu_2) &= \frac{1}{\sigma} (1 - \mu_1 - \mu_2) + \lambda \left( \frac{1}{\sigma \mu_1} + \frac{1}{\mu_2} - 200 \right) + \varphi_1 \left( \frac{1}{2} - \sigma \right) \\ &+ \varphi_2 (\mu_1 - 0.1) + \varphi_3 (0.01 - \mu_1) + \varphi_4 (\mu_2 - 0.1) \\ &+ \varphi_5 (0.005 - \mu_2) \end{aligned} \quad (20)$$

At the optimum

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = 0 = -\frac{1}{\sigma} - \frac{\lambda}{\sigma \mu_1^2} + \varphi_2 - \varphi_3 \quad (21)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} = 0 = -\frac{1}{\sigma} - \frac{\lambda}{\mu_2^2} + \varphi_4 - \varphi_5 \quad (22)$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} = 0 = -\frac{1 - \mu_1 - \mu_2}{\sigma^2} - \frac{\lambda}{\sigma^2 \mu_1} - \varphi_1 \quad (23)$$

$$\frac{1}{\sigma \mu_1} + \frac{1}{\mu_2} - 200 = 0 \quad (24)$$

It is easily seen by setting all the dual variables  $\varphi$  equal to zero that equations (21) through (23) do not have an acceptable solution, e.g. the optimum must be located on one or more constraints. After a few trial calculations, it can be guessed that the solution is located on the constraint  $\sigma = \frac{1}{2}$ . The optimization equations are thereafter resolved to determine  $\lambda$ ,  $\mu_1$  and  $\mu_2$  and to check this solution by verifying that  $\varphi_1 < 0$ , since this is a necessary condition for the optimum. The result is

$$\begin{aligned} \sigma &= \frac{1}{2} & \varphi_1 &= -3.952 \\ \mu_1 &= 0.01707 & V &= 1.9416 \\ \mu_2 &= 0.0121 & \lambda &= -2.91 \cdot 10^{-4} \end{aligned} \quad (25)$$

For the second example, we assume that the lower bound on  $\sigma$  is zero and we allow the possibility of both equipments failing. Then

$$V = \frac{1}{\sigma} (1 - \mu_1 - \mu_2 + \mu_1 \mu_2) \quad (26)$$

and

$$\mathcal{L}(\sigma, \mu_1, \mu_2) = \frac{1}{\sigma} (1 - \mu_1 - \mu_2 + \mu_1 \mu_2) + \lambda \left( \frac{1}{\sigma \mu_1} + \frac{1}{\mu_2} - 200 \right)$$



$$+ \varphi_2(\mu_1 - 0.1) + \varphi_3(0.01 - \mu_1) + \varphi_4(\mu_2 - 0.1) + \varphi_5(0.005 - \mu_2) \quad (27)$$

The optimization equations are now

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = 0 = -\frac{1}{\sigma} + \frac{\mu_2}{\sigma} - \frac{\lambda}{\sigma \mu_1^2} + \varphi_2 - \varphi_3 \quad (28)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} = 0 = \frac{-1}{\sigma} + \frac{\mu_1}{\sigma} - \frac{\lambda}{\mu_2} + \varphi_4 - \varphi_5$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} = 0 = \frac{-(1 - \mu_1 - \mu_2 + \mu_1 \mu_2)}{\sigma^2} - \frac{\lambda}{\mu_1 \sigma^2} \quad (30)$$

As an initial guess, we try  $\mu_1 = 0.1$ ,  $\mu_2 = 0.1$  and find that  $\varphi_4$  is positive, indicating that the upper bound of  $\mu_2$  is not reached.

For  $\mu_1 = 0.1$ , the optimization equations yield the following optimum solution:

$$\begin{aligned} \mu_1 &= 0.1 \\ \mu_2 &= 0.071 \\ \sigma &= 0.054 \\ V &= 15.4 \\ \lambda &= -0.0826 \\ \varphi_2 &= -136 \end{aligned}$$

It should be noted that the sensitivity about the optimum point is quite small; for example, a choice of  $\mu_1 = \mu_2 = 0.1$  leads to a value of 15.2 instead of 15.4

The dual variable  $\varphi_1$  measures the sensitivity of  $V$  with respect to the constraint  $\sigma^m = \frac{1}{2}$ , that is

$$\varphi_1 = \frac{\Delta V}{\Delta \sigma^m} \quad (26)$$

Similarly, the dual variable  $\lambda$  measures the sensitivity of  $V$  with respect to the weight constraint  $C$ , that is

$$\lambda = - \frac{\Delta V}{\Delta C} \quad (27)$$

#### V COMPUTATIONAL PROCEDURES FOR NONLINEAR PROGRAMMING

Clearly, if the number of variables becomes larger, it is increasingly difficult to guess which variables must lie on constraints. There now exist efficient computational procedures capable of handling up to several hundred variables. The most straightforward approach is to apply Newton's method to the linearized optimization equations (21) through (24), replacing the primary variables  $x_i$  and  $\mu_i$  by their duals once a constraint is reached. The number of unknowns is thus always equal to the number of optimization equations.

## VI EXTENSIONS

The design procedure outlined above can be extended by relaxing some of the restrictive assumptions made. In particular:

1. It is possible to include partial equipment failures by means of the probability density function  $p(x_i)$ , where the variable  $x_i$  ranges over all possible values below nominal possibly in discrete steps. It is then necessary to calculate the function  $Q_i(\underline{x})$  for the whole range of variation of  $x_i$  below nominal capacity. The expected performance for  $x_i$  below nominal is thus

$$\int_{x_i} Q_i(\underline{x}) p(x_i) dx_i \quad (28)$$

2. It is also possible to consider multiple failures involving the equipments  $i$  and  $j$  by defining the remaining performance  $Q_{ij}$  which occurs with probability  $\mu_i \mu_j$  for complete failures. In the case of partial failures, the corresponding expected performance for both  $x_i$  and  $x_j$  below nominal is then

$$\int_{x_i} \int_{x_j} Q_{ij} r(x_i, x_j) dx_i dx_j \quad (29)$$

This process can easily be extended to any number of simultaneous failures.

## VII SHORTCOMINGS

At the present stage, the following shortcomings are noted:

1. The failure probabilities  $\mu_i$  are often not available except for certain military equipment.

2. The calculation of  $Q_i(\underline{x})$  may be quite involved in those cases where the failure of equipment  $i$  entails the failure of adjacent equipment or where the function of equipment  $i$  can be taken over partly or wholly by adjacent equipment.

3. The functions  $c_i(x_i, \mu_i)$  are usually not continuous since the designer only has a limited number of equipment alternatives. To handle this situation with nonlinear programming, he would first obtain the function  $c_i(x_i, \mu_i)$  by fitting a curve to the available data, then obtain the optimum point, and finally round off to the nearest alternative. This rounding off procedure does not necessarily lead to an optimum solution in terms of the available alternatives, but should be close.

## VIII DYNAMIC PROGRAMMING APPROACH

In general, the value  $V$  is a function of the form

$$V(\underline{x}, \underline{\mu}) = Q(\underline{x}) (1 - \sum_i \mu_i) + \sum_i \mu_i Q_i(\underline{x}) \quad (30)$$

Under special circumstances,  $V$  reduces to a function of the form

$$V = \sum_i f_i(x_i, \mu_i) \quad (31)$$

in which case dynamic programming can be applied conveniently as follows:<sup>3</sup>

It is desired to maximize  $V$  by selecting  $\underline{x}$  and  $\underline{\mu}$  subject to

$$C_N = \sum_i c_i(x_i, \mu_i) \leq C \quad (32)$$

We define

$$\begin{aligned} I(C_N, N) &= \max_{\underline{x}, \underline{\mu}} \left\{ \sum_i f_i(x_i, \mu_i) \right\} \\ &= \max_{x_1, \mu_1} \max_{\substack{x_j, \mu_j \\ j=2, \dots, N}} \left\{ f_1(x_1, \mu_1) + \sum_j f_j(x_j, \mu_j) \right\} \end{aligned} \quad (33)$$

$$= \max_{x_1, \mu_1} \left\{ f_1(x_1, \mu_1) + I(C_N - c_1, N-1) \right\} \quad (34)$$

At stage  $\alpha$ , the recursive relation (34) is written

$$I(C_N - \sum_{j=1}^{\alpha-1} c_j, \alpha-1) = \max_{x_\alpha, \mu_\alpha} \left\{ f(x_\alpha, \mu_\alpha) + I(C_N - \sum_{j=1}^{\alpha} c_j, \alpha) \right\} \quad (35)$$

Equation (35) constitutes a dynamic programming algorithm in the single state variable  $C_N - \sum_{j=1}^{\alpha-1} c_j$ , the amount left at stage  $\alpha$ , and hence presents no computational problems of any significance. It not only gives the best allocation policy and value for the allowable cost  $C$ , but for all cost  $0 < C_N \leq C$ ; this is valuable additional information.

In the general case,  $V$  depends on  $\underline{x}$  and  $\underline{\mu}$  as in (30). Under those circumstances, the maximization (33) does not separate and a recursive

relation of the form (35) cannot be written. From simple examples treated in the course of this study, it appears, however, that repeated application of the recursive equation

$$I(C_N - \sum_{\alpha-1}^{\alpha} c_j, \alpha-1) = \max_{x_\alpha, \mu_\alpha} \{ V(x_\alpha, \mu_\alpha; \underline{x}^*, \underline{\mu}^*) + I(C_N - \sum_{\alpha}^{\alpha} c_j, \alpha) \} \quad (36)$$

where  $\underline{x}^* = \{x_1, \dots, x_{\alpha-1}, x_{\alpha+1}, \dots, x_N\}$  and

$$\underline{\mu}^* = \{\mu_1, \dots, \mu_{\alpha-1}, \mu_{\alpha+1}, \dots, \mu_N\}$$

denote the optimizing values found in the previous iteration, converges toward the optimum after a few iterations when certain precautions are taken.

## IX ILLUSTRATIVE EXAMPLE

The example previously treated by nonlinear programming is now treated by dynamic programming. To illustrate the ability of dynamic programming to handle discrete variables, we assume that the equipment available is characterized as follows:

### Transmitter:

	<u>Capacity</u>	<u>Failure Probability</u> ( $\mu_1$ )	<u>Cost</u>
A	1	.0055	180
B	1	.0067	150
*C	1	.01	100
D	1	.02	50
E	1	.05	20
F	1	.10	10

### Measuring Equipment

	<u>Standard Deviation</u> ( $\sigma$ )	<u>Failure Probability</u> ( $\mu_1$ )	<u>Cost</u>
A'	2	.01	50
B'	2	.025	20
C'	2	.05	10
D'	1	.01	100
E'	1	.02	50
F'	1	.05	20
G'	1	.10	10
H'	0.5	.011	180
*I'	0.5	.02	100
J'	0.5	.05	40
K'	0.5	.10	20

For this example problem involving two subsystems only, the recursion equation (35) can be carried out easily by hand. The resulting optimum combinations of equipment together with V are shown in terms of the total cost  $C_N$  in Fig. 1.

It is seen that the equipment combination C, I' ( $\mu_1 = 0.02$ ,  $\mu_2 = 0.01$ ,  $\sigma = 0.5$ ) is optimum for  $C_N = C = 200$ , resulting in  $V = 1.94$ .

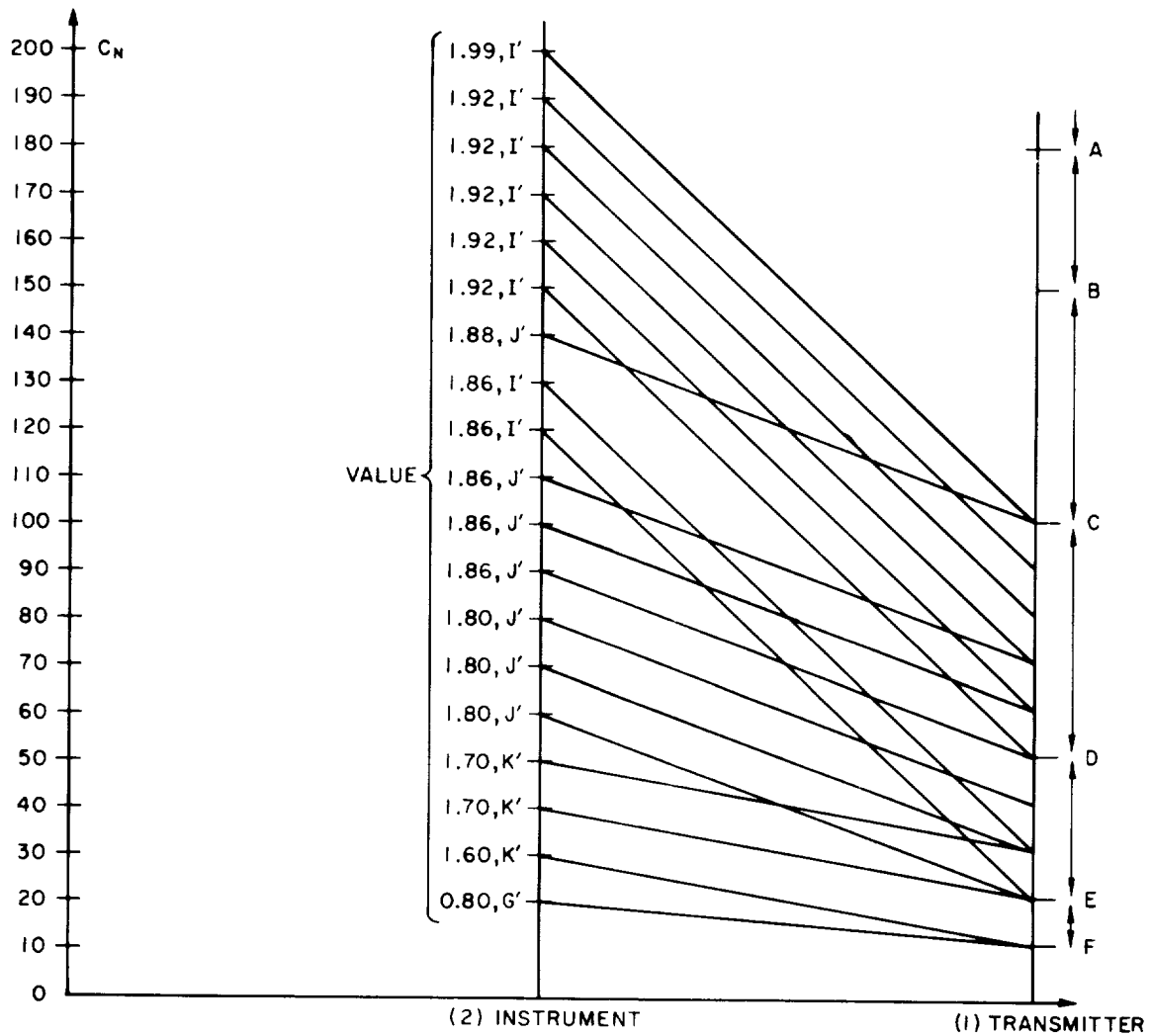


FIG. 1 OPTIMIZATION BY DYNAMIC PROGRAMMING

At stage (1), the cost of transmitter F through A is marked. At stage (2), the most appropriate instrument-transmitter combination is selected for a given total cost  $C_N$ . The inclined lines joining stage (2) to stage (1) identify the optimum equipment combinations, for increments of 10 of  $C_N$ . Some of these inclined lines terminate at stage 1 at points not marked by any equipment; the significance of this is that some of the allowed weight  $C_N$  is not used.



This should be compared with the results previously attained by nonlinear programming, i.e.,  $\mu_1 = 0.01707$ ,  $\mu_2 = 0.0121$ ,  $\sigma = 0.5$ ,  $V = 1.9416$ , under the assumption that the variables  $\mu_1$ ,  $\mu_2$ ,  $\sigma$  are continuous.

It is furthermore seen from Fig. 1. that the sensitivity of  $V$  with respect to  $C_N$  is very small; thus, by selecting, for example, equipment J' and E, the value drops from 1.94 to 1.80, but the weight drops from 200 to 60. This information is also provided by the Kuhn and Tucker theorem--see (27)--but only for small variation  $\Delta C$ .

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MEMORANDUM 4

RELIABLE OPERATION OF SYSTEMS



## MEMORANDUM 4

### RELIABLE OPERATION OF SYSTEMS

This memorandum considers the problem of operating a system so that it performs its mission in a reliable manner. Of particular concern will be the amount of redundancy that should be built into the system and how this redundancy can be best used in the case of a failure.

#### I INTRODUCTION

The prime question to be treated is: In case of a failure, how is the remaining operative equipment best utilized? For this question to have any meaning, either the remaining equipment must be capable of performing, at least partially, the functions of the failed equipment or there must be alternative objectives which the system may perform without the failed equipment. In other words, the system must have redundancy built into it and no system without this redundancy will operate satisfactorily when failures occur no matter how much effort is exerted toward adapting to changed circumstances.

To make use of any available redundancy, three tasks must be performed:

1. Detection of failures.
2. Decision of what to do about failures.
3. Implementation of these decisions.

The similarity of these functions with the functions of measurement, control decision, and actuation in a control problem suggest the use of a state space model to describe the system. Such a model is developed in Sec. II and its use is illustrated in Sec. III.

Of equal importance with the problem of using available redundancy of a given system is the problem of designing that redundancy into the system.

What equipment should be duplicated or backed up, what alternative mission objectives should be considered, how extensive should the failure detection equipment be and what alternative modes of operation should be allowed?

Section IV considers the use of the mathematical model developed in Sec. II to answer these questions as well as the operational questions posed above.

## II MATHEMATICAL MODEL

The behavior of a system can be described in terms of its state, which by definition completely summarizes its past history. The operating status of its components and the successful occurrence of certain events are of prime importance to the operation of a system. A state space description of both these system properties is presented in the present section along with equations describing their change in time.

Let  $x_i(t)$  give the operating status of the  $i$ th component at time  $t$ , where each different mode of operation is assigned a number. For example, if the first component is the sequencer, then  $x_1(t) = 0$  might represent sequencer failed at time  $t$ ,  $x_1(t) = 1$  sequencer turned off, and  $x_1(t) = 2$  sequencer turned on. Note that failure, which implies a random breakdown which is difficult or impossible to reverse is quite different from turned off, which is an easily changed condition.

Most components will have at least the three modes of operation described above; however, many components may have more modes of operation. For example, a transmitter may be capable of operating at two power levels and hence may have five possible modes: 0, completely failed; 1, turned off; 2, low power due to failure; 3, low power by choice; and 4, high power. It should be emphasized that when talking about the status of a component, we are not restricting ourselves to failure modes but are describing the complete operating status.

Let  $y_i(t)$  describe the status of the  $i$  th event at time  $t$ . In most cases,  $y_i(t)$  will have two possible values, 0 and 1, indicating that the event has not yet occurred or that it has occurred. (Note that once  $y_i(t)$  goes from zero to one it remains there.) In some cases, however, it may be convenient to define compound events and then  $y_i(t)$  can take on more than two values. In general  $y_i(t)$  will increase by one as each part of the compound event occurs successfully.

Events, failures, or directed changes in operation may take place at any time; however, for purposes of description and computation it is convenient to quantize time. The rate of sampling may vary greatly as a function of the phases of the mission. For example, a very high sampling rate is likely during a midcourse maneuver whereas a low sampling rate is reasonable during cruise.

It is now possible to describe the operation of the system in terms of  $x_i$  and  $y_i$ . There are two basic ways in which the operating status of a component may be changed: a random event may cause a failure in the component, or the component may be switched by command. Let the different types of random failures be numbered and let  $w_j(t)$  be 0 or 1 according to whether or not the  $j$  ' th failure has taken place. Similarly, set  $u_i(t_{k+1}) = 1$  if the  $i$  ' th command is given during the interval  $(t_k, t_{k+1})$  and zero otherwise. Then

$$x(t_{k+1}) = f_k[x(t_k), y(t_{k+1}), w(t_{k+1}), u(t_{k+1})] \quad (1)$$

where the letters without subscripts are one dimensional arrays whose elements are the corresponding subscripted variables.

Two comments about (1) are in order. First, the variables and functions are not the ordinary continuous variables and functions of standard control

theory; rather, the variables are discrete and the functions are logical in nature -- in most cases best defined by tables. Second, in  $u(t_k)$  are included only those commands which may be given in response to unforeseen occurrences. Changes in operation which occur as a function of time or phase of the mission are manifest in the form of  $f_k$ .

Given (1) and the probabilities of the  $w_i$  going from 0 to 1 in the interval  $t_k$  to  $t_{k+1}$  it is possible to calculate\*

$$F_k(a; b; c, d) = P_r^{\Delta} [x(t_{k+1}) = a/x(t_k) = b, u(t_{k+1}) = c, y(t_{k+1}) = d] \quad (2)$$

Hence the operation of the system components is governed by a time varying Markov process:

$$P_{k+1}(a) = \sum_b F_k(a; b; c, d) P_k(b) \quad (3)$$

where

$$P_k(a) = P_r^{\Delta} [x(t_k) = a] \quad (4)$$

A model very similar to that given by (2), (3) and (4) is used by Sandler<sup>1,2</sup> where, however, only failure modes are considered.

The major part of classical reliability theory is concerned with determining the probabilities of failures occurring (i.e. the  $w_i$ 's going from 0 to 1). Since in this memorandum the concern is with how these probabilities are used rather than how they are determined, the present theory compliments rather than replaces classical reliability theory.

The occurrence of a particular event depends upon the previous occurrence

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\* It is also possible to allow the possibility of random repair by giving a non-zero probability of  $w_i$  going from 1 to 0.



of other prerequisite events and on the proper operation of the needed components; hence

$$y(t_{k+1}) = g[y(t_k), x(t_{k+1}), k+1] \quad (5)$$

In order to make a decision (i.e. choose  $u(t_k)$ ), it is necessary to make measurements on the system. These measurements are embodied in the vector  $z(t)$  where

$$z(t) = h[x(t), y(t), t] \quad (6)$$

Decisions are made on the basis of the information contained in the past history of these measurements. A major simplifying assumption is that  $x(t_k)$  and  $y(t_k)$  can be completely determined from  $z(t)$  for  $t < t_k$ ; then the decision may be made as a function of  $x(t_k)$  and  $y(t_k)$ .

To complete the model the following performance function is defined:

$$J = \sum_{k=1}^N \ell[x(t_k), k] + L[y(t_N)] \quad (7)$$

where  $\ell[x(t_k), k]$  is the value of the system components being in operating status  $\underline{x}$  during the interval  $t_{k-1}$  to  $t_k$  and  $L[y(t_N)]$  is the value of the occurrence of the events specified by  $y(t_N)$ , which tells what events have occurred and failed to occur in the course of the mission. It is assumed that for an event to be counted as occurring, it must occur at the proper time. If an event can occur at varying times with varying value or effect on the remainder of the mission then a separate  $y_i$  must be assigned for each possible time of occurrence.

### III ILLUSTRATION

To illustrate the application of the mathematical model developed in the previous section we consider a somewhat contrived example of the flyby mission whose primary purpose is to take TV pictures of Mars with a secondary goal of

performing several experiments in route. The system consists of the following subsystems and modes of operation:

1. Sequencer
  - 0 failed
  - 1 turned off
  - 2 working
  
2. Command System
  - 0 failed
  - 1 turned off
  - 2 working
  
3. Power System (Two Solar Panels and Batteries)
  - 0 failed
  - 1 only battery
  - 2 battery and 1 panel
  - 3 completely working
  
4. Communication
  - 0 completely failed
  - 1 turned off
  - 2 low power due to failure
  - 3 low power due to command
  - 4 high power
  
5. TV System
  - 0 failed
  - 1 turned off
  - 2 operating

6. Interplanetary Science

- 0 failed
- 1 turned off
- 2 minor experiments only failed
- 3 minor experiments turned off
- 4 all experiments working

7. Attitude Control

- 0 failed
- 1 turned off
- 2 working - acquisition mode
- 3 working - cruise mode
- 4 working - midcourse mode

8. Guidance (midcourse motor)

- 0 failed
- 1 turned off
- 2 operating

The mission takes place in the following phases:

1. Launch
2. Cruise
3. Midcourse
4. Cruise
5. Terminal

The following important events should occur during the mission:

1. Successful launch
2. Successful separation of launch vehicle and space craft
3. Successful midcourse

#### 4. Successful flyby

Rather than giving all the state equations for the system, it is sufficient to give examples: If  $x_3$  becomes 0 at any time (i.e. if the power system fails) then all other state variables become 1 (i.e. turned off). If  $x_3$  is not zero then in the time interval  $t_k$  to  $t_{k+1}$ ,  $x_1$  changes from 2 to 1 (i.e. the sequencer fails) if random event 1 occurs. If random event 1 occurs at a constant rate  $\sigma$  then

$$P_r [w_1(t_{k+1}) = 0 / w_1(t_k) = 0] = e^{-\sigma(t_{k+1} - t_k)} \quad (8)$$

To illustrate the dependence of the operation of components upon previous events, note that if launch fails all subsystems will be failed (i.e. 0) and that if separation fails  $x_3$  (the power system) will be limited to 0 or 1;  $x_4$  (communications) to 0, 1, or 2; and  $x_5$  (TV),  $x_6$  (science),  $x_7$  (attitude control), and  $x_8$  (guidance) to 0. Furthermore, it is clear that the occurrence of each event depends upon the successful occurrence of the previous event. Occurrence of the midcourse maneuver requires that  $x_2 = 2$ ,  $x_3 = 1$ ,  $x_4 = 2$  or 3,  $x_1 = 4$  and  $x_8 = 2$ , if it is assumed that the midcourse maneuver is or can be controlled from ground.

As examples of how built in redundancy may allow partial mission completion, consider the following: if one solar panel fails and if the remaining panel is capable of supplying power for the major, but not all, interplanetary science experiments, then the minor experiments may be turned off to allow continued operation of the major experiments without draining the battery. If both solar panels fail, but if the battery is capable of storing enough energy to take the terminal TV picture, then everything except communications and command might be turned off in order to conserve this energy and still complete the major goal of the mission.

#### IV APPLICATION OF THE MODEL

Once a description of a system in terms of the model presented in Section II has been developed, then by use of optimization procedures such as dynamic programming the optimum decisions for the operation of the system under all circumstances and the optimum expected performance can be determined. With this information the following may be accomplished:

1. Automatic systems for implementing the optimum decisions may be installed on board the space craft or on the ground. Alternatively these decisions may be determined on the ground in real time.

2. By perturbing the design of the given system, those changes which yield most improvement in performance can be determined. In this way the design of the system may be improved and critical areas may be delineated.

In addition to providing a reasonable method of determining optimum decisions and performance, the model developed in this memorandum has the following important properties:

1. It considers the possibility and value of partial successes.

2. It takes into account the effect not only of chance occurrences but also changes in operation caused by decision.

3. It is quite flexible in the allowed complexity of system description. For example, in the early stages of design a very simple model is likely to be used, with an order of complexity at the level of the illustration. In more advanced stages of design and during operation of the system a considerably more detailed model would be required.

4. It is in a form easily amenable to computer programming.

To make such a mathematical model a useful tool for the design and operation of space systems, two important tasks need to be accomplished:

1. Development of a computer program embodying the mathematical model and

suitable optimization technique.

2. Use of this computer program on a suitable realistic example.

It would seem most efficient if these two tasks were performed simultaneously since the experience in trying to model a real life situation should have great effect on the programming techniques used.

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MEMORANDUM 5

A SIMPLIFIED TECHNIQUE FOR THE SYNTHESIS  
OF MODEL-REFERENCED ADAPTIVE CONTROL SYSTEMS



## MEMORANDUM 5

### A SIMPLIFIED TECHNIQUE FOR THE SYNTHESIS OF MODEL-REFERENCED ADAPTIVE CONTROL SYSTEMS

The purpose of this research is to derive analytically an adaption technique that is extremely simple to implement for use with model-referenced adaptive control systems. This feature is a distinct advantage, compared to other techniques that have been described in the literature, and makes the simplified adaption technique very attractive for practical applications. The approach used in this study employs state-space methods. Some of the results of an extensive stability analysis, which employs Lyapunov's second method, are presented in order to give an indication as to the performance capabilities of the simplified adaption technique. A simple example is discussed for the purpose of illustrating certain important aspects of this study.

#### I INTRODUCTION

In recent years a great deal of effort has been devoted to the study of adaptive control systems. An extensive survey and review of the literature in this field is presented in Quarterly Report 1 under Contract NAS 12-59.<sup>1</sup>

The interest in adaptive control systems has been largely motivated by a sizable class of problems for which conventional techniques for synthesizing the controller have proven inadequate. Specifically, a controller having fixed parameters may not be capable of achieving the desired system performance with a given plant. Such a situation may

occur when the parameters which describe the plant vary over a wide range of values during the operation of the system (i.e., when the dynamic characteristics of the plant change markedly). These parameter variations may be deterministic, stochastic, or wholly unpredictable. To make the problem more complex, the entire system may be directly affected by an environment that varies drastically over the range of operation. In addition, the performance criterion may vary as the system encounters different operating conditions which necessitate different control policies. Also, the description of the plant may be incomplete or imprecise because of the nature of the problem. Practical examples of considerable importance in this class of problems are found in the design of high-performance aircraft and missiles.

The concept of model-referenced adaptive control systems evolved from work done by Whitaker, et al<sup>2</sup>; a block diagram description is shown in Fig. 1. The model-referenced approach is "closed-loop" with respect to system performance; i.e., the performance criterion is periodically or continuously monitored and, using this information the parameters of the adaptive controller\* are adjusted to extremize the performance measure. Additionally, this approach has the advantage of avoiding the plant identification problem essential to other methods. Adaption techniques employed in conjunction with this approach have been developed by Osburn<sup>3</sup> and Donalson<sup>4</sup>. The performance criterion  $P$  that is employed is an even function of  $e(t)$  in Ref. 3 and is a function of  $e(t)$  and its derivatives in Ref. 4, where the performance error  $e(t)$  is the difference

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\*The parameters of the adaptive controller are also referred to as the adaptive parameters.

between the adaptive control system output  $c(t)$  and the reference model output  $c_D(t)$ . Each adaptive parameter is adjusted at a rate directly proportional to the partial derivative of  $P$  with respect to that parameter. Hence, the adaption proceeds toward a minimum approximately in the direction of the gradient of  $P$  (with respect to the adaptive parameters); i.e., the adaption technique approximates a surface search along the path of steepest descent. Osburn and Donalson generate the necessary partial derivatives of  $P$  by procedures which are similar to each other. However, in order to obtain these partial derivatives, both procedures require a separate mechanization of the reference model for each adaptive parameter in the system.

Although the model-referenced approach has wide applicability, the complexity associated with the implementation of the adaption technique, as employed in Refs. 3 and 4, is a distinct drawback because of practical considerations. It is the purpose of this study, therefore to derive a simplified adaption technique for model-referenced adaptive control systems which does not involve the complexity of implementation inherent in the techniques developed by Osburn and by Donalson.

## II PROBLEM FORMULATION

The model-referenced adaptive control system that is considered in this report is illustrated in Fig. 1. The study considers the class of dynamical systems that can be described by linear, ordinary differential equations (i.e., linear, lumped-parameter systems).

### A. Adaptive Control Systems

The adaptive control system (plant plus adaptive controller) is described by the linear equations with time-varying coefficients:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{F}_s(t)\mathbf{x}(t) + \mathbf{D}_s(t)\mathbf{u}(t), \\ c(t) &= \mathbf{M}^T\mathbf{x}(t),\end{aligned}\tag{1}$$

where

$\mathbf{x}(t)$  = n-dimensional state vector,

$\mathbf{u}(t)$  = q-dimensional input vector,

$c(t)$  = scalar output

$\mathbf{F}_s(t)$  = n x n feedback matrix,

$\mathbf{D}_s(t)$  = n x q input matrix,

$\mathbf{M}$  = n x 1 output matrix.

The plant (physical) process to be controlled contains an arbitrary number of physical parameters that vary in an unknown manner; i.e., there is no explicit knowledge of the behavior of the time-varying plant parameters. However, it is assumed that the basic differential equation description of the plant is known, and that the necessary plant states are available. In practice, the plant might represent a high-performance aircraft or a missile, whose parameters vary markedly over various flight conditions (e.g., altitude and velocity). Indeed, for some flight conditions the aircraft or missile may actually be unstable. In Fig. 1,  $\mathbf{x}_p(t)$  is the state vector of the plant and  $\mathbf{u}_p(t)$  is the input (or control) vector to the plant.

To achieve the desired performance, it is necessary to provide the plant with the appropriate adaptive controller. This means that there exist values at which the adaptive parameters (these are the parameters of the adaptive controller) can be set so that the differential equations

representing the adaptive control system are identical with the differential equations of the reference model for any values the plant parameters may assume. That is, any elements of the matrices in (1) which contain time-varying plant parameters will also contain adaptive parameters providing the required compensation.

In general, it is possible to choose the state variables of the adaptive control system in such a manner that only  $F_S(t)$  and  $D_S(t)$  contain the time-varying parameters, while  $M$  is a constant matrix. In (1) it is assumed that  $x(t)$  has been so chosen.

#### B. Reference Model

The desired performance can be expressed in terms of certain criteria (e.g., response time, overshoot, stability), according to classical control theory. Alternatively, these criteria can be formulated as a set of differential equations which yield the desired input-output relationships (desired system). In this study it will be assumed that the desired performance is expressed in terms of a set of differential equations which can be considered as an implicit characterization of the performance criteria. The desired system, which will be referred to as the reference model, is described by the following linear differential equations with constant coefficients:

$$\begin{aligned} \dot{y}(t) &= F_D y(t) + D_D u(t) , \\ c_D(t) &= M^T y(t) , \end{aligned} \tag{2}$$

where

$y(t)$  = n-dimensional state vector,

$u(t)$  = q-dimensional input vector,

$c_D(t)$  = scalar output ,

$F_D$  =  $n \times n$  feedback matrix,

$D_D$  =  $n \times q$  input matrix,

$M$  =  $n \times 1$  output matrix.

It is assumed that the reference model is of the same dimension ( $n$ ) as the adaptive control system. In many instances, one is concerned with problems in which the model is of smaller dimension than the adaptive control system. However, the derivation of the simplified adaption technique to be presented in Section III is based upon the assumption that the model and the adaptive control system are of the same dimension. This situation can be met by augmenting the model with extra states so that it is of dimension  $n$  . These extra states are chosen in such a manner that their effect on the behavior of the model is negligible.

Since the output of the reference model  $c_D(t)$  corresponds to the desired output for the adaptive control system when both are subjected to the same input  $u(t)$  , the design objective is to adjust the adaptive parameters so that the adaptive control system output  $c(t)$  approximates  $c_D(t)$  despite variations in the plant parameters. This objective can be accomplished by minimizing the magnitude of the performance error  $e(t)$  , which is given by

$$e(t) = c(t) - c_D(t) = M^T [x(t) - y(t)] . \quad (3)$$

If the initial conditions for the adaptive control system and the reference model are equal [i.e., if  $x(t_0) = y(t_0)$  , where  $t_0 =$  initial time] , then by maintaining  $F_s(t) \equiv F_D$  and  $D_s(t) \equiv D_D$  for



$t \geq t_0$  , it would follow from (1) and (2)  $x(t) \equiv y(t)$  for  $t \geq t_0$  .  
 Then, from (3),  $e(t) \equiv 0$  for  $t \geq t_0$  . However, it is not reasonable  
 to assume that  $F_s(t) \equiv F_D$  and  $D_s(t) \equiv D_D$  , since such an assumption  
 supposes that the time-varying behavior of the plant parameters is known  
 precisely, and that the adaptive parameters are capable of perfectly  
 compensating for the variations of the plant parameters. Unfortunately,  
 the behavior of the plant parameters is not explicitly known. Further-  
 more, practical considerations for the implementation of an adaption  
 mechanism make it impossible to change the adaptive parameters instan-  
 taneously. Hence, a realistic objective is to develop a procedure for  
 adjusting the adaptive parameters when the performance error  $e(t) \neq 0$  .

### III DERIVATION OF THE ADAPTION EQUATIONS

The approach to be used in deriving the simplified adaption  
 technique consists of obtaining an expression for  $e(t)$  that shows its  
 explicit functional dependence on the adaptive parameters, all of which  
 are contained in  $F_s(t)$  and  $D_s(t)$  . This expression for  $e(t)$  can  
 be obtained from (3) by finding  $x(t)$  , the solution to the differential  
 equations of (1), and  $y(t)$  , the solution to the differential equa-  
 tions of (2) .

It is a well-known fact that the solution to (2) is

$$y(t) = \Phi_D(t-t_0)y(t_0) + \int_{t_0}^t \Phi_D(t-\zeta)D_D u(\zeta) d\zeta \quad t \in [t_0, \infty) , \quad (4)$$

where

$\Phi_D(v) = \exp F_D v = n \times n$  fundamental matrix of the reference  
 model.

In order to obtain an explicit relation for  $x(t)$ , the matrices  $F_s(t)$  and  $D_s(t)$  are decomposed into constant and time-varying components as follows:

$$\begin{aligned} F_s(t) &= F_D + \delta F_\delta(t) , \\ D_s(t) &= D_D + \delta D_\delta(t) , \end{aligned} \tag{5}$$

where  $\delta$  is a scalar constant. The matrices  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  contain the adaptive parameters and the time-varying portion of the plant parameters, and can be considered as perturbations of the adaptive control system matrices from the desired matrices (i.e., the reference-model matrices)  $F_D$  and  $D_D$ . The scalar  $\delta$  has been introduced to aid in the ensuing analysis, and to provide an explicit measure of the perturbations, where  $F_\delta(t)$  and  $D_\delta(t)$  are normalized in some sense.

Substituting (5) into (1),

$$\dot{x}(t) = [F_D + \delta F_\delta(t)]x(t) + [D_D + \delta D_\delta(t)]u(t) . \tag{6}$$

This substitution enables (6) to be put into integral form,

$$x(t) = \Phi_D(t-t_o)x(t_o) + \int_{t_o}^t \Phi_D(t-\tau) \left\{ \delta F_\delta(\tau)x(\tau) + [D_D + \delta D_\delta(\tau)]u(\tau) \right\} d\tau .$$

To obtain  $x(t)$  explicitly, the method of successive approximations\* is applied to the above expression, and yields

$$x(t) = \Phi_D(t-t_0)x(t_0) + \int_{t_0}^t \Phi_D(t-\tau)D_D u(\tau) d\tau \quad (7)$$

$$+ \delta \int_{t_0}^t \Phi_D(t-\tau) \left[ D_\delta(\tau)u(\tau) + F_\delta(\tau) \left\{ \Phi_D(\tau-t_0)x(t_0) + \int_{t_0}^{\tau} \Phi_D(\tau-\sigma)D_D u(\sigma) d\sigma \right\} \right] d\tau + o(\delta^2) ,$$

for  $t \in [t_0, \infty)$ , and where  $o(\delta^2)$  and represents those terms that contain  $\delta$  of second degree and higher.

Consider the initial conditions of the adaptive control system and the reference model to be related as follows:

$$x(t_0) - y(t_0) = \delta \mathcal{E}_\delta(t_0) \quad . \quad (8)$$

Combining (7) and (8),

$$x(t) = \Phi_D(t-t_0)\delta \mathcal{E}_\delta(t_0) + \left\{ \Phi_D(t-t_0)y(t_0) + \int_{t_0}^t \Phi_D(t-\tau)D_D u(\tau) d\tau \right\} \quad (9)$$

$$+ \delta \int_{t_0}^t \Phi_D(t-\tau) \left[ D_\delta(\tau)u(\tau) + F_\delta(\tau) \left\{ \Phi_D(\tau-t_0)y(t_0) + \int_{t_0}^{\tau} \Phi_D(\tau-\sigma)D_D u(\sigma) d\sigma \right\} \right] d\tau + o(\delta^2) .$$

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\*The method of successive approximations<sup>6</sup> is a recursive procedure for obtaining solutions to differential equations. If  $F_\delta(t)$ ,  $D_\delta(t)$ , and  $u(t)$  are continuous with respect to  $t$  on the interval  $[t_0, \infty)$ , then this is sufficient to ensure that the recursive procedure converges to the unique solution  $x(t)$  defined on  $[t_0, \infty)$ . The assumption of continuity is quite reasonable for physical systems and introduces no severe limitations on the scope of the analysis.

Referring to (4), it can be seen that the expressions in the braces { } of (9) are equivalent to  $y(t)$  and  $y(\tau)$ , respectively. This observation results in a considerable simplification of (9):

$$x(t) = \Phi_D(t-t_0)\delta\mathcal{E}_\delta(t_0)+y(t)+\delta\int_{t_0}^t \Phi_D(t-\tau)\left[D_\delta(\tau)u(\tau)+F_\delta(\tau)y(\tau)\right]d\tau+o(\delta^2) . \quad (10)$$

Finally, the expression for the performance error  $e(t)$  is obtained by substituting (10) into (3),

$$e(t) = M^T\Phi_D(t-t_0)\delta\mathcal{E}_\delta(t_0)+M^T\int_{t_0}^t \Phi_D(t-\tau)\left[\delta D_\delta(\tau)u(\tau)+\delta F_\delta(\tau)y(\tau)\right]d\tau+o(\delta^2) . \quad (11)$$

This expression makes sense intuitively; that is, if  $\delta F_\delta \equiv 0$  and  $\delta D_\delta \equiv 0$  for  $t \geq t_0$  [ $F_s(t) \equiv F_D$  and  $D_s(t) \equiv D_D$  for  $t \geq t_0$ ] and if  $\delta\mathcal{E}_\delta(t_0) = 0$  [ $x(t_0) = y(t_0)$ ], then (11) indicates that  $e(t) \equiv 0$  for  $t \geq t_0$ .

At this point in the derivation, two basic assumptions are introduced.

Assumption 1:  $\delta$  is sufficiently small so that the term  $o(\delta^2)$  in (11) can be neglected; that is,  $F_s(t) \cong F_D$  and  $D_s(t) \cong D_D$  for all  $t \geq t_0$ , and  $x(t_0) \cong y(t_0)$ .

Assumption 2: The rates of change of the plant parameters, and of  $\Phi_D(t)$ ,  $u(t)$ , and  $y(t)$  are negligible compared with those of the adaptive parameters.

These two assumptions have been made for the purpose of achieving mathematical rigor in the derivation, and it might appear that they would severely restrict the general applicability of the simplified adaption

technique. However, it is demonstrated in Ref. 5, by an extensive stability analysis and computer simulations, that Assumptions 1 and 2 may be relaxed appreciably without impairing the performance of the adaptive system. The example presented in Section V also serves to illustrate this point.

Employing Assumption 1, the closed-form expression for  $e(t)$  is obtained from (11) as

$$e(t) = M^T \Phi_D(t-t_0) \delta \mathcal{E}_\delta(t_0) + M^T \int_{t_0}^t \Phi_D(t-\tau) \left[ \delta F_\delta(\tau) y(\tau) + \delta D_\delta(\tau) u(\tau) \right] d\tau . \quad (12)$$

The explicit functional dependence of the performance error  $e(t)$  on the adaptive parameters (via  $\delta F_\delta$  and  $\delta D_\delta$ ) as given by (12) is the expression that has been sought in order to derive the simplified adaptation technique.

Now, it is essential to have a measure of the change in the performance error produced by adjusting the adaptive parameters. The incremental error,  $\Delta e(t)$ , is defined by

$$\Delta e(t) \triangleq e(t + \Delta t) - e(t) , \quad (13)$$

where  $\Delta t$  is positive and is chosen sufficiently small so that, by virtue of Assumption 2, any change  $\Delta e(t)$  is essentially due only to the adjustment of the adaptive parameters. The adaptive parameters are to be adjusted based upon the value of the performance error, that is, if at some time,  $t_1$ , the error  $e(t_1)$  is not zero, the adaptive parameters will be adjusted so as to reduce the magnitude of the error for

$t > t_1$  . Therefore, in terms of the incremental error  $\Delta e(t_1)$  , the design objective for the adaptive system can be expressed as follows:

$$\begin{aligned} \Delta e(t_1) &> 0 \text{ if } e(t_1) < 0 \\ \Delta e(t_1) &< 0 \text{ if } e(t_1) > 0 \\ \Delta e(t_1) &= 0 \text{ if } e(t_1) = 0 \end{aligned} \quad (14)$$

where  $t_1$  is in the interval  $[t_0, \infty)$  .

Substituting (12) into (13), and rearranging terms in order to isolate the effect of the adaption procedure upon the incremental error  $\Delta e(t_1)$  :\*

$$\Delta e(t_1) = h(t_1) + \Delta_A e(t_1) , \quad (15)$$

where  $h(t_1)$  contains those quantities that are not affected by adjusting the adaptive parameters for  $t \geq t_1$  , and

$$\Delta_A e(t_1) = \frac{(\Delta t)^2}{2} \left[ \sum_{i=1}^n \sum_{j=1}^n a_i \dot{f}'_{ij}(t_1) y_j(t_1) + \sum_{m=1}^n \sum_{\ell=1}^q a_m \dot{d}'_{m\ell}(t_1) u_\ell(t_1) \right] , \quad (16)$$

in which  $f'_{ij}(t)$  and  $d'_{m\ell}(t)$  are elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  , respectively, and  $a_i$  is an element of the n-dimensional vector  $M^T \exp F_D \Delta t$  .

The term  $\Delta_A e(t_1)$  contains those quantities that can be affected by adjustment of the adaptive parameters for  $t \geq t_1$  . Thus to obtain the appropriate  $\Delta e(t_1)$  by the adaption procedure it is only necessary

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\*The several manipulations used to obtain (15) are quite involved; for the details, see Ref. 5.

to consider  $\Delta_A e(t_1)$ . Those elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  which do not contain any adaptive parameters are identically equal to zero. Hence, the time derivatives of these elements are identically zero, and they vanish in (16). The elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  that do contain the adaptive parameters will be denoted by  $f_{ij}(t)$  and  $d_{m\ell}(t)$ , respectively. Only those terms in (16) corresponding to the  $f_{ij}(t)$  and  $d_{m\ell}(t)$  can be affected by the adaption procedure.

The design objective for the adaptive system, as given by (14) is expressed more concisely as

$$\Delta e(t_1) e(t_1) \leq 0 . \quad (17)$$

Now, consider the following relation:

$$\Delta_A e(t_1) e(t_1) \leq 0 . \quad (18)$$

Investigation of (16) reveals that (18) is satisfied if

$$\begin{aligned} \dot{f}_{ij}(t_1) &= -\mu_{F_{ij}} a_i y_j(t_1) e(t_1) , \\ \dot{d}_{m\ell}(t_1) &= -\mu_{D_{m\ell}} a_m u_\ell(t_1) e(t_1) , \end{aligned} \quad (19)$$

for the appropriate  $i$ ,  $j$ ,  $m$ , and  $\ell$ ; where the  $\mu_{F_{ij}}$  and  $\mu_{D_{m\ell}}$  are positive constants.\* The elements  $a_i$  are functions of  $\Delta t$ , whose value is arbitrarily chosen. Hence  $\mu_{F_{ij}}$  and  $a_i$  may be combined and considered as a single constant  $\mu'_{F_{ij}}$ . Similarly,  $\mu_{D_{m\ell}}$  and  $a_m$  may

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\* It should be noted that the expressions in (19) are by no means the only expressions that enable (18) to be satisfied.

be combined as  $\mu'_{D_{m\ell}}$ . These new constants are referred to as "the adaptive loop gains" and take the sign of either  $a_i$  or  $a_m$ . By choosing the  $\mu'_{F_{ij}}$  and  $\mu'_{D_{m\ell}}$  to be sufficiently large,  $\Delta_A e(t_1)$  will tend to dominate the right-hand side of (15), so that it is possible by satisfying (18) to imply that (17) will be satisfied. Therefore, (19) defines the adaption procedure to be applied at time  $t_1$  in order that the magnitude of the performance error for  $t > t_1$  will be reduced.

The assumption that  $\Delta_A e(t_1)$  dominates the right-hand side of (15) is equivalent to assuming that any change in the performance error during a small interval of time ( $\Delta t$ ) is primarily caused by the adjustment of the adaptive parameters (this was noted previously in conjunction with Assumption 2). It is possible to realize this condition by making the adaptive loop gains suitably large. However, in general there are bounds on the  $\mu'_{F_{ij}}$  and  $\mu'_{D_{m\ell}}$  because of stability considerations. This matter is discussed in considerable detail in Ref. 5, where it is also shown that the rates of convergence in the adaptive system and the size of stability regions are affected by the adaptive loop gains. The example presented in Section V illustrates some of these points.

Since  $t_1$  was chosen arbitrarily, it may be replaced in (19) by  $t$ , where  $t \in [t_0, \infty)$ . Recalling that the rates of change of the plant parameters are assumed negligible compared to those of the adaptive parameters, (19) can be considered as defining the time derivatives for those elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  which contain the adaptive parameters. Hence, the result that has been sought is given as follows:



### Adaption Equations

$$\begin{aligned} \dot{f}_{ij}(t) &= -\mu'_{F_{ij}} y_j(t) e(t) , \\ \dot{d}_{m\ell}(t) &= -\mu'_{D_{m\ell}} u_\ell(t) e(t) , \end{aligned} \tag{20}$$

where

$$\mu'_{F_{ij}} \triangleq a_i \mu_{F_{ij}} \quad \text{and} \quad \mu'_{D_{m\ell}} \triangleq a_m \mu_{D_{m\ell}} .$$

The adaption equations show that the adaptive parameters are adjusted continuously at a rate proportional to the product of the instantaneous values of  $e(t)$  and the appropriate model state variable  $y_j(t)$  or input variable  $u_\ell(t)$ . The various  $y_j(t)$  are readily available from the actual mechanization of the model. The  $u_\ell(t)$  are also available, since they are the inputs to the system. The adaptive loop gains ( $\mu'_{F_{ij}}$  and  $\mu'_{D_{m\ell}}$ ) are free to be chosen in order to satisfy the particular requirements of each problem. The model-referenced adaptive control system with the adaption mechanism that implements (20) is illustrated in Fig. 2.

The adaption technique, based on (20), is extremely simple to implement compared to the techniques developed in Refs. 3 and 4. For practical applications, this feature is a distinct advantage and makes the simplified adaption technique, which has been derived in this study, very attractive for the synthesis of model-referenced adaptive control systems.

#### IV STABILITY ANALYSIS

In this section the results of an extensive stability analysis, which is described in more detail in Ref. 5, will be discussed. The stability investigation was undertaken in order to verify the theoretical results that have been obtained and to demonstrate that the simplified adaption technique is capable of providing satisfactory system performance over a wide range of operating conditions. The stability problem in adaptive systems has received scant attention in the literature.

In order to obtain a mathematical description of the adaptive system, Fig. 2, one must consider the interaction (coupling) between the adaption mechanism which implements (20), and the adaptive control system described by (1), with  $F_s(t)$  and  $D_s(t)$  defined in (5). The operation of the adaptive system is described by those elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  containing the adaptive parameters and by  $x(t)$ , the state vector of the adaptive control system.

The elements of  $\delta F_\delta(t)$  and  $\delta D_\delta(t)$  which contain the adaptive parameters are the  $f_{ij}(t)$  and  $d_{m\ell}(t)$ . Assuming that there are  $k$  adaptive parameters, define a  $k$ -dimensional vector  $\rho(t)$  containing these parameters as follows:

$$\rho(t) \triangleq \begin{bmatrix} \vdots \\ f_{ij}(t) \\ \vdots \\ d_{m\ell}(t) \\ \vdots \end{bmatrix} . \quad (21)$$

Instead of the state vector of the adaptive control system  $x(t)$ , one can consider the difference between  $x(t)$  and  $y(t)$ , since the model state vector  $y(t)$  is known. Define this difference by the  $n$ -dimensional vector

$$\mathcal{E}(t) \triangleq x(t) - y(t) . \quad (22)$$

Hence, the state of the adaptive system is defined by the  $(k + n)$ -dimensional vector

$$\beta(t) \triangleq \begin{bmatrix} \rho(t) \\ \mathcal{E}(t) \end{bmatrix} . \quad (23)$$

The differential equation description of the adaptive system is obtained in the following manner: Differentiating (21) with respect to time, substituting (20), and noting that  $e(t) = M^T \mathcal{E}(t)$ , yields

$$\dot{\rho}(t) = \begin{bmatrix} \vdots \\ -\mu'_{F_{ij}} y_j(t) M^T \mathcal{E}(t) \\ \vdots \\ -\mu'_{D_{m\ell}} u_\ell(t) M^T \mathcal{E}(t) \\ \vdots \end{bmatrix} = H(t) \mathcal{E}(t) , \quad (24)$$

where  $H(t)$  is a  $k \times n$  matrix. Differentiating (22) with respect to time and substituting (1), (2), and (5),

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \delta F_\delta(t) y(t) + \delta D_\delta(t) u(t) + F_D \mathcal{E}(t) + \delta F_\delta(t) \mathcal{E}(t) \\ &= B(t) \rho(t) + F_D \mathcal{E}(t) + \delta F_\delta(t) \mathcal{E}(t) , \end{aligned} \quad (25)$$

where  $B(t)$  is an  $n \times k$  matrix whose elements  $[b_{vw}(t)]$  are either zero or the appropriate  $y_j(t)$  and  $u_\ell(t)$  as defined below:

If the  $r$ -th element of  $\rho(t)$  is  $f_{ij}(t)$ ,

$$b_{ir}(t) = y_j(t), \text{ and } b_{vr}(t) = 0 \quad \text{for } v \neq i.$$

Similarly, if the  $s$ -th element of  $\rho(t)$  is  $d_{m\ell}(t)$ ,

$$b_{ms}(t) = u_\ell(t), \text{ and } b_{vs}(t) = 0 \quad \text{for } v \neq m.$$

Combining (23), (24), and (25),

$$\begin{aligned} \dot{\beta} &= \begin{bmatrix} 0 & H(t) \\ B(t) & F_D \end{bmatrix} \beta + \begin{bmatrix} 0 \\ \delta F_\delta \mathcal{E} \end{bmatrix} \\ &= \psi(t) \beta + \eta(\beta), \end{aligned} \tag{26}$$

where the explicit dependence of  $\beta$  on  $t$  is understood, but has been omitted as a matter of convenience. The adaptive system is governed by the nonstationary, nonlinear differential equations described in (26), where  $\eta(\beta)$  contains the nonlinear terms.

From (26),  $\beta = 0$  implies that  $\dot{\beta}(t) \equiv 0$ ; the state  $\beta = 0$  is referred to as an equilibrium state of (26). For  $\beta = 0$  it follows that (a) the matrices of the adaptive control system [see (1) and (5)] and the reference model [see (2)] are equal, i.e.  $\delta F_\delta = 0$  and  $\delta D_\delta = 0$ ; and (b) the states of the adaptive control system and the reference model are equal, i.e. from (22)  $\mathcal{E} = 0$ . Ideally,  $\beta = 0$  is the desired state of the adaptive system. However, as this problem has

been formulated, the desired performance of the adaptive system corresponds, not to the equilibrium state  $\beta = 0$ , but to  $e = 0$ . From (3) and (22),  $\beta = 0$  implies that the performance error  $e = 0$ ; but the converse is not true. Therefore, an investigation of the stability properties of the equilibrium state  $\beta = 0$  of (26) will provide information pertaining to the performance of the adaptive system.

Before proceeding any further, it is necessary to define certain stability concepts that are pertinent to this analysis. The stability analysis in this section will consider perturbations at time  $t_0$  of the adaptive system from its equilibrium state  $\beta = 0$ . That is, the subsequent behavior of the adaptive system when it has been perturbed from its equilibrium state at  $t_0$  will be investigated. This analysis considers the case when the plant parameters take arbitrary constant values for  $t \geq t_0$  and are not time-varying functions.

It is assumed that the differential equations of (26) possess the appropriate "smoothness" properties in order to guarantee that a solution to (26), with arbitrary initial conditions at  $t_0$ , exists and is unique for all  $t \geq t_0$ . This solution is given formally by  $\beta(t; \beta_0, t_0)$ , where  $\beta_0$  is the value of  $\beta$  at time  $t_0$ . Based on the work of Hahn,<sup>7</sup> the following definitions are introduced:

Definition 1: The equilibrium state  $\beta = 0$  of (26) is weakly stable if for every number  $\xi_1 > 0$  there exists a number  $\xi_2 > 0$ , depending (in general) on  $\xi_1$  and  $t_0$ , such that  $\|\beta_0\| < \xi_2$  implies

$$\|\beta(t; \beta_0, t_0)\| < \xi_1 \quad \text{for all } t > t_0.$$



It should be noted that for  $F_D$  and  $M$  as given above, the model is completely observable (the concept of observability was introduced by Kalman<sup>8</sup>). The results of this theorem, which holds for arbitrary  $k$  and  $n$ , are as follows: If (26) is stationary\*, then there exist values for the adaptive loop gains  $(\mu'_{F_{ij}}, \mu'_{D_{ml}})$  such that  $\beta = 0$  of (26) is weakly stable. If (26) is quasi-stationary\*\* and  $k = 1$ , then there exist values for the adaptive loop gains such that  $\beta = 0$  of (26) is asymptotically stable. For the case when (26) is stationary and  $\beta = 0$  is weakly stable, it should be noted that there are equilibrium states other than  $\beta = 0$ ; i.e., there are also nonzero states  $\beta$  for which  $\dot{\beta}(t) = 0$ . In general, the equilibrium state that the adaptive system arrives at is a function of the initial conditions (see Ref. 5). However, (26) reveals that  $e = 0$  at any of these equilibrium states ( $e = 0$  corresponds to the desired performance of the adaptive system).

#### V EXAMPLE

In order to illustrate the application of the simplified adaption technique derived in Section III, a simple example is discussed. This is a 1-dimensional system with its block diagram shown in Fig. 3. The differential equation that describes the plant and its adaptive controller is given by

$$\dot{x}(t) = F_s(t)x(t) + D_s(t)u(t) = [-\alpha(t) - \tilde{f}(t)]x(t) + u(t), \quad (27)$$

---

\*The term stationary means that  $\psi(t)$  is constant for all  $t \geq t_0$ , which implies that  $u(t)$  and  $y(t)$  are constant for all  $t \geq t_0$ .

\*\*The term quasi-stationary means that  $\psi(t)$  approaches a constant, steady state value as  $t \rightarrow \infty$ , which implies that  $u(t)$  and  $y(t)$  approach constant, steady state values as  $t \rightarrow \infty$ .

where

$\tilde{f}(t)$  = plant parameter,

$\alpha(t)$  = adaptive parameter.

Consider the reference model described by the following differential equation:

$$\dot{y}(t) = F_D y(t) + D_D u(t) = -\hat{f} y(t) + u(t) , \quad (28)$$

where

$\hat{f} > 0$  , which implies that  $x(t)$  is bounded for bounded inputs  $u(t)$ . The transfer function of the model is given by

$$\frac{Y(s)}{U(s)} = \frac{1}{s + \hat{f}} .$$

The state variables  $x(t)$  and  $y(t)$  correspond to the outputs of the control system and the reference model, respectively. Thus, the performance error is

$$e(t) = x(t) - y(t) .$$

Rewriting (27) in a form that is analogous to (6),

$$\dot{x}(t) = [-\hat{f} + f(t)] x(t) + u(t) , \quad (29)$$

where

$$f(t) = -\alpha(t) - \tilde{f}(t) + \hat{f} .$$

Differentiating  $f(t)$  with respect to time, and assuming that  $\dot{\tilde{f}}(t) \cong 0$  (since by Assumption 2 the rate of change of the plant parameter is assumed negligible compared to that of the adaptive parameter),



$$\dot{f}(t) = -\dot{\alpha}(t) . \quad (30)$$

From the adaption equations (20), the above equation yields

$$\dot{\alpha}(t) = \mu_F' y(t) e(t) , \quad (31)$$

where  $\mu_F'$  is the adaptive loop gain. This is the adaption equation to be implemented by the adaption mechanism in Fig. 3. Since  $\mu_F' = a\mu_F$  and  $a = \exp(-\hat{f}\Delta t) > 0$ , it follows that  $\mu_F'$  is a positive constant. The manner in which the proper value for the adaptive loop gain is chosen is discussed below.

From (23), the operation of the adaptive system is described by the 2-dimensional state vector

$$\beta(t) = \begin{bmatrix} f(t) \\ e(t) \end{bmatrix} .$$

Combining (26), (30), and (31),

$$\dot{\beta} = \begin{bmatrix} 0 & -\mu_F' y(t) \\ y(t) & -\hat{f} \end{bmatrix} \beta + \begin{bmatrix} 0 \\ f e \end{bmatrix} , \quad (32)$$

where  $\mu_F' > 0$ .

In the design of adaptive control systems, it is essential to determine the extent of the stability regions. That is, one must find the size of perturbations for which the state  $\beta = 0$  is weakly or asymptotically stable. Application of Lyapunov's second (or direct) method<sup>9,10</sup> enables one to estimate the size of these stability regions. This approach has found wide applicability in the stability analysis of

differential equations; it attempts to draw conclusions concerning the stability behavior of an equilibrium state without having knowledge of the solutions to the differential equations. This feature makes the approach powerful since it is not possible, in general, to evaluate explicitly the solutions to differential equations.

In the stability analysis of this example, it is assumed that the plant parameter  $\tilde{f}$  takes arbitrary constant values and is not a time-varying function. Examples with time-varying plant parameters are investigated by computer simulations in Ref. 5.

Consider the following quadratic Lyapunov function having no explicit time dependence:

$$V(\beta) = \mu_F' e^2 + f^2 . \quad (33)$$

Differentiating (33) with respect to time and substituting (32),

$$\begin{aligned} \dot{V}(\beta) &= \left. \frac{dV}{dt} \right|_{\substack{\text{along} \\ \text{solution} \\ \text{of (32)}}} = \text{grad } V \cdot \dot{\beta} \quad (32) \\ &= -2\mu_F' e^2 (\hat{f} - f) . \quad (34) \end{aligned}$$

Therefore,  $f < \hat{f}$  implies  $\dot{V}(\beta) \leq 0$ . From (33) and (34) it can be shown that  $\dot{V}(\beta) \leq 0$  in the region bounded by the ellipse  $V(\beta) = \hat{f}^2$ . It follows (from a theorem presented in Ref. 9) that  $\beta = 0$  of (26) is weakly stable in the region bounded by  $V(\beta) = \hat{f}^2$ . That is, all motions

originating in this region will remain in it indefinitely. This result holds for arbitrary  $y(t)$  and all  $\mu_F' > 0$ .

For the case when  $y(t)$  is a periodic function and not identically zero, it is possible to demonstrate asymptotic stability using the Lyapunov function of (33). In the stability region bounded by  $V(\beta) = \hat{f}^2$ ,  $\dot{V}(\beta) = 0$  corresponds to  $e = 0$  as shown by (34). For  $e = 0$  and  $f \neq 0$ , it follows from (32) that  $\dot{\beta}(t) \neq 0$  and, in particular,  $\dot{e} \neq 0$  for some time  $t' \in [t_0, \infty)$  since  $y(t) \neq 0$ . It can be shown (using a theorem presented in Ref. 10) that  $\beta = 0$  of (26) is asymptotically stable in the region bounded by  $V(\beta) = \hat{f}^2$ . That is, all motions originating in this region decay to  $(f = 0, e = 0)$ , if  $y(t)$  is a periodic function of time.

In the stability region,  $|f| < \hat{f}$  and from (29) it follows that the feedback coefficient of the adaptive control system may be perturbed by nearly  $\pm 100\%$  from the corresponding desired coefficient  $(\hat{f})$  of the model. In addition,  $e$  may be perturbed by an arbitrarily large amount if  $\mu_F'$  is chosen suitably small. It should be noted that the region bounded by  $V(\beta) = \hat{f}^2$  is an estimate to the region of stability.

If  $y(t)$  is a constant ( $y \neq 0$ ), which is a trivial kind of periodic function, the result proven above still holds. However, in this case one can establish the actual region of asymptotic stability by a graphical technique known as the method of isoclines (for the details see Ref. 5). In Figs. 4 and 5, the state space portraits in the  $f, e$  plane (trajectories corresponding to various initial conditions) have been constructed for  $y = .1$  and  $\hat{f} = 2$  with  $\mu_F' = 50$  and  $400$ ,

respectively. The arrows that are affixed to the trajectories indicate the direction of motion for increasing time, and  $t$  appears only implicitly as a parameter that changes value along each trajectory. The dashed line in each figure is referred to as the separatrix: for  $\mu'_F > 0$  all motions starting below the separatrix decay to the origin ( $\beta = 0$ ), and all motions starting above the separatrix grow undoundedly, as shown in Figs. 4 and 5. Thus, the state  $\beta = 0$  of (26) is asymptotically stable in the region below the separatrix.

Investigation of the figures reveals that the size of the region of asymptotic stability increases for decreasing values of  $\mu'_F$ . However, (31) indicates that by decreasing  $\mu'_F$  the rate of adjustment of the adaptive parameter  $\alpha(t)$  is decreased, which implies that the rate of convergence of the adaptive system is slower. Thus, a compromise between the size of the region of asymptotic stability and the rate of convergence of the adaptive system to  $\beta = 0$  must be arrived at in choosing  $\mu'_F$ .

It can be shown that the lower asymptote of the separatrix (for all  $\hat{f}$ ,  $\mu'_F > 0$ ) is the line  $-e = y$ . Hence from Figs. 4 and 5, any motion having the initial conditions  $e_0 \geq -y$  and  $f_0$  arbitrary will decay to the equilibrium state  $\beta = 0^*$ . That is,  $x$  (where  $e = x - y$ ) may be perturbed from its desired value,  $y$ , by an amount -100% or greater (algebraically) with  $f$  assuming any value. Additionally,  $x$  may deviate from  $y$  by less (algebraically) than -100% for  $f$  sufficiently negative. As shown in Figs. 4 and 5,  $f$  may be perturbed

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\*  $e_0$  and  $f_0$  are the values for  $e$  and  $f$  at  $t_0$ , respectively.

by an arbitrarily large amount. For  $f_o > \hat{f}$ , (27) and (29) show that  $F_s$  is positive at time  $t_o$ , which implies that the adaptive control system is initially unstable (where initially unstable means that the motions would be unbounded if  $F_s$  were to remain positive).

This example demonstrates that Assumption 1 need not be satisfied in order for the simplified adaption technique to perform satisfactorily and, indeed, the adaptive control system may even be initially unstable, yet achieve asymptotically stable behavior after adaption. As pointed out by Fraser<sup>11</sup>, the restriction that the adaptive control system not be initially unstable has limited the applicability of many adaptive techniques that have appeared in the literature.

The stability region obtained by employing the Lyapunov function of (33) applies for arbitrary  $y(t)$  [or arbitrary  $u(t)$ ]. In particular, it is not necessary for the rate of change of  $y(t)$  to be negligible compared to that of the adaptive parameter  $\alpha(t)$ . In addition,  $\hat{f}$  is an arbitrary positive constant which implies that  $\phi_D(t) = \exp(-\hat{f}t)$  has a rate of change that is not necessarily negligible compared to that of  $\alpha(t)$ . Hence, the simplified adaption technique will operate satisfactorily despite the violation of Assumption 2.

## VI CONCLUSION

An adaption technique for the synthesis of model-referenced adaptive control systems has been derived analytically. A somewhat more direct approach to the problem was taken, employing state space methods. It was shown that the adaption equations (20) are extremely simple to implement, which is a definite advantage for practical applications. The results of an extensive stability analysis<sup>5</sup> were discussed in order to evaluate the performance of model-referenced adaptive control systems utilizing the simplified adaption technique.

For the purpose of illustrating certain important aspects of this study, a simple example was discussed. This example demonstrated that the rate of convergence of the adaptive system and the size of the stability region are dependent upon the adaptive loop gain. In addition, it was shown that the adaptive control system may even be initially unstable, yet achieve asymptotically stable behavior after adaption.

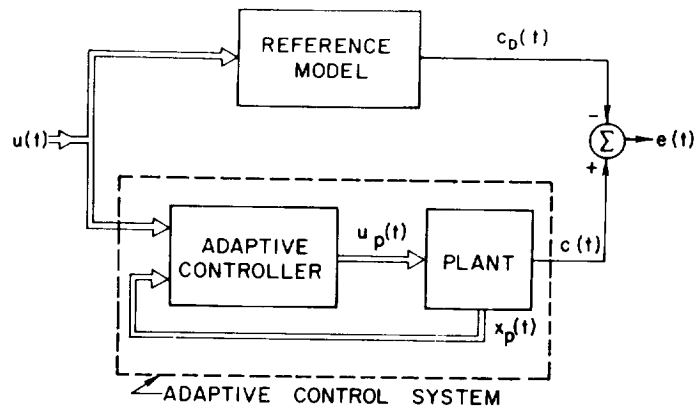


FIG. 1 MODEL-REFERENCED ADAPTIVE CONTROL SYSTEM

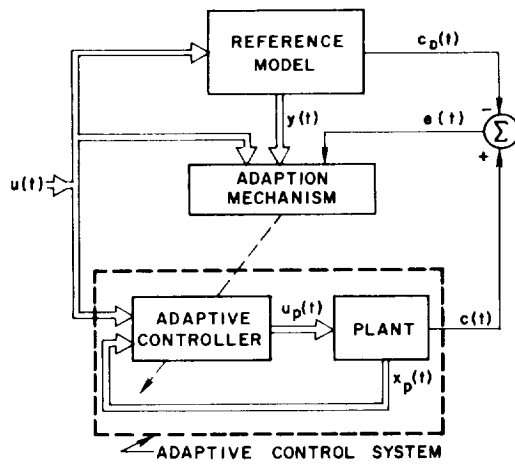


FIG. 2 MODEL-REFERENCED ADAPTIVE CONTROL SYSTEM WITH ADAPTION MECHANISM

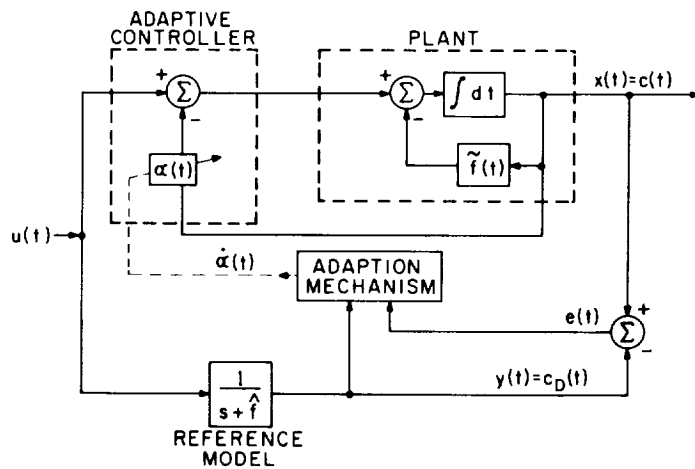


FIG. 3 BLOCK DIAGRAM FOR THE EXAMPLE

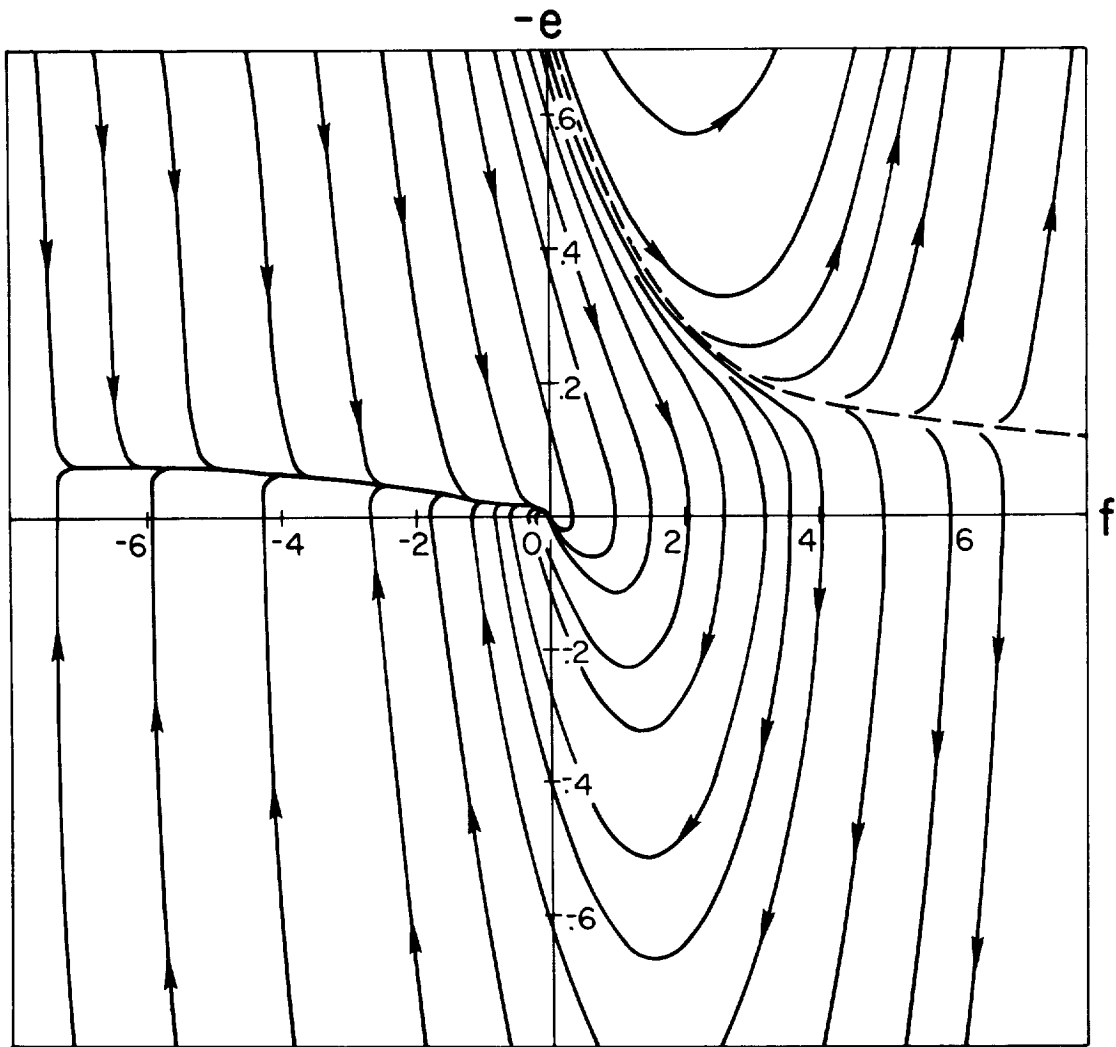


FIG. 4 STATE SPACE TRAJECTORIES FOR  $\mu_F' = 50$ , WITH  $\gamma = 0.1$ ,  $\hat{f} = 2$



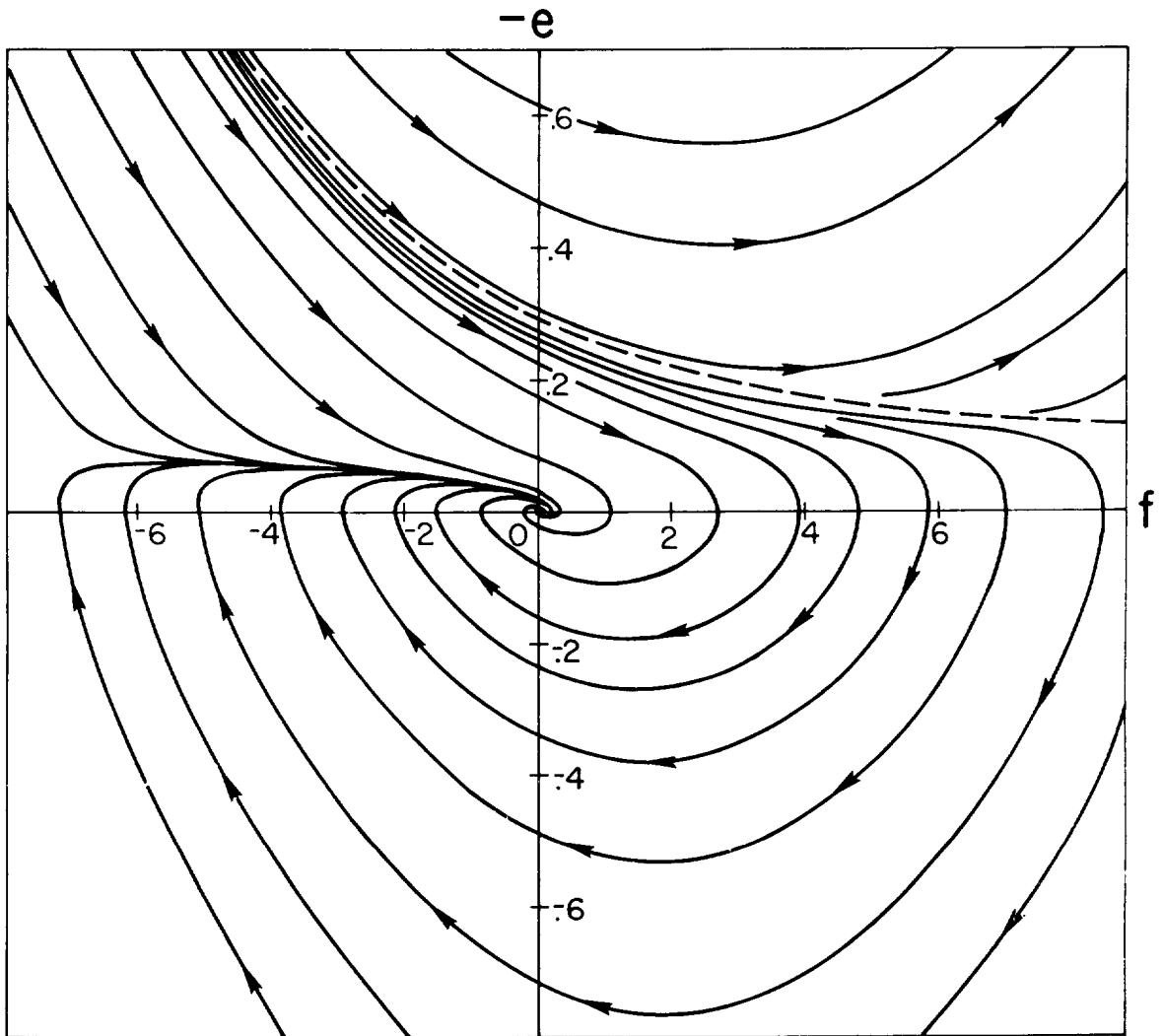


FIG. 5 STATE SPACE TRAJECTORIES FOR  $\mu_F' = 400$ , WITH  $\gamma = 0.1$ ,  $\hat{f} = 2$

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MEMORANDUM 6

ADAPTIVE CONTROL AND THE COMBINED  
OPTIMIZATION PROBLEM



## MEMORANDUM 6

### ADAPTIVE CONTROL AND THE COMBINED OPTIMIZATION PROBLEM

#### I INTRODUCTION

This memorandum places the design of adaptive control systems on a firm theoretical base by formulating the adaptive control problem in terms of a combined optimization problem. This formulation, presented in Sec. II, consists of considering uncertainty in the structural parameters of a linear plant by augmenting the state vector, thus converting the identification problem into an estimation problem.

Estimation, described in Sec. III, is performed by linearization about the estimate of the present and next state, and use of linear estimation theory to update the prediction when the next measurement occurs. If the system is initially at rest and if there is no measurement noise, this procedure is optimal because no multiplication of random variables occurs.

Control, considered in Sec. IV, consists of using the control that would be optimal if the present estimate of the present and future values of the plant parameters were exact. Both low sensitivity (no identification) and analysis-synthesis systems are considered. The major effect of parameter uncertainty is equivalent to an additional term in the loss function of the performance index.

#### II PROBLEM FORMULATION

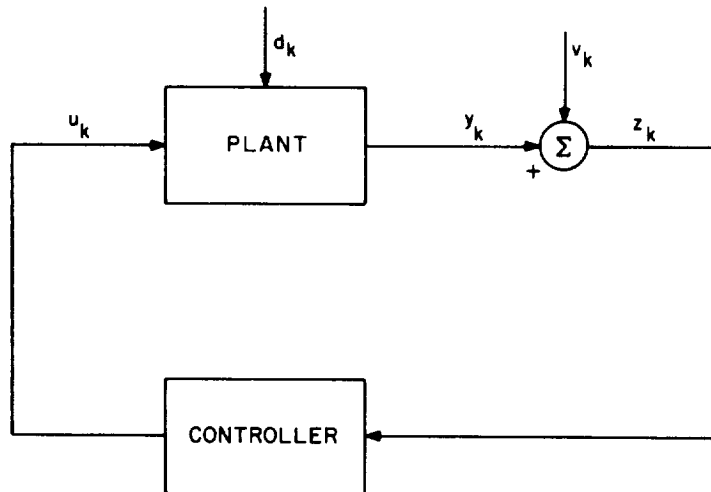
In this section the linear adaptive control problem is defined and shown to be a special case of the combined optimization problem. This latter problem has been the subject of considerable theoretical study\*<sup>1,2</sup> by SRI under a contract with the NASA Ames Research Center; this problem provides the theoretical basis for the development to be taken up in the remainder of the memorandum.

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\* References are listed at the end of the memorandum.

## A. STATEMENT OF THE LINEAR ADAPTIVE CONTROL PROBLEM

Consider Fig. 1 with the plant linear, and the disturbance  $d_k$  and the noise  $v_k$  white Gaussian. If the performance index is quadratic and if the system parameters are known exactly, then the optimum controller is linear and may be found by application of well-known procedures.<sup>1,2</sup>



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FIG. 1 LINEAR ADAPTIVE CONTROL PROBLEM

However, in many situations the parameters are not known exactly and change in a random manner due to environmental effects. In other situations the plant may actually be nonlinear; thus the linearization parameters change as the operating point shifts. It would be desirable to find optimum or near optimum controllers for these situations. This problem in essence is the linear adaptive control problem.

Stated mathematically, the problem is:

*Linear Adaptive Control Problem*

*Given*

- (1) The input/output relation\*

$$y_k = a_{1k}y_{k-1} + \dots + a_{nk}y_{k-n} + b_{1k}u_{k-1} + \dots + b_{nk}u_{k-n} + d_{k-1} + c_{2k}d_{k-2} + \dots + c_{nk}d_{k-n} \quad (1)$$

\* This is the most general input/output relation for a  $n$ th order system with one control input, one disturbance input, and one output. For multiple-input systems, more terms appear on the right-hand side; for multiple outputs, there will be more than one equation.

where

$y_k$  is the scalar output

$u_k$  is the scalar control input

$d_k$  is the scalar disturbance input, white in time, and

$a_{ik}$ ,  $b_{ik}$ ,  $c_{ik}$  are parameters;  $c_{ik}$  known.

(2) The parameter equations:

$$\begin{aligned}\phi_{k+1} &= F_k^\phi \phi_k + \eta_k \\ a_{ik} &= a_{ik}^o + \underline{a}_{ik}^T \phi_k \\ b_{ik} &= b_{ik}^o + \underline{b}_{ik}^T \phi_k\end{aligned}\tag{2}$$

where

$\phi_k$  is the parameter state vector

$\eta_k$  is the parameter disturbance noise, white in time

$F_k^\phi$  is a known matrix

$\underline{a}_{ik}$ ,  $\underline{b}_{ik}$  are known vectors

$a_{ik}^o$ ,  $b_{ik}^o$  are the nominal values of  $a_{ik}$  and  $b_{ik}$ .

(3) The measurement equation

$$z_k = y_k + v_k\tag{3}$$

where

$z_k$  is the scalar measurement

$v_k$  is the scalar measurement noise, white in time.

(4) The statistics

$$\begin{aligned}(y_{1-n}, \dots, y_0) &\sim N(\hat{y}_{1-n/-1}, \dots, \hat{y}_{0/-1}, \hat{P}_{0/-1}^D) \\ d_k &\sim N(0, \hat{q}_k) \\ v_k &\sim N(0, \hat{r}_k) \\ \phi_0 &\sim N(\phi_{0/-1}, \hat{P}_{0/-1}^\phi) \\ \eta_k &\sim N(0, Q_k^\phi)\end{aligned}\tag{4}$$

where

$x \sim N(\hat{x}, \hat{P})$  means  $x$  is normally distributed with mean  $\hat{x}$  and covariance  $\hat{P}$ .

(5) The performance index\*

$$J = \sum_{k=0}^N (q_k y_k^2 + r_k u_k^2) \quad (5)$$

where

$q_k$  and  $r_k$  are given scalars.

*Find:* The controller which determines  $u_k$  as a function of  $Z_k \triangleq (z_0, \dots, z_k)$  for each  $k$  in such a manner as to minimize  $E(J)$ .

Note that the assumption that the  $c_{ik}$  are known implies that the statistics of random effects on the system are known. Only uncertainty in the structure of the system is considered in this memorandum.

#### B. STATEMENT OF THE COMBINED OPTIMIZATION PROBLEM

At this point the combined optimization problem and its solution in terms of iterative equations is stated in preparation for a demonstration that the linear adaptive control problem is a special case. The combined optimization problem is illustrated in Fig. 2.

*Combined Optimization Problem*

*Given*

(1) A plant, described by

$$x_{k+1} = f(x_k, u_k, w_k, k) \quad (6)$$

where

$x_k$  is the state vector

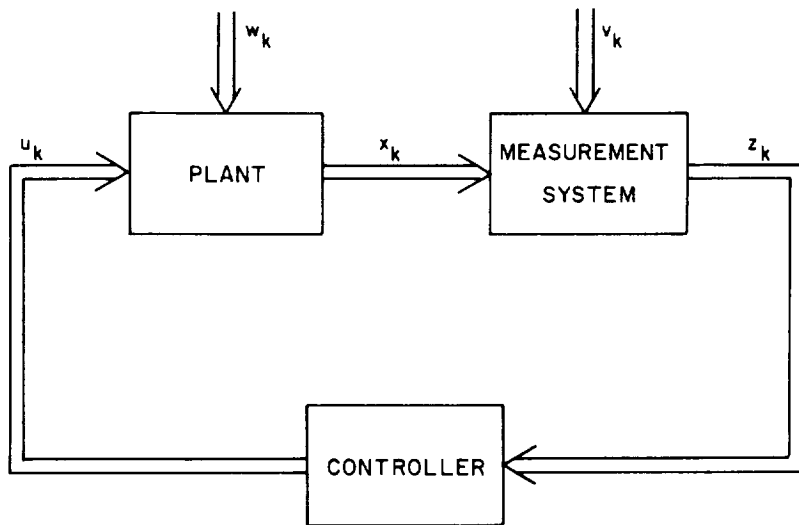
$u_k$  is the control or input vector

$w_k$  is the disturbance vector, assumed to be white.

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\* More general quadratic cost functions involving up to the last  $n - 1$  outputs at a given time may be treated with little increase in complexity.





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FIG. 2 COMBINED ESTIMATION PROBLEM

(2) A measurement system, described by

$$z_k = h(x_k, v_k, k) \quad (7)$$

where

$z_k$  is the measurement vector

$v_k$  is the measurement noise vector, assumed to be white.

(3) The probability distributions

$$(a) \quad p(x_0)$$

$$(b) \quad p(w_i) \quad i = 0, \dots, N$$

$$(c) \quad p(v_i) \quad i = 0, \dots, N \quad . \quad (8)$$

(4) The performance index

$$J = E \left\{ \sum_{i=0}^N l(x_i, u_i, i) \right\} \quad (9)$$

(5) The admissibility constraint

$$u_i \in \Omega_i \quad (10)$$

Find the admissible controller that minimizes  $J$ , where

- (1) A controller is defined as any algorithm that at time  $k$  generates  $u_k$  as a function of the present and all past measurements  $(z_k, \dots, z_0)$ .
- (2) An admissible controller is defined as any controller which, when used in the closed-loop system shown in Fig. 2, yields admissible  $u_i$ .

It can be shown that the optimum controller can be broken into two parts: an estimator, which calculates the condition probability density  $\mathbb{P}_k \triangleq p(x_k/Z_k, U_{k-1})$ , and a control law  $u_k = u_k(\mathbb{P}_k)$ .<sup>1,2</sup> The estimator is governed by the equation

$$p(x_{k+1}/Z_{k+1}, U_k) = \frac{p(z_{k+1}/x_{k+1}) \int_{x_k} p(x_{k+1}/x_k, u_k) p(x_k/Z_k, U_{k-1}) dx_k}{\int_{x_{k+1}} p(z_{k+1}/x_{k+1}) \int_{x_k} p(x_{k+1}/x_k, u_k) p(x_k/Z_k, U_{k-1}) dx_k dx_{k+1}} \quad k > 0$$

$$p(x_0/Z_0, U_{-1}) = \frac{p(z_0/x_0)p(x_0)}{\int_{x_0} p(z_0/x_0)p(x_0) dx_0} \quad (11)$$

and the control law is found by solution of

$$I^*(\mathbb{P}_k, k) = \min_{u_k} \left( L(\mathbb{P}_k, u_k, k) + z_{k+1} E \left\{ I^* \left[ \mathbb{P}_k(\mathbb{P}_k, u_k, z_{k+1}), k+1 \right] \right\} \right) \quad k < N$$

$$I^*(\mathbb{P}_N, N) = \min_{u_N} L(\mathbb{P}_N, u_N, N) \quad (12)$$

where

$$I^*(\mathbb{P}_k, k) \triangleq \min_{u_k, \dots, u_N} E \left[ \sum_{i=k}^N L(\mathbb{P}_i, u_i, i) / \mathbb{P}_k \right]$$

$$L(\mathbb{P}_i, u_i, i) \triangleq E_{x_i} [l(x_i, u_i, i) / \mathbb{P}_i]$$

$$= E_{x_i} [l(x_i, u_i, i) / Z_i U_{i-1}] \quad (13)$$

and  $\mathbb{P}_k$  is defined by Eq. (11).

C. FORMULATION OF THE LINEAR ADAPTIVE CONTROL PROBLEM  
AS A COMBINED OPTIMIZATION PROBLEM

To show that the linear adaptive control problem is a combined optimization problem, it is sufficient to make the following definitions:

$$x_k \triangleq \begin{bmatrix} \overline{y_k^*} \\ y_k \\ \overline{u_k^*} \\ d_k^* \\ \overline{\phi^k} \end{bmatrix} \quad w_k = \begin{bmatrix} \overline{0} \\ 0 \\ \overline{0} \\ 0 \\ \vdots \\ 0 \\ d_k \\ \overline{\eta_k} \end{bmatrix} \quad (14)$$

where

$$\alpha_k^* \triangleq \begin{bmatrix} \alpha_{k-n-1} \\ \vdots \\ \alpha_{k-1} \end{bmatrix} \quad \text{for any scalar time function } \alpha_k \quad .$$

The state equation, measurement equation and performance function may now be written in terms of these two vectors and the scalars  $u_k$  and  $v_k$  previously defined, by use of obvious identities and Eqs. (1), (2), and (3). The results are cumbersome (although simple) and unedifying and hence are not reproduced here, but are given in Appendix A. It is only necessary to note that the linear adaptive situation may be described in terms of equations of the form (6), (7), and (9), if the definitions of state and disturbance noise given in (14) are used.

Two comments are in order at this point:

- (1) The dynamic behavior of the system is described by the dynamic state vector

$$x_k^D \triangleq \begin{bmatrix} y_k^* \\ \hline y_k \\ \hline u_k^* \\ \hline d_k^* \end{bmatrix} \quad (15)$$

of dimension  $3n - 2$ . This vector has almost three times the minimum number of  $n$  dimensions that are needed to describe the behavior of an  $n$ th order dynamic system. The additional dimensions are necessary to facilitate identification.

- (2) The unknown parameters of the system are handled by augmenting the dynamic state vector with the vector  $\phi_k$ .

### III ESTIMATION

In this section, approximate solution of the estimation equation (11) is considered. The development is based upon application of perturbation theory and linear estimation theory.

Battin<sup>3</sup> and Schmidt<sup>4</sup> were the first workers to apply linear estimation theory to nonlinear estimation by linearization of the system equations about the present estimate. They considered application to satellite tracking. Farison<sup>5</sup> and Kopp and Orford<sup>6</sup> considered the use of such linearized estimators in the identification or analysis half of analysis-synthesis systems. The present work is based upon some of the ideas developed by Lee in Chapter 4 of his research monograph.<sup>7</sup> Such techniques have also been applied by SRI successfully to (enemy) missile tracking problems, including identification of unknown ballistic coefficients.<sup>8</sup>

#### A. THE "EXTENDED KALMAN FILTER"

In this paragraph the theory of the so called "extended Kalman filter," is presented. As a first step the state space formulation of linear estimation developed by Kalman<sup>9</sup> and commonly referred to as the Kalman filter is briefly described. Consider a system with state equation

$$x_{k+1} = F_k x_k + w_k \quad (16)$$

and measurement equation

$$z_k = H_k^T x_k + v_k \quad (17)$$

where  $w_k$  and  $v_k$  are uncorrelated white, Gaussian random processes with mean zero and covariances  $\hat{Q}_k$  and  $\hat{R}_k$  respectively. If the *a priori* distribution of  $x_0$  is Gaussian, then all conditional distributions of  $x_k$  given  $Z_k$  will be Gaussian and it is sufficient to find equations to update the mean and variance. This can be accomplished, among other methods, by use of the estimation equation (11) (see Ref. 2 for details).

The resulting equations may be broken into two sets:

(1) The prediction equations

$$\begin{aligned}\hat{x}_{k+1/k} &= F_k \hat{x}_{k/k} \\ \hat{P}_{k+1/k} &= F_k \hat{P}_{k/k} F_k^T + \hat{Q}_k \quad , \quad \text{and}\end{aligned}\tag{18}$$

(2) The regression equations

$$\begin{aligned}\hat{x}_{k+1/k+1} &= \hat{x}_{k+1/k} + \hat{P}_{k+1/k} H_{k+1}^T (H_{k+1}^T \hat{P}_{k+1/k} H_{k+1} + \hat{R}_{k+1})^{-1} (z_{k+1} - H_{k+1}^T \hat{x}_{k+1/k}) \\ \hat{P}_{k+1/k+1} &= \hat{P}_{k+1/k} - \hat{P}_{k+1/k} H_{k+1}^T (H_{k+1}^T \hat{P}_{k+1/k} H_{k+1} + \hat{R}_{k+1})^{-1} H_{k+1} \hat{P}_{k+1/k} \quad ,\end{aligned}\tag{19}$$

where

$$\begin{aligned}\hat{x}_{i/j} &\triangleq E(x_i / Z_j) \\ \hat{P}_{i/j} &= E[(x_i - \hat{x}_{i/j})(x_i - \hat{x}_{i/j})^T / Z_j] \quad .\end{aligned}$$

Now consider the state and measurement equations\*

$$\begin{aligned}x_{k+1} &= f(x_k, u_k, k) + w_k \\ z_k &= h(x_k, k) + v_k \quad .\end{aligned}\tag{20}$$

---

\* These equations need not be linear in  $w_k$  and  $v_k$  but for simplicity only this case is treated; the extension is trivial.

Prediction is investigated first. Linearization of  $f$  about  $\hat{x}_{k/k}$  yields

$$x_{k+1} \approx f(\hat{x}_{k/k}, u_k, k) + f_x(\hat{x}_{k/k}, u_k, k)(x_k - \hat{x}_{k/k}) + w_k \quad (21)$$

where the gradient  $g_x(x)$  of a vector function  $g(x)$  is the matrix defined by

$$g_x^{(i)(j)} \triangleq \frac{\partial g^{(i)}}{\partial x^{(j)}} \quad (22)$$

with the superscripts denoting components.

Letting

$$\tilde{x}_{k+1} = x_{k+1} - f(\hat{x}_{k/k}, u_k, k) \quad \text{and} \quad \tilde{x}_k = x_k - \hat{x}_{k/k} \quad ,$$

Eq. (20) takes the form of (15); therefore

$$\hat{\tilde{x}}_{k+1/k} = f_x(\hat{x}_{k/k}, u_k, k) \hat{\tilde{x}}_{k/k} = 0$$

$$\hat{\tilde{P}}_{k+1/k} = f_x(\hat{x}_{k/k}, u_k, k) \hat{P}_{k/k} f_x^T(\hat{x}_{k/k}, u_k, k) + \hat{Q}_k$$

or

$$\hat{\tilde{x}}_{k+1/k} = f(\hat{x}_{k/k}, u_k, k)$$

$$\hat{\tilde{P}}_{k+1/k} = f_x(\hat{x}_{k/k}, u_k, k) \hat{P}_{k/k} f_x^T(\hat{x}_{k/k}, u_k, k) + \hat{Q}_k \quad (23)$$

These are the approximate prediction equations.

Now consider regression; if (20) is linearized about  $\hat{x}_{k+1/k}$  then

$$z_{k+1} \approx h(\hat{x}_{k+1/k}, k+1) + h_x(\hat{x}_{k+1/k}, k+1)(x_{k+1} - \hat{x}_{k+1/k}) + v_{k+1} \quad (24)$$

Letting

$$\tilde{z}_{k+1} \triangleq z_{k+1} - h(\hat{x}_{k+1/k}, k+1)$$

(24) takes the form of (17); hence

$$\begin{aligned}
 \hat{x}_{k+1/k+1} &= \hat{x}_{k+1/k} + \hat{P}_{k+1/k} h_x^T (h_x \hat{P}_{k+1/k} h_x^T + R_{k+1})^{-1} \\
 &\quad \cdot [z_{k+1} - h(\hat{x}_{k+1/k}, k+1)] \\
 \hat{P}_{k+1/k+1} &= \hat{P}_{k+1/k} - \hat{P}_{k+1/k} h_x^T (h_x \hat{P}_{k+1/k} h_x^T + R_{k+1})^{-1} h_x \hat{P}_{k+1/k}
 \end{aligned}
 \tag{25}$$

where the argument of  $h_x(\hat{x}_{k+1/k}, k+1)$  has been suppressed for simplicity. These are the approximate regression equations.

The essence of the extended Kalman filter is presented in Fig. 3. In words, the filter operates basically as follows: From the present estimate, the nonlinear state and measurement equations are used to predict the next measurement under the assumption of zero noise and disturbance. This prediction is compared with the actual measurement and the

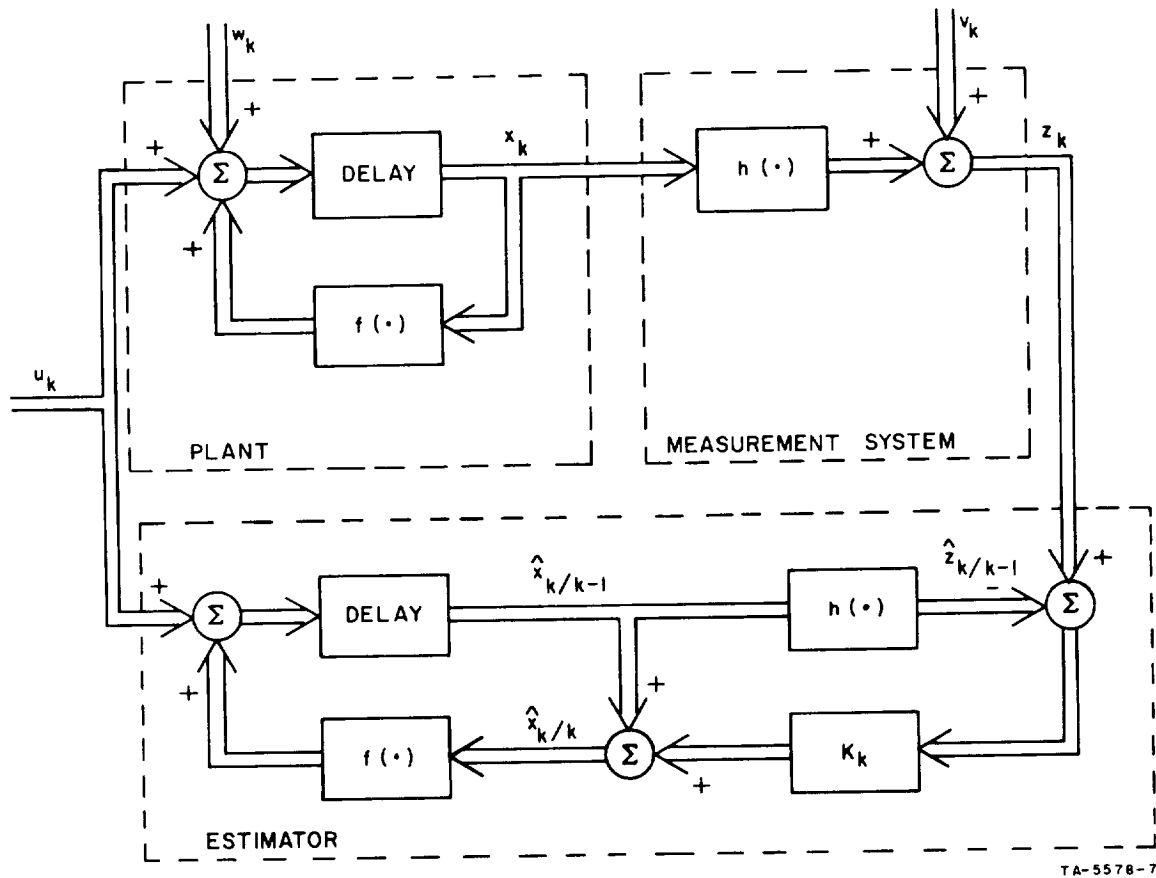


FIG. 3 EXTENDED KALMAN FILTER

estimate corrected by a linear function of their difference. Linear estimation theory and appropriate linearization are used to determine this linear function. Viewed in this light the extended Kalman Filter is an eminently reasonable method of estimation.

## B. USE OF THE "EXTENDED KALMAN FILTER" FOR IDENTIFICATION

The equations given in Appendix A describing the linear adaptive control problem have the form of (20) with the simplification that the measurement equation is linear. Hence by calculating the gradient  $f_x$  and substituting directly into (24) and (25) we can derive the equations that simultaneously estimate the dynamic state of the plant and identify its parameters. The details of such a derivation are presented in Appendix B, the results are:

### *Prediction Equations*

$$\begin{aligned}
 \hat{x}_{k+1/k}^D &= F_k^D \hat{x}_{k/k}^D + G_k^D u_k \\
 \hat{\phi}_{k+1/k} &= F_k^\phi \hat{\phi}_{k/k} \\
 \hat{P}_{k+1/k}^D &= F_k^D \hat{P}_{k/k}^D F_k^{D T} + \hat{Q}_k^* + \hat{Q}_k \\
 \hat{P}_{k+1/k}^{\phi D} &= F_k^\phi \hat{P}_{k/k}^{\phi D} F_k^{D T} + F_k^\phi \hat{P}_{k/k}^{\phi \phi} F_k^{D \phi T} \\
 \hat{P}_{k+1/k}^{\phi \phi} &= F_k^\phi \hat{P}_{k/k}^{\phi \phi} F_k^{\phi T} \quad , \quad (26)
 \end{aligned}$$

where

$$\hat{Q}_k^* = F_k^D \hat{P}_{k/k}^{\phi D} F_k^{D T} + F_k^D \hat{P}_{k/k}^{\phi \phi} F_k^{D \phi T} + F_k^D \hat{P}_{k/k}^{\phi \phi} F_k^{D \phi T} \quad (27)$$



$$\hat{Q}_k = q_k \cdot \begin{bmatrix} & 0 & & \\ & \vdots & & \\ & 0 & 0 & 0 \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 & \hat{\Lambda}^T \\ \hline & & & 0 & & & & \\ & & & \vdots & & & & \\ & & & 0 & & & & \\ \hline 0 & & & \hat{\Lambda} & & & 0 & \hat{\Lambda}\hat{\Lambda}^T \end{bmatrix}$$

$F_k^D$  = transition matrix for the plant if  $\phi_k = \hat{f}_{k/k}$

$G_k^D$  = distribution matrix for the plant if  $f_k = \hat{f}_{k/k}$

$\hat{P}_{i/j}^D$  = covariance of  $x_i^D$  given  $Z_j$

other quantities are defined in Appendix B.

### Regression Equations

$$\begin{aligned} \hat{x}_{k+1/k+1}^D &= \hat{x}_{k+1/k}^D + \hat{p}_{k+1/k}^{Dy} (\hat{p}_{k+1/k}^{yy} + \hat{r}_k)^{-1} (z_{k+1} - \hat{y}_{k+1/k}) \\ \hat{\phi}_{k+1/k+1} &= \hat{\phi}_{k+1/k} + \hat{p}_{k+1/k}^{\phi y} (\hat{p}_{k+1/k}^{yy} + \hat{r}_k)^{-1} (z_{k+1} - \hat{y}_{k+1/k}) \\ \hat{P}_{k+1/k+1}^D &= \hat{P}_{k+1/k}^D - \hat{p}_{k+1/k}^{Dy} (\hat{p}_{k+1/k}^{yy} + \hat{r}_k)^{-1} \hat{p}_{k+1/k}^{Dy T} \\ \hat{P}_{k+1/k+1}^{\phi} &= \hat{P}_{k+1/k}^{\phi} - \hat{p}_{k+1/k}^{\phi y} (\hat{p}_{k+1/k}^{yy} + \hat{r}_k)^{-1} \hat{p}_{k+1/k}^{\phi y T} \\ \hat{P}_{k+1/k+1}^{\phi y} &= \hat{P}_{k+1/k}^{\phi y} - \hat{p}_{k+1/k}^{\phi y} (\hat{p}_{k+1/k}^{yy} + \hat{r}_k)^{-1} \hat{p}_{k+1/k}^{\phi y T} \end{aligned} \quad (28)$$

where

$p_{i/j}^{yy}$  is the variance of  $y_i$  given  $Z_j$

$p_{i/j}^{Dy}$  is the covariance between  $x_i^D$  and  $y_i$  given  $Z_j$

$p_{i/j}^{\phi y}$  is the covariance between  $\phi_i$  and  $y_i$  given  $Z_j$ .

Figure 4 is a diagram of an adaptive control system using the extended Kalman filter. Note that the present estimate of the parameter state is used to update the plant model and to vary the control law. Derivation of the control law is treated in the next section. The gains  $K_1$  and  $K_2$  are determined by solution of the variance equations. Equation (26c) implies that the effect of the parameter uncertainty on estimation of the dynamic state  $x_{k+1}^D$  is equivalent to a random disturbance with covariance  $Q_k^*$ .

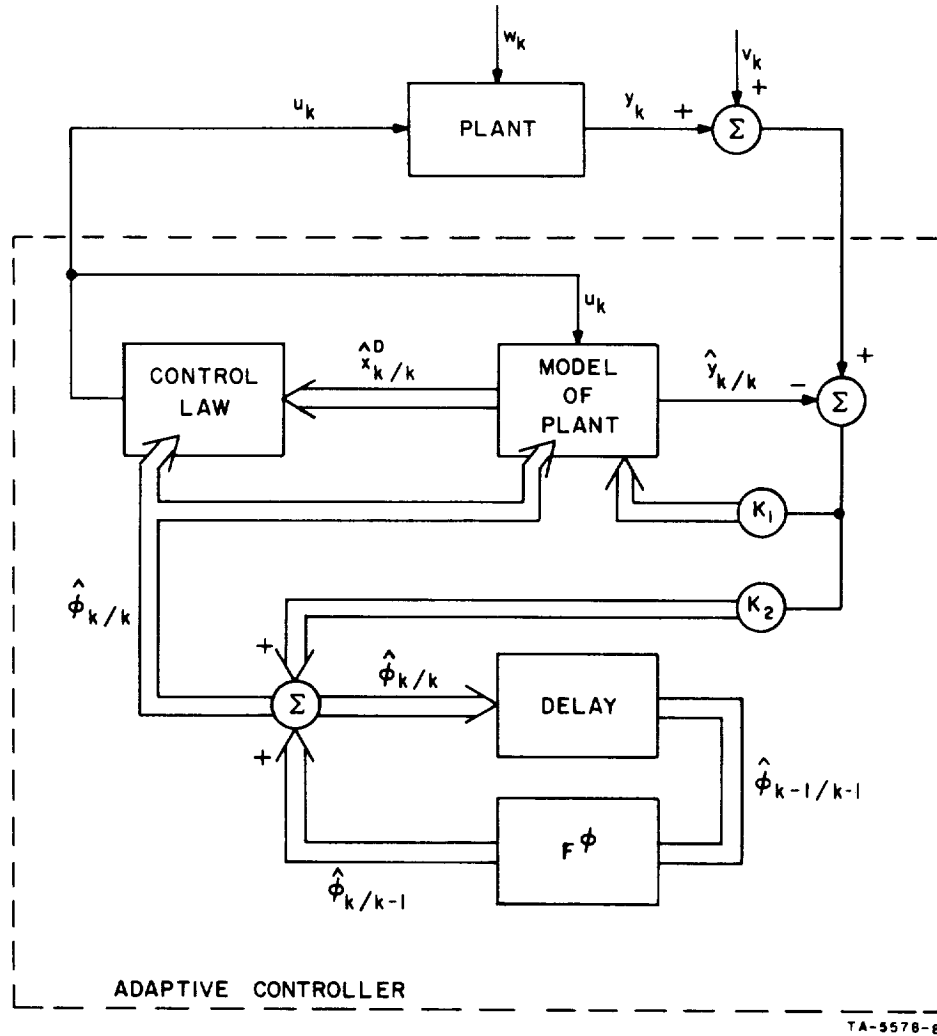


FIG. 4 ADAPTIVE CONTROL SYSTEM

Can the use of the extended Kalman filter which is heuristically valid be justified theoretically? One approach is to solve (11) approximately and compare the results with the extended Kalman filter. Bucy<sup>10</sup> has done this for the continuous time analog of (11) which is a

generalized Fokker-Planck equation. His results contain terms that are not present in the continuous version of the extended Kalman filter (which may be obtained by limiting arguments from the results of the previous paragraph). Similar results have also been obtained for the discrete time case in unpublished work by the author. Thus, to justify the use of the extended Kalman filter for identification, one must show that these additional terms are negligible in this case.

The procedure just mentioned gives as an estimate of the present state an approximation of the most probable present state. Alternatively, one may seek as an estimate the most recent state on the most probable trajectory. In the linear case these two estimates are equal, but in general they will not be the same. The problem of finding the most likely trajectory may be converted to a nonlinear control problem and treated by dynamic programming.<sup>11</sup> Unpublished work by Luenberger and a paper by Detchmendy and Sridhar<sup>12</sup> indicate that an approximate solution to this problem is similar to the extended Kalman filter but again with extra terms. However, these terms disappear in the identification problem presented here; hence, to justify the extended Kalman filter on this basis requires justification of using the most probable trajectory rather than most probable present state for estimation.

Rather than following either of the above approaches to justification of the extended Kalman filter, a third approach is taken herein. Special cases are found in which the extended Kalman filter equations are exact, then it is assumed that for situations closely approximating these cases the equations are good approximations. Two such cases are considered in the next two paragraphs.

### C. LITTLE UNCERTAINTY ABOUT PARAMETERS

One obvious case where the linearized equations are exact is the case where the parameters are known exactly; hence one can expect that the extended Kalman filter would work well when the amount of uncertainty about the system is small.

The final term of the regression Eq. (28b) for updating the estimate of the parameter state contains  $\underline{p}_{k+1/k}^{\phi y}$  as a multiplicative factor. When the parameters are well known, this covariance is small and the estimate of the parameters is essentially the *a priori* estimate; hence, it is reasonable to consider not updating the parameter estimates. If this is

done (*i.e.*, identification is not performed and estimation of the dynamic state is based upon the *a priori* estimate of the structure) then the estimator still obeys the equations given in Part B above, except that

$$\begin{aligned}\hat{\phi}_{k+1/k+1} &= \hat{\phi}_{k+1/k} \\ \hat{P}_{k+1/k+1}^{\phi D} &= \hat{P}_{k+1/k}^{\phi D} \\ \hat{P}_{k+1/k+1}^{\phi} &= \hat{P}_{k+1/k}^{\phi} \quad .\end{aligned}\tag{29}$$

This observation, which is true any time the extended Kalman filter can be justified, will prove of great use in the analysis of passive adaptive systems.

#### D. LITTLE MEASUREMENT NOISE

A second major case in which the extended Kalman filter is exact is when the measurement noise is zero and the system is known to be initially at rest. In this case  $y_k$  and  $y_k^*$  are known exactly initially and can be measured exactly for all future time. Suppose that  $p(x_k/Z_k)$  is Gaussian. Since  $y_k$  and  $y_k^*$  are known, the linearization of Eq. (21) is exact; therefore the prediction Eq. (23) is exact and  $p(x_{k+1}/Z_k)$  is Gaussian. Since the measurement equation is linear, the regression equations are exact and  $p(x_{k+1}/Z_{k+1})$  is Gaussian. By induction, the extended Kalman filter is thus exact when  $\hat{r}_k = 0$ .

### IV CONTROL AND PERFORMANCE

The subject of this section is the approximate solution of the control equation (12) by application of linear optimal control theory. The results from this solution are twofold: a determination of the control law and a calculation of the performance. This development of this section is similar to that of Farison.<sup>5</sup> It is assumed for this section that the measurement noise is zero.

#### A. PASSIVE ADAPTIVE SYSTEMS

Suppose that, in Fig. 4, the gain  $K_2$  is set equal to zero. In this case no identification is performed and the *a priori* estimate of the system parameters is used in designing the estimator and determining the control law. Such a system can be called a passive adaptive system—

passive because no active adaption procedures are used and adaptive be-  
cause normal feedback provides some insensitivity to parameter variations.

In Sec. III-B it was pointed out that the effect of parameter un-  
certainty on the plant was equivalent to a disturbance noise with co-  
variance  $\hat{Q}_k^*$ . For  $\hat{r}_k = 0$ , i.e., no measurement noise,

$$\hat{Q}_k^* = F^{D\phi} P_k^{\hat{\phi}} F^{D\phi} \quad (30)$$

In Appendix B it is shown that

$$F^{D\phi} = \begin{bmatrix} 0 \\ \hline x_k^{DT} M_k^T + u_k \underline{n}_k^T \\ \hline 0 \\ \hline 0 \end{bmatrix} \quad (31)$$

where  $M_k$  and  $\underline{n}_k$  are given in that appendix. Therefore,

$$\hat{Q}_k^* = \begin{bmatrix} & 0 & & \\ & \vdots & & \\ & \vdots & & \\ & 0 & & \\ \hline 0 \dots 0 & \hat{q}_k^y & 0 \dots 0 & 0 \dots 0 \\ \hline & 0 & & \\ & \vdots & & \\ & \vdots & & \\ & 0 & & \\ \hline & 0 & & \\ & \vdots & & \\ & \vdots & & \\ & 0 & & \end{bmatrix}, \quad (32)$$

where

$$\begin{aligned} \hat{q}_k^y &= x_k^{DT} M_k^T P_k^{\hat{\phi}} M_k x_k^D + 2x_k^{DT} M_k^T P_k^{\hat{\phi}} \underline{n}_k u_k + u_k \underline{n}_k^T P_k^{\hat{\phi}} \underline{n}_k u_k \\ &\triangleq x_k^{DT} Q_k^{D*} x_k^D + 2x_k^{DT} \underline{s}_k^{D*} u_k + u_k r_k^* u_k \end{aligned} \quad (33)$$

Note that since  $P_k^{\hat{\phi}}$  can be calculated *a priori* (since no identification  
takes place)  $Q_k^{D*}$ ,  $\underline{s}_k^{D*}$  and  $r_k^*$  can be determined *a priori*.

Even though  $\hat{Q}_k^*$  is a function of the dynamic state and control, it is of such a form that linear theory can still be applied. The development begins with the assumption that

$$I(\hat{\rho}_k, k) = \hat{x}_k^{D^T} P_k x_k^D + b_k \quad . \quad (34)$$

Substitution of (26), (32), and (34) into control equation (12) yields

$$\begin{aligned} I_k(\hat{\rho}_k, k) &= \min_{u_k} E\{(y_k^2 q_k + u_k^2 r_k + x_{k+1}^{D^T} P_{k+1} x_{k+1}^D + b_{k+1})/Z_k\} \\ &= \min_{u_k} [y_k^2 q_k + u_k^2 r_k + (F_k^{D^D} x_{k/k}^D + G_k^D u_k)^T P_{k+1} (F_k^{D^D} x_{k/k}^D + G_k^D u_k) \\ &\quad + p_{k+1}^y (x_k^{D^T} Q_k^{D^*} x_k + 2x_k^{D^T} \underline{s}_k^{D^*} u_k + r_k^* u^2) \\ &\quad + \text{tr} [P_{k+1} (F_k^{D^D} P_{k/k}^{D^D} F_k^{D^T} + \hat{Q}_k)] + b_{k+1}] \quad , \quad (35) \end{aligned}$$

where  $p_{k+1}^y$  is component of  $P_{k+1}$  corresponding to  $y_k^2$ .

Note that this recursion equation is the same as would be obtained if the *a priori* estimate of the plant were exact, but the performance index were

$$J' = E \sum_{k=0}^N (x_k^{D^T} Q_k^{D'} x_k^D + 2x_k^{D^T} \underline{s}_k^{D'} u_k + r_k' u^2) \quad , \quad (36)$$

where

$$Q_k^{D'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \hline 0 \dots 0 & q_k & 0 \dots 0 & 0 \dots 0 \\ \hline 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} + p_{k+1}^y Q_k^{D^*}$$

$$\underline{s}_k^{D'} = p_{k+1}^y s_k^{D*}$$

$$r_k' = r_k + p_{k+1}^y r_k^* \quad .$$

The primed quantities cannot be calculated before the minimization; however,  $p_{k+1}^y$  will be available in time to compute  $Q_k^{D'}$ ,  $\underline{s}_k^{D'}$ , and  $r_k^{D'}$  when they are needed.

The minimization of (35) can be carried out by completion of squares; Ref. 2 contains the details. The results are

$$u_k = -K_k^D \hat{x}_k^D \quad , \quad (37)$$

where

$$K_k^D = (r_k' + G_k^{D T} P_{k+1} G_k^D)^{-1} (G_k^{D T} P_{k+1} F_k^D + \underline{s}_k^{D' T})$$

and

$$P_k = Q_k^{D'} + F_k^{D T} P_{k+1} F_k^D - (F_k^{D T} P_{k+1} G_k^D + \underline{s}_k^{D'}) K_k^D$$

$$b_k = \text{tr} [P_{k+1} \hat{F}_k^D P_{k/k}^D F_k^{D T} + \hat{Q}_k] + b_{k+1} \quad . \quad (38)$$

Performance is given by

$$J = E[I(\hat{\theta}_0, 0)]$$

$$= \hat{x}_{0/-1}^{D T} P_0 \hat{x}_{0/-1}^D + \text{tr} (P_0 \hat{Q}_{-1}) + \sum_{k=0}^{N-1} \Delta \beta_k \quad (39)$$

where

$$\hat{\beta}_k^2 = \text{tr} [P_{k+1} \hat{Q}_k + P_{k+1}^* V_k]$$

$$P_{k+1}^* = Q_k^{D'} + F_k^{D T} P_{k+1} F_k^D - P_k^D \quad .$$

The details of this derivation are also in Ref. 2.

It is important to note that in performing the calculations of this section that (29) must be used in the updating of  $\hat{P}^\phi$ . These computations result in the optimal control law for a minimum sensitivity design.<sup>13</sup>

## B. ANALYSIS-SYNTHESIS ADAPTIVE CONTROL SYSTEMS

If  $K_2$  in Fig. 4 is not set equal to zero, then the filter identifies the system parameters. Such a system is called an analysis-synthesis system; analysis refers to the process of identifying the system parameters and synthesis to the process of determining the control law and plant model on the basis of these parameters.

The situation in this case is more complicated than for the passive adaptive system because the estimate of the system parameters is not made *a priori*. This implies among other things that  $\hat{P}_k^\phi$  cannot be calculated *a priori*; hence, linear analysis will not provide the optimum control law. One reasonable approximation in calculation of the control at time  $k$  is to assume that the estimate of  $\phi_k$  at time  $k$  is exact and that  $\eta_i$  is zero for  $i \geq k$  and to ignore the effect of  $u_k$  on  $\hat{P}_k^\phi$ . Under these assumptions the linear theory of the previous section may be applied, and the equations given there hold, except that (26) is used for updating  $P_k^\phi$  rather than (29).

To realize such a system it is necessary either to compute  $K_k^D$  for each possible  $\phi_k$  *a priori* and store the results or to compute  $K_k^D$  for the estimated  $\phi_k$  in real time. An approximate computation of performance can be made by use of the nominal values of the parameters. The major difference between analysis-synthesis and passive-adaptive systems is that, for the former, the additional cost terms in the performance index are less because  $\hat{P}_k^\phi$  is less since (26) rather than (29) holds.

In two special cases, the procedure outlined above is optimal. If  $\phi_k$  is known exactly initially and  $\eta_k$  is zero, then the procedure is obviously optimal. If  $\phi_0$  is known and if  $\phi_k$  changes slowly, then the identifier should be able to follow  $\phi_k$  very closely and the above procedure should be close to optimal. If  $F_k^\phi \equiv 0$  then the variation of parameters is random and no identification is possible; hence the above procedure is again optimal.<sup>14</sup> Thus for very rapid changes in parameters the approximation should be very good.



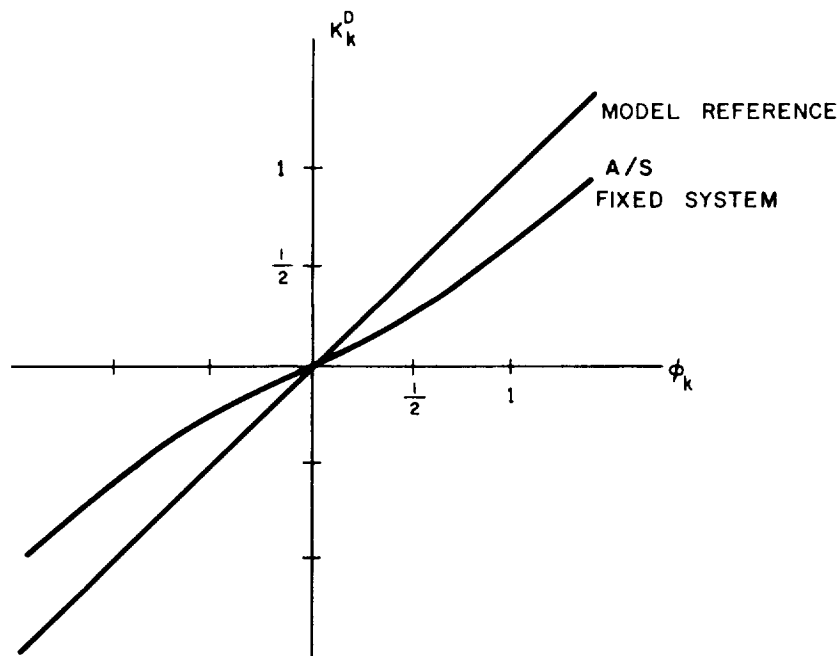
### C. MODEL REFERENCE SYNTHESIS

In general,  $K_k^D$  is a complicated function of  $\phi_k$ ; hence, realization of the synthesis procedure given above may be complicated. An alternate method is to calculate  $K_k^D$  using the nominal values of the system parameters; from this, the optimum closed-loop system can be determined. Synthesis consists of picking  $K_k^D$  so that the closed-loop system matches this system for the estimated values of the parameters. For this synthesis procedure,  $K_k^D$  is a simple linear function of  $\phi_k$ .

In general, the difference between the two methods of synthesis depends upon the cost of control. If the control cost is low they are very similar; if it is high they differ considerably. Figure 5 is a graph of  $K_k^D$  as a function of  $\phi_k$  for the scalar system with no disturbance or noise:

$$x_{k+1}^D = \phi_k x_k^D + u_k$$

$$J = \sum_{L=D}^N x_k^{D^2} + u_k^2 \quad (40)$$



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FIG. 5 TWO METHODS OF SYNTHESIS

## V. CONCLUSIONS

The development presented in this memorandum was based on three assumptions:

- (1) The problem would be a linear problem if the parameters were known (*i.e.*, linear equations, Gaussian random processes, quadratic costs, no constraints.)
- (2) The disturbance statistics are known
- (3) The measurement noise is small.

The first of these assumptions is most important to the development, since nonlinear problems are very hard to handle in general even without the difficulties introduced by parameter uncertainty. Fortunately many important problems satisfy this linearization assumption. Nonquadratic cost and/or constraints on the control will not affect the estimation procedures but will complicate the control.

With these assumptions, the following results may be obtained:

- (1) The adaptive control problem is a combined optimization problem, in general nonlinear. Adaptive control can be viewed as an approximation to solving this combined optimization problem, whose solution is generally incomputable. (This conclusion does not depend upon the above assumptions.)
- (2) The simplest approximation consists of designing the system to have low sensitivity to the parameter variations. Estimation in this case is the Kalman filter, which consists of the *a priori* model of the plant with the state being updated by a linear function of the difference between the predicted and actual measurements.
- (3) If the low sensitivity design has inadequate performance, then a better approximation to combined optimization is an analysis-synthesis system in which the plant parameters are identified on the basis of the available measurements. The extended Kalman filter is a good approximate technique of estimating the dynamic state of the system and identifying its parameters; in fact it is the optimal estimator and identifier when the measurement noise is zero and the system is initially at rest. The filter consists of a model of the plant based on the present estimate of parameters and a model of the parameter behavior, both of which are updated by linear functions of the difference between predicted and actual measurements.

- (4) For either the low sensitivity or the analysis-synthesis system, the major effect of parameter uncertainty is equivalent to an additional term in the loss function. A linear control law, which is optimal in the low sensitivity case and very close to optimal in the analysis-synthesis case, may be found by solution of a linear control problem without parameter uncertainty but with the modified performance index. The primary effect of identification is to reduce the size of the added cost terms.
- (5) Realization of the control law in the analysis-synthesis situation may be simplified by use of a model reference in synthesis at a cost in performance.

In conclusion, a standard and systematic procedure, based upon optimal linear system theory, has been developed for the design of low sensitivity and analysis-synthesis adaptive control systems. The resulting systems are close to optimum in important situations and their performance can be analyzed in these situations. In particular, it is possible to calculate the gain in performance resulting from parameter identification.



*APPENDIX A*

**SYSTEM EQUATIONS FOR THE LINEAR ADAPTIVE PROBLEM**

APPENDIX A

SYSTEM EQUATIONS FOR THE LINEAR ADAPTIVE PROBLEM

From (1), (2), (14), and (6) the following state equation may be generated

$$x_{k+1} = f(x_k, u_k, w_k, k) = \begin{bmatrix} Dy_k^* + \Delta y_k \\ \hline g_k \\ \hline Du_k^* + \Delta u_k \\ \hline Dd_k^* + \Delta d_k \\ \hline F\phi_k + \eta_k \end{bmatrix} \quad (A-1)$$

where

$$D = n - 1 \times n - 1 \text{ matrix} = \begin{bmatrix} 0 & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ 0 & & & 1 \\ & & & & 0 \end{bmatrix}$$

$$\Delta = n - 1 \times 1 \text{ matrix} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

and

$$g_k = \phi_k^T H_k^0 + \underline{a}_{k+1}^0 y_k^* + a_{1k+1}^0 y_k + \underline{b}_{k+1}^0 u_k^* + b_{1k+1}^0 u_k + \underline{c}_k^T d_k^* + d_k$$

$$H_k^0 = A_k y_k^* + \underline{a}_{1k+1} y_k + B_k u_k^* + \underline{b}_{1k+1} u_k \quad (A-2)$$

with

$$\begin{aligned}
 A_k^T &\triangleq \begin{bmatrix} a_{nk+1}^T \\ \vdots \\ a_{2k+1}^T \end{bmatrix} & \underline{a}_k^o &= \begin{bmatrix} a_{nk+1}^o \\ \vdots \\ a_{2k+1}^o \end{bmatrix} \\
 B_k^T &\triangleq \begin{bmatrix} b_{nk+1}^T \\ \vdots \\ b_{2k+1}^T \end{bmatrix} & \underline{b}_k^o &= \begin{bmatrix} a_{nk+1}^o \\ \vdots \\ a_{2k+1}^o \end{bmatrix} \\
 \underline{c}_k &= \begin{bmatrix} c_{nk+1} \\ \vdots \\ c_{2k+1} \end{bmatrix}
 \end{aligned} \tag{A-3}$$

The measurement equation is simply

$$y_k = h_k(x_k) + v_k = H_k^T x_k = y_k + v_k$$

where

$$H_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{A-4}$$

*APPENDIX B*

**DERIVATION OF THE LINEARIZED KALMAN FILTER EQUATIONS  
FOR THE LINEAR ADAPTIVE PROBLEM**





and if

$$G_k^D \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline b_{1k+1}^o + \phi_k b_{1k+1} \\ \hline \Delta \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} = \text{distribution matrix for dynamic state} \\ \text{if } \phi_k \text{ is known,} \tag{B-3}$$

then the extended Kalman filter equations presented in Sec. III-B may be written down by substitution into (23) and (25).

From (A-2) and (B-1),  $F^{D\phi}$  has the form given in (31), where

$$M_k = \begin{bmatrix} A_k^T \\ \hline a_{1k+1}^T \\ \hline B_k^T \\ \hline 0 \end{bmatrix} ,$$

$$\underline{n}_k = \underline{b}_{1k+1} \tag{B-4}$$

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MEMORANDUM 7

APPLICATION OF OPTIMUM ESTIMATION AND CONTROL  
THEORY TO SATELLITE TRACKING PROBLEMS



## MEMORANDUM 7

# APPLICATION OF OPTIMUM ESTIMATION AND CONTROL THEORY TO SATELLITE TRACKING PROBLEMS

## I INTRODUCTION

The purpose of this memorandum is to derive optimum (approximately) estimation and control techniques for the satellite tracking problem. The problem is nonlinear, as will become apparent in subsequent sections. Some preliminary studies are described in Refs. 1 and 2.\* The present study has resulted in the development of a digital computer program that implements the operation of the optimal estimator and controller in conjunction with the satellite tracking system.

A solution to the problem can be obtained by solving the estimation and control portions separately. Since the satellite tracking problem is nonlinear, the assumption that the estimation and control portions separate may not be optimal in the strictest sense;<sup>3</sup> however, since the estimation and control portions are weakly coupled (as will be seen in subsequent sections), the assumption of separation is quite reasonable.

The estimator, which generates an optimum estimate of the present state of the system (satellite and antenna control system), is derived in Sec. III. The estimation problem is solved by employing the extended Kalman filter, which necessitates the linearization of the satellite equations and the measurement equations.

The estimate of the system state is then employed in the controller to compute the optimum control with respect to the given performance criterion. The control problem is solved in Sec. IV by making the appropriate linearization and applying some new results in the theory of linear optimal control.

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\* References are listed at the end of the memorandum.

## II PROBLEM FORMULATION

Figure 1 is a block diagram of the satellite tracking system. The mathematical models for the various parts of the system are given below.

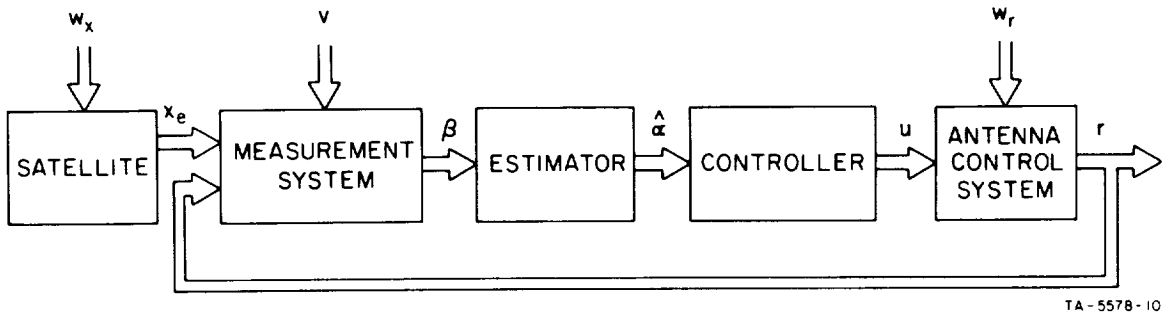


FIG. 1 SATELLITE TRACKING SYSTEM

### A. SATELLITE\*

$$\begin{aligned}\ddot{x}_{1e} &= -\frac{\mu_e x_{1e}}{r_e^3}, \\ \ddot{x}_{2e} &= -\frac{\mu_e x_{2e}}{r_e^3}, \\ \ddot{x}_{3e} &= -\frac{\mu_e x_{3e}}{r_e^3},\end{aligned}\quad (1)$$

where

$$r_e = \left(x_{1e}^2 + x_{2e}^2 + x_{3e}^2\right)^{1/2},$$

\* The term "satellite" does not necessarily mean a near-earth satellite; it could, for instance, refer to a deep-space probe.



$\mu_e$  = the product of the universal gravitational constant and the mass of the earth,  
 $x_{1_e}, x_{2_e}, x_{3_e}$  = the position coordinates of the satellite with respect to an earth-centered Cartesian coordinate system. (The  $3_e$  axis is coincident with the earth's polar axis, and the  $1_e$  and  $2_e$  axes lie in the equatorial plane, completing a right-handed orthogonal set.)

It should be noted that the above differential equations (1) merely give an approximate description of the motion of the satellite, and are used only to obtain the solutions to the estimation and control portions of the problem. The actual\* trajectory of the satellite is generated by a more exact computer program model developed at NASA Ames Research Center, Mountain View, California.

The differential equations (1) can be put into state variable form upon definition of the following variables:

$$\begin{aligned}
 \alpha_1 &= x_{1_e} \quad , \\
 \alpha_2 &= x_{2_e} \quad , \\
 \alpha_3 &= x_{3_e} \quad , \\
 \alpha_4 &= \dot{x}_{1_e} \quad , \\
 \alpha_5 &= \dot{x}_{2_e} \quad , \\
 \alpha_6 &= \dot{x}_{3_e} \quad .
 \end{aligned}
 \tag{2}$$

Combining Eqs. (1) and (2) yields

$$\begin{aligned}
 \dot{\alpha}_1 &= \alpha_4 \quad , \\
 \dot{\alpha}_2 &= \alpha_5 \quad , \\
 \dot{\alpha}_3 &= \alpha_6 \quad ,
 \end{aligned}$$

---

\* The term "actual" refers to the trajectory to be tracked by the antenna in the computer simulation.

$$\begin{aligned}\dot{\alpha}_4 &= \frac{-\mu_e \alpha_1}{r_e^3} \quad , \\ \dot{\alpha}_5 &= \frac{-\mu_e \alpha_2}{r_e^3} \quad , \\ \dot{\alpha}_6 &= \frac{-\mu_e \alpha_3}{r_e^3} \quad ,\end{aligned}\tag{3}$$

where

$$r_e = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2} \quad .$$

After defining the six-dimensional state vector of the satellite as

$$x_e = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_6 \end{bmatrix} \quad ,\tag{4}$$

the differential equations (3) can be rewritten concisely as

$$\dot{x}_e = f(x_e) \quad ,\tag{5}$$

where  $f(x_e)$  is a six-dimensional vector function of  $x_e$  as given by Eqs. (3).

Equation (5) is a nonlinear differential equation; however, in order to take advantage of certain results in the theory of linear estimation and control, it is necessary to linearize this equation. This concept will be clarified in Secs. III and IV. Linearization of Eq. (5) is achieved by considering  $x_e$  to be composed of some nominal trajectory  $x_e^o$  and a perturbation from the nominal  $\tilde{x}_e$ :

$$x_e = x_e^o + \tilde{x}_e \quad .\tag{6}$$

Upon expanding Eq. (5) in a Taylor series about  $x_e^o$  and neglecting second and higher-order terms, the linear perturbation equation is found to be:

$$\dot{\tilde{x}}_e = \mathfrak{J}(x_e^o) \tilde{x}_e \quad ,\tag{7}$$

where

$$\dot{x}_e^o = f(x_e^o) \quad ,$$

$$H(x_e^o) = \left. \frac{\partial f}{\partial x_e} \right|_{x_e^o} = \left[ \frac{\partial f_i}{\partial x_j} \right]_{x_e^o} .$$

$$H(x_e^o) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\mu_e (2\alpha_1^2 - \alpha_2^2 - \alpha_3^2)}{r_e^5} & \frac{3\mu_e \alpha_1 \alpha_2}{r_e^5} & \frac{3\mu_e \alpha_1 \alpha_3}{r_e^5} & 0 & 0 & 0 \\ \frac{3\mu_e \alpha_2 \alpha_1}{r_e^5} & \frac{\mu_e (2\alpha_2^2 - \alpha_1^2 - \alpha_3^2)}{r_e^5} & \frac{3\mu_e \alpha_2 \alpha_3}{r_e^5} & 0 & 0 & 0 \\ \frac{3\mu_e \alpha_3 \alpha_1}{r_e^5} & \frac{3\mu_e \alpha_3 \alpha_2}{r_e^5} & \frac{\mu_e (2\alpha_3^2 - \alpha_1^2 - \alpha_2^2)}{r_e^5} & 0 & 0 & 0 \end{bmatrix}_{x_e^o} \quad (8)$$

Since the problem is to be simulated on a digital computer, it is essential to convert the differential equation (7) into an equivalent difference equation. This can be done by noting that the time derivative is approximately given by

$$\dot{\tilde{x}}_e(k-1) = \frac{\tilde{x}_e(k) - \tilde{x}_e(k-1)}{\Delta t} \quad ,$$

or

$$\tilde{x}_e(k) = \tilde{x}_e(k-1) + \dot{\tilde{x}}_e(k-1)\Delta t \quad , \quad (9)$$

where

$\tilde{x}_e(k\Delta t)$  is defined as  $\tilde{x}_e(k)$ , and  $\Delta t$  is the time increment.

Substituting Eq. (7) into (9) gives

$$\tilde{x}_e(k) = \{I + \mathfrak{F}[x_e^o(k-1)]\Delta t\}\tilde{x}_e(k-1), \quad (10)$$

where the transition matrix is given by

$$\Phi_x(k-1) = I + \mathfrak{F}[x_e^o(k-1)]\Delta t. \quad (11)$$

Since Eq. (10) is only an approximate mathematical model of the satellite motion, a random forcing term will be included as follows in order to account for the imprecise nature of this model:

$$\tilde{x}_e(k) = \Phi_x(k-1)\tilde{x}_e(k-1) + \Gamma_x(k-1)w_x(k-1), \quad (12)$$

where

$$\Gamma_x(k-1) = \Gamma_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad w_x(k-1) = \begin{bmatrix} w_{x_1}(k-1) \\ w_{x_2}(k-1) \\ w_{x_3}(k-1) \end{bmatrix}.$$

It is assumed that the random forcing term  $w_x(k-1)$  is white\* gaussian noise with zero mean and covariance  $Q_x(k-1) = E[w_x(k-1)w_x^T(k-1)]$ .

## B. ANTENNA CONTROL SYSTEM

The antenna control system consists of two channels—elevation and azimuth. The elevation channel, which includes the antenna dynamics, is illustrated schematically in Fig. 2; the azimuth channel has a similar configuration. In this study, the analysis is carried through for an electric drive; a hydraulic drive could be considered in an analogous manner.

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\* The statement that a random quantity  $z$  is white implies that  $E[z(i)z^T(j)] = 0$  for  $i \neq j$ ; i.e.,  $z$  is uncorrelated for different sample times.

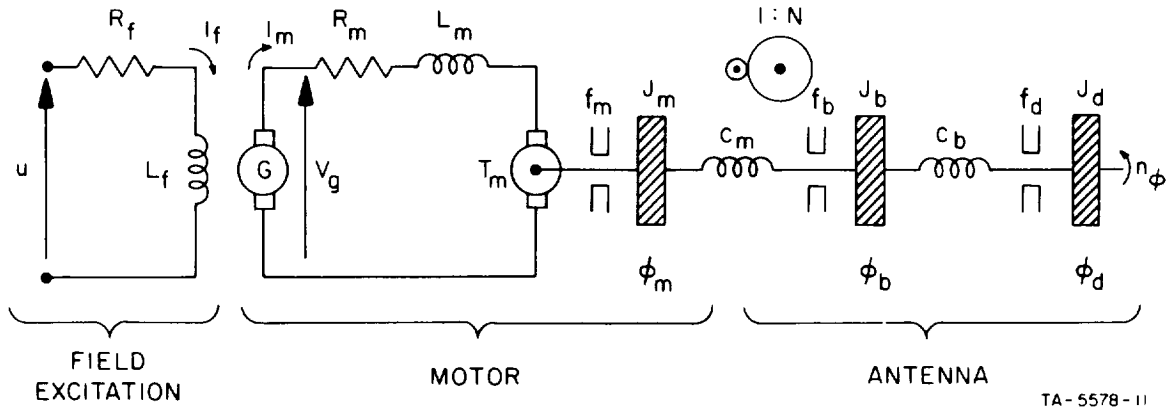


FIG. 2 SCHEMATIC DIAGRAM OF ANTENNA CONTROL SYSTEM (Elevation Channel)

It is assumed that the elevation channel is linear (for suitably small signals) and is described by the following:

$$\begin{aligned}
 L_f \dot{I}_f + R_f I_f &= u_\phi \quad , \\
 V_g &= k_f I_f \quad , \\
 L_m \dot{I}_m + R_m I_m &= V_g - k_m \dot{\phi}_m \quad , \\
 T_m &= k_m I_m \quad , \\
 J_m \ddot{\phi}_m + f_m \dot{\phi}_m + c_m \left( \phi_m - N \phi_b \right) &= T_m \quad , \\
 J_b \ddot{\phi}_b + f_b \dot{\phi}_b + c_b \left( \phi_b - \phi_d \right) + N^2 c_m \left( \phi_b - \frac{1}{N} \phi_m \right) &= 0 \quad , \\
 J_d \ddot{\phi}_d + f_d \dot{\phi}_d + c_b \left( \phi_d - \phi_b \right) &= n_\phi \quad , \quad (13)
 \end{aligned}$$

where

- $u_\phi$  = control variable
- $L_f$  = field inductance
- $R_f$  = field resistance
- $I_f$  = field current
- $k_f$  = field proportionality constant
- $V_g$  = generator voltage
- $L_m$  = motor inductance
- $R_m$  = motor resistance
- $I_m$  = motor current

- $k_m$  = motor proportionality constant
- $T_m$  = motor torque
- $J_m$  = moment of inertia of motor
- $f_m$  = damping of motor
- $c_m$  = gear spring constant of motor (referred to the motor shaft)
- $N$  = gear ratio
- $\phi_m$  = motor angle
- $J_b$  = moment of inertia of antenna base
- $f_b$  = damping of antenna base
- $c_b$  = spring constant of antenna
- $\phi_b$  = angle of antenna base or angle of antenna's mechanical axis
- $J_d$  = moment of inertia of antenna dish
- $f_d$  = damping of antenna dish
- $\phi_d$  = angle of antenna dish or angle of antenna's electrical axis
- $n_\phi$  = random disturbance (noise) due to wind gusts.

It should be noted that this model of the antenna considers the first bending mode. For large antennas this effect is quite significant.

It has been shown<sup>4</sup> that the power spectral density of the wind disturbance  $n_\phi$  is approximately equal to

$$\mathcal{S}_{n_\phi}(s) = \frac{1}{-s^2 + a_\phi^2} .$$

The noise  $n_\phi$  can be considered as the output of a filter having the transfer function

$$\frac{1}{s + a_\phi}$$

and subjected to white noise  $w_\phi$ , where  $\mathcal{S}_{w_\phi}(s) = 1$ . This step is necessary in order to put the problem in the appropriate form for the relevant theory. In the time domain,  $n_\phi$  and  $w_\phi$  are related by

$$\dot{n}_\phi = -a_\phi n_\phi + w_\phi . \tag{14}$$

The differential equations (13) and (14) can be put into state variable form by defining the following variables:

$$\begin{aligned}
 \alpha_7 &= I_f \\
 \alpha_8 &= I_m \\
 \alpha_9 &= \phi_m \\
 \alpha_{10} &= \dot{\phi}_m \\
 \alpha_{11} &= \phi_b \\
 \alpha_{12} &= \dot{\phi}_b \\
 \alpha_{13} &= \phi_d \\
 \alpha_{14} &= \dot{\phi}_d \\
 \alpha_{15} &= n_\phi \quad .
 \end{aligned} \tag{15}$$

Upon denoting the nine-dimensional state vector of the elevation channel as

$$r_\phi = \begin{bmatrix} \alpha_7 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{15} \end{bmatrix} , \tag{16}$$

the differential equations (13) and (14) can be rewritten concisely as

$$\dot{r}_\phi = F_\phi r_\phi + D_\phi u_\phi + G_\phi w_\phi \quad , \tag{17}$$

where

$$F_{\phi} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{k_f}{L_m} & -\frac{R_m}{L_m} & 0 & -\frac{k_m}{L_m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_m}{J_m} & -\frac{c_m}{J_m} & -\frac{f_m}{J_m} & -\frac{c_m N}{J_m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{c_m N}{J_b} & 0 & -\frac{N^2 c_m + c_b}{J_b} & -\frac{f_b}{J_b} & -\frac{c_b}{J_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{c_b}{J_d} & 0 & -\frac{c_b}{J_d} & -\frac{f_d}{J_d} & \frac{1}{J_d} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{\phi} \end{bmatrix}$$

$$D_{\phi} = \begin{bmatrix} L_f^{-1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad G_{\phi} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The digital computer simulation of the problem necessitates the conversion of the differential equation (17) into an equivalent difference equation. This can be accomplished by solving Eq. (17) with an arbitrary initial condition  $r_j(t')$ :

$$r_A(t) = \exp [F_{\phi}(t-t')] r_j(t') + \int_{t'}^t \exp [F_{\phi}(t-\tau)] D_{\phi} u_j(\tau) d\tau + \int_{t'}^t \exp [F_{\phi}(t-\tau)] G_{\phi} w_{\phi}(\tau) d\tau \quad (18)$$



With  $t = k\Delta t$  and  $t' = (k - 1)\Delta t$ , and with  $u_\phi$  and  $w_\phi$  assumed to be constant over the time interval  $[(k - 1)\Delta t, k\Delta t]$ , Eq. (18) becomes

$$r_\phi(k) = \Phi_\phi(k - 1)r_\phi(k - 1) + \Lambda_\phi(k - 1)u_\phi(k - 1) + \Gamma_\phi(k - 1)w_\phi(k - 1) \quad , \quad (19)$$

where

$$\begin{aligned} \Phi_\phi(k - 1) &= \Phi_\phi = \exp [F_\phi \Delta t] = \sum_{i=0}^{\infty} \frac{F_\phi^i \cdot (\Delta t)^i}{i!} \quad , \\ \Lambda_\phi(k - 1) &= \Lambda_\phi = \int_{(k-1)\Delta t}^{k\Delta t} \exp [F_\phi(k\Delta t - \tau)] d\tau D_\phi \\ &= \left[ \sum_{i=0}^{\infty} \frac{F_\phi^i \cdot (\Delta t)^{i+1}}{(i + 1)!} \right] D_\phi \quad , \\ \Gamma_\phi(k - 1) &= \Gamma_\phi = \int_{(k-1)\Delta t}^{k\Delta t} \exp [F_\phi(k\Delta t - \tau)] d\tau G_\phi \\ &= \left[ \sum_{i=0}^{\infty} \frac{F_\phi^i \cdot (\Delta t)^{i+1}}{(i + 1)!} \right] G_\phi \quad , \end{aligned} \quad (20)$$

and  $r_\phi(k\Delta t)$  is defined as  $r_\phi(k)$ .

The azimuth channel has the same form as the elevation channel, which is described by the differential equation (17). The only difference between the two channels is that the moments of inertia of the antenna (in azimuth) are functions of the elevation angles. Since the rates of change for these moments of inertia are slow with respect to the control system time constants, it will be assumed that they can be treated as time-varying functions. Hence, the azimuth channel can be described by a differential equation that is analogous to Eq. (17):

$$\dot{r}_\theta = F_\theta(t)r_\theta + D_\theta u_\theta + G_\theta w_\theta \quad , \quad (21)$$

where

$$r_\theta = \begin{bmatrix} \alpha_{16} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{24} \end{bmatrix} \quad (22)$$

is the nine-dimensional state vector of the azimuth channel and is entirely analogous to  $r_\phi$  as defined by Eqs. (15) and (16). The matrices  $F_\theta(t)$ ,  $D_\theta$ , and  $G_\theta$  have the identical form of the corresponding matrices defined in Eq. (17), the time dependence in  $F_\theta(t)$  being due to the time-varying moments of inertia.

The differential equation (21) can be converted into an equivalent difference equation by assuming that  $F_\theta$ , in addition to  $u_\theta$  and  $w_\theta$ , is constant over the interval  $[(k-1)\Delta t, k\Delta t]$ :

$$r_\theta(k) = \Phi_\theta(k-1)r_\theta(k-1) + \Delta_\theta(k-1)u_\theta(k-1) + \Gamma_\theta(k-1)w_\theta(k-1) \quad , \quad (23)$$

where

$$\begin{aligned} \Phi_\theta(k-1) &= \sum_{i=0}^{\infty} \frac{F_\theta^i(k-1) \cdot (\Delta t)^i}{i!} \\ \Delta_\theta(k-1) &= \left[ \sum_{i=0}^{\infty} \frac{F_\theta^i(k-1) \cdot (\Delta t)^{i+1}}{(i+1)!} \right] D_\theta \\ \Gamma_\theta(k-1) &= \left[ \sum_{i=0}^{\infty} \frac{F_\theta^i(k-1) \cdot (\Delta t)^{i+1}}{(i+1)!} \right] G_\theta \quad , \end{aligned} \quad (24)$$

and  $r_\theta(k\Delta t)$  is defined as  $r_\theta(k)$ .

Hence, the antenna control system (elevation and azimuth channels) is described by

$$r(k) = \Phi_r(k-1)r(k-1) + \Delta_r(k-1)u(k-1) + \Gamma_r(k-1)w_r(k-1) \quad , \quad (25)$$

where

$$\begin{aligned} r(k-1) &= \begin{bmatrix} r_\phi(k-1) \\ r_\theta(k-1) \end{bmatrix} \\ \Phi_r(k-1) &= \begin{bmatrix} \Phi_\phi & 0 \\ 0 & \Phi_\theta(k-1) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
u(k-1) &= \begin{bmatrix} u_\phi(k-1) \\ u_\theta(k-1) \end{bmatrix} \\
\Lambda_r(k-1) &= \begin{bmatrix} \Lambda_\phi & 0 \\ 0 & \Lambda_\theta(k-1) \end{bmatrix} \\
w_r(k-1) &= \begin{bmatrix} w_\phi(k-1) \\ w_\theta(k-1) \end{bmatrix} \\
\Gamma_r(k-1) &= \begin{bmatrix} \Gamma_\phi & 0 \\ 0 & \Gamma_\theta(k-1) \end{bmatrix} .
\end{aligned}$$

In addition, it is assumed that  $w_r(k-1)$  is white gaussian noise with zero mean and covariance  $Q_r(k-1) = E[w_r(k-1)w_r^T(k-1)]$ . The matrices  $\Phi_r$ ,  $\Lambda_r$ , and  $\Gamma_r$  can be computed with an arbitrary degree of accuracy by taking a suitably large (but finite) number of terms in the series expansions of Eqs. (20) and (24).

### C. MEASUREMENT SYSTEM

The state of the satellite tracking system, which consists of the satellite ( $x_e$ ) and the antenna control system ( $r$ ), may be defined by the 24-dimensional vector

$$\alpha = \begin{bmatrix} x_e \\ r \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{24} \end{bmatrix} . \quad (26)$$

The measurement system, which includes the monopulse receiver\*, is defined by the 15-dimensional measurement vector

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\* The monopulse receiver and its associated demodulating equipment measures the elevation and azimuth components of the difference between the angle of the antenna's electrical axis and the satellite angle. It is assumed that this difference is suitably small so that the operation of the monopulse receiver is linear.

$$\beta(k) = \begin{bmatrix} \alpha_7(k) \\ \cdot \\ \cdot \\ \alpha_{12}(k) \\ \alpha_{16}(k) \\ \cdot \\ \cdot \\ \alpha_{21}(k) \\ \phi_d(k) - \phi_s[\alpha(k), k] \\ \theta_d(k) - \theta_s[\alpha(k), k] \\ \dot{\rho}_s[\alpha(k), k] \end{bmatrix} + v(k) = h[\alpha(k), k] + v(k) \quad , \quad (27)$$

where

$$\begin{aligned} \phi_s[\alpha(k), k] &= \text{elevation angle of satellite} \\ \theta_s[\alpha(k), k] &= \text{azimuth angle of satellite} \\ \dot{\rho}_s[\alpha(k), k] &= \text{range rate of satellite} \\ \dot{\phi}_d(k) &= \alpha_{13}(k) \quad , \quad \theta_d(k) = \alpha_{22}(k) \\ v(k) &= \text{measurement noise, which is assumed} \\ &\quad \text{to be a white gaussian random pro-} \\ &\quad \text{cess with zero mean and covariance} \\ &\quad R(k) = E[v(k)v^T(k)]. \end{aligned}$$

The expressions  $\phi_s$ ,  $\theta_s$ , and  $\dot{\rho}_s$  (which are time-varying, nonlinear functions) are derived in Appendix A and given by Eqs. (A-6), (A-7), and (A-9), respectively. Figures A-1, A-2, and A-3 in Appendix A illustrate the geometry of the satellite tracking problem. It should be pointed out that this study considers the relative motion of the antenna with respect to the satellite as the earth rotates on its axis.

Since the measurement equation (27) is nonlinear, it is necessary, as before, to perform a linearization. Consider  $\alpha$  to be composed of some nominal trajectory  $\alpha^o$  and a perturbation from the nominal  $\tilde{\alpha}$ :

$$\alpha = \alpha^o + \tilde{\alpha} \quad . \quad (28)$$

Similarly, let  $\beta$  be given as follows:

$$\beta = \beta^o + \tilde{\beta} \quad . \quad (29)$$

Expanding Eq. (27) in a Taylor series about  $\alpha^o$  and neglecting second and higher-order terms, the linear perturbation equation is

$$\tilde{\beta}(k) = H[\alpha^o(k), k] \tilde{\alpha}(k) + v(k) \quad , \quad (30)$$

where

$$\beta^o(k) = h[\alpha^o(k), k] \quad ,$$

$$H[\alpha^o(k), k] = \left. \frac{\partial h}{\partial \alpha} \right|_{\alpha^o(k), k} = \left[ \frac{\partial h_i}{\partial \alpha_j} \right]_{\alpha^o(k), k} \quad .$$

From Eqs. (26) and (27), it can be demonstrated that

$$H[\alpha^o(k), k] \triangleq H(k) = \begin{array}{c} \left[ \begin{array}{ccc|cc} \bigcirc & \text{I} & \bigcirc & & \\ (6 \times 6) & (6 \times 6) & (6 \times 12) & & \\ \hline & \bigcirc & \text{I} & \bigcirc & \\ & (6 \times 15) & (6 \times 6) & (6 \times 3) & \\ \hline a_1 & a_2 & a_3 & 0 & 0 & 0 & | & 0 & \dots & 0 & 1 & 0 & 0 & | & 0 & \dots & \dots & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 & | & 0 & \dots & \dots & \dots & 0 & 1 & 0 & 0 & | & 0 & \dots & \dots & 0 \\ c_1 & c_2 & c_3 & d_1 & d_2 & d_3 & | & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & | & \dots & \dots & \dots & 0 \end{array} \right]_{\alpha^o(k), k} \end{array} \quad (31)$$

where the  $a_i$ ,  $b_i$ ,  $c_i$ , and  $d_i$  are derived in Appendix B and given by Eqs. (B-5) through (B-8).

#### D. ESTIMATOR AND CONTROLLER

The function of the estimator is to generate an optimum estimate of the present state  $\alpha$  from the measurement  $\beta$ , which is corrupted by noise. This estimate is then employed in the controller to compute the optimum control with respect to the given performance criterion. The estimation and control equations are obtained in Secs. III and IV, respectively.

### III ESTIMATION EQUATIONS

In this section the estimation problem is solved by employing the extended Kalman filter. This concept is an application to nonlinear systems of work done by Kalman in linear estimation theory.<sup>5</sup> The derivation of the extended (or linearized) Kalman filter is presented in Memorandum 6<sup>6</sup> and hence will not be repeated here. This approach has been successfully applied at SRI to missile tracking problems, including the identification of unknown aerodynamic parameters.<sup>7</sup>

From Eqs. (12) and (25), the random disturbance acting upon the satellite tracking system is given by the five-dimensional vector

$$w(k) = \begin{bmatrix} w_x(k) \\ w_r(k) \end{bmatrix} ,$$

which is white gaussian noise with

$$E[w(k)] = 0 ,$$
$$E[w(k)w^T(k)] = Q(k) = \begin{bmatrix} Q_x(k) & 0 \\ 0 & Q_r(k) \end{bmatrix} .$$

The measurement noise  $v(k)$  has been defined in Eq. (27). The initial state  $\alpha(0)$  is a gaussian random variable with

$$E[\alpha(0)] = \hat{\alpha}(0/0) ,$$
$$E[\{\alpha(0) - \hat{\alpha}(0/0)\}\{\alpha(0) - \hat{\alpha}(0/0)\}^T] = P(0/0) .$$

Furthermore, it is assumed that  $w(k)$ ,  $v(k)$ , and  $\alpha(0)$  are uncorrelated.

The resulting estimation equations can be considered as consisting of two parts: prediction and correction (or regression).\*

### A. PREDICTION

Given the estimate of the system state at the  $k-1$ th instant  $[\hat{\alpha}(k-1/k-1)]$ , the predicted system state for the  $k$ th instant  $[\hat{\alpha}(k/k-1)]$  is obtained from Eqs. (9) and (25):

$$\hat{\alpha}(k/k-1) \begin{cases} \hat{x}_e(k/k-1) = \hat{x}_e(k-1/k-1) + f[\hat{x}_e(k-1/k-1)]\Delta t & , \dagger \\ \hat{r}(k/k-1) = \Phi_r(k-1)\hat{r}(k-1/k-1) + \Delta_r(k-1)u(k-1) & , \end{cases} \quad (32)$$

with the covariance of the error in this prediction given by

$$P(k/k-1) = \Phi(k-1)P(k-1/k-1)\Phi^T(k-1) + \Gamma(k-1)Q(k-1)\Gamma^T(k-1) \quad , \quad (33)$$

where

$$\Gamma(k-1) = \begin{bmatrix} \Gamma_x & 0 \\ 0 & \Gamma_r(k-1) \end{bmatrix} \quad ,$$

$$\Phi(k-1) = \begin{bmatrix} \Phi_x(k-1) & 0 \\ 0 & \Phi_r(k-1) \end{bmatrix} \quad ,$$

and  $\Phi_x(k-1)$  is obtained from Eqs. (7), (8), and (11) by linearization about the estimate  $\hat{\alpha}(k-1/k-1)$  [or  $\hat{x}_e(k-1/k-1)$ ]; *i.e.*,

$$\Phi_x(k-1) = I + \mathfrak{B}[\hat{x}_e(k-1/k-1)]\Delta t \quad .$$

It should be noted that  $w(k-1) = 0$  in Eq. (32), since  $E[w(k-1)] = 0$ .

---

\* The following notation will be employed:

$$\hat{\alpha}(i/j) \triangleq E[\alpha(i)/\beta(j), \dots, \beta(1), u(j-1), \dots, u(0)] \quad ,$$

$$P(i/j) \triangleq E\{[\alpha(i) - \hat{\alpha}(i/j)]\{[\alpha(i) - \hat{\alpha}(i/j)]^T/\beta(j), \dots, \beta(1), u(j-1), \dots, u(0)]\} \quad .$$

These expectations are conditioned on the previous measurements and inputs.

† The nonlinear differential equation for  $x_e$  may be integrated by a more accurate method if necessary.

## B. CORRECTION

The prediction  $\hat{\alpha}(k/k-1)$  is then "corrected" by using the actual measurement at the  $k$ th instant  $[\beta(k)]$  and the predicted measurement for the  $k$ th instant  $[\hat{\beta}(k/k-1)]$ , which is obtained from Eq. (27):

$$\hat{\beta}(k/k-1) = h[\hat{\alpha}(k/k-1), k] \quad . \quad (34)$$

It should be noted that  $v(k) = 0$  in Eq. (34), since  $E[v(k)] = 0$ .

Hence, the estimate of the system state at the  $k$ th instant is given by

$$\hat{\alpha}(k/k) = \hat{\alpha}(k/k-1) + W(k)[\beta(k) - \hat{\beta}(k/k-1)] \quad , \quad (35)$$

where the weighting matrix

$$W(k) = P(k/k-1)H^T(k)[R(k) + H(k)P(k/k-1)H^T(k)]^{-1} \quad , \quad (36)$$

and  $H(k)$  is obtained from Eqs. (30) and (31) by linearization about the prediction  $\hat{\alpha}(k/k-1)$ ; *i.e.*,

$$H(k) = H[\hat{\alpha}(k/k-1), k] = \left. \frac{\partial h}{\partial \alpha} \right|_{\hat{\alpha}(k/k-1), k} \quad . \quad *$$

The covariance of the error in the estimate  $\hat{\alpha}(k/k)$  is

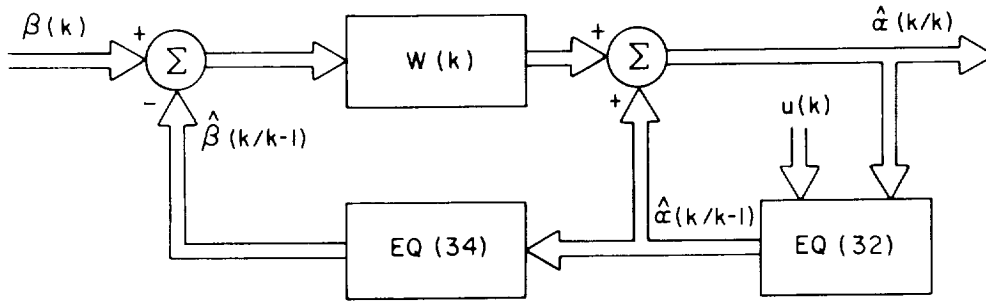
$$\begin{aligned} P(k/k) &= [I - W(k)H(k)]P(k/k-1) \\ &= P(k/k-1) - P(k/k-1)H^T(k)[R(k) + H(k)P(k/k-1)H^T(k)]^{-1}H(k)P(k/k-1) \quad . \end{aligned} \quad (37)$$

The extended Kalman filter [Eqs. (32) through (37)], which is depicted in Fig. 3, gives the solution to the estimation problem. Obviously, this solution can be readily implemented on a digital computer. However, since the overall system is not linear, the solution is

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\* From Eqs. (B-5) through (B-8), together with Eq. (31), it is obvious that  $H(k)$  is actually evaluated at the prediction  $\hat{x}_p(k/k-1)$ .





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FIG. 3 BLOCK DIAGRAM OF THE ESTIMATOR

suboptimal. Intuitively, this approach seems to be quite reasonable, but its validity has not been rigorously established. The extent to which this solution to the estimation problem differs from the optimum is mainly dependent upon the accuracy of the linearization of Eq. (5), the differential equation for  $x_e$ , and of Eq. (27), the measurement equation. There are many questions pertaining to this subject that remain to be answered.

It should be noted that in the derivation of the extended Kalman filter, the nonlinear equations (5) and (27) were used in Eqs. (32) and (34) to obtain the predicted state and the predicted measurement. The linearization of Eqs. (5) and (27), in order to obtain  $\Phi_x$  and  $H$ , is only employed to calculate the covariance matrices  $P$  and the weighting matrix  $W$ .

#### IV CONTROL EQUATIONS

In this section the control problem is solved by application of linear optimal control theory. Consider the performance criterion

$$J = E \left[ \sum_{k=0}^M \{ [\phi_b(k) - \phi_s(k)]^2 + [\theta_b(k) - \theta_s(k)]^2 + \gamma [u_\phi^2(k) + u_\theta^2(k)] \} \right] \quad (38)$$

This performance criterion corresponds to tracking for the purpose of gathering satellite position data. The cost associated with control (where  $\gamma \geq 0$ ) is essential in order to guarantee that  $u_\phi$  and  $u_\theta$  do not become too large, which, in turn, could cause certain state variables of the antenna control system to exceed their permissible range of values (e.g., the motor speed and torque are bounded because of physical considerations). However, the actual performance of the satellite tracking system is determined by the first two terms in Eq. (38).

To use the results of linear optimal control theory, it is necessary for the performance criterion  $J$  to be quadratic in the system state  $\alpha$ . However, this condition is not satisfied, since  $\phi_s$  and  $\theta_s$  are nonlinear functions of  $\alpha$  (or  $x_e$ ), as shown by Eqs. (A-6) and (A-7). The criterion  $J$  can be put into the appropriate form by linearization of  $\phi_s(k)$  about the estimate  $\hat{\alpha}(k/k)$  [or  $\hat{x}_e(k/k)$ ]. After writing Eqs. (A-6) and (A-7) as Taylor series expansions about  $\hat{x}_e(k/k)$  and neglecting second and higher-order terms,

$$\begin{aligned} \phi_s(k) &= \hat{\phi}_s(k/k) + \underline{a}^T(k) \tilde{x}_e(k) \\ \theta_s(k) &= \hat{\theta}_s(k/k) + \underline{b}^T(k) \tilde{x}_e(k) \end{aligned} \quad (39)$$

where

$$x_e(k) = \hat{x}_e(k/k) + \tilde{x}_e(k) \quad (40)$$

$$\hat{\phi}_s(k/k) = \phi_s[\hat{x}_e(k/k), k]$$

$$\hat{\theta}_s(k/k) = \theta_s[\hat{x}_e(k/k), k] \quad ,^*$$

$$\underline{a}(k) = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\hat{x}_e(k/k), k}, \quad \underline{b}(k) = \begin{bmatrix} -b_1 \\ -b_2 \\ -b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\hat{x}_e(k/k), k},$$

in which the  $a_i$  and  $b_i$  are given by Eqs. (B-5) and (B-6).

The state of the antenna control system can be defined by the 26-dimensional vector

$$\underline{\alpha} = \begin{bmatrix} \hat{\phi}_s \\ \hat{\theta}_s \\ \tilde{x}_e \\ r \end{bmatrix}, \quad (41)$$

which contains  $\alpha$  (with  $x_e$  linearized) and is augmented by  $\hat{\phi}_s$  and  $\hat{\theta}_s$ . The dynamics of  $\tilde{x}_e$  and  $r$  are given by Eqs. (12) and (25), respectively; there are no dynamics associated with  $\hat{\phi}_s$  and  $\hat{\theta}_s$ . Therefore,

$$\underline{\alpha}(k+1) = \underline{\Phi}(k)\underline{\alpha}(k) + \underline{\Delta}(k)u(k) + \underline{\Gamma}(k)w(k), \quad (42)$$

---

\* It should be noted that because of the nonlinearity of Eqs. (A-6) and (A-7),  $\hat{\phi}_s$  and  $\hat{\theta}_s$  are not optimal estimates in the usual sense.

where

$$\underline{\Phi}(k) = \begin{bmatrix} \Phi_1(k) & 0 \\ 0 & \Phi_r(k) \end{bmatrix}$$

$$\Phi_1(k) = \begin{bmatrix} I & 0 \\ 0 & \Phi_x(k) \end{bmatrix}, \quad \Phi_x(k) = I + \mathfrak{B}[\hat{x}_e(k/k)]\Delta t$$

$$\underline{\Delta}(k) = \begin{bmatrix} 0 \\ \Delta_r(k) \end{bmatrix}$$

$$\underline{\Gamma}(k) = \begin{bmatrix} \circ & 0 \\ \Gamma_x & 0 \\ 0 & \Gamma_r(k) \end{bmatrix}.$$

Substituting Eq. (39) into Eq. (38) and rewriting  $J$  according to the standard formulation gives

$$J = E \left[ \sum_{k=0}^M \{ \underline{\alpha}^T(k) A(k) \underline{\alpha}(k) + u^T(k) B(k) u(k) \} \right], \quad (43)$$

where

$$B(k) = B = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$$

$$A(k) = \begin{bmatrix} A_1(k) & A_3(k) \\ A_3^T(k) & A_2 \end{bmatrix},$$

in which  $A_1(k)$  is  $8 \times 8$ ;  $A_2$  is  $18 \times 18$ ;  $A_3(k)$  is  $8 \times 18$ ; and  $A(k)$  is symmetric and positive semidefinite. After comparing Eq. (38) with Eq. (43), it is a straightforward matter to determine  $A(k)$ :

$$A_1(k) = \left[ \begin{array}{cc|cc} & & & -\underline{a}^T(k) \\ & I & & -\underline{b}^T(k) \\ \hline -\underline{a}(k) & -\underline{b}(k) & & \underline{a}(k)\underline{a}^T(k) + \underline{b}(k)\underline{b}^T(k) \end{array} \right]$$

$$A_3(k) = \left[ \begin{array}{c|c|c|c|c} & -1 & & 0 & \\ & 0 & & -1 & \\ \hline \text{O} & -a(k) & \text{O} & -b(k) & \text{O} \\ \hline (8 \times 4) & & (8 \times 8) & & (8 \times 4) \end{array} \right]$$

$$A_2 = \left[ \begin{array}{cccccccc} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \\ & & & & & & & 1 & \\ & & & & & & & & 0 \\ & & & & & & & & & 0 \end{array} \right]$$

(the 5th and 14th elements on the diagonal of  $A_2$  are equal to one).

The design objective is to find the sequence of controls

$$[u(0), u(1), \dots, u(M)]$$

that minimize  $J$ . The control equations will be derived by applying some results obtained by Larson;<sup>8</sup> this work is an extension of results in linear optimal control theory.<sup>3</sup>

The optimal control  $u(k)$  is given by

$$u(k) = -K(k) \hat{\underline{\alpha}}(k/k) \quad , \quad (44)$$

where the gain matrix  $K(k)$  is denoted by

$$K(k) = [B + \underline{\Delta}^T(k)P_c(k+1)\underline{\Delta}(k)]^{-1}\underline{\Delta}^T(k)P_c(k+1)\underline{\Phi}(k) \quad , \quad (45)$$

and  $P_c$  satisfies the discrete Riccati equation

$$P_c(k) = A(k) + \underline{\Phi}^T(k)P_c(k+1)\underline{\Phi}(k) - \underline{\Phi}^T(k)P_c(k+1)\underline{\Delta}(k)[B + \underline{\Delta}^T(k)P_c(k+1)\underline{\Delta}(k)]^{-1}\underline{\Delta}^T(k)P_c(k+1)\underline{\Phi}(k) \quad , \quad 0 \leq k < M \quad (46)$$

$$P_c(M) = A(M).$$

For convenience,  $P_c(k)$  will be rewritten in a form entirely analogous to  $A(k)$ :

$$P_c(k) = \begin{bmatrix} P_1(k) & P_3(k) \\ P_3^T(k) & P_2(k) \end{bmatrix} \quad ,$$

where  $P_c(k)$  is symmetric and positive semidefinite.

Upon performance of the indicated matrix multiplications, the optimal control in Eq. (44) becomes

$$u(k) = -[B + \underline{\Delta}_r^T(k)P_2(k+1)\underline{\Delta}_r(k)]^{-1}\underline{\Delta}_r^T(k)P_3^T(k+1)\underline{\Phi}_1(k)\hat{\underline{x}}(k/k) - [B + \underline{\Delta}_r^T(k)P_2(k+1)\underline{\Delta}_r(k)]^{-1}\underline{\Delta}_r^T(k)P_2(k+1)\underline{\Phi}_r(k)\hat{\underline{r}}(k/k) \quad , \quad (47)$$

where

$$\hat{\underline{x}}(k/k) = \begin{bmatrix} \hat{\phi}_s(k/k) \\ \hat{\theta}_s(k/k) \\ \hat{x}_e(k/k) \end{bmatrix} \quad . \quad (48)$$

The Riccati equation (46) can be partitioned into separate equations for  $P_1$ ,  $P_2$ , and  $P_3$ :

$$\begin{aligned}
 P_1(k) &= A_1(k) + \Phi_1^T(k)P_1(k+1)\Phi_1(k) \\
 &\quad - \Phi_1^T(k)P_3^T(k+1)\Delta_r(k)[B + \Delta_r^T(k)P_2(k+1)\Delta_r(k)]^{-1}\Delta_r^T(k)P_3(k+1)\Phi_1(k) \\
 &\qquad\qquad\qquad 0 \leq k \leq M \quad (49)
 \end{aligned}$$

$$P_1(M) = A_1(M);$$

$$\begin{aligned}
 P_2(k) &= A_2 + \Phi_r^T(k)P_2(k+1)\Phi_r(k) \\
 &\quad - \Phi_r^T(k)P_2(k+1)\Delta_r(k)[B + \Delta_r^T(k)P_2(k+1)\Delta_r(k)]^{-1}\Delta_r^T(k)P_2(k+1)\Phi_r(k) \\
 &\qquad\qquad\qquad 0 \leq k \leq M \quad (50)
 \end{aligned}$$

$$P_2(M) = A_2$$

$$\begin{aligned}
 P_3(k) &= A_3(k) + \Phi_1^T(k)P_3(k+1)\Phi_r(k) \\
 &\quad - \Phi_1^T(k)P_3(k+1)\Delta_r(k)[B + \Delta_r^T(k)P_2(k+1)\Delta_r(k)]^{-1}\Delta_r^T(k)P_2(k+1)\Phi_r(k) \\
 &\qquad\qquad\qquad 0 \leq k \leq M \quad (51)
 \end{aligned}$$

$$P_3(M) = A_3(M).$$

Equation (50) can be solved for  $P_2$  independently of Eqs. (49) and (51). Hence, the dimension of the Riccati equation to be solved has been reduced from  $26 \times 26$  to  $18 \times 18$ . It should be noted that Eq. (50) is the Riccati equation for the antenna control system of Eq. (25) with the performance criterion

$$E \left[ \sum_{k=0}^M \{r^T(k)A_2 r(k) + u^T(k)Bu(k)\} \right].$$

Once  $P_2$  has been found, it is substituted into Eq. (51), which is a linear equation in  $P_3$  (of dimension  $8 \times 18$ ) and very easy to solve. Since  $P_1$  does not enter into the control equation (47) or the calculation of  $P_2$  and  $P_3$ , it is not necessary to solve Eq. (49).

Thus, the computational requirements have been reduced markedly. Instead of solving Eq. (46) for  $P_c$ , it will suffice to solve Eq. (50) for  $P_2$  and calculate  $P_3$  from Eq. (51). The optimal control  $u$  is then obtained by substituting  $P_2$  and  $P_3$  into Eq. (47). Equations (47), (50), and (51), together with  $\alpha$ , give the solution to the control problem.

## A. STEADY-STATE APPROXIMATION

Suppose that the antenna control system of Eq. (25) is stationary (*i.e.*, the matrices  $\Phi_r$ ,  $\Delta_r$ , and  $\Gamma_r$  are constant), which is equivalent to assuming that  $F_\theta$  of Eq. (21) is constant. This assumption is fairly reasonable over a substantial time interval, since the rate of change of  $F_\theta$  is slow with respect to the control system time constants. Additionally, it will be assumed that the summation in the performance criterion  $J$  of Eq. (43) is over an infinite time interval (*i.e.*,  $M = \infty$ ). This assumption is quite reasonable, since the interval of time during which the antenna is tracking the satellite will be appreciably larger than the control system time constants. With these two assumptions, computation of the optimal control  $u(k)$  is greatly simplified, as will be shown below. Formulation of the control problem in this manner will be referred to as the "steady-state approximation."

The Riccati equation (50) becomes

$$P_2 = A_2 + \Phi_r^T P_2 \Phi_r - \Phi_r^T P_2 \Delta_r [B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T P_2 \Phi_r \quad . \quad (52)$$

The above is a nonlinear algebraic equation in the steady-state matrix  $P_2$ . In general, Eq. (52) is very difficult to solve. The most straightforward way to obtain  $P_2$  is by the iterative solution of Eq. (50). That is, let  $P_2(k+1)$  be some positive definite matrix and then solve Eq. (50) iteratively until it converges to a steady-state solution.

Instead of solving Eq. (51) for  $P_3$ , consider the following quantity from the first term of Eq. (47):

$$P_3^T(k+1) \Phi_1(k) \hat{\underline{x}}(k/k) \quad . \quad (53)$$

It will be shown that this approach simplifies the computation of the optimal control  $u(k)$ . From Eq. (40) it can be seen that

$$\hat{\underline{x}}_e(k/k) = 0 \quad ;$$



hence, Eq. (48) yields

$$\hat{\underline{x}}(k/k) = \begin{bmatrix} \hat{\phi}_s(k/k) \\ \hat{\theta}_s(k/k) \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \quad (54)$$

From Eqs. (42) and (54), it follows that

$$\Phi_1(k) \hat{\underline{x}}(k/k) = \hat{\underline{x}}(k/k) . \quad (55)$$

In effect, the linearization in Eqs. (39) and (40) enables the left-hand side of Eq. (55) to be rewritten as

$$\Phi_1(k) \hat{\underline{x}}(k/k) = \hat{\underline{x}}(k + 1/k + 1) . \quad (56)$$

Transposing Eq. (51) and multiplying by  $\hat{\underline{x}}(k/k)$  yields

$$\begin{aligned} P_3^T(k) \hat{\underline{x}}(k/k) &= A_3^T(k) \hat{\underline{x}}(k/k) + \Phi_r^T P_3^T(k + 1) \Phi_1(k) \hat{\underline{x}}(k/k) \\ &\quad - \Phi_r^T P_2 \Delta_r [B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T P_3^T(k + 1) \Phi_1(k) \hat{\underline{x}}(k/k) . \end{aligned} \quad (57)$$

For convenience, define

$$\eta(k) \stackrel{\Delta}{=} P_3^T(k) \hat{\underline{x}}(k/k) . \quad (58)$$

Hence, substitution of Eqs. (56) and (58) into Eq. (57) gives

$$\begin{aligned} \eta(k) &= A_3^T \hat{\underline{x}}(k/k) + \Phi_r^T \eta(k + 1) \\ &\quad - \Phi_r^T P_2 \Delta_r [B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T \eta(k + 1) . \end{aligned} \quad (59)$$

From Eqs. (43) and (54), it can be shown that

$$A_3^T(k)\hat{\underline{x}}(k/k) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -\hat{\phi}_s(k/k) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -\hat{\theta}_s(k/k) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (60)$$

( $-\hat{\phi}_s$  and  $-\hat{\theta}_s$  are the 5th and 14th elements, respectively).

From Eqs. (55) and (56), it can be seen that the 18-dimensional vector in Eq. (60) is effectively constant. Therefore, the steady-state solution to Eq. (59) is given by

$$\eta = [I - \Psi]^{-1}A_3^T(k)\hat{\underline{x}}(k/k) \quad , \quad (61)$$

where

$$\Psi = \Phi_r^T - \Phi_r^T P_2 \Delta_r [B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T \quad . \quad (62)$$

If the state  $r(k)$  were known exactly,  $\Psi^T$  would correspond to the closed-loop transition matrix of the antenna control system. For a control law that is asymptotically stable,  $|\lambda_i(\Psi)| < 1$ , where the  $\lambda_i(\Psi)$  are the eigenvalues of  $\Psi$ . With this condition satisfied it can be demonstrated that the inverse of  $[I - \Psi]$  exists.

From Eqs. (56) and (58), it can be seen that the expression in Eq. (53) is equivalent to  $\eta$ ; therefore, the optimal control is

$$\begin{aligned}
u(k) &= -[B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T \eta \\
&\quad - [B + \Delta_r^T P_2 \Delta_r]^{-1} \Delta_r^T P_2 \Phi_r \hat{r}(k/k) \quad . \quad (63)
\end{aligned}$$

As the succeeding estimates of the satellite state ( $\hat{x}_e$ ) are computed,  $\hat{\phi}_s$  and  $\hat{\theta}_s$  will actually change. Thus, Eqs. (60) and (61) show that it is necessary to update  $\eta$  at each discrete time and then substitute it into the first term of Eq. (63). Since  $\hat{\phi}_s$  and  $\hat{\theta}_s$  (the estimates of the elevation and azimuth angles of the satellite) change slowly with respect to the control system time constants, use of the steady-state solution to compute the optimal control is quite reasonable. In addition, as successive estimates of the antenna control system state ( $\hat{r}$ ) are calculated, they are substituted into the second term of Eq. (63). Equations (52), (61), and (63), together with  $\hat{\alpha}$ , give the solution to the control problem under the steady-state approximation.

As a further refinement to this approximation, the time-varying nature of  $F_\theta$  can be taken into account as follows: Update  $F_\theta$  periodically and recalculate  $\Phi_r$ ,  $\Delta_r$ , and  $\Gamma_r$  of Eq. (25). With these new matrices,  $P_2$  [the solution to Eq. (52)] and  $\eta$  [the solution to Eq. (61)] are recomputed. Finally,  $u$  is obtained from Eq. (63) by substituting these updated matrices. Thus, a nonstationary problem is solved as a series of different, stationary problems. It is not necessary to repeat this procedure at every discrete instant  $k\Delta t$ , since the rate of change of  $F_\theta$  is slow with respect to the control system time constants.

The solutions to the control problem can be readily implemented on a digital computer. Although these solutions are suboptimal, the approach used seems quite reasonable. The validity of these results remains to be investigated.

## V CONCLUSION

The digital computer program for implementation of the operation of the (approximately) optimal estimator and controller in conjunction with the satellite tracking system has been written and is now functioning properly. The program has been organized so that it will be sufficiently general and flexible enough for the proposed applications.

This program is a valuable study tool for the investigation of several important topics. Primarily, it will provide a way of evaluating existing tracking techniques; *i.e.*, it will be a yardstick for comparing system performance.

An important question relates to the linearizations employed in Secs. III and IV in order to obtain solutions to the estimation and control problems. Since the satellite tracking problem is nonlinear, the solutions obtained in this manner are suboptimal. Although this approach is intuitively reasonable, its validity has not been rigorously established. The extent to which these solutions differ from the optimum will be studied by computer simulations in conjunction with analytical investigations.

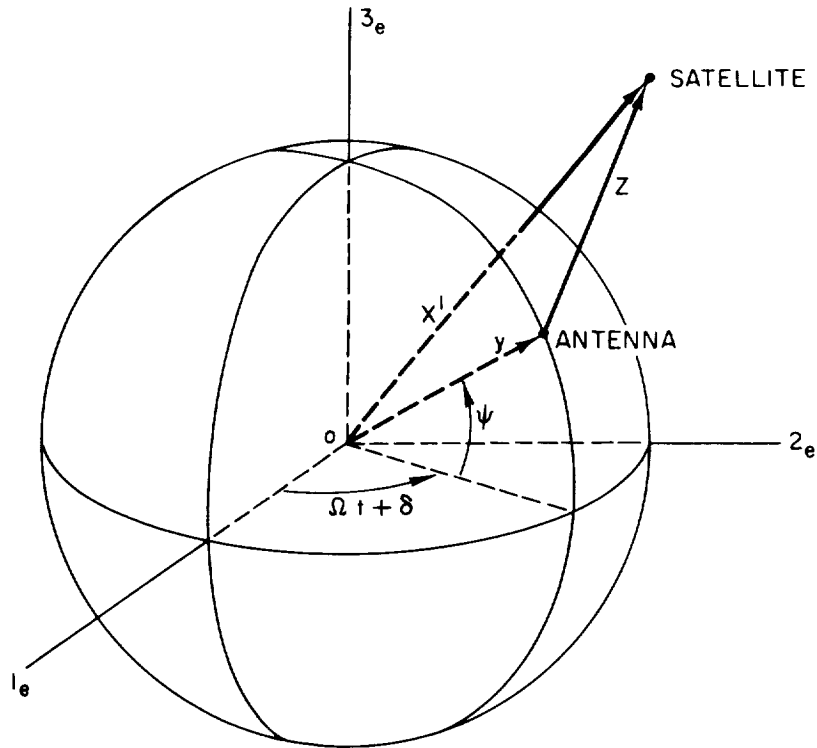
*APPENDIX A*

**DETERMINATION OF  $\varphi_s$ ,  $\theta_s$ ,  $\dot{\rho}_s$**

APPENDIX A

DETERMINATION OF  $\phi_s, \theta_s, \dot{\rho}_s$

The equations of motion of the satellite, as given by Eqs. (1) or (3), are expressed in terms of an earth-centered Cartesian coordinate system. However, the actual operation of the antenna control system is in terms of radar coordinates—elevation, azimuth, and range. In fact, the measurement system [Eq. (27)] observes the elevation and azimuth components of the difference between the angle of the antenna's electrical axis and the satellite angle, in addition to the range rate of the satellite.



TA-5578-13

FIG. A-1 GEOMETRY OF THE SATELLITE TRACKING PROBLEM

The geometry of the satellite tracking problem is illustrated in Fig. A-1. The  $1_e$  and  $2_e$  axes, which lie in the equatorial plane of the

earth, and the  $3_e$  axis, which is coincident with the earth's polar axis, comprise an earth-centered Cartesian coordinate system. The position of the antenna is given by the three-dimensional vector

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} .$$

In the  $1_e, 2_e, 3_e$  coordinate system,

$$y_e \begin{cases} y_{1_e} = R_e \cos \psi \cos (\Omega t + \delta) \\ y_{2_e} = R_e \cos \psi \sin (\Omega t + \delta) \\ y_{3_e} = R_e \sin \psi \end{cases} , \quad (\text{A-1})$$

where

$R_e$  = radius of the earth,

$\Omega$  = angular rate of rotation of the earth,

$\delta$  = an arbitrary angle.

The position of the satellite is given by the three-dimensional vector

$$x' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .$$

In the  $1_e, 2_e, 3_e$  coordinate system,  $x'_e$  consists of the  $x_{1_e}$  defined in Eqs. (1). Now, the vector from the antenna to the satellite is denoted by

$$z = x' - y = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}^* . \quad (\text{A 2})$$

Before proceeding any further, it is necessary to define certain terminology that will be used:

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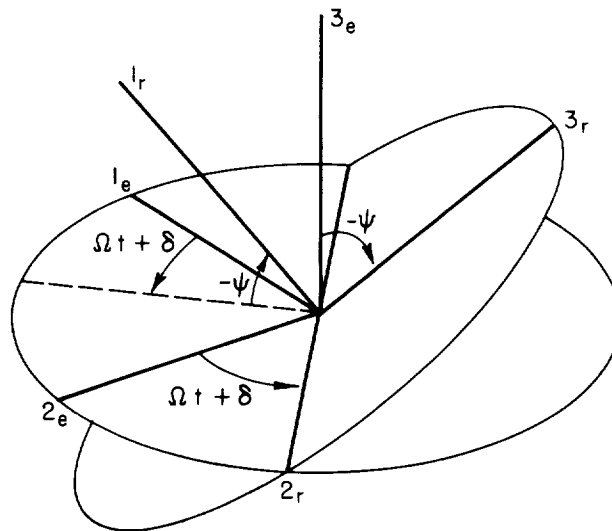
\* This study considers the relative motion of the antenna with respect to the satellite as the earth rotates on its axis.

- Azimuth plane - plane tangent to the earth at the antenna site; this plane is perpendicular to  $y$
- Zero-azimuth line - perpendicular projection onto the azimuth plane of a great circle passing through the North Pole and the antenna site
- Azimuth line - perpendicular projection of  $z$  onto the azimuth plane
- Elevation angle  $\phi_s$  - the angle between the azimuth line and  $z$
- Azimuth angle  $\theta_s$  - the angle between the zero-azimuth line and the azimuth line.

In the  $1_e, 2_e, 3_e$  coordinate system, Eq. (A-2) yields

$$z_e = x'_e - y_e \quad . \quad (A-3)$$

The expressions for the satellite angles  $\phi_s$  and  $\theta_s$  can be obtained from Eq. (A-3) by expressing  $z$  in terms of the  $1_r, 2_r, 3_r$  coordinate system depicted in Fig. A-2. The  $1_r, 2_r, 3_r$  and the  $1_e, 2_e, 3_e$  coordinate systems are related by the following two rotations (or orthogonal transformations):



TA-5578-14

FIG. A-2 RELATION OF THE  $1_e, 2_e, 3_e$  AND  $1_r, 2_r, 3_r$  COORDINATE SYSTEMS VIA THE TRANSFORMATION  $R_{r/e}$



(1) rotation about the  $3_e$  axis by the angle  $\Omega t + \delta$ ; (2) rotation about the displaced  $2_e$  axis by the angle  $-\psi$ . It can be seen that the  $1_r$  axis is perpendicular to the azimuth plane, while the  $2_r$  and  $3_r$  axes lie in the azimuth plane and the  $3_r$  axis is coincident with the zero-azimuth line. The resulting orthogonal transformation can be represented by

$$\begin{aligned}
 R_{r/e} &= \begin{bmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{bmatrix} \begin{bmatrix} \cos (\Omega t + \delta) & \sin (\Omega t + \delta) & 0 \\ -\sin (\Omega t + \delta) & \cos (\Omega t + \delta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos \psi \cos (\Omega t + \delta) & \cos \psi \sin (\Omega t + \delta) & \sin \psi \\ -\sin (\Omega t + \delta) & \cos (\Omega t + \delta) & 0 \\ -\sin \psi \cos (\Omega t + \delta) & -\sin \psi \sin (\Omega t + \delta) & \cos \psi \end{bmatrix} . \quad (\text{A-4})
 \end{aligned}$$

Thus, in the  $1_r, 2_r, 3_r$  coordinate system,

$$z_r = R_{r/e} z_e . \quad (\text{A-5})$$

Inspection of Fig. A-3 enables one to readily determine  $\phi_s$  and  $\theta_s$ , which are given by

$$\phi_s = \sin^{-1} \left[ \frac{z_{1_r}}{|z_r|} \right] , \quad (\text{A-6})$$

$$\theta_s = \cos^{-1} \left[ \frac{z_{3_r}}{\left( z_{2_r}^2 + z_{3_r}^2 \right)^{1/2}} \right] , \quad (\text{A-7})$$

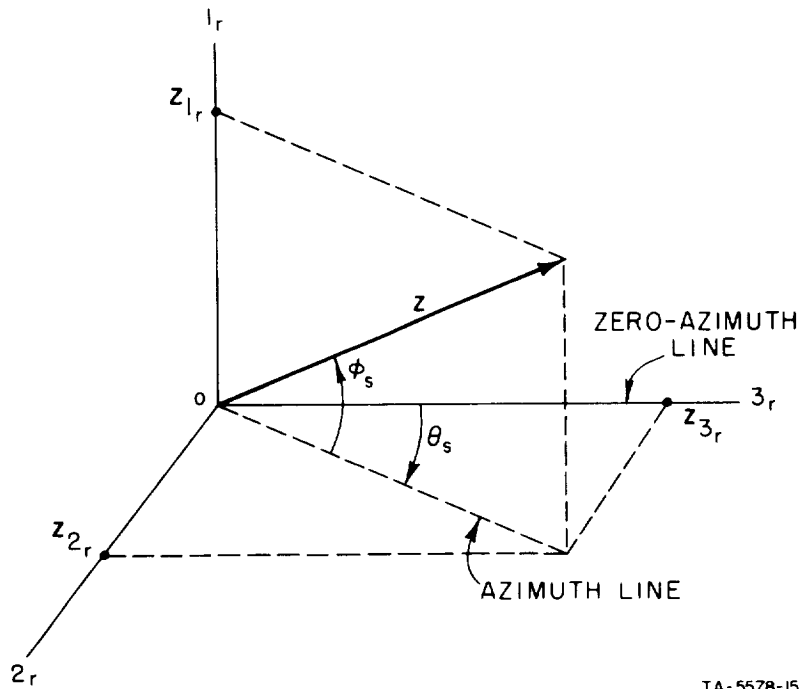
where

$$|z_r| = |z_e| = \left[ \sum_{i=1}^3 (x_{i_e} - y_{i_e})^2 \right]^{1/2} .$$

Finally,  $\phi_s$  and  $\theta_s$  can be expressed in terms of the  $x_{i_e}$  and  $y_{i_e}$  by substituting from Eqs. (A-5) and (A-3). The  $x_{i_e}$  are contained in the state vector  $\alpha$  of Eq. (26) [see Eqs. (2)], and the  $y_{i_e}$  are known time-varying

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\* It should be noted that the magnitude of a vector is independent of the coordinate system.



TA-5578-15

FIG. A-3 DETERMINATION OF  $\phi_s$  AND  $\theta_s$

functions [see Eqs. (A-1)]. Hence, the satellite elevation and azimuth angles are time-varying, nonlinear functions— $\phi_s[\alpha(k), k]$  and  $\theta_s[\alpha(k), k]$ .

The range  $\rho_s$ , the distance from the antenna to the satellite, is given by

$$\rho_s = |z_e| \quad . \quad (A-8)$$

From Eqs. (A-8) and (A-3),

$$\dot{\rho}_s = \frac{\sum_{i=1}^3 (x_{i_e} - y_{i_e})(\dot{x}_{i_e} - \dot{y}_{i_e})}{\left[ \sum_{i=1}^3 (x_{i_e} - y_{i_e})^2 \right]^{1/2}} \quad (A-9)$$

The  $x_{i_e}$  and  $\dot{x}_{i_e}$  are contained in  $\alpha$  of Eq. (26) [see Eqs. (2)], and the  $y_{i_e}$  and  $\dot{y}_{i_e}$  are known time-varying functions [the  $\dot{y}_{i_e}$  are readily obtained from Eqs. (A-1)]. Thus, the satellite range rate is a time-varying, nonlinear function— $\dot{\rho}_s[\alpha(k), k]$ .

*APPENDIX B*

**CALCULATION OF THE  $a_i$ ,  $b_i$ ,  $c_i$ ,  $d_i$**

APPENDIX B

CALCULATION OF THE  $a_i, b_i, c_i, d_i$

The matrices  $H(k)$  of Eq. (31) and  $\underline{a}(k)$  and  $\underline{b}(k)$  of Eqs. (39) contain the partial derivatives of  $\phi_s, \theta_s,$  and  $\dot{\rho}_s$  with respect to the elements of  $\alpha$  [or  $x_e$ —see Eqs. (2) and (4)]; *i. e.*,

$$-a_i = \frac{\partial \phi_s}{\partial x_{i_e}} = \frac{\partial \phi_s}{\partial \alpha_i} \quad (\text{B-1})$$

$$-b_i = \frac{\partial \theta_s}{\partial x_{i_e}} = \frac{\partial \theta_s}{\partial \alpha_i} \quad (\text{B-2})$$

$$c_i = \frac{\partial \dot{\rho}_s}{\partial x_{i_e}} = \frac{\partial \dot{\rho}_s}{\partial \alpha_i} \quad (\text{B-3})$$

$$d_i = \frac{\partial \dot{\rho}_s}{\partial \dot{x}_{i_e}} = \frac{\partial \dot{\rho}_s}{\partial \alpha_{3+i}} \quad (\text{B-4})$$

for  $i = 1, 2, 3$ .

Applying the chain rule for differentiation to Eqs. (A-5), (A-6), and (A-7), one can express the terms in Eqs. (B-1) and (B-2) in the compact form:

$$\begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} = R_{r/e}^T \begin{bmatrix} \partial \phi_s / \partial z_{1_r} \\ \partial \phi_s / \partial z_{2_r} \\ \partial \phi_s / \partial z_{3_r} \end{bmatrix} \quad (\text{B-5})$$

where

$$\begin{aligned} \frac{\partial \phi_s}{\partial z_{1_r}} &= \frac{\left(z_{2_r}^2 + z_{3_r}^2\right)^{1/2}}{\left|z_r\right|^2} \\ \frac{\partial \phi_s}{\partial z_{2_r}} &= \frac{-z_{1_r} z_{2_r}}{\left(z_{2_r}^2 + z_{3_r}^2\right)^{1/2} \left|z_r\right|^2} \\ \frac{\partial \phi_s}{\partial z_{3_r}} &= \frac{-z_{1_r} z_{3_r}}{\left(z_{2_r}^2 + z_{3_r}^2\right)^{1/2} \left|z_r\right|^2} \end{aligned} ;$$

and

$$\begin{bmatrix} -b_1 \\ -b_2 \\ -b_3 \end{bmatrix} = R_{r/e}^T \begin{bmatrix} \partial \theta_s / \partial z_{1_r} \\ \partial \theta_s / \partial z_{2_r} \\ \partial \theta_s / \partial z_{3_r} \end{bmatrix} \quad , \quad (\text{B-6})$$

where

$$\begin{aligned} \frac{\partial \theta_s}{\partial z_{1_r}} &= 0 \\ \frac{\partial \theta_s}{\partial z_{2_r}} &= \frac{z_{3_r}}{z_{2_r}^2 + z_{3_r}^2} \\ \frac{\partial \theta_s}{\partial z_{3_r}} &= \frac{-z_{2_r}}{z_{2_r}^2 + z_{3_r}^2} \end{aligned} .$$

The above results make use of the fact that

$$\frac{\partial z_{i_e}}{\partial x_{j_e}} = \begin{cases} 1 & , \text{ for } i = j \\ 0 & , \text{ for } i \neq j \end{cases}$$

and that the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $R_{r/e}^T$  corresponds to  $\partial z_{j_r} / \partial z_{i_e}$ . Finally, the  $\partial \phi_s / \partial z_{j_r}$  and  $\partial \theta_s / \partial z_{j_r}$  can be expressed in terms of the  $x_{i_e}$  and  $y_{i_e}$  by substitution from Eqs. (A-5) and (A-3).

From Eq. (A-11), it is a straightforward matter to show that the terms in Eqs. (B-3) and (B-4) are given by

$$c_i = \frac{(\dot{x}_{i_e} - \dot{y}_{i_e}) \sum_{l \neq i} (x_{l_e} - y_{l_e})^2}{\left[ \sum_{j=1}^3 (x_{j_e} - y_{j_e})^2 \right]^{3/2}} \quad (\text{B-7})$$

$$d_i = \frac{x_{i_e} - y_{i_e}}{\left[ \sum_{j=1}^3 (x_{j_e} - y_{j_e})^2 \right]^{1/2}} \quad (\text{B-8})$$

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