THEORETICAL STUDIES OF PROPELLANT BEHAVIOR IN THE STATE OF WEIGHTLESSNESS

by

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ABSTRACT

Two analyses are given for predicting the motion of a heavy liquid enclosed in a partly filled prismatic cylinder which is itself in motion, under the joint action of effective body forces, surface and interfacial tension forces. One is strictly numerical in nature and yields the free surface, velocity potential, etc., as functions of time. The other is a perturbation approach whereby the free surface, velocity potential and motion of the tank are assumed to be small deviations from a known equilibrium reference state. Extension of the latter analysis to vessels possessing rotational symmetry but otherwise arbitrary seems to be possible.

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I. INTRODUCTION

With the advent of space flight, scientists and engineers have evidenced an interest in the dynamics of solids containing liquid cavities in the state of weightlessness. Of particular importance is the problem of predicting the role which the liquid plays in the attitude stability of space vehicles in such an environment. The subject of liquids in the condition of weightlessness has been responsible for innumerable papers, and their number seems to be increasing at an undiminished rate. Most of the previous research on low-gravity fluid mechanics is reviewed in [1], which also contains a lengthy list of references. Reports by Reynolds et al [2, 3] are especially recommended. More recent developments are given in [4, 5, 6]. Contents of [4, 6] are of special interest because they give some background which forms the basis of this study.

<u>Benedikt et al [4]</u>

This report gives a technique for determining the motion of a heavy liquid enclosed in a partly filled rigid vessel which is under the joint action of gravitational, surface and interfacial tension forces. The procedure consists of expanding the free surface equation in a series of orthogonal functions, and regarding the coefficients of the series expansion as generalized coordinates describing the liquid system. With appropriate transformations, the velocity potential is also expressed as a series of these coefficients. The kinetic and potential (gravitational and surface) energy of the liquid and thus the Lagrangian function, modified by the requirement of the constancy of the volume of liquid, is computed. With this function known, the Lagrangian equations of motion are derived. Theoretically, the integration of these equations provides a time history of the generalized coordinates. The motion of the liquid, shape of the free surface and distribution of pressure can thus be determined. The effect of an arbitrary translational motion of the tank is indicated by a slight modification of the Lagrangian function.

The technique is applied to a heavy liquid enclosed in a semi-infinite circular cylinder, initially at rest, under the action of terrestrial gravity which is suddenly removed. Results of this study are compared with experimental data. The correspondence appears to be favorable with the exception of the contact angle. This anomaly arises from an implied assumption restricting the analysis to 90° contact angles.

Dodge et al [6]

This report gives an analysis for liquid sloshing in a rigid cylindrical tank under conditions of moderately low gravitational acceleration; the theory is valid for Bond numbers that are greater than 10. The results are cast in the form of an "equivalent" mechanical model, similar to that of high-g sloshing. A series of experiments conducted to determine the sloshing force and the natural frequency for Bond numbers between 10 and 200 are reported. Test results are compared to the theoretical predictions of the mechanical model; favorable correlation between theory and experiment is displayed.

The procedure consists of perturbing the free surface, velocity potential and motion of the tank about a reference equilibrium state. This leads to an infinite set of ordinary linear differential equations which are then solved in the usual manner.

In this study, we shall attempt to describe the motion of a heavy liquid enclosed in a partly filled prismatic cylinder (which is itself in planar motion) under the joint action of gravitational, surface and interfacial tension forces. To this end, we shall parallel the analyses of [2, 6] as closely as possible, since they are of practical significance. We shall not repeat any non-essential detail that may be found in these analyses.

II. FUNDAMENTAL FORMULAE FROM HYDRODYNAMICS

A. Definitions and Coordinate System

In this study we consider the motion of a frictionless liquid enclosed in a partly filled prismatic cylinder which is itself in motion, Figure 1. To describe the motion of the system, take a cartesian frame of reference fixed relatively to the vessel, say an origin 0 and three axes ox, oy, oz. Reference oxyz is orientated in such a manner that oz is measured positively upward along the normal to the free surface if it were a plane.

Let

$$z = \zeta(x, y, t) = f(x, y) + w(x, y, t) \qquad \text{(II.1)}$$

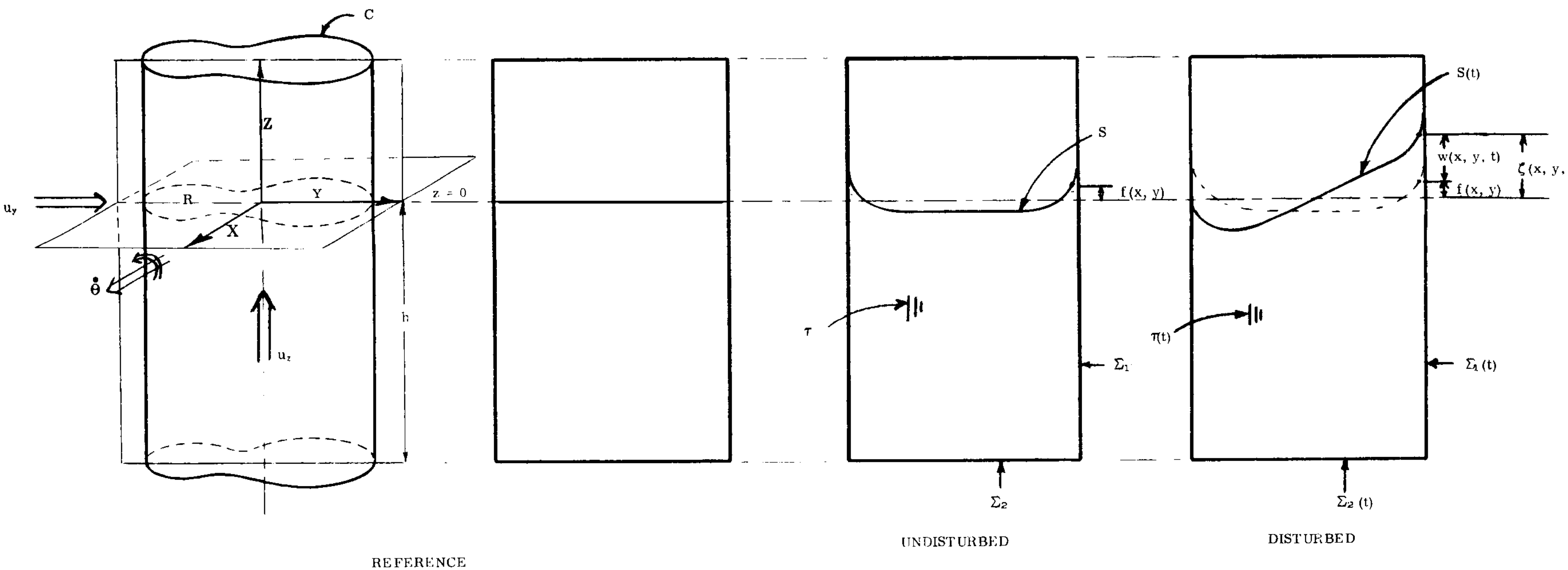
be the equation of the free surface, denoted by S(t), when it is displaced. Function $f(x, y)$ is the height of the undisturbed free surface, denoted by S, measured positively above $z = 0$; function $w(x, y, t)$ is the displacement of the free surface measured from the undisturbed free surface, and not from $z = 0$. Denote by $\Sigma(t)$ the wetted surface of the vessel, and by $\tau(t)$ the variable volume enclosed by S(t) and $\Sigma(t)$. Let Σ and τ represent $\Sigma(t)$ and $\tau(t)$ in the undisturbed position. All surfaces are assumed to be piecewise smooth.

Suppose that at time t the vessel is coincident with inertial space and that it is moving relatively to inertial space with motion described by an observer in inertial space as a velocity $\bar{u} = (0, u_y, u_z)$ of 0 and an angular velocity $\bar{\omega} = (\dot{\theta}, 0, 0)$. Then the position vector $\bar{r} = (x, y, z)$ of a particular liquid particle $P \in \tau(t)$ at time t is the same for an observer moving with the vessel as it is for an observer in inertial space.

The point P, if rigidly attached to the moving frame of reference oxyz, has the velocity

$$\bar{V} = (0,\ u_y - \dot{\theta} z,\ u_z + \dot{\theta} y)\ . \qquad \text{(II.2)}$$

Thus, if P is fixed in inertial space instead of in oxyz, it will appear to an observer in oxyz to move with velocity $-\bar{V}$.



Notes:

1. $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$

2. $\zeta(x, y, t) = w(x, y, t) + f(x, y)$

Figure 1. Coordinate System and Notation

Denote by $\bar{q}(P, t)$, $\bar{v}(P, t)$ the velocities of the liquid particle at point $P(x, y, z) \in \tau(t)$ at time t as estimated by observers in inertial space and oxyz respectively. Then

$$\bar{q} = \bar{V} + \bar{v}, \quad \bar{v} \equiv \frac{d\bar{r}}{dt}, \tag{II.3}$$

in which the position vector $\bar{r}$ is referred to the moving reference oxyz.

B. Basic Equations

Assume the liquid to be homogeneous and incompressible throughout the motion. Moreover, let the absolute velocity of liquid particles be irrotational. Then, from the fundamental formulae of theoretical hydrodynamics, we have [7]:

Velocity of Liquid Particles

$$\bar{q} = \nabla\varphi + (0,\ u_y - \dot{\theta} z,\ u_z - \dot{\theta} y), \quad P \in \tau(t),$$

$$\bar{v} = \nabla\varphi + (0,\ 0,\ -2\dot{\theta} y), \quad P \in \tau(t), \tag{II.4}$$

Continuity of Liquid Motion

$$\nabla \cdot \bar{q} = \nabla \cdot \bar{v} = 0, \quad P \in \tau(t),$$

$$\Delta\varphi = 0, \quad P \in \tau(t), \tag{II.5}$$

Boundary Conditions (Kinematical)

$$\frac{\partial\varphi}{\partial n} = \begin{cases} 2\dot{\theta} y \cos(n, z), & P \in \Sigma(t), \\ 2\dot{\theta} y \cos(n, z) + \zeta_t \cos(n, z), & P \in S(t) \end{cases} \tag{II.6}$$

Pressure

$$\frac{p}{\rho} + \frac{\partial\varphi}{\partial t} - \ddot{\theta}\, y z + a_y\, y + a_z\, z + \Omega + \tfrac{1}{2} v^2 - \tfrac{1}{2}\dot{\theta}^2 (y^2 + z^2) = C(t), \quad P \in \tau(t), \tag{II.7}$$

in which

$$\bar{g} = -\nabla\Omega \tag{II.8}$$

is the vector of body forces per unit mass,

$$\bar{g} = (0,\ g_y,\ g_z);$$

ρ is the mass density; p is the pressure intensity;

$$\bar{a} = (0, a_y, a_z) = (0, \dot{u}_y - \dot{\theta} u_z, \dot{u}_z + \dot{\theta} u_y) \tag{II.9}$$

is the absolute acceleration of 0 as measured by an observer in oxyz; cos (n, z) denotes the cosine of the angle between the outward directed normal to the surface under consideration and axis oz; C(t) is an instantaneous constant independent of position.

C. Surface Tension (Mean Curvature)

The free surface is presumed to act like a membrane. We require that the normal stress across the free surface be discontinuous by an amount proportional to the product of the interfacial tension, T, and the mean surface curvature. This gives rise to the expression

$$p_g - p = T \left\{ \frac{\partial}{\partial x} \frac{\frac{\partial \zeta}{\partial x}}{\left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]^{\frac{1}{2}}} + \frac{\partial}{\partial y} \frac{\frac{\partial \zeta}{\partial y}}{\left[1 + \left(\frac{\partial \zeta}{\partial x}\right)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2\right]^{\frac{1}{2}}} \right\} \tag{II.10}$$

in which p_g is the ullage gas pressure. The quantity enclosed in the braces is the mean curvature of the surface.

D. Contact Angle

The angle at which the free surface meets the vessel wall is defined by

$$\theta_c = \cot^{-1} \left\{ \pm \frac{\nabla \zeta \cdot \bar{n}}{[1 + (\nabla \zeta \cdot \bar{t})^2]^{\frac{1}{2}}} \right\} \tag{II.11}$$

in which $\bar{n}$ is the outward directed unit normal of $\Sigma(t)$ (or C), and $\bar{t}$ is the unit tangent vector of curve C. This formula is applicable to prismatic cylinders.

E. Kinetic Energy

The kinetic energy of the liquid is

$$\text{K.E.} = \tfrac{1}{2} \rho \iiint_{\tau(t)} (\bar{q})^2 \, d\tau = \tfrac{1}{2} \rho \iiint_{\tau(t)} |\bar{V} + \bar{v}|^2 \, d\tau ,$$

which, in light of (II.4), takes on the form

$$\text{K.E.} = \tfrac{1}{2}\rho \iiint_{\tau(t)} (\nabla\varphi)^2 \, d\tau + \tfrac{1}{2}\rho\,\dot{\theta}^2 \iiint_{\tau(t)} (y^2 + z^2)\, d\tau + \tfrac{1}{2}\rho\,(u_y^2 + u_z^2) \iiint_{\tau(t)} d\tau$$

$$- \rho\,\dot{\theta}\,\Big\{ \iiint_{\tau(t)} \nabla\varphi \cdot \nabla z\, y \, d\tau + u_y \iiint_{\tau(t)} z \, d\tau + u_z \iiint_{\tau(t)} y \, d\tau \Big\}$$

$$+ \rho \iiint_{\tau(t)} \nabla\varphi \cdot \nabla(\bar{u}\cdot\bar{r})\, d\tau \,. \tag{II.12}$$

Using Green's theorem, conditions (II.6) and theory of surfaces, we find that

$$\tfrac{1}{2}\rho \iiint_{\tau(t)} (\nabla\varphi)^2 \, d\tau = \rho\,\dot{\theta} \iint_R y\,[\varphi(x, y, \zeta, t) - \varphi(x, y, -h, t)]\, dx\, dy$$

$$+ \tfrac{1}{2}\rho \iint_R \varphi(x, y, \zeta, t)\, \zeta_t \, dx\, dy \,,$$

$$-\rho\,\dot{\theta} \iiint_{\tau(t)} \nabla\varphi \cdot \nabla z\, y\, d\tau = -2\rho\,\dot{\theta}^2 \iint_R y^2 \zeta\, dx\, dy - 2\rho\,\dot{\theta}^2 h \iint_R y^2\, dx\, dy$$

$$- \rho\,\dot{\theta} \iint_R y\,\zeta\,\zeta_t\, dx\, dy \,,$$

$$\rho \iiint_{\tau(t)} \nabla\varphi \cdot \nabla(\bar{u}\cdot\bar{r})\, d\tau = 2\rho\,\dot{\theta}\, u_z \iint_R y\,\zeta\, dx\, dy + \rho\, u_y \iint_R y\,\zeta_t\, dx\, dy$$

$$+ \rho\, u_z \iint_R \zeta\,\zeta_t\, dx\, dy \,,$$

$$\Big(\iint_R y\, dx\, dy = 0 \Big) \,,$$

$$\rho \iiint_{\tau(t)} d\tau = \rho\, h \iint_R dx\, dy + \rho \iint_R \zeta\, dx\, dy$$

$$\rho \iiint_{\tau(t)} y\, d\tau = \rho\, h \iint_R y\, dx\, dy + \rho \iint_R y\,\zeta\, dx\, dy$$

$$\rho \iiint_{\tau(t)} z\, d\tau = -\rho\, h/2 \iint_R dx\, dy + \rho/2 \iint_R \zeta^2\, dx\, dy$$

$$\rho \iiint_{\tau(t)} (y^2 + z^2)\, d\tau = \rho \iint_R (y^2 h + h^3/3)\, dx\, dy + \rho \iint_R (y^2\zeta + \zeta^3/3)\, dx\, dy$$

The region of integration R is the cross section of the cylinder bounded by curve C. Thus the kinetic energy becomes

$$
\begin{aligned}
K.E. = {} & \tfrac{1}{2}\rho \iint_R \varphi(x, y, \zeta, t)\, \zeta_t \, dx\, dy + \rho\, u_y \iint_R y\, \zeta_t \, dx\, dy + \rho\, u_z \iint_R \zeta\, \zeta_t \, dx\, dy \\
& + \tfrac{1}{2}\rho\,(u_y^2 + u_z^2)\Big\{ \iint_R \zeta \, dx\, dy + h \iint_R dx\, dy \Big\} + \tfrac{1}{2}\rho\, \dot{\theta}^2 \Big\{ \iint_R (\zeta^3/3 - 3\, y^2 \zeta)\, dx\, dy \\
& + \iint_R (h^3/3 - 3\, h\, y^2)\, dx\, dy + \rho\, \dot{\theta} \Big\{ \iint_R y\, [\varphi(x, y, \zeta, t) - \varphi(x, y, -h, t)]\, dx\, dy \\
& - \iint_R y\, \zeta \zeta_t \, dx\, dy - u_y \iint_R \zeta^2/2 \, dx\, dy + u_z \iint_R y\, \zeta \, dx\, dy + h/2\, u_y \iint_R dx\, dy \Big\} . \qquad \text{(II.13)}
\end{aligned}
$$

F. Surface Energy

The surface energy of the liquid is given by the expression

$$\Pi_s = \iint_{S(t)+\Sigma(t)} T \, ds$$

in which T indicates the surface or interfacial tension typical of the pertinent surface/vapor or surface (solid) boundary. Now

$$
\begin{aligned}
\Pi_s &= \iint_{S(t)+\Sigma(t)} T \, ds = T_1 \iint_{\Sigma_1(t)} ds + T_2 \iint_{\Sigma_2(t)} ds + T \iint_{S(t)} ds \\
&= T_2 \iint_R dx\, dy + T_1 \oint_C dL \int_{-h}^{\zeta} dz + T \iint_R [1 + (\nabla \zeta)^2]^{\frac{1}{2}} \, dx\, dy \\
&= T \iint_R [1 + (\nabla \zeta)^2]^{\frac{1}{2}} \, dx\, dy + T_1 \oint_C \zeta(x(s), y(s), t)\, dL + \text{const.}, \qquad \text{(II.14)}
\end{aligned}
$$

where

$$x = x(s)$$

$$y = y(s)$$

is the parametric representation of the boundary C.

G. Gravitational Energy

The gravitational energy of the liquid is expressed by

$$\Pi_g = \rho \iiint\limits_{\tau(t)} \Omega \, d\tau . \qquad \text{(II.15)}$$

In most applications we can assume that

$$\Omega = g_y \, y + g_z \, z \ ,$$

so that

$$\begin{aligned} \Pi_g &= \rho \, g_y \iiint\limits_{\tau(t)} y \, d\tau + \rho \, g_z \iiint\limits_{\tau(t)} z \, d\tau \qquad \text{(II.16)} \\ &= \rho \, g_y \iint\limits_{R} y \, \zeta \, dx \, dy + \rho/2 \, g_z \iint\limits_{R} \zeta^2 \, dx \, dy + \text{const.} \end{aligned}$$

H. Forces and Moments

Forces and moments resulting from the action of the liquid motion on the vessel surface $\Sigma(t)$ may be computed in an elementary fashion by integrating the unsteady pressure equation (II.7). This computation can be found in [7]. The forces and moments resulting from the interfacial tension forces acting on $\Sigma(t)$ is another matter however. To illustrate how this is done, consider the determination of the forces on $\Sigma(t)$ due to the membrane action of the free surface.

Since the free surface is assumed to be a membrane, we may consider it stretched over a certain simply connected planar region R which is bounded by a rectifiable curve C. Choose the coordinate system so that C lies in the xy plane. (This is in keeping with our conventions implied earlier). The vertical displacement of the membrane at the point (x, y) at the time t is denoted by $\zeta(x, y, t)$, as before. We assume that the tension per unit length has the constant value T along the boundary C. If we represent the tension by a vector $\bar{T}$, then $|\bar{T}| = T$ and $\bar{T}$ is normal to C but lies within the tangent plane to the membrane. If the membrane is in equilibrium, the tension is constant over the entire surface; i.e., along any arbitrary cut through the membrane, the tension is transferred from one side of the cut to the other side without change of magnitude. Thus, the tension along any arbitrary cut through the membrane can be represented by a vector of the constant magnitude T and a direction which is perpendicular to the cut and lies in the tangent plane of the surface.

Assume that there are no external forces besides the tension present. We now proceed to establish an expression for the transfer of the external force due to the tension T to the surface $\Sigma(t)$.

Consider for this purpose a surface element ΔS of the membrane which is arbitrarily cut out. ΔS is encompassed by the rectifiable curve C' and its projection into the x, y plane is ΔR. Let

$$x = x(s)$$

$$y = y(s)$$

$$\zeta = \zeta(s)$$

be the parametric representation of the boundary C', where S stands for the arc length on C'. If P is a point on C', then the tension vector $\bar{T}$ at P is perpendicular to the tangent vector $\bar{t}$ to C' at P and lies in the tangent plane to the surface at P. (See Figure 2).

To find the components of $\bar{T}$ exerted upon C', we proceed as follows: The surface is represented by $\zeta = \zeta(x, y, t)$ at any time t. If we let $\Phi = \zeta(x, y, t) - \zeta$, we can represent the membrane by

$$\Phi(x, y, \zeta, t) = 0$$

and

$$\bar{n} = \nabla\Phi = \left(\frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y}, -1\right)$$

is a vector orthogonal to ΔS. The tangent vector $\bar{t}$ to C' is

$$\bar{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{d\zeta}{ds}\right)$$

and in view of s being the arc length on C', we have $|\bar{t}| = 1$. Let $(\quad)_p$ denote "at P". Then

$$(\bar{T})_p = T \cdot \frac{\bar{n} \times \bar{t}}{|\bar{n}|}$$

(where $|\bar{n}| = \left[1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{\frac{1}{2}}$) is the tension vector in P, because it is perpendicular to $\bar{n}$ and thus lies in the tangent plane to ΔS, is perpendicular to $\bar{t}$, and

$$\left| T \cdot \frac{\bar{n} \times \bar{t}}{|\bar{n}|} \right| = T .$$

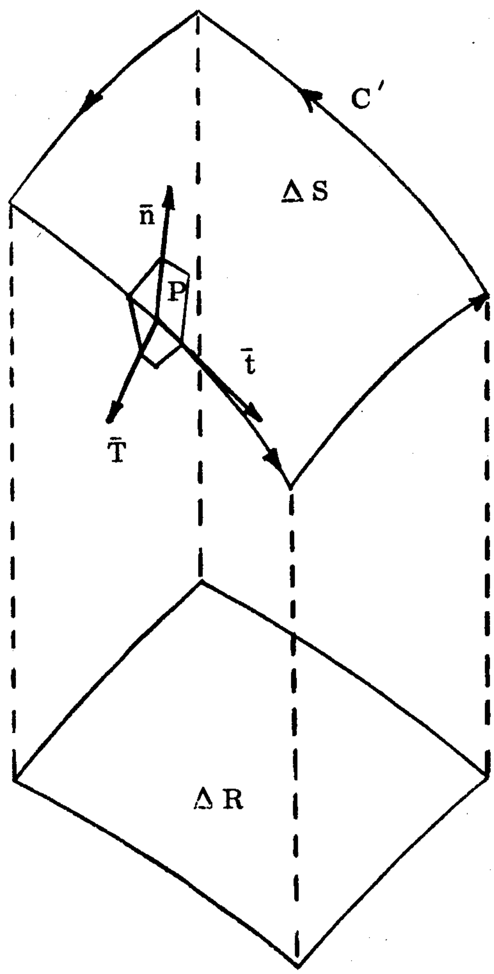


Figure 2. Membrane Element

Now

$$\bar{T} = T \frac{\bar{n} \times \bar{t}}{|n|} = \frac{T}{[1 + (\frac{\partial \zeta}{\partial x})^2 + (\frac{\partial \zeta}{\partial y})^2]^{\frac{1}{2}}} \cdot [(\frac{\partial \zeta}{\partial y} \frac{d\zeta}{ds} + \frac{dy}{ds}),$$

$$-(\frac{\partial \zeta}{\partial x} \frac{d\zeta}{ds} + \frac{dx}{ds}), (\frac{\partial \zeta}{\partial x} \frac{dy}{ds} - \frac{\partial \zeta}{\partial y} \frac{dx}{ds})],$$

which represents the external force per unit length which is exerted upon the boundary C' of ΔS. In order to find the external force exerted upon the entire boundary C' of ΔS, we have to take the line integral along C':

$$\bar{T}(C') = \oint_{C'} \bar{T} \, ds \quad .$$

But, since C' is arbitrary, we can extend this integral along C, getting

$$\bar{T}(C) = T \oint_C \frac{1}{[1 + (\frac{\partial \zeta}{\partial x})^2 + (\frac{\partial \zeta}{\partial y})^2]^{\frac{1}{2}}} [(\frac{\partial \zeta}{\partial y} \frac{d\zeta}{ds} + \frac{dy}{ds}), -(\frac{\partial \zeta}{\partial x} \frac{d\zeta}{ds} + \frac{dx}{ds}),$$

$$(\frac{\partial \zeta}{\partial x} \frac{dy}{ds} - \frac{\partial \zeta}{\partial y} \frac{dx}{ds})] \, ds \qquad \text{(II. 17)}$$

where

$$\frac{d\zeta}{ds} = \frac{\partial \zeta}{\partial x} \frac{dx}{ds} + \frac{\partial \zeta}{\partial y} \frac{dy}{ds} \quad .$$

The moment of the external force may be obtained in a similar manner.

I. Constraint

The volume $\tau(t)$ occupied by the liquid in a given configuration must be subjected to the constraint expressing the constancy of volume. This follows from the divergence theorem which states

$$\iiint_{\tau(t)} \nabla \cdot \bar{v} \, d\tau = \iint_{S(t)} \zeta_t \cos(n, z) \, ds$$

$$= \iint_R \zeta_t \, dx \, dy = 0 \, , \qquad \text{(II. 18)}$$

in virtue of (II. 5).

III. ANALYSIS (1)

In this analysis we present a method for determining the motion of a heavy liquid enclosed in a partly filled prismatic cylinder which is itself in planar motion under the joint action of gravitational, surface and interfacial tension forces. The procedure consists of expanding the free surface equation in a series of orthogonal functions, and regarding the coefficients of the series expansion as generalized coordinates describing the liquid system. With appropriate transformations, the velocity potential is also expressed as a series of these coefficients. The kinetic and potential energy of the liquid and consequently the Lagrange function, modified by the requirement of the constancy of volume of liquid, is constructed. With this function known, the Lagrangian equations of motion are derived. Theorectically, the integration of these equations provides a time history of the generalized coordinates. The motion of liquid, shape of the free surface, pressure distribution, forces and moments thus can be determined.

In some respects this method is more than a mere specialization of the generalized analysis given in [4]; it is an extension of it to include translational and rotational motion of the vessel.

A. Determination of Kinematics of Liquid

The solution of (II. 5) satisfying boundary conditions (II. 6), that is,

$$\Delta\varphi = 0 \ , \quad P \in \tau(t) \ ;$$

$$\frac{\partial\varphi}{\partial n} = 0 \ , \quad P \in \Sigma_1(t) \ , \tag{III. 1}$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial z} = 2\,\dot{\theta}\,y \ , \quad P \in \zeta(t) \ , \ z = -h \ ,$$

$$\frac{\partial\varphi}{\partial n} = (2\,\dot{\theta}\,y + \zeta_t)\cos(n, z) \ , \quad P \in S(t) \ , \tag{III. 2}$$

may be taken in the form of a series

$$\varphi(x, y, z, t) = \sum_{1}^{\infty} (\alpha_i(t) \cosh k_i (z + h) + \gamma_i \sinh k_i z) \varphi_i (x, y) + \alpha_0(t) \; . \qquad (III.3)$$

This function is harmonic as indicated by (III. 1), and satisfies (III.2_1) if the infinitely many values k_i^2 (i = 1, 2, . . .) are the values of k^2 (eigenvalues) for which the two-dimensional scalar Helmholtz equation

$$\Delta \varphi + k^2 \varphi = \; , \quad P \in R \; , \qquad (III.4)$$

has a non-zero solution satisfying

$$\frac{\partial \varphi}{\partial n} = 0 \; , \quad P \in C \; . \qquad (III.5)$$

Functions φ_i (i = 1, 2. . .) are the corresponding solutions (eigenfunctions) of (III.4).

We point out, without proof, several important properties of the system of functions φ_i and the numbers k_i^2:

(1) Of the infinite number of numbers k_i^2, all are real and positive;

(2) The set of functions φ_i is orthogonal

$$(\varphi_i \; , \; \varphi_j) \equiv \iint_R \varphi_i \varphi_j \, dx \, dy = \begin{cases} 0, & i \neq j \\ \|\varphi_i\|^2 \; , & i = j \end{cases} \; , \qquad (III.6)$$

and can be normalized;

(3) Any function $\mu(x, y)$ which has continuous second-order derivatives in and on the boundary of R and which is orthogonal to a constant:

$$(\mu \, , \; 1) = \iint_R \mu(x, y) \, dx \, dy = 0 \; ,$$

and which satisfies the boundary conditions, may be expressed as a uniformly convergent series of eigenfunctions

$$\mu(x, y) = \sum_{1}^{\infty} C_i \varphi_i (x, y) \; ; \qquad (III.7)$$

(4) If function μ which is continuously twice-differentiable satisfies the condition (III. 5), then the series (III. 7) not only converges uniformly to μ, but the series obtained from it by termwise differentiation also converges in the mean to the corresponding derivative of μ;

(5) In addition to the condition for orthogonality, written above, the following equations hold:

$$\iint_R \nabla \varphi_i \nabla \varphi_j \, dx\, dy = \begin{cases} 0, & i \neq j \\ k_i^2 \|\varphi_i\|^2, & i = j \end{cases} \qquad \text{(III. 8)}$$

We are now in a position to expand $2 y \dot{\theta}$ and $\zeta(x, y, t)$ as series of the form

$$2 \dot{\theta} y = 2 \dot{\theta} \{ \sum_1^{\infty} \eta_i (t) \varphi_i (x, y) + \eta_0 (t) \},$$

$$\zeta(x, y, t) = \sum_1^{\infty} \xi_i (t) \varphi_i (x, y) + \xi_0 (t) . \qquad \text{(III. 9)}$$

Such expansions are possible, since for a fixed value of t functions $2 \dot{\theta} (y - \eta_0)$ and $\zeta(x, y, t) - \xi_0(t)$ are both developable in series of the form (III. 7). The coefficients of these series depend, in general, on t. Choosing η_0 and ξ_0 so that $y - \eta_0$ and $\zeta - \xi_0$ are orthogonal to a constant (unity), we get

$$\eta_i = \frac{(y, \varphi_i)}{\|\varphi_i\|^2}$$

$$\eta_0 = \frac{(y, 1)}{A} = 0, \quad (y, 1) = 0 , \qquad \text{(III. 10)}$$

and

$$\xi(t) = \frac{(\zeta, \varphi_i)}{\|\varphi_i\|^2} , \qquad \text{(III. 11)}$$

$$\xi_0(t) = \frac{(1, \zeta)}{A} ,$$

respectively, using the condition for orthogonality (III. 6). Here A is the area of the cross section of the cylinder, i.e., the area of R. Now if

$$\frac{\partial \zeta}{\partial n} = 0 , \quad P \in C ,$$

the series obtained from (III. 9_2) by termwise differentiation converges in the mean to the corresponding derivative of ζ. However, this implies a 90° contact angle, as can be seen from Formula (II. 11). It should be noted that this restriction is implied in [4] for the case of a semi-infinite circular cylinder (see formulae (2-1) - (2-6) of [4]). The authors, upon comparing theoretical results and experimental data for the circular cylinder problem, noted that the theoretically computed free surface "rounded-off" at the wall of the tank (i.e. attempted to maintain a 90° contact angle) and did not coincide with the actual free surface there (see page 32 of [4]). This discrepency was attributed to truncation and numerical errors incurred in the solution process. The authors further stated, "At time $t = 0$ (initial configuration), if the largest number of generalized coordinates are used, this error decreases appreciably but does not vanish". In other words, the theoretically computed interface and actual interface were closer in the equilibrium position. But, this is what one would expect because, for their problem, the contact angle of the actual interface was closer to 90° at $t = 0$ than for any other time thereafter. This apparent anomaly was "reconciled" by neglecting the round-off and extending the computed curve of the free surface near the wall until it intersected the wall. The authors then stated that the resulting curve was a good representation of the actual contact angle to within ± 0.5 percent. It was then concluded that the rounding-off of the computed menisci at the container wall was, in reality, the contact angle calculated to an accuracy of approximately 5 percent! Perhaps this is so, but their mathematics indicates that the computed contact angle should be 90°, and above all, the curves show a contact angle of 90°!

For the time being, let us go along with the 90° contact angle assumption, and extend [4] to include finite prismatic cylinder.

For (III.3) to satisfy the boundary condition (III.2_2) it is necessary that

$$\gamma_i(t) = \frac{2\,\dot{\theta}(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \tag{III.12}$$

to satisfy the boundary condition (III.2_3) it is necessary that

$$\sum_1^\infty \{k_i \varphi_i \sinh k_i(\zeta + h) - \nabla\varphi_i \cdot \nabla\zeta \cosh k_i(\zeta + h)\}\alpha_i(t) + \sum_1^\infty \{k_i \varphi_i \cosh k_i \zeta$$

$$- \nabla\varphi_i \cdot \nabla\zeta \sinh k_i \zeta\}\gamma_i(t) = 2\,\dot{\theta}\, y + \zeta_t = \sum_1^\infty \gamma_i(t)\, \varphi_i\, k_i \cosh k_i h + \zeta_t .$$

However,

$$\varphi_i = -\frac{1}{k_i^2}\Delta\varphi_i = -\frac{1}{k_i^2}\nabla\cdot(\nabla\varphi_i)$$

so that this expression can be written, on application of a known vector identity,

$$\sum_1^\infty \alpha_i(t)\, \nabla\cdot\left\{\frac{\sinh k_i(\zeta + h)}{k_i}\nabla\varphi_i\right\} + \sum_1^\infty \gamma_i(t)\nabla\cdot\left\{\frac{\cosh k_i\zeta - \cosh k_i h}{k_i}\nabla\varphi_i\right\}$$

$$= -\zeta_t = -\sum_1^\infty \dot{\xi}_i(t)\,\varphi_i - \dot{\xi}_o(t)\,, \tag{III.13}$$

using (III. 9_2). Multiplying both sides of this equality by φ_j and integrating over R, we get

$$- \|\varphi_i\|^2 \dot{\xi}_i = \sum_1^\infty \alpha_j \iint_R \varphi_i \nabla \cdot \{ \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \} \, dx \, dy$$

$$+ \sum_1^\infty \gamma_j \iint_R \varphi_i \nabla \cdot \{ \frac{\cosh k_i \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \} \, dx \, dy \, .$$

But

$$\varphi_i \nabla \cdot \{ \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \} = \nabla \cdot \{ \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \}$$

$$- \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_i \cdot \nabla \varphi_j \, ,$$

$$\varphi_i \nabla \cdot \{ \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \} = \nabla \cdot \{ \varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \}$$

$$- \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_i \cdot \nabla \varphi_j \, ,$$

and, moreover

$$\iint_R \nabla \cdot \{ \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \} \, dx \, dy = \oint_C \varphi_i \frac{\sinh k_j (\zeta + h)}{k_j} \frac{\partial \varphi_j}{\partial n} \, dS = 0 \, ,$$

$$\iint_R \nabla \cdot \{ \varphi_i \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \} \, dx \, dy = 0 \, ,$$

in accordance with the divergence theorem and boundary condition (III. 5). Therefore, it follows

$$\dot{\xi}_i = \sum_1^\infty C_{ij} \alpha_j + 2 \dot{\theta} \beta_i \qquad \text{(III. 14)}$$

in which

$$C_{ij} = \frac{1}{\|\varphi_i\|^2} \iint_R \frac{\sinh k_j (\zeta + h)}{k_j} \nabla \varphi_j \cdot \nabla \varphi_i \, dx \, dy \, , \qquad \text{(III. 15)}$$

$$\beta_i = \frac{1}{\|\varphi_i\|^2} \sum_1^\infty \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_R \frac{\cosh k_j \zeta - \cosh k_j h}{k_j} \nabla \varphi_j \cdot \nabla \varphi_i \, dx \, dy \, .$$

If we integrate both sides of (III. 13) over R we obtain the constraint expressing the constancy of volume. To be sure

$$\iint_R \nabla \cdot \left\{ \frac{\sinh k_i (\zeta + h)}{k_i} \nabla \varphi_i \right\} dx\, dy = \oint_C \frac{\sinh k_i (\zeta + h)}{k_i} \frac{\partial \varphi_i}{\partial n} dS = 0 ,$$

$$\iint_R \nabla \cdot \left\{ \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \nabla \varphi_i \right\} dx\, dy = \oint_C \frac{\cosh k_i \zeta - \cosh k_i h}{k_i} \frac{\partial \varphi_i}{\partial n} dL = 0 ,$$

and

$$\sum_1^{\infty} \dot{\xi}_i \iint_R \varphi_i \, dx\, dy + \dot{\xi}_0 \iint_R dx\, dy = A \dot{\xi}_0 ,$$

so that

$$A\dot{f}_0 = 0 .$$

But, the volume is given by

$$\tau(t) = \iint_R dx\, dy \int_{-h}^{\zeta} dz = h \iint_R dx\, dy + \iint_R \zeta \, dx\, dy = \text{Constant} + A\xi_0 ,$$

and

$$\dot{\tau}(t) = 0 = A \dot{\xi}_0$$

as adduced.

If the transformation (III. 14) has an inverse, we can write

$$\alpha_i = \sum_1^{\infty} \Gamma_{ij} \dot{\xi}_j - 2 \dot{\theta} \sum_1^{\infty} \Gamma_{ij} \beta_j \qquad \text{(III.16)}$$

$$= \sum_1^{\infty} \Gamma_{ij} \dot{\xi}_j - 2 \dot{\theta} \beta_i^* ,$$

$$\Gamma_{ij} = C_{ij}^{-1}, \quad \beta_i^* = \Sigma \Gamma_{ij} \beta_j$$

B. Kinetic Energy

Using (II.13) and (III.3, 9, 16), we find that the kinetic energy is given by

$$K.E. = \tfrac{1}{2}\rho \sum_i \sum_j \sum_l \dot{\xi}_j \, \dot{\xi}_l \iint_R \Gamma_{lj}\, \varphi_i \varphi_l \cosh k_i(\zeta + h)\, dx\, dy \tag{III.17}$$

$$+ \rho \dot{\theta} \{ \sum_i \sum_j \dot{\xi}_j \Big[\frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \iint_R \varphi_i \varphi_j \sinh k_i \zeta \; dx\, dy$$

$$- \beta_i^* \iint_R \varphi_i \; \varphi_j \; \cosh k_i \; (\zeta + h)\; dx\, dy \Big] - \sum_i \sum_j \dot{\xi}_i \; \xi_j \iint_R y\, \varphi_i \varphi_j \, dx\, dy$$

$$+ \sum_i \sum_j \Gamma_{ij}\, \xi_j \iint_R y\, \varphi_i \,(\cosh k_i (\zeta + h) - 1)\, dx\, dy + u_z \sum_i (y, \varphi_i)\, \xi_i$$

$$- \tfrac{1}{2} u_y \sum_i \|\varphi_i\|^2 \xi_i^2 + \tfrac{1}{2} h\, u_y A \}$$

$$+ \tfrac{1}{2} \rho \dot{\theta}^2 \{ 4 \sum_i \Big[\frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \iint_R y\, \varphi_i \,(\sinh k_i \zeta + \sinh k_i h)\, dx\, dy$$

$$- \beta_i^* \iint_R y\, \varphi_i (\cosh k_i (\zeta + h) - 1)\, dx\, dy \Big] + \iint_R (\zeta^3/3 - 3\, y^2 \zeta)\, dx\, dy$$

$$+ \iint_R (h^3/3 - 3\, h\, y^2)\, dx\, dy \} + \beta\, u_y \sum_i (y, \varphi_i)\, \dot{\xi}_i$$

$$+ \rho\, u_z \sum_i \|\varphi_i\|^2 \dot{\xi}_i \; \xi_i + \tfrac{1}{2} \rho (u_y^2 + u_z^2)\, [A\, \xi_0 + A\, h]$$

C. Potential Energy

Using (II.16) and (III.9_2), we get for the potential energy

$$\Pi_g = \rho\, g_y \sum_i (y, \varphi_i)\, \xi_i$$

$$+ \tfrac{1}{2} \rho \; g_z \sum_i \|\varphi_i\|^2 \xi_i^2 \tag{III.18}$$

D. Surface Energy

With (II.14) and (III.9_2), we have for the surface energy

$$\Pi_s = T \iint_R [1 + [\sum_i^\infty (\nabla \varphi_i)\, \xi_i]^2]^{\frac{1}{2}} \; dx\, dy$$

$$+ T_1 \oint \sum_i^\infty \varphi_i (x(s),\, y(s))\, \xi_i\, (t)\, dL \tag{III.19}$$

E. Lagrangian Potential

To simplify the formulation, we rewrite the kinetic energy in the form

$$K.E. = \tfrac{1}{2}\rho \sum_i \sum_j V_{ij}\, \dot{\xi}_i\, \dot{\xi}_j + \rho\, \dot{\theta}\, \{ \sum_i W_i \dot{\xi}_i - \sum_i \sum_j Y_{ij}\, \dot{\xi}_i\, \xi_j - \tfrac{1}{2} u_y \sum_i \|\varphi_i\|^2 \xi_i^2 \tag{III.20}$$

$$+ u_z \sum_i (y, \varphi_i)\, \xi_i \} + \tfrac{1}{2} \rho\, \dot{\theta}^2 \{ Z + \sum_i \sum_j \sum_l M_{ijl}\, \xi_i \xi_j \xi_l - 3 \sum_i (y^2, \varphi_i)\, \xi_i \}$$

$$+ \rho\, u_y \sum_i (y, \varphi_i)\, \dot{\xi}_i + \rho\, u_z \sum_i \|\varphi_i\|^2\, \dot{\xi}_i\, \xi_i + \tfrac{1}{2} \rho\, \dot{\theta}\, h\, u_y\, A$$

$$+ \tfrac{1}{2} \rho\, \dot{\theta}^2 \iint_R (h^3/3 - 3\, h y^2)\, dx\, dy + \tfrac{1}{2} \rho (u_y^2 + u_z^2)\, (A\, \xi_0 + Ah)\ ,$$

where

$$V_{ij} = \iint_R \sum_l \Gamma_{lj}\, \varphi_l\, \varphi_i \cosh k_l (\zeta + h)\, dx\, dy\ ,$$

$$W_i = \sum_j \Big[\frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_R \varphi_i\, \varphi_j \sinh k_j \zeta\ dx\, dy - \beta_j^* \iint_R \varphi_i\, \varphi_j \cosh k_j (\zeta + h)\, dx\, dy$$

$$+ \Gamma_{ji} \iint_R y\, \varphi_j\, (\cosh k_j (\zeta + h) - 1)\, dx\, dy \Big]\ ,$$

$$Y_{ij} = \iint_R y\, \varphi_i\, \varphi_j\, dx\, dy\ ,$$

$$Z = 4 \sum_i \Big[\frac{(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \iint_R y\, \varphi_i\, (\sinh k_i \zeta + \sinh k_i h)\, dx\, dy$$

$$- \beta_i^* \iint_R y\, \varphi_i\, (\cosh k_i\, (\zeta + h) - 1)\, dx\, dy \Big]\ ,$$

$$M_{ijl} = \frac{1}{3} \iint_R \varphi_i\, \varphi_j\, \varphi_l\, dx\, dy\ .$$

The Lagrangian function for our dynamic process is given by

$$L = K.E. - \Pi_g - T \iint_R [\, 1 + [\sum_i (\nabla \varphi_i)\, \xi_i\,]^2\,]^{\frac{1}{2}}\, dx\, dy\ .$$

Following Hamilton, the motion of the liquid must occur in such a way that

$$L\, [\int_0^t L\, dt + T_1 \sum_i \mu_i \xi_i]\,. \tag{III.21}$$

In addition, the volume of the liquid $\tau(t)$ must be constant, thus requiring the introduction of the constraint

$$\dot{\tau}(t) = 0 = A\, \dot{\xi}_0$$

on the values of the generalized coordinates ξ_i. This requirement can be satisfied by introducing

$$L^* = L + \lambda\, \tau(t) \tag{III.22}$$

so that the variational principle (III.21) becomes

$$L\, [\int_0^t L^* dt + T_1 \sum_i \mu_i \xi_i] = 0 \tag{III.23}$$

in which λ is a Lagrangian multiplier ultimately determinables from the volume constraint.

F. Equations of Motion

If formula (III.23) is to be an extremum, L^* must satisfy the system of differential equations,

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{\xi}_i}\right) - \frac{\partial L^*}{\partial \xi_i} = - T_1 \mu_i \tag{III.24}$$

which, on performing the indicated operations, become

$$\sum_j (V_{ij} + V_{ji})\, \ddot{\xi}_j + \sum_j (\dot{V}_{ij} + \dot{V}_{ji})\, \dot{\xi}_j - \sum_j \sum_l \frac{\partial V_{lj}}{\partial \xi_i}\, \dot{\xi}_j\, \dot{\xi}_l + 2\, \ddot{\theta}\, (W_i - \sum_j Y_{ij}\, \xi_j)$$

$$+ 2\, \dot{\theta}\, (\dot{W}_i - \sum_j \frac{\partial W_j}{\partial \xi_i}\, \dot{\xi}_j + \sum_j Y_{ji}\, \dot{\xi}_j - \sum_j Y_{ij}\, \dot{\xi}_j) - \dot{\theta}^2 (\frac{\partial Z}{\partial \xi_i} + 3 \sum_j \sum_l M_{ijl}\, \xi_j\, \xi_l$$

$$- 3\, (y^2, \varphi_i)) + 2\, \|\varphi_i\|^2\, (a_z + g_z)\, \dot{\xi}_i + 2\, (y, \varphi_i)\, (a_y + g_y)$$

$$+ \frac{2T}{\rho} \sum_j \iint_R \frac{\nabla \varphi_i\, \nabla \varphi_j}{[1 + [\Sigma \nabla \varphi_l\, \xi_l]^2]^{\frac{1}{2}}}\, dx\, dy + 2\, \lambda\, \frac{A}{\rho}\, \delta_{0i}$$

$$- (u_y^2 + u_z^2)\, A\, \delta_{0i} + \frac{2T_1}{\rho}\, \mu_i = 0\,, \tag{III.25}$$

in which

$$a_y = \dot{u}_y - \dot{\theta}\, u_z \, ,$$

$$a_z = \dot{u}_z + \dot{\theta}\, u_y \, .$$

These equations can be solved for the ξ_i's, analytically (in principle), or numerically as in [4]. Note that it is necessary to know the initial configuration and state of the free surface. The manner in which the equations are integrated is discussed in detail in [4] and need not be repeated here. Once the generalized coordinates are known the problem is completely solved.

G. Discussion

The above analysis yields reasonably accurate results for a stationary, semi-infinite right circular cylinder, as indicated in [4], and can be extended to the class of problems considered here without any difficulty. Note that in this approach, the generalized coordinates and consequently forces, moments, etc., would be computed as functions of time. As noted previously, the analysis is valid only for 90° contact angles.

IV. ANALYSIS (2)

In this analysis we present a method for determining the motion of a heavy liquid enclosed in a partly filled prismatic cylinder which is itself in planar motion under the joint action of moderately low gravitational (acceleration) and surface tension forces. The procedure consists of perturbing the free surface, velocity of the liquid and motion of the tank about a finite equilibrium state. In this way products of the dependent variables and their derivatives may be assumed small. The analysis is valid for low Bond numbers, $10 \leq$ Bond number ≤ 1000. Although the effective body forces are dominant for these Bond numbers, surface tension causes the undisturbed free surface to depart significantly from a flat surface, as commonly occurs in high-G problems. Bond numbers of this size occur in propellant tanks of large boost stages in coasting orbits.

A similar analysis for simple harmonic translation of a semi-infinite, circular cylinder is given in [6]. Also included in this study, are experimental results which appear to agree well with theory.

A. Basic Equations

Assume the liquid to be homogeneous and incompressible throughout the motion. Moreover, let the absolute velocity of liquid particles be irrotational. Then from (II.4-6), we have:

Velocity of Liquid Particles

$$\bar{q} = \nabla \varphi + (0,\ u_y - \dot{\theta} z,\ u_z - \dot{\theta} y),\ P \epsilon \tau(t), \qquad (IV.1)$$

$$\bar{v} = \nabla \varphi + (0,\ 0,\ -2 \dot{\theta} y),\ P \epsilon \tau(t),$$

Continuity of Liquid Motion

$$\nabla \cdot \bar{q} = \nabla \cdot \bar{v} = 0,\ P \epsilon \tau(t), \qquad (IV.2)$$

$$\nabla \varphi = 0,\ P \epsilon \tau(t),$$

Boundary Conditions (Kinematical)

$$\frac{\partial\varphi}{\partial n} = 0\ , \quad P \epsilon\ \Sigma_1(t) \text{ or } P \epsilon\ C\ ,$$

$$\frac{\partial\varphi}{\partial n} \equiv \frac{\partial\varphi}{\partial z} = 2\,\dot{\theta}\,y\ , \ z = -h\ , \ P \epsilon\ \Sigma_2(t)$$

$$\frac{\partial\varphi}{\partial n} = (2\,\dot{\theta}\,y + \zeta_t)\cos(n, z), \quad z = \zeta, \quad P \epsilon\ S(t)\ . \tag{IV.3}$$

The dynamic condition to be satisfied at the free surface is obtained from (II.7, 8):

$$\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} - \ddot{\theta}\,y\,\zeta + (a_y + g_y)\,y + (a_z + g_z)\,\zeta + \tfrac{1}{2}\,v^2 - \tfrac{1}{2}\,\dot{\theta}^2(y^2 + \zeta^2) = C(t), \ z = \zeta\ . \tag{IV.4}$$

The discontinuity of the normal stress across the free surface is given by (II.10),

$$p_g - p = T\left\{\frac{\partial}{\partial x}\ \frac{\frac{\partial\zeta}{\partial x}}{[1 + (\nabla\zeta)^2]^{\frac{1}{2}}} + \frac{\partial}{\partial y}\ \frac{\frac{\partial\zeta}{\partial y}}{[1 + (\nabla\zeta)^2]^{\frac{1}{2}}}\right\}\ , \tag{IV.5}$$

in which T is the tension, p_g the ullage gas pressure.

Finally, the angle at which the free surface intersects the vessel wall is given by (II.11),

$$\pm\cot\theta_c = \frac{\frac{\partial\zeta}{\partial n}}{[1 + (\frac{\partial\zeta}{\partial s})^2]^{\frac{1}{2}}} \tag{IV.6}$$

where

$$\nabla\zeta\cdot\bar{n} = \frac{\partial\zeta}{\partial n}\ , \quad \nabla\zeta\cdot\bar{t} = \frac{\partial\zeta}{\partial s}\ .$$

We assume that the contact angle measured in the liquid, for the undisturbed equilibrium surface, is constant throughout the motion. In this regard, see [6]. Let us see what are the implications of this assumption.

The static value of the contact angle is, from (IV.6),

$$\pm\cot\theta_c^* = \frac{\frac{\partial f}{\partial n}}{[1 + (\frac{\partial f}{\partial s})^2]^{\frac{1}{2}}} \tag{IV.7}$$

Also, since

$$\frac{\partial\zeta}{\partial n} = \frac{\partial f}{\partial n} + \frac{\partial w}{\partial n} ,$$

$$\frac{\partial\zeta}{\partial s} = \frac{\partial f}{\partial s} + \frac{\partial w}{\partial s} ,$$

we have

$$\pm \cot \theta_c = \frac{\frac{\partial f}{\partial n} + \frac{\partial w}{\partial n}}{\{[1 + (\frac{\partial f}{\partial s})^2] + \frac{\partial w}{\partial s}(\frac{\partial w}{\partial s} + 2\frac{\partial f}{\partial s})\}^{\frac{1}{2}}} .$$

Expanding and linearizing this expression with respect to w, we get

$$\pm \cot \theta_c = \cot \theta_c^* + \frac{\frac{\partial w}{\partial n}}{[1 + (\frac{\partial f}{\partial s})^2]^{\frac{1}{2}}} - \frac{(\frac{\partial f}{\partial n})(\frac{\partial f}{\partial s})(\frac{\partial w}{\partial s})}{[1 + (\frac{\partial f}{\partial s})^2]^{1.5}} , \qquad \text{(IV. 8)}$$

using (IV. 7). Now, if we assume

$$\cot \theta_c \approx \cot \theta_c^*$$

for all time, then formula (IV. 8) requires that

$$\frac{\partial w}{\partial n} = \frac{(\frac{\partial f}{\partial n})(\frac{\partial f}{\partial s})}{1 + (\frac{\partial f}{\partial s})^2} \frac{\partial w}{\partial s} . \qquad \text{(IV. 9)}$$

For an axisymmetric undisturbed equilibrium surface expression (IV. 9) becomes

$$\frac{\partial w}{\partial n} = 0 . \qquad \text{(IV. 10)}$$

In most practical applications, the equilibrium free surface is axisymmetric, and we assume (IV. 10) to hold in the analysis.

B. Equilibrium Interface

The shape of the equilibrium free surface, $f(x, y)$, must be computed as an input to the analysis. This can be done in several ways. One such method, for a circular cross section, is given in [6]. This is well suited for the range of Bond numbers considered here. We shall return to this point when we consider an example.

Let

$$\varphi = \ddot{\theta} = (a_y + g_y) = v^2 = \dot{\theta}^2 = 0 ,$$

$$C(t) = \frac{p_0}{\rho} ,$$

in formula (IV.4), where p_0 is the liquid pressure at some known point $P(x^*, y^*, z^*)$. Then the pressure at any other point on the free surface of the liquid is

$$p = p_0 - P (a_z + g_z) f , \qquad \text{(IV.11)}$$

and the interface tension-curvature relation (IV.5) is

$$p = p_g - T \left\{ \frac{\partial}{\partial x} \frac{\frac{\partial f}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} + \frac{\partial}{\partial y} \frac{\frac{\partial f}{\partial y}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \right\} . \qquad \text{(IV.12)}$$

Combining formulae (IV.11, 12), we get

$$\left\{ \frac{\partial}{\partial x} \frac{\frac{\partial f}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} + \frac{\partial}{\partial y} \frac{\frac{\partial y}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \right\}_{P(x^*, y^*, z^*)}$$

$$- \left\{ \frac{\partial}{\partial x} \frac{\frac{\partial f}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} + \frac{\partial}{\partial y} \frac{\frac{\partial y}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \right\} + \frac{\rho (a_z + g_z)}{T} f = 0 . \qquad \text{(IV.13)}$$

This equation together with boundary conditions

$$\frac{\frac{\partial f}{\partial n}}{[1 + (\frac{\partial f}{\partial s})^2]^{\frac{1}{2}}} = \pm \cot \theta_c^*$$

and

$$\frac{\partial f}{\partial n} = C \text{ at } P(x^*, y^*, z^*)$$

completely determines $f(x, y)$. Since the surface is axisymmetric C can be taken to be zero by a suitable adjustment of the coordinates.

C. Linearized "Sloshing" Equations

Potential φ, wave height w, angular rotation $\dot{\theta}$ are now assumed to be infinitesimal and the appropriate equations are linearized with respect to them (see [7]). a_z is assumed constant and $f(x, y)$ and $\zeta(x, y, t)$ are not necessarily small. On combining formulae (IV.4, 5) and subtracting (IV.13), we get the linearized condition expressing the constancy of pressure at the free surface, namely,

$$\frac{\partial \varphi}{\partial t} + (a_z + g_z)\zeta - \frac{T}{\rho}\left\{\frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} - \frac{(\frac{\partial f}{\partial x})(\nabla f \cdot \nabla w)}{[1 + (\nabla f)^2]^{1.5}}\right]\right. \tag{IV.14}$$

$$\left. + \frac{\partial}{\partial y}\left[\frac{\frac{\partial f}{\partial y}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} - \frac{(\frac{\partial f}{\partial y})(\nabla f \cdot \nabla w)}{[1 + (\nabla f)^2]^{1.5}}\right]\right\} + (a_y + g_y)\, y = 0 ,$$

$$z = f(x, y) ,$$

where it is evaluated at the undisturbed free surface. Similarly, formula (IV.3_3) assumes the form

$$\nabla \varphi \quad \nabla(z - f) = 2\,\dot{\theta}\, y + w_t , \quad P \epsilon S , \quad z = f . \tag{IV.15}$$

The complete set of linearized equations is summarized below:

$$\nabla \varphi = 0 \quad , \quad P \epsilon \tau \tag{IV.16}$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad , \quad P \epsilon \Sigma_1 \text{ (or C)}$$

$$\frac{\partial \varphi}{\partial n} \equiv \frac{\partial \varphi}{\partial z} = 2\,\dot{\theta} , \quad P \epsilon \Sigma_2 , \; z = - h , \tag{IV.17}$$

$$\nabla \varphi \cdot \nabla(z - f) = 2\, y\, \dot{\theta} , \quad P \epsilon S , \; z = f ,$$

$$\frac{\partial \varphi}{\partial t} + (a_z + g_z)\zeta - \frac{T}{\rho}\left\{ \frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{[1+(\nabla f)^2]^{\frac{1}{2}}} - \frac{(\frac{\partial f}{\partial x})(\nabla f \cdot \nabla w)}{[1+(\nabla f)^2]^{1.5}} \right] \right. \tag{IV.18}$$

$$\left. + \frac{\partial}{\partial y}\left[\frac{\frac{\partial f}{\partial y}}{[1+(\nabla f)^2]^{\frac{1}{2}}} - \frac{(\frac{\partial f}{\partial y})(\nabla f \cdot \nabla w)}{[1+(\nabla f)^2]^{1.5}} \right] \right\} + (a_y + g_y)\, y = 0 ,$$

$$\frac{\partial w}{\partial n} = 0 , \quad P \epsilon C . \tag{IV.19}$$

These equations may be nondimensionalized to reflect the dependence on Bond numbers. This is done in [6], and we will not repeat it here.

D. Variational Formulation

To indicate how this is done, we shall consider only the case of a stationary vessel. The extension to a moving container poses no problems other than a great deal of routine algebra.

The kinetic energy of the liquid is

$$\text{K.E.} = \tfrac{1}{2}\rho \iiint_{\mathcal{T}(t)} (\nabla\varphi)^2 \, d\tau = \tfrac{1}{2}\rho \iint_{S(t)} \varphi(x, y, \zeta, t)\, \zeta_t \cos(n, z)\, dS \tag{IV.20}$$

$$= \tfrac{1}{2}\rho \iint_R \varphi(x, y, \zeta, t)\zeta_t \, dx\, dy$$

$$\approx \tfrac{1}{2}\rho \iint_R \varphi(x, y, f, t)\, w_t \, dx\, dy \ .$$

Similarly, the potential energy is

$$\Pi_g = \tfrac{1}{2}\rho\, g_z \iint_R \zeta^2 \, dx\, dy \tag{IV.21}$$

$$\Pi_s = T \left\{ \iint_{S(t)} dS - \iint_S dS \right\}$$

$$= T \left\{ \iint_R [1 + (\nabla\zeta)^2]^{\frac{1}{2}} \, dx\, dy - \iint_R [1 + (\nabla f)^2]^{\frac{1}{2}} \, dx\, dy \right\}$$

$$\approx \frac{1}{2} T \left\{ \iint_R \frac{(\nabla w)^2}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \, dx\,dy - \iint_R \frac{(\nabla f \cdot \nabla w)}{[1 + (\nabla f)^2]^{1.5}} \, dx\,dy \right\}$$

The Lagrangian can now be constructed and the usual variational methods may be used to obtain solutions.

E. Solution

The solution to (IV.16) satisfying boundary condition (IV.17) may be taken in the form

$$\varphi(x, y, z, t) = \sum_i (\alpha_i(t) \cosh k_i (z + h) + \gamma_i(t) \sinh k_i z) \, \varphi_i(x, y) \, , \qquad \text{(IV.22)}$$

if the infinitely many values k_i^2 ($i = 1, 2, \ldots$) are the values of k^2 (eigenvalues) for which the two-dimensional Helmholtz Equation

$$\Delta\varphi + k^2 \varphi = 0 \, , \quad P \epsilon R \qquad \text{(IV.23)}$$

has a non-zero solution satisfying

$$\frac{\partial\varphi}{\partial n} = 0 \, , \quad P \epsilon C \; . \qquad \text{(IV.24)}$$

Functions φ_i ($i = 1, 2, \ldots$) are the corresponding solutions (eigenfunctions) of (IV.23). Properties of the system of functions φ_i and the numbers k_i^2 are listed in (III.6-8).

Boundary condition (IV.19) is satisfied if we expand w in the series

$$w(x, y, t) = \sum_i \varphi_i(x, y) \, \xi_i(t) \; . \qquad \text{(IV.25)}$$

For (IV.22) to satisfy the boundary condition (IV.17_2) it is necessary that

$$\gamma_i(t) = \frac{2\,\theta\,(y, \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h}$$

For (IV.22) to satisfy the boundary condition (IV.17_3) it is necessary that

$$\dot{\xi}_i = \sum_j A_{ij} \, \alpha_j + 2\,\dot{\theta}\, B_i \, , \qquad \text{(IV.26)}$$

$$A_{ij} = \frac{1}{\|\varphi_i\|^2} \iint_R \frac{\sinh k_j (f + h)}{k_j} \nabla \varphi_i \cdot \nabla \varphi_j \, dx\, dy \ ,$$

$$B_i = \frac{1}{\|\varphi_i\|^2} \sum_j \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_R \frac{\cosh k_j f - \cosh k_j h}{k_j} \nabla \varphi_i \cdot \nabla \varphi_j \, dx\, dy \ .$$

Substituting (IV.22) into (IV.18), we find, after considerable manipulation,

$$\xi_i + \sum_j C_{ij} \alpha_j + \sum_j D_{ij} \xi_j + 2 \ddot{\theta} F_i + (a_y + g_y) G_i = 0 \ , \qquad \text{(IV.27)}$$

$$C_{ij} = \frac{1}{(a_z + g_z)\|\varphi_i\|^2} \iint_R \varphi_i \varphi_j \cosh k_j (f + h) \, dx\, dy \ ,$$

$$D_{ij} = \frac{T}{\rho} \frac{1}{(a_z + g_z)\|\varphi_i\|^2} \iint_R \left\{ \frac{\partial}{\partial x} \left[\frac{(\frac{\partial f}{\partial x})(\nabla f \cdot \nabla \varphi_j)}{[1 + (\nabla f)^2]^{1.5}} - \frac{\frac{\partial \varphi_j}{\partial x}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \right] + \frac{\partial}{\partial y} \left[\frac{(\frac{\partial f}{\partial y})(\nabla f \cdot \nabla \varphi_j)}{[1 + (\nabla f)^2]^{1.5}} - \frac{\frac{\partial \varphi_j}{\partial y}}{[1 + (\nabla f)^2]^{\frac{1}{2}}} \right] \right\} \varphi_i \, dx\, dy \ ,$$

$$F_i = \frac{1}{(a_z + g_z)\|\varphi_i\|^2} \sum_j \frac{(y, \varphi_j)}{k_j \|\varphi_j\|^2 \cosh k_j h} \iint_R \varphi_i \varphi_j \sinh k_j f \, dx\, dy \ ,$$

$$G_i = \frac{(y, \varphi_i)}{(a_z + g_z)\|\varphi_i\|^2} \ , \quad (a_z \text{ assumed constant}) \ .$$

Formulae (IV.26$_1$) and (IV.27$_1$) are sufficient to determine the generalized coordinates ξ_i, α_i. We could eliminate ξ_i by differentiating (IV.27$_1$) and substituting for $\dot{\xi}_i$, etc., from (IV.26$_1$). However, this procedure increases the order of the time derivative of $\ddot{\theta}$ and a_y. To avoid this, we can solve for α_i in (IV.26$_1$) and substitute the results in (IV.27$_1$). We assume A_{ij} to possess an inverse; this can be shown to be true if we consider a finite number of terms. The resulting equations can then be cast into the form

$$\ddot{\xi}_i + a_i \xi_i + \sum_{\substack{j \\ i \neq j}} (c_{ij} \ddot{\xi}_j + d_{ij} \xi_j) + 2 \ddot{\theta} f_i + a_y g_i = 0 \ , \qquad \text{(IV.28)}$$

$i = 1, 2, \ldots,$

in which a_i, c_{ij}, ... are numerical constants, and we suppose $g_y = 0$.

Let us compute the steady state response of system (IV.28) with driving function a_y, $\ddot{\theta}$ defined by

$$a_y = \ddot{X}\ , \quad X = X_0 \cos \omega t\ , \quad \ddot{X} = -\omega^2 X_0 \cos \omega t$$

$$\theta = \theta_0 \cos \omega t\ , \quad \ddot{\theta} = -\omega^2 \theta_0 \cos \omega t\ .$$

Then, letting $\xi_i = \xi_i^* \cos \omega t$, and solving for the ξ_i^*, we obtain the following general form:

$$\xi_i^* = \left\{ \frac{H_{1i}\, \omega^{2i} + H_{2i}\, \omega^{2i-2} + \ldots + H_{mi}\, \omega^2}{\omega^{2i} + H_1\, \omega^{2i-2} + \ldots + H_{m-1}\, \omega^2 + H_m} \right\} 2\, \theta_0$$

$$+ \left\{ \frac{K_{1i}\, \omega^{2i} + K_{2i}\, \omega^{2i-2} + \ldots + K_{mi}\, \omega^2}{\omega^{2i} + H_1\, \omega^{2i-2} + \ldots + H_{m-1}\, \omega^2 + H_m} \right\} X_0 \qquad \text{(IV.29)}$$

H_{ij}, K_{ij}, H_i are various products of constants occurring in (IV.28). Formula (IV.29) can be put in a more meaningful form in accordance with the theory of partial fractions:

$$\xi_i^* = \left\{ \frac{Q_{1i}}{\omega^2 - \omega_1^2} + \frac{Q_{2i}}{\omega^2 - \omega_2^2} + \ldots + \frac{Q_{mi}}{\omega^2 - \omega_m^2} \right\} 2\, \omega^2\, \theta_0$$

$$+ \left\{ \frac{P_{1i}}{\omega^2 - \omega_1^2} + \frac{P_{2i}}{\omega^2 - \omega_2^2} + \ldots + \frac{P_{mi}}{\omega^2 - \omega_m^2} \right\} \omega^2 X_0 \qquad \text{(IV.30)}$$

ω_i^2 is the square of the natural frequency of the i^{th} sloshing mode. These frequencies are ordered in such a way that $\omega_1^2 < \omega_2^2 < \ldots < \omega_m^2$, and they are real, $\omega_i^2 > 0$.

Substituting these results in formula (IV.22) gives the velocity potential

$$\varphi = -2\, \omega\, \theta_0 \sin \omega t \sum_i \left\{ \omega^2 \sum_j \sum_n \Gamma_{ij} \frac{Q_{nj}}{\omega^2 - \omega_j^2}\, \varphi_i \cosh k_i (z + h) \right.$$

$$+\varphi_i \left[B_i^* \cosh k_i (z+h) + \frac{(y,\ \varphi_i)}{k_i \|\varphi_i\|^2 \cosh k_i h} \sinh k_i z \right] \}$$

$$-\omega^3 X_0 \sin \omega t \sum_i \sum_j \sum_n \Gamma_{ij} \frac{P_{nj}}{\omega^2 - \omega_j^2} \varphi_i \cosh k_i (z+h) , \qquad \text{(IV.31)}$$

where Γ_{ij} is the inverse of A_{ij} in (IV.26$_1$) and $B_i^* = \sum_j \Gamma_{ij} B_j$.

With the velocity potential known, it is a routine job to compute the pressure distribution, forces and moments. A mechanical model which duplicates the forces and moments resulting from the action of the liquid on the vessel walls may be devised in a manner similar to that of [6].

F. Case of a Circular Cylinder

To illustrate the use of the theory, consider a right circular cylinder.

Introduce cylindrical coordinates $(r,\ \beta,\ z)$, such that

$$x = r \cos \beta ,$$

$$y = r \sin \beta ,$$

$$z = z .$$

Then, the problem is completely defined by the following differential system:

$$\Delta \varphi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \beta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 , \quad P \epsilon \tau , \qquad \text{(IV.32)}$$

$$\frac{\partial \varphi}{\partial r} = 0, \quad r = R_0 ,$$

$$\frac{\partial \varphi}{\partial z} = 2 \dot{\theta} r \sin \beta , \quad z = -h ,$$

$$\frac{\partial \varphi}{\partial z} - \frac{df}{dr} \left(\frac{\partial \varphi}{\partial r} \right) = w_t + 2 \dot{\theta} r \sin \beta , \quad z = f ,$$

$$\frac{\partial \varphi}{\partial t} + (a_z + g_z) w - \frac{T}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[\frac{r \frac{\partial w}{\partial r}}{[1 + (\frac{df}{dr})^2]^{1.5}} \right] + \frac{1}{r^2} \frac{\partial}{\partial \beta} \left[\frac{\frac{\partial w}{\partial \beta}}{[1 + (\frac{df}{dr})^2]^{\frac{1}{2}}} \right] \right\}$$

$$+ a_y \, r \sin\beta = 0 \quad , \; z = f \; ,$$

in which the axisymmetric equilibrium interface is determined from the solution of

$$\left\{ \frac{1}{r} \frac{d}{dr} \left[\frac{r \frac{df}{dr}}{[1 + (\frac{df}{dr})^2]^{\frac{1}{2}}} \right] \right\}_{r=0} - \frac{1}{r} \frac{d}{dr} \left[\frac{r \frac{df}{dr}}{[1 + (\frac{df}{dr})^2]^{\frac{1}{2}}} \right] + \rho \frac{(a_z + g_z)}{T} f = 0 \; ,$$

$$f = \frac{df}{dr} = 0 \; , \quad r = 0 \; ,$$

$$\frac{df}{dr} = \cot\theta_c \; , \quad r = R_O \; . \qquad \text{(IV.33)}$$

If we assume that the contact angle is zero,

$$\frac{df}{dr} = \cot\,\theta_c = \infty \;, \; r = R_O \;,$$

then we can approximate the solution of (IV.33) by

$$f(r) = \beta \, R_O \, [1 - (1 - \frac{r^3}{R_O^3})^{\frac{1}{2}}] \qquad \text{(IV.34)}$$

with

$$\beta^3 N_{bo} - \beta^2 - 2/3 = 0 \; ,$$

where

$$N_{bo} = \frac{\rho (a_z + g_z) R_O^2}{T} \quad \text{(Bond number)} \; ,$$

as indicated in [6] . . This solution is valid for $N_{bo} > 10$.

The solution of (IV.32_1) satisfying condition (IV.32_2) is

$$\varphi(r, \beta, z, t) = \sum_i (\alpha_i(t) \cosh k_i (z + h) + \gamma_i(t) \sinh k_i z) J_1 (\lambda_i r) \sin\beta$$

where $J_1'(\lambda_i R_O) = 0$. The remainder of the analysis is straightforward and it is not necessary to repeat it here.

V. CONCLUDING REMARKS

A general analytical method (Analysis (1)) has been formulated for determining the motion of a heavy liquid enclosed in a prismatic cylinder (which is itself in motion) under the joint action of gravitational, surface, and interfacial tension forces. The method is an extension of [4], and is valid only for 90° contact angles. An automatic numerical procedure can be developed to integrate the resulting ordinary non-linear differential equations in exactly the same manner as in [4]. This would give the motion of the liquid, free surface, pressure distribution, forces and moments as functions of time. It appears that the analysis can be extended to include contact angles other than 90°. It should be emphasized that the entire analysis is numerical and is extremely time consuming.

In addition, an analysis (Analysis (2)) is given for a liquid sloshing in a prismatic cylinder under the state of moderately low effective body forces. The procedure consisted of perturbing the free surface, velocity of the liquid and motion of the tank about a finite equilibrium state. This gives rise to a system of ordinary linear differential, which may be solved in a routine manner. The results may be cast in a form of an equivalent mechanical model. Following [6], the theory is valid for Bond number > 10. However, by using different laws for the equilibrium interface, the analysis would be valid for lower Bond numbers. The method may readily be extended to vessels possessing rotational symmetry (but otherwise arbitrary), and programmed in a fashion similar to that of [8] for high gravity problems.

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