Equations for Runge-Kutta Formulas Through the Eighth Order

by
H. A. Luther and J. A. Smitherman
Mathematics Department, Texas A&M University

A Report Supported by
The National Aeronautics and Space Administration
Research Grant No. NGR-44-001-024

Equations for Runge-Kutta Formulas Through the Eighth Order

by
H. A. Luther and J. A. Smitherman
Mathematics Department, Texas A&M University

A Report Supported by
The National Aeronautics and Space Administration
Research Grant No. NGR-44-001-024

Equations for Runge-Kutta Formulas Through the Eighth Order*

H. A. Luther and J. A. Smitherman†

The Runge-Kutta expressions considered are to be both the explicit and the implicit. The notation chosen is as follows:

$$y_{n+1} = y_n + \sum_{i=1}^{\nu} R_i k_i ,$$

$$k_i = h\, f(x_n + a_i\, h,\ y_n + \sum_{j=1}^{\nu} b_{ij} k_j) \tag{1}$$

$$a_i = \sum_{j=1}^{\nu} b_{ij} .$$

The differential system is of course

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{2}$$

Here y is a column matrix of length m, as is f(x,y), so that we are considering m simultaneous first order ordinary equations in the variable x. The constant ν may be any integer as large, or larger, than m.

The requirement $\nu \geq m$ should be considered as pertaining only to formal presentation. If (1) is to properly approximate an explicit solution of (2) in the Runge-Kutta sense, for $m > 4$ we must have $\nu > m$. A more precise description has been given by Butcher [2].

The following tables are based on the exposition of Butcher [1]; the elementary weights such as $[\phi]$, $[[\phi]^2]$, etc., found in the tables, are those defined by him.

Table One can be used to write down the equations which the k_i, R_i, and b_{ij} must satisfy. Each elementary weight is to be equated to the proper constant as described in [1]. For a specific choice of R_i, k_i, and b_{ij} in (1), and for a remainder of order p+1 (a Runge-Kutta formula of order p) all relations listed through the first p orders must be

* This report was supported by the National Aeronautics and Space Administration under Research Grant No. NGR-44-001-024.

† Texas A&M University, College Station, Texas

satisfied. Thus to get a third order R-K formula we must satisfy the restrictions implied by the first-order relation, the second-order relation, and the two third-order relations, namely

$$1 = \sum_{i_1=1}^{\nu} R_{i_1}\,; \quad 1/2 = \sum_{i_1=1}^{\nu} R_{i_1} a_{i_1}\,; \quad 1/6 = \sum_{i_1=1}^{\nu} \sum_{i_2=1}^{\nu} R_{i_1} b_{i_1 i_2} a_{i_2}\,;$$

$$1/3 = \sum_{i_1=1}^{\nu} R_{i_1} a_{i_1}^{\,2}.$$

The summation convention is of course employed in the table, and all indices extend from 1 to ν. It is observed that these equations serve to define an implicit formulation, in general. That is, in applying (1), the k_i are defined implicitly.

If we require $b_{ij} = 0$ for $j \geq i$, then (1) defines each k_i explicitly. Table Two is based on this assumption. For this table, ν has been arbitrarily taken as 9. As explained in [2], this choice gives more parameters than needed for orders one through six. It gives a minimum number for order seven, and is not sufficient for order eight. However, the pattern seems clear, and there is no difficulty in increasing (or decreasing) the value of ν for the orders shown. Certain abbreviations for Table Two are found at its beginning. The equations listed are for orders one through seven only.

REFERENCES

[1]. Butcher, J. C., Coefficients for the Study of Runge-Kutta Integration Processes, *J. Austral. Math. Soc.* 3(1963), 185-201.

[2]. _____, On the Attainable Order of Runge-Kutta Methods, *Math. Comp.* 19(1965), 408-417.

TABLE ONE

The Elementary Weights for the Runge-Kutta Formulas
of the First Eight Orders

FIRST ORDER

1. $\phi = \sum_{i_1=1}^{\nu} R_{i_1}$

SECOND ORDER

1. $[\phi] = R_{i_1} a_{i_1}$

THIRD ORDER

1. $[{}_2\phi]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}$

2. $[\phi^2] = R_{i_1} a_{i_1}{}^2$

FOURTH ORDER

1. $[{}_3\phi]_3 = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_3}$

2. $[{}_2\phi^2]_2 = R_{i_1} b_{i_1 i_2} a_{i_2}{}^2$

3. $[[\phi]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}$

4. $[\phi^3] = R_{i_1} a_{i_1}{}^3$

FIFTH ORDER

1. $[{}_4\phi]_4 = R_{i_1}{}^{b_{i_1 i_2}} {}^{b_{i_2 i_3}} {}^{b_{i_3 i_4}} {}^{a_{i_4}}$

2. $[{}_3\phi^2]_3 = R_{i_1}{}^{b_{i_1 i_2}} {}^{b_{i_2 i_3}} {}^{a_{i_3}}{}^2$

3. $[{}_2[\phi]\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}} {}^{b_{i_2 i_3}} {}^{a_{i_2}} {}^{a_{i_3}}$

4. $[{}_2\phi^3]_2 = R_{i_1}{}^{b_{i_1 i_2}} {}^{a_{i_2}}{}^3$

5. $[[{}_2\phi]_2\phi] = R_{i_1}{}^{b_{i_1 i_2}} {}^{b_{i_2 i_3}} {}^{a_{i_1}} {}^{a_{i_3}}$

6. $[[\phi^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}} {}^{a_{i_1}} {}^{a_{i_2}}{}^2$

7. $[[\phi]^2] = R_{i_1}{}^{(b_{i_1 i_2}} {}^{a_{i_2}})^2$

8. $[[\phi]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}} {}^{a_{i_1}}{}^2 {}^{a_{i_2}}$

9. $[\phi^4] = R_{i_1}{}^{a_{i_1}}{}^4$

SIXTH ORDER

1. $[_5\phi]_5 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, b_{i_3 i_4}\, b_{i_4 i_5}\, a_{i_5}$

2. $[_4\phi^2]_4 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, b_{i_3 i_4}\, a_{i_4}^{\,2}$

3. $[_3[\phi]\phi]_3 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, b_{i_3 i_4}\, a_{i_3}\, a_{i_4}$

4. $[_3\phi^3]_3 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, a_{i_3}^{\,3}$

5. $[_2[_2\phi]_2]_2 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, b_{i_3 i_4}\, a_{i_2}\, a_{i_4}$

6. $[_2[\phi^2]\phi]_2 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, a_{i_2}\, a_{i_3}^{\,2}$

7. $[_2[\phi]^2]_2 = R_{i_1}\, b_{i_1 i_2}\, (b_{i_2 i_3}\, a_{i_3})^2$

8. $[_2[\phi]\phi^2]_2 = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, a_{i_2}^{\,2}\, a_{i_3}$

9. $[_2\phi^4]_2 = R_{i_1}\, b_{i_1 i_2}\, a_{i_2}^{\,4}$

10. $[[_3\phi]_3\phi] = R_{i_1}\, b_{i_1 i_2}\, b_{i_2 i_3}\, b_{i_3 i_4}\, a_{i_1}\, a_{i_4}$

11. $[[_2\phi^2]_2\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_3}^2$

12. $[[[\phi]\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1} a_{i_2} a_{i_3}$

13. $[[\phi^3]\phi] = R_{i_1} b_{i_1 i_2} a_{i_1} a_{i_2}^3$

14. $[[_2\phi]_2[\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3} a_{i_4}$

15. $[[\phi^2][\phi]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_2}^3 a_{i_3}$

16. $[[_2\phi]_2\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^2 a_{i_3}$

17. $[[\phi^2]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}^2 a_{i_2}^2$

18. $[[\phi]^2\phi] = R_{i_1} a_{i_1} (b_{i_1 i_2} a_{i_2})^2$

19. $[[\phi]\phi^3] = R_{i_1} b_{i_1 i_2} a_{i_1}^3 a_{i_2}$

20. $[\phi^5] = R_{i_1} a_{i_1}^5$

SEVENTH ORDER

1. $[_6\phi]_6 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{b}{}_{i_5 i_6}{}^{a}{}_{i_6}$

2. $[_5\phi^2]_5 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_5}{}^{2}$

3. $[_4[\phi]\phi]_4 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_4}{}^{a}{}_{i_5}$

4. $[_4\phi^3]_4 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_4}{}^{3}$

5. $[_3[_2\phi]_2\phi]_3 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_3}{}^{a}{}_{i_5}$

6. $[_3[\phi^2]\phi]_3 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_3}{}^{a}{}_{i_4}{}^{2}$

7. $[_3[\phi]^2]_3 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}\ (b_{i_3 i_4}{}^{a}{}_{i_4})^2$

8. $[_3[\phi]\phi^2]_3 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_3}{}^{2}{}^{a}{}_{i_4}$

9. $[_3\phi^4]_3 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{a}{}_{i_3}{}^{4}$

10. $[_2[_3\phi]_3\phi]_2 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_2}{}^{a}{}_{i_5}$

11. $[_2[_2\phi^2]_2\phi]_2 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_2}{}^{a}{}_{i_4}{}^{2}$

12. $[_2[[\phi]\phi]\phi]_2 = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_2}{}^{a}{}_{i_3}{}^{a}{}_{i_4}$

13. $[_2[\phi^3]\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{3}$

14. $[_2[_2\phi]_2[\phi]]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_4}}{}^{a_{i_5}}$

15. $[_2[\phi^2][\phi]]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}$

16. $[_2[_2\phi]_2\phi^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_2}}{}^{2}{}^{a_{i_4}}$

17. $[_2[\phi^2]\phi^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{2}$

18. $[_2[\phi]^2\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_2}}(b_{i_2 i_3}{}^{a_{i_3}})^2$

19. $[_2[\phi]\phi^3]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{3}{}^{a_{i_3}}$

20. $[_2\phi^5]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_2}}{}^{5}$

21. $[[_4\phi]_4\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_1}}{}^{a_{i_5}}$

22. $[[_3\phi^2]_3\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{a_{i_4}}{}^{2}$

23. $[[_2[\phi]\phi]_2\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{a_{i_3}}{}^{a_{i_4}}$

24. $[[_2\phi^3]_2\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_3}}{}^{3}$

25. $[[[_2\phi]_2\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{a_{i_4}}$

26. $[[[\phi^2]\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{2}$

27. $[[[\phi]^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}(b_{i_2 i_3}{}^{a_{i_3}})^2$

28. $[[[\phi]\phi^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{2}a_{i_3}$

29. $[[\phi^4]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{4}$

30. $[[_3\phi]_3[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{a_{i_5}}$

31. $[[_2\phi^2]_2[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{2}$

32. $[[[\phi]\phi][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_4}}$

33. $[[\phi^3][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{a_{i_2}}{}^{3}a_{i_3}$

34. $[[_3\phi]_3\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{2}a_{i_4}$

35. $[[_2\phi^2]_2\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{2}a_{i_3}{}^{2}$

36. $[[[\phi]\phi]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{2}a_{i_2}{}^{a_{i_3}}$

37. $[[\phi^3]\phi^2] = R_{i_1} b_{i_1 i_2} a_{i_1}{}^2 a_{i_2}{}^3$

38. $[[{}_2\phi]_2{}^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} b_{i_3 i_5} a_{i_4} a_{i_5}$

39. $[[{}_2\phi]_2[\phi^2]] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_3}{}^2 a_{i_4}$

40. $[[\phi^2]^2] = R_{i_1} (b_{i_1 i_2} a_{i_2}{}^2)^2$

41. $[[{}_2\phi]_2[\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} b_{i_2 i_4} a_{i_1} a_{i_3} a_{i_4}$

42. $[[\phi^2][\phi]\phi] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_1} a_{i_2}{}^2 a_{i_3}$

43. $[[{}_2\phi]_2\phi^3] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}{}^3 a_{i_3}$

44. $[[\phi^2]\phi^3] = R_{i_1} b_{i_1 i_2} a_{i_1}{}^3 a_{i_2}{}^2$

45. $[[\phi]^3] = R_{i_1} (b_{i_1 i_2} a_{i_2})^3$

46. $[[\phi]^2\phi^2] = R_{i_1} a_{i_1}{}^2 (b_{i_1 i_2} a_{i_2})^2$

47. $[[\phi]\phi^4] = R_{i_1} b_{i_1 i_2} a_{i_1}{}^4 a_{i_2}$

48. $[\phi^6] = R_{i_1} a_{i_1}{}^6$

EIGHTH ORDER

1. $[_7\phi]_7 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{b_{i_6 i_7}}{}^{a_{i_7}}$

2. $[_6\phi^2]_6 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_6}}{}^{2}$

3. $[_5[\phi]\phi]_5 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_5}}{}^{a_{i_6}}$

4. $[_5\phi^3]_5 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_5}}{}^{3}$

5. $[_4[_2\phi]_2\phi]_4 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_4}}{}^{a_{i_6}}$

6. $[_4[\phi^2]\phi]_4 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_4}}{}^{a_{i_5}}{}^{2}$

7. $[_4[\phi]^2]_4 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{(b_{i_4 i_5}}{}^{a_{i_5}})^2$

8. $[_4[\phi]\phi^2]_4 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_4}}{}^{2}{}^{a_{i_5}}$

9. $[_4\phi^4]_4 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_4}}{}^{4}$

10. $[_3[_3\phi]_3\phi]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_3}}{}^{a_{i_6}}$

11. $[_3[_2\phi^2]_2\phi]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{a_{i_5}}{}^{2}$

12. $[_3[[\phi]\phi]\phi]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{a_{i_5}}$

13. $[_3[\phi^3]\phi]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{3}$

14. $[_3[_2\phi]_2[\phi]]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_3 i_5}}{}^{b_{i_4 i_6}}{}^{a_{i_5}}{}^{a_{i_6}}$

15. $[_3[\phi^2][\phi]]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_4}}{}^{2}{}^{a_{i_5}}$

16. $[_3[_2\phi]_2\phi^2]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{2}{}^{a_{i_5}}$

17. $[_3[\phi]^2\phi^2]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}{}^{2}$

18. $[_3[\phi]^2\phi]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_3}}(b_{i_3 i_4}{}^{a_{i_4}})^2$

19. $[_3[\phi]\phi^3]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_3}}{}^{3}{}^{a_{i_4}}$

20. $[_3\phi^5]_3 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_3}}{}^{5}$

21. $[_2[_4\phi]_4\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_2}}{}^{a_{i_6}}$

22. $[_2[_3\phi^2]_3\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_2}}{}^{a_{i_5}}{}^{2}$

23. $[_2[_2[\phi]\phi]_2\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_2}}{}^{a_{i_4}}{}^{a_{i_5}}$

24. $[_2[_2\phi^3]_2\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_2}}{}^{a_{i_4}}{}^{3}$

25. $[_2[[_2\phi]_2\phi]\phi]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{b_{i_4}i_5}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_5}}$

26. $[_2[[\phi^2]\phi]\phi]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{2}$

27. $[_2[[\phi]^2]\phi]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{a_{i_2}}{}^{(b_{i_3}i_4}{}^{a_{i_4}})^2$

28. $[_2[[\phi]\phi^2]\phi]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{a_{i_2}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}$

29. $[_2[\phi^4]\phi]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{a_{i_2}}{}^{a_{i_3}}{}^{4}$

30. $[_2[_3\phi]_3[\phi]]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_2}i_4}{}^{b_{i_3}i_5}{}^{b_{i_5}i_6}{}^{a_{i_4}}{}^{a_{i_6}}$

31. $[_2[_2\phi^2]_2[\phi]]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_2}i_4}{}^{b_{i_3}i_5}{}^{a_{i_4}}{}^{a_{i_5}}{}^{2}$

32. $[_2[[\phi]\phi][\phi]]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_2}i_4}{}^{b_{i_3}i_5}{}^{a_{i_3}}{}^{a_{i_4}}{}^{a_{i_5}}$

33. $[_2[\phi^3][\phi]]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_2}i_4}{}^{a_{i_3}}{}^{3}{}^{a_{i_4}}$

34. $[_2[_3\phi]_3\phi^2]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{b_{i_4}i_5}{}^{a_{i_2}}{}^{2}{}^{a_{i_5}}$

35. $[_2[_2\phi^2]_2\phi^2]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{a_{i_2}}{}^{2}{}^{a_{i_4}}{}^{2}$

36. $[_2[[\phi]\phi]\phi^2]_2 = R_{i_1}{}^{b_{i_1}i_2}{}^{b_{i_2}i_3}{}^{b_{i_3}i_4}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{a_{i_4}}$

37. $[_2[\phi^3]\phi^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{3}$

38. $[_2[_2\phi]_2{}^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{b_{i_4 i_6}}{}^{a_{i_5}}{}^{a_{i_6}}$

39. $[_2[_2\phi]_2[\phi^2]]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_4}}{}^{2}{}^{a_{i_5}}$

40. $[_2[\phi^2]^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}{}^{2}$

41. $[_2[_2\phi]_2[\phi]\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_2}}{}^{a_{i_4}}{}^{a_{i_5}}$

42. $[_2[\phi^2][\phi]\phi]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}$

43. $[_2[_2\phi]_2\phi^3]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_2}}{}^{3}{}^{a_{i_4}}$

44. $[_2[\phi^2]\phi^3]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{3}{}^{a_{i_3}}{}^{2}$

45. $[_2[\phi]^3]_2 = R_{i_1}{}^{b_{i_1 i_2}}(b_{i_2 i_3}{}^{a_{i_3}})^3$

46. $[_2[\phi]^2\phi^2]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_2}}{}^{2}(b_{i_2 i_3}{}^{a_{i_3}})^2$

47. $[_2[\phi]\phi^4]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_2}}{}^{4}{}^{a_{i_3}}$

48. $[_2\phi^6]_2 = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_2}}{}^{6}$

49. $[[_5\phi]_5\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{b}{}_{i_5 i_6}{}^{a}{}_{i_1}{}^{a}{}_{i_6}$

50. $[[_4\phi^2]_4\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_1}{}^{a}{}_{i_5}{}^{2}$

51. $[[_3[\phi]\phi]_3\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_1}{}^{a}{}_{i_4}{}^{a}{}_{i_5}$

52. $[[_3\phi^3]_3\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_4}{}^{3}$

53. $[[_2[_2\phi]_2\phi]_2\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{a}{}_{i_5}$

54. $[[_2[\phi^2]\phi]_2\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{a}{}_{i_4}{}^{2}$

55. $[[_2[\phi]^2]_2\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{a}{}_{i_1}(b_{i_3 i_4}{}^{a}{}_{i_4})^2$

56. $[[_2[\phi]\phi^2]_2\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{2}{}^{a}{}_{i_4}$

57. $[[_2\phi^4]_2\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{4}$

58. $[[[_3\phi]_3\phi]\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_1}{}^{a}{}_{i_2}{}^{a}{}_{i_5}$

59. $[[[_2\phi^2]_2\phi]\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_2}{}^{a}{}_{i_4}{}^{2}$

60. $[[[[\phi]\phi]\phi]\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_2}{}^{a}{}_{i_3}{}^{a}{}_{i_4}$

61. $[[[\phi^3]\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{3}$

62. $[[[{}_2\phi]_2[\phi]]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_1}}{}^{a_{i_4}}{}^{a_{i_5}}$

63. $[[[\phi^2][\phi]]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_1}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}$

64. $[[[{}_2\phi]_2\phi^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{2}{}^{a_{i_4}}$

65. $[[[\phi^2]\phi^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{2}$

66. $[[[\phi]^2\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}{}^{a_{i_2}}(b_{i_2 i_3}{}^{a_{i_3}})^2$

67. $[[[\phi]\phi^3]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{3}{}^{a_{i_3}}$

68. $[[\phi^5]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{5}$

69. $[[{}_4\phi]_4[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_4 i_5}}{}^{b_{i_5 i_6}}{}^{a_{i_3}}{}^{a_{i_6}}$

70. $[[{}_3\phi^2]_3[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{a_{i_5}}{}^{2}$

71. $[[{}_2[\phi]\phi]_2[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{a_{i_5}}$

72. $[[{}_2\phi^3]_2[\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{3}$

73. $[[[_2\phi]_2\phi][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_5}}$

74. $[[[\phi^2]\phi][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_4}{}^2}$

75. $[[[\phi]^2][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{a_{i_3}}(b_{i_2 i_4}{}^{a_{i_4}})^2$

76. $[[[\phi]\phi^2][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{a_{i_4}}$

77. $[[\phi^4][\phi]] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{a_{i_2}}{}^{4}{}^{a_{i_3}}$

78. $[[_4\phi]_4\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{b_{i_4 i_5}}{}^{a_{i_1}}{}^{2}{}^{a_{i_5}}$

79. $[[_3\phi^2]_3\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{2}{}^{a_{i_4}}{}^{2}$

80. $[[_2[\phi]\phi]_2\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{2}{}^{a_{i_3}}{}^{a_{i_4}}$

81. $[[_2\phi^3]_2\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{2}{}^{a_{i_3}}{}^{3}$

82. $[[[_2\phi]_2\phi]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{2}{}^{a_{i_2}}{}^{a_{i_4}}$

83. $[[[\phi^2]\phi]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{2}{}^{a_{i_2}}{}^{a_{i_3}}{}^{2}$

84. $[[[\phi]^2]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}{}^{2}(b_{i_2 i_3}{}^{a_{i_3}})^2$

85. $[[[\phi]\phi^2]\phi^2] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_2 i_3}{}^{a}{}_{i_1}{}^{2}{}_{a}{}_{i_2}{}^{2}{}_{a}{}_{i_3}$

86. $[[\phi^4]\phi^2] = R_{i_1}{}^{a}{}_{i_1}{}^{3}{}_{a}{}_{i_2}{}^{4}$

87. $[[_3\phi]_3[_2\phi]_2] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{b}{}_{i_4 i_5}{}^{b}{}_{i_3 i_6}{}^{a}{}_{i_5}{}^{a}{}_{i_6}$

88. $[[_3\phi]_3[\phi^2]] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_3}{}^{2}{}_{a}{}_{i_5}$

89. $[[_2\phi^2]_2[_2\phi]_2] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{b}{}_{i_3 i_5}{}^{a}{}_{i_4}{}^{2}{}_{a}{}_{i_5}$

90. $[[_2\phi^2]_2[\phi^2]] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{a}{}_{i_3}{}^{2}{}_{a}{}_{i_4}{}^{2}$

91. $[[[\phi]\phi][_2\phi]_2] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{b}{}_{i_3 i_5}{}^{a}{}_{i_2}{}^{a}{}_{i_4}{}^{a}{}_{i_5}$

92. $[[[\phi]\phi][\phi^2]] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{a}{}_{i_2}{}^{a}{}_{i_3}{}^{2}{}_{a}{}_{i_4}$

93. $[[\phi^3][_2\phi]_2] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_3 i_4}{}^{a}{}_{i_2}{}^{3}{}_{a}{}_{i_4}$

94. $[[\phi^3][\phi^2]] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{a}{}_{i_2}{}^{3}{}_{a}{}_{i_3}{}^{2}$

95. $[[_3\phi]_3[\phi]\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{b}{}_{i_4 i_5}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{a}{}_{i_5}$

96. $[[_2\phi^2]_2[\phi]\phi] = R_{i_1}{}^{b}{}_{i_1 i_2}{}^{b}{}_{i_1 i_3}{}^{b}{}_{i_2 i_4}{}^{a}{}_{i_1}{}^{a}{}_{i_3}{}^{a}{}_{i_4}{}^{2}$

97. $[[[\phi]\phi][\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{a_{i_3}}{}^{a_{i_4}}$

98. $[[\phi^3][\phi]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{a_{i_1}}{}^{a_{i_2}}{}^{3}{}^{a_{i_3}}$

99. $[[{}_3\phi]_3\phi^3] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{b_{i_3 i_4}}{}^{a_{i_1}}{}^{3}{}^{a_{i_4}}$

100. $[[{}_2\phi^2]_2\phi^3] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{3}{}^{a_{i_3}}{}^{2}$

101. $[[[\phi]\phi]\phi^3] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_2 i_3}}{}^{a_{i_1}}{}^{3}{}^{a_{i_2}}{}^{a_{i_3}}$

102. $[[\phi^3]\phi^3] = R_{i_1}{}^{b_{i_1 i_2}}{}^{a_{i_1}}{}^{3}{}^{a_{i_2}}{}^{3}$

103. $[[{}_2\phi]_2{}^2\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{b_{i_3 i_5}}{}^{a_{i_1}}{}^{a_{i_4}}{}^{a_{i_5}}$

104. $[[{}_2\phi]_2[\phi^2]\phi] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_1}}{}^{a_{i_3}}{}^{2}{}^{a_{i_4}}$

105. $[[\phi^2]^2\phi] = R_{i_1}{}^{a_{i_1}}(b_{i_1 i_2}{}^{a_{i_2}}{}^{2})^2$

106. $[[{}_2\phi]_2[\phi]^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_1 i_4}}{}^{b_{i_2 i_5}}{}^{a_{i_3}}{}^{a_{i_4}}{}^{a_{i_5}}$

107. $[[\phi^2][\phi]^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_1 i_4}}{}^{a_{i_2}}{}^{2}{}^{a_{i_3}}{}^{a_{i_4}}$

108. $[[{}_2\phi]_2[\phi]\phi^2] = R_{i_1}{}^{b_{i_1 i_2}}{}^{b_{i_1 i_3}}{}^{b_{i_2 i_4}}{}^{a_{i_1}}{}^{2}{}^{a_{i_3}}{}^{a_{i_4}}$

109. $[[\phi^2][\phi]\phi^2] = R_{i_1} b_{i_1 i_2} b_{i_1 i_3} a_{i_1}^{\ 2} a_{i_2}^{\ 2} a_{i_3}$

110. $[[{}_2\phi]_2\phi^4] = R_{i_1} b_{i_1 i_2} b_{i_2 i_3} a_{i_1}^{\ 4} a_{i_3}$

111. $[[\phi^2]\phi^4] = R_{i_1} b_{i_1 i_2} a_{i_1}^{\ 4} a_{i_2}^{\ 2}$

112. $[[\phi]^3\phi] = R_{i_1} a_{i_1} (b_{i_1 i_2} a_{i_2})^3$

113. $[[\phi]^2\phi^3] = R_{i_1} a_{i_1}^{\ 3} (b_{i_1 i_2} a_{i_2})^2$

114. $[[\phi]\phi^5] = R_{i_1} b_{i_1 i_2} a_{i_1}^{\ 5} a_{i_2}$

115. $[\phi^7] = R_{i_1} a_{i_1}^{\ 7}$

TABLE TWO

Runge-Kutta Relations
for Explicit Formulas Through Order Seven

The following notation is employed:

$$c_i = \sum_{j=2}^{i+1} a_j b_{i+2,j} \qquad d_i = \sum_{j=2}^{i+1} a_j^2 b_{i+2,j}$$

$$e_i = \sum_{j=2}^{i+1} a_j^3 b_{i+2,j} \qquad f_i = \sum_{j=2}^{i+1} a_j^4 b_{i+2,j}$$

$$g_i = \sum_{j=2}^{i+1} a_j^5 b_{i+2,j}$$

FIRST ORDER

$$\phi = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 + R_9 = 1$$

SECOND ORDER

$$[\phi] = a_2R_2 + a_3R_3 + a_4R_4 + a_5R_5 + a_6R_6 + a_7R_7 + a_8R_8 + a_9R_9 = \frac{1}{2}$$

THIRD ORDER

$$[_2\phi]_2 = c_1R_3 + c_2R_4 + c_3R_5 + c_4R_6 + c_5R_7 + c_6R_8 + c_7R_9 = \frac{1}{6}$$

$$[\phi^2] = a_2{}^2R_2 + a_3{}^2R_3 + a_4{}^2R_4 + a_5{}^2R_5 + a_6{}^2R_6 + a_7{}^2R_7 + a_8{}^2R_8 + a_9{}^2R_9 = \frac{1}{3}$$

FOURTH ORDER

$$[{}_3\phi]_3 = c_1b_{43}R_4 + (c_1b_{53} + c_2b_{54})R_5 + (c_1b_{63} + c_2b_{64} + c_3b_{65})R_6 + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})R_7 + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})R_8 + (c_1b_{93} + c_2b_{94} + c_3b_{95} + c_4b_{96} + c_5b_{97} + c_6b_{98})R_9 = \frac{1}{24}$$

$$[{}_2\phi^2]_2 = d_1R_3 + d_2R_4 + d_3R_5 + d_4R_6 + d_5R_7 + d_6R_8 + d_7R_9 = \frac{1}{12}$$

$$[[\phi]\phi] = a_3c_1R_3 + a_4c_2R_4 + a_5c_3R_5 + a_6c_4R_6 + a_7c_5R_7 + a_8c_6R_8 + a_9c_7R_9 = \frac{1}{8}$$

$$[\phi^3] = a_2{}^3R_2 + a_3{}^3R_3 + a_4{}^3R_4 + a_5{}^3R_5 + a_6{}^3R_6 + a_7{}^3R_7 + a_8{}^3R_8 + a_9{}^3R_9 = \frac{1}{4}$$

FIFTH ORDER

$$[{}_4\phi]_4 = c_1b_{43}b_{54}R_5 + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]R_6 + [c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]R_7 + [c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]R_8 + [c_1b_{43}b_{94} + (c_1b_{53} + c_2b_{54})b_{95} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})b_{98}]R_9 = \frac{1}{120}$$

$$[{}_3\phi^2]_3 = b_{43}d_1R_4 + (b_{53}d_1 + b_{54}d_2)R_5 + (b_{63}d_1 + b_{64}d_2 + b_{65}d_3)R_6 + (b_{73}d_1 + b_{74}d_2 + b_{75}d_3 + b_{76}d_4)R_7 + (b_{83}d_1 + b_{84}d_2 + b_{85}d_3 + b_{86}d_4 + b_{87}d_5)R_8 + (b_{93}d_1 + b_{94}d_2 + b_{95}d_3 + b_{96}d_4 + b_{97}d_5 + b_{98}d_6)R_9 = \frac{1}{60}$$

$$[{}_2[\phi]\phi]_2 = a_3b_{43}c_1R_4 + (a_3b_{53}c_1 + a_4b_{54}c_2)R_5 + (a_3b_{63}c_1 + a_4b_{64}c_2 + a_5b_{65}c_3)R_6 + (a_3b_{73}c_1 + a_4b_{74}c_2 + a_5b_{75}c_3 + a_6b_{76}c_4)R_7 + (a_3b_{83}c_1 + a_4b_{84}c_2 + a_5b_{85}c_3 + a_6b_{86}c_4 + a_7b_{87}c_5)R_8 + (a_3b_{93}c_1 + a_4b_{94}c_2 + a_5b_{95}c_3 + a_6b_{96}c_4 + a_7b_{97}c_5 + a_8b_{98}c_6)R_9 = \frac{1}{40}$$

$$[{}_2\phi^3]_2 = e_1R_3 + e_2R_4 + e_3R_5 + e_4R_6 + e_5R_7 + e_6R_8 + e_7R_9 = \frac{1}{20}$$

$$\begin{aligned}[[{}_2\phi]_2\phi] = {} & a_4b_{43}c_1R_4 + a_5(b_{53}c_1 + b_{54}c_2)R_5 + a_6(b_{63}c_1 + b_{64}c_2 + b_{65}c_3)R_6 \\ & + a_7(b_{73}c_1 + b_{74}c_2 + b_{75}c_3 + b_{76}c_4)R_7 + a_8(b_{83}c_1 + b_{84}c_2 + b_{85}c_3 \\ & + b_{86}c_4 + b_{87}c_5)R_8 + a_9(b_{93}c_1 + b_{94}c_2 + b_{95}c_3 + b_{96}c_4 \\ & + b_{97}c_5 + b_{98}c_6)R_9 = \frac{1}{30}\end{aligned}$$

$$\begin{aligned}[[\phi^2]\phi] = {} & a_3d_1R_3 + a_4d_2R_4 + a_5d_3R_5 + a_6d_4R_6 + a_7d_5R_7 + a_8d_6R_8 \\ & + a_9d_7R_9 = \frac{1}{15}\end{aligned}$$

$$[[\phi]^2] = c_1{}^2R_3 + c_2{}^2R_4 + c_3{}^2R_5 + c_4{}^2R_6 + c_5{}^2R_7 + c_6{}^2R_8 + c_7{}^2R_9 = \frac{1}{20}$$

$$\begin{aligned}[[\phi]\phi^2] = {} & a_3{}^2c_1R_3 + a_4{}^2c_2R_4 + a_5{}^2c_3R_5 + a_6{}^2c_4R_6 + a_7{}^2c_5R_7 + a_8{}^2c_6R_8 \\ & + a_9{}^2c_7R_9 = \frac{1}{10}\end{aligned}$$

$$[\phi^4] = a_2{}^4R_2 + a_3{}^4R_3 + a_4{}^4R_4 + a_5{}^4R_5 + a_6{}^4R_6 + a_7{}^4R_7 + a_8{}^4R_8 + a_9{}^4R_9 = \frac{1}{5}$$

SIXTH ORDER

$$[{}_5\phi]_5 = c_1b_{43}b_{54}b_{65}R_6 + \{c_1b_{43}b_{54}b_{75} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{76}\}R_7 + \{c_1b_{43}b_{54}b_{85} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{86} + [c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}R_8 + \{c_1b_{43}b_{54}b_{95} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{96} + [c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97} + [c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{720}$$

$$[{}_4\phi^2]_4 = d_1b_{43}b_{54}R_5 + [d_1b_{43}b_{64} + (d_1b_{53} + d_2b_{54})b_{65}]R_6 + [d_1b_{43}b_{74} + (d_1b_{53} + d_2b_{54})b_{75} + (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{76}]R_7 + [d_1b_{43}b_{84} + (d_1b_{53} + d_2b_{54})b_{85} + (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{86} + (d_1b_{73} + d_2b_{74} + d_3b_{75} + d_4b_{76})b_{87}]R_8 + [d_1b_{43}b_{94} + (d_1b_{53} + d_2b_{54})b_{95} + (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{96} + (d_1b_{73} + d_2b_{74} + d_3b_{75} + d_4b_{76})b_{97} + (d_1b_{83} + d_2b_{84} + d_3b_{85} + d_4b_{86} + d_5b_{87})b_{98}]R_9 = \frac{1}{360}$$

$$[_3[\phi]\phi]_3 = a_3c_1b_{43}b_{54}R_5 + [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]R_6$$
$$+ [a_3c_1b_{43}b_{74} + (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64}$$
$$+ a_5c_3b_{65})b_{76}]R_7 + [a_3c_1b_{43}b_{84} + (a_3c_1b_{53} + a_4c_2b_{54})b_{85}$$
$$+ (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{86} + (a_3c_1b_{73} + a_4c_2b_{74}$$
$$+ a_5c_3b_{75} + a_6c_4b_{76})b_{87}]R_8 + [a_3c_1b_{43}b_{94} + (a_3c_1b_{53}$$
$$+ a_4c_2b_{54})b_{95} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{96}$$
$$+ (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{97} + (a_3c_1b_{83}$$
$$+ a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} + a_7c_5b_{87})b_{98}]R_9 = \frac{1}{240}$$

$$[_3\phi^3]_3 = b_{43}e_1R_4 + (b_{53}e_1 + b_{54}e_2)R_5 + (b_{63}e_1 + b_{64}e_2 + b_{65}e_3)R_6$$
$$+ (b_{73}e_1 + b_{74}e_2 + b_{75}e_3 + b_{76}e_4)R_7 + (b_{83}e_1 + b_{84}e_2 + b_{85}e_3$$
$$+ b_{86}e_4 + b_{87}e_5)R_8 + (b_{93}e_1 + b_{94}e_2 + b_{95}e_3 + b_{96}e_4 + b_{97}e_5$$
$$+ b_{98}e_6)R_9 = \frac{1}{120}$$

$$[_2[_2\phi]_2\phi]_2 = a_4c_1b_{43}b_{54}R_5 + [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]R_6$$
$$+ [a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64}$$
$$+ c_3b_{65})b_{76}]R_7 + [a_4c_1b_{43}b_{84} + a_5(c_1b_{53} + c_2b_{54})b_{85}$$
$$+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75}$$
$$+ c_4b_{76})b_{87}]R_8 + [a_4c_1b_{43}b_{94} + a_5(c_1b_{53} + c_2b_{54})b_{95}$$
$$+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + a_7(c_1b_{73} + c_2b_{74}$$
$$+ c_3b_{75} + c_4b_{76})b_{97} + a_8(c_1b_{83} + c_2b_{84} + c_3b_{85}$$
$$+ c_4b_{86} + c_5b_{87})b_{98}]R_9 = \frac{1}{180}$$

$$[_2[\phi^2]\phi]_2 = a_3b_{43}d_1R_4 + (a_3b_{53}d_1 + a_4b_{54}d_2)R_5 + (a_3b_{63}d_1 + a_4b_{64}d_2$$
$$+ a_5b_{65}d_3)R_6 + (a_3b_{73}d_1 + a_4b_{74}d_2 + a_5b_{75}d_3 + a_6b_{76}d_4)R_7$$
$$+ (a_3b_{83}d_1 + a_4b_{84}d_2 + a_5b_{85}d_3 + a_6b_{86}d_4 + a_7b_{87}d_5)R_8$$
$$+ (a_3b_{93}d_1 + a_4b_{94}d_2 + a_5b_{95}d_3 + a_6b_{96}d_4 + a_7b_{97}d_5$$
$$+ a_8b_{98}d_6)R_9 = \frac{1}{90}$$

$$[_2[\phi]^2]_2 = c_1^2b_{43}R_4 + (c_1^2b_{53} + c_2^2b_{54})R_5 + (c_1^2b_{63} + c_2^2b_{64} + c_3^2b_{65})R_6$$
$$+ (c_1^2b_{73} + c_2^2b_{74} + c_3^2b_{75} + c_4^2b_{76})R_7 + (c_1^2b_{83} + c_2^2b_{84}$$
$$+ c_3^2b_{85} + c_4^2b_{86} + c_5^2b_{87})R_8 + (c_1^2b_{93} + c_2^2b_{94} + c_3^2b_{95}$$
$$+ c_4^2b_{96} + c_5^2b_{97} + c_6^2b_{98})R_9 = \frac{1}{120}$$

$$[_2[\phi]\phi^2]_2 = a_3^2 c_1 b_{43} R_4 + (a_3^2 b_{53} c_1 + a_4^2 b_{54} c_2) R_5 + (a_3^2 b_{63} c_1 + a_4^2 b_{64} c_2$$
$$+ a_5^2 b_{65} c_3) R_6 + (a_3^2 b_{73} c_1 + a_4^2 b_{74} c_2 + a_5^2 b_{75} c_3 + a_6^2 b_{76} c_4) R_7$$
$$+ (a_3^2 b_{83} c_1 + a_4^2 b_{84} c_2 + a_5^2 b_{85} c_3 + a_6^2 b_{86} c_4 + a_7^2 b_{87} c_5) R_8$$
$$+ (a_3^2 b_{93} c_1 + a_4^2 b_{94} c_2 + a_5^2 b_{95} c_3 + a_6^2 b_{96} c_4 + a_7^2 b_{97} c_5$$
$$+ a_8^2 b_{98} c_6) R_9 = \frac{1}{60}$$

$$[_2\phi^4]_2 = f_1 R_3 + f_2 R_4 + f_3 R_5 + f_4 R_6 + f_5 R_7 + f_6 R_8 + f_7 R_9 = \frac{1}{30}$$

$$[[_3\phi]_3\phi] = a_5 c_1 b_{43} b_{54} R_5 + a_6 [c_1 b_{43} b_{64} + (c_1 b_{53} + c_2 b_{54}) b_{65}] R_6$$
$$+ a_7 [c_1 b_{43} b_{74} + (c_1 b_{53} + c_2 b_{54}) b_{75} + (c_1 b_{63} + c_2 b_{64}$$
$$+ c_3 b_{65}) b_{76}] R_7 + a_8 [c_1 b_{43} b_{84} + (c_1 b_{53} + c_2 b_{54}) b_{85}$$
$$+ (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{86} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75}$$
$$+ c_4 b_{76}) b_{87}] R_8 + a_9 [c_1 b_{43} b_{94} + (c_1 b_{53} + c_2 b_{54}) b_{95}$$
$$+ (c_1 b_{63} + c_2 b_{64} + c_3 b_{65}) b_{96} + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75}$$
$$+ c_4 b_{76}) b_{97} + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86}$$
$$+ c_5 b_{87}) b_{98}] R_9 = \frac{1}{144}$$

$$[[{}_2\phi^2]_2\phi] = a_4d_1b_{43}R_4 + a_5(d_1b_{53} + d_2b_{54})R_5 + a_6(d_1b_{63} + d_2b_{64} + d_3b_{65})R_6 + a_7(d_1b_{73} + d_2b_{74} + d_3b_{75} + d_4b_{76})R_7 + a_8(d_1b_{83} + d_2b_{84} + d_3b_{84} + d_3b_{85} + d_4b_{86} + d_5b_{87})R_8 + a_9(d_1b_{93} + d_2b_{94} + d_3b_{95} + d_4b_{96} + d_5b_{97} + d_6b_{98})R_9 = \frac{1}{72}$$

$$[[[\phi]\phi]\phi] = a_3a_4c_1b_{43}R_4 + a_5(a_3c_1b_{53} + a_4c_2b_{54})R_5 + a_6(a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})R_6 + a_7(a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})R_7 + a_8(a_3c_1b_{83} + a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} + a_7c_5b_{87})R_8 + + a_9(a_3c_1b_{93} + a_4c_2b_{94} + a_5c_3b_{95} + a_6c_4b_{96} + a_7c_5b_{97} + a_8c_6b_{98})R_9 = \frac{1}{48}$$

$$[[\phi^3]\phi] = a_3e_1R_3 + a_4e_2R_4 + a_5e_3R_5 + a_6e_4R_6 + a_7e_5R_7 + a_8e_6R_8 + a_9e_7R_9 = \frac{1}{24}$$

$$[[{}_2\phi]_2[\phi]] = c_1c_2b_{43}R_4 + (c_1b_{53} + c_2b_{54})c_3R_5 + (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4R_6 + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})c_5R_7 + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})c_6R_8 + (c_1b_{93} + c_2b_{94} + c_3b_{95} + c_4b_{96} + c_5b_{97} + c_6b_{98})c_7R_9 = \frac{1}{72}$$

$$[[\phi^2][\phi]] = c_1d_1R_3 + c_2d_2R_4 + c_3d_3R_5 + c_4d_4R_6 + c_5d_5R_7 + c_6d_6R_8 + c_7d_7R_9 = \frac{1}{36}$$

$$[[{}_2\phi]_2\phi^2] = a_4{}^2c_1b_{43}R_4 + a_5{}^2(b_{53}c_1 + b_{54}c_2)R_5 + a_6{}^2(b_{63}c_1 + b_{64}c_2 + b_{65}c_3)R_6 + a_7{}^2(b_{73}c_1 + b_{74}c_2 + b_{75}c_3 + b_{76}c_4)R_7 + a_8{}^2(b_{83}c_1 + b_{84}c_2 + b_{85}c_3 + b_{86}c_4 + b_{87}c_5)R_8 + a_9{}^2(b_{93}c_1 + b_{94}c_2 + b_{95}c_3 + b_{96}c_4 + b_{97}c_5 + b_{98}c_6)R_9 = \frac{1}{36}$$

$$[[\phi^2]\phi^2] = a_3{}^2d_1R_3 + a_4{}^2d_2R_4 + a_5{}^2d_3R_5 + a_6{}^2d_4R_6 + a_7{}^2d_5R_7 + a_8{}^2d_6R_8 + a_9{}^2d_7R_9 = \frac{1}{18}$$

$$[[\phi]^2\phi] = a_3c_1{}^2R_3 + a_4c_2{}^2R_4 + a_5c_3{}^2R_5 + a_6c_4{}^2R_6 + a_7c_5{}^2R_7 + a_8c_6{}^2R_8 + a_9c_7{}^2R_9 = \frac{1}{24}$$

$$[[\phi]\phi^3] = a_3{}^3c_1R_3 + a_4{}^3c_2R_4 + a_5{}^3c_3R_5 + a_6{}^3c_4R_6 + a_7{}^3c_5R_7 + a_8{}^3c_6R_8$$
$$+ a_9{}^3c_7R_9 = \frac{1}{12}$$

$$[\phi^5] = a_2{}^5R_2 + a_3{}^5R_3 + a_4{}^5R_4 + a_5{}^5R_5 + a_6{}^5R_6 + a_7{}^5R_7 + a_8{}^5R_8$$
$$+ a_9{}^5R_9 = \frac{1}{6}$$

SEVENTH ORDER

$$[{}_6\phi]_6 = c_1b_{43}b_{54}b_{65}b_{76}R_7 + \{c_1b_{43}b_{54}b_{65}b_{86} + [c_1b_{43}b_{54}b_{75}$$

$$+ (c_1b_{43}b_{64} + \overline{c_1b_{53} + c_2b_{54}}\ b_{65})b_{76}]b_{87}\}R_8 + \{c_1b_{43}b_{54}b_{65}b_{96}$$

$$+ [c_1b_{43}b_{54}b_{75} + (c_1b_{43}b_{64} + \overline{c_1b_{53} + c_2b_{54}}\ b_{65})b_{76}]b_{97}$$

$$+ [c_1b_{43}b_{54}b_{85} + (c_1b_{43}b_{64} + \overline{c_1b_{53} + c_2b_{54}}\ b_{65})b_{86}$$

$$+ (c_1b_{43}b_{74} + \overline{c_1b_{53} + c_2b_{54}}\ b_{75}$$

$$+ \overline{c_1b_{63} + c_2b_{64} + c_3b_{65}}\ b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{5040}$$

$$[{}_5\phi^2]_5 = d_1b_{43}b_{54}b_{65}R_6 + \{d_1b_{43}b_{54}b_{75} + [d_1b_{43}b_{64} + (d_1b_{53}$$

$$+ d_2b_{54})b_{65}]b_{76}\}R_7 + \{d_1b_{43}b_{54}b_{85} + [d_1b_{43}b_{64} + (d_1b_{53}$$

$$+ d_2b_{54})b_{65}]b_{86} + [d_1b_{43}b_{74} + (d_1b_{53} + d_2b_{54})b_{75} + (d_1b_{63}$$

$$+ d_2b_{64} + d_3b_{65})b_{76}]b_{87}\}R_8 + \{d_1b_{43}b_{54}b_{95} + [d_1b_{43}b_{64}$$

$$+ (d_1b_{53} + d_2b_{54})b_{65}]b_{96} + [d_1b_{43}b_{74} + (d_1b_{53} + d_2b_{54})b_{75}$$

$$+ (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{76}]b_{97} + [d_1b_{43}b_{84} + (d_1b_{53}$$

$$+ d_2b_{54})b_{85} + (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{86} + (d_1b_{73} + d_2b_{74}$$

$$+ d_3b_{75} + d_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{2520}$$

$$[_4[\phi]\phi]_4 = a_3c_1b_{43}b_{54}b_{65}R_6 + \{a_3c_1b_{43}b_{54}b_{75} + [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_3c_1b_{43}b_{54}b_{85} + [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{86} + [a_3c_1b_{43}b_{74} + (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{76}]b_{87}\}R_8 + a_3c_1b_{43}b_{54}b_{95} + [a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]b_{96} + [a_3c_1b_{43}b_{74} + (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{76}]b_{97} + [a_3c_1b_{43}b_{84} + (a_3c_1b_{53} + a_4c_2b_{54})b_{85} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{86} + (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{1680}$$

$$[_4\phi^3]_4 = e_1b_{43}b_{54}R_5 + [e_1b_{43}b_{64} + (e_1b_{53} + e_2b_{54})b_{65}]R_6 + [e_1b_{43}b_{74} + (e_1b_{53} + e_2b_{54})b_{75} + (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{76}]R_7 + [e_1b_{43}b_{84} + (e_1b_{53} + e_2b_{54})b_{85} + (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{86} + (e_1b_{73} + e_2b_{74} + e_3b_{75} + e_4b_{76})b_{87}]R_8 + [e_1b_{43}b_{94} + (e_1b_{53} + e_2b_{54})b_{95} + (e_1b_{63} + e_2b_{64} + e_3b_{65})b_{96} + (e_1b_{73} + e_2b_{74} + e_3b_{75} + e_4b_{76})b_{97} + (e_1b_{83} + e_2b_{84} + e_3b_{85} + e_4b_{86} + e_5b_{87})b_{98}]R_9 = \frac{1}{840}$$

$$[_3[_2\phi]_2\phi]_3 = a_4c_1b_{43}b_{54}b_{65}R_6 + \{a_4c_1b_{43}b_{54}b_{75} + [a_1c_4b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_4c_1b_{43}b_{54}b_{85} + [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]b_{86} + [a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}R_8 + \{a_4c_1b_{43}b_{54}b_{95} + [a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]b_{96} + [a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97} + [a_4c_1b_{43}b_{84} + a_5(c_1b_{53} + c_2b_{54})b_{85} + a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]b_{98}\}R_9 = \frac{1}{1260}$$

$$[_3[\phi^2]\phi]_3 = a_3d_1b_{43}b_{54}R_5 + [a_3d_1b_{43}b_{64} + (a_3d_1b_{53} + a_4d_2b_{54})b_{65}]R_6 + [a_3d_1b_{43}b_{74} + (a_3d_1b_{53} + a_4d_2b_{54})b_{75} + (a_3d_1b_{63} + a_4d_2b_{64} + a_5d_3b_{65})b_{76}]R_7 + [a_3d_1b_{43}b_{84} + (a_3d_1b_{53} + a_4d_2b_{54})b_{85} + (a_3d_1b_{63} + a_4d_2b_{64} + a_5d_3b_{65})b_{86} + (a_3d_1b_{73} + a_4d_2b_{74} + a_5d_3b_{75} + a_6d_4b_{76})b_{87}]R_8 + [a_3d_1b_{43}b_{94} + (a_3d_1b_{53} + a_4d_2b_{54})b_{95} + (a_3d_1b_{63} + a_4d_2b_{64} + a_5d_3b_{65})b_{96} + (a_3d_1b_{73} + a_4d_2b_{74} + a_5d_3b_{75} + a_6d_4b_{76})b_{97} + (a_3d_1b_{83} + a_4d_2b_{84} + a_5d_3b_{85} + a_6d_4b_{86} + a_7d_5b_{87})b_{98}]R_9 = \frac{1}{630}$$

$$[_3[\phi]^2]_3 = c_1^2 b_{43} b_{54} R_5 + [c_1^2 b_{43} b_{64} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{65}] R_6$$
$$+ [c_1^2 b_{43} b_{74} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{75} + (c_1^2 b_{63} + c_2^2 b_{64}$$
$$+ c_3^2 b_{65}) b_{76}] R_7 + [c_1^2 b_{43} b_{84} + (c_1^2 b_{53} + c_2^2 b_{54}) b_{85}$$
$$+ (c_1^2 b_{63} + c_2^2 b_{64} + c_3^2 b_{65}) b_{86} + (c_1^2 b_{73} + c_2^2 b_{74}$$
$$+ c_3^2 b_{75} + c_4^2 b_{76}) b_{87}] R_8 + [c_1^2 b_{43} b_{94} + (c_1^2 b_{53}$$
$$+ c_2^2 b_{54}) b_{95} + (c_1^2 b_{63} + c_2^2 b_{64} + c_3^2 b_{65}) b_{96}$$
$$+ (c_1^2 b_{73} + c_2^2 b_{74} + c_3^2 b_{75} + c_4^2 b_{76}) b_{97} + (c_1^2 b_{83}$$
$$+ c_2^2 b_{84} + c_3^2 b_{85} + c_4^2 b_{86} + c_5^2 b_{87}) b_{98}] R_9 = \frac{1}{840}$$

$$[_3[\phi]\phi^2]_3 = a_3^2 c_1 b_{43} b_{54} R_5 + [a_3^2 c_1 b_{43} b_{64} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{65}] R_6$$
$$+ [a_3^2 c_1 b_{43} b_{74} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{75} + (a_3^2 c_1 b_{63}$$
$$+ a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{76}] R_7 + [a_3^2 c_1 b_{43} b_{84} + (a_3^2 c_1 b_{53}$$
$$+ a_4^2 c_2 b_{54}) b_{85} + (a_3^2 c_1 b_{63} + a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{86}$$
$$+ (a_3^2 c_1 b_{73} + a_4^2 c_2 b_{74} + a_5^2 c_3 b_{75} + a_6^2 c_4 b_{76}) b_{87}] R_8$$
$$+ [a_3^2 c_1 b_{43} b_{94} + (a_3^2 c_1 b_{53} + a_4^2 c_2 b_{54}) b_{95} + (a_3^2 c_1 b_{63}$$
$$+ a_4^2 c_2 b_{64} + a_5^2 c_3 b_{65}) b_{96} + (a_3^2 c_1 b_{73} + a_4^2 c_2 b_{74}$$
$$+ a_5^2 c_3 b_{75} + a_6^2 c_4 b_{76}) b_{97} + (a_3^2 c_1 b_{83} + a_4^2 c_2 b_{84}$$
$$+ a_5^2 c_3 b_{85} + a_6^2 c_4 b_{86} + a_7^2 c_5 b_{87}) b_{98}] R_9 = \frac{1}{420}$$

$$[{}_3\phi^4]_3 = b_{43}f_1R_4 + (b_{53}f_1 + b_{54}f_2)R_5 + (b_{63}f_1 + b_{64}f_2 + b_{65}f_3)R_6$$
$$+ (b_{73}f_1 + b_{74}f_2 + b_{75}f_3 + b_{76}f_4)R_7 + (b_{83}f_1 + b_{84}f_2 + b_{85}f_3$$
$$+ b_{86}f_4 + b_{87}f_5)R_8 + (b_{93}f_1 + b_{94}f_2 + b_{95}f_3 + b_{96}f_4 + b_{97}f_5$$
$$+ b_{98}f_6)R_9 = \frac{1}{210}$$

$$[{}_2[{}_3\phi]_3\phi]_2 = a_5c_1b_{43}b_{54}b_{65}R_6 + \{a_5c_1b_{43}b_{65}b_{75} + a_6[c_1b_{43}b_{64} + (c_1b_{53}$$
$$+ c_2b_{54})b_{65}]b_{76}\}R_7 + \{a_5c_1b_{43}b_{54}b_{85} + a_6[c_1b_{43}b_{64} + (c_1b_{53}$$
$$+ c_2b_{54})b_{65}]b_{86} + a_7[c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75}$$
$$+ (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}R_8 + \{a_5c_1b_{43}b_{54}b_{95}$$
$$+ a_6[c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{96} + a_7[c_1b_{43}b_{74}$$
$$+ (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97}$$
$$+ a_8[c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64}$$
$$+ c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]b_{98}\}R_9$$
$$= \frac{1}{1008}$$

$$[_2[_2\phi^2]_2\phi]_2 = a_4d_1b_{43}b_{54}R_5 + [a_4d_1b_{43}b_{64} + a_5(d_1b_{53} + d_2b_{54})b_{65}]R_6$$
$$+ [a_4d_1b_{43}b_{74} + a_5(d_1b_{53} + d_2b_{54})b_{75} + a_6(d_1b_{63} + d_2b_{64}$$
$$+ d_3b_{65})b_{76}]R_7 + [a_4d_1b_{43}b_{84} + a_5(d_1b_{53} + d_2b_{54})b_{85}$$
$$+ a_6(d_1b_{63} + d_2b_{64} + d_3b_{65})b_{86} + a_7(d_1b_{73} + d_2b_{74}$$
$$+ d_3b_{75} + d_4b_{76})b_{87}]R_8 + [a_4d_1b_{43}b_{94} + a_5(d_1b_{53}$$
$$+ d_2b_{54})b_{95} + a_6(d_1b_{63} + d_2b_{64} + d_3b_{65})b_{96} + a_7(d_1b_{73}$$
$$+ d_2b_{74} + d_3b_{75} + d_4b_{76})b_{97} + a_8(d_1b_{83} + d_2b_{84} + d_3b_{85}$$
$$+ d_4b_{86} + d_5b_{87})b_{98}]R_9 = \frac{1}{504}$$

$$[_2[[\phi]\phi]\phi]_2 = a_3a_4c_1b_{43}b_{54}R_5 + [a_3a_4c_1b_{43}b_{64} + a_5(a_3c_1b_{53}$$
$$+ a_4c_2b_{54})b_{65}]R_6 + [a_3a_4c_1b_{43}b_{74} + a_5(a_3c_1b_{53} + a_4c_2b_{54})b_{75}$$
$$+ a_6(a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{76}]R_7 + [a_3a_4c_1b_{43}b_{84}$$
$$+ a_5(a_3c_1b_{53} + a_4c_2b_{54})b_{85} + a_6(a_3c_1b_{63} + a_4c_2b_{64}$$
$$+ a_5c_3b_{65})b_{86} + a_7(a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75}$$
$$+ a_6c_4b_{76})b_{87}]R_8 + [a_3a_4c_1b_{43}b_{94} + a_5(a_3c_1b_{53}$$
$$+ a_4c_2b_{54})b_{95} + a_6(a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{96}$$
$$+ a_7(a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{97}$$
$$+ a_8(a_3c_1b_{83} + a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86}$$
$$+ a_7c_5b_{87})b_{98}]R_9 = \frac{1}{336}$$

$$[_2[\phi^3]\phi]_2 = a_3b_{43}e_1R_4 + (a_3b_{53}e_1 + a_4b_{54}e_2)R_5 + (a_3b_{63}e_1 + a_4b_{64}e_2 + a_5b_{65}e_3)R_6 + (a_3b_{73}e_1 + a_4b_{74}e_2 + a_5b_{75}e_3 + a_6b_{76}e_4)R_7 + (a_3b_{83}e_1 + a_4b_{84}e_2 + a_5b_{85}e_3 + a_6b_{86}e_4 + a_7b_{87}e_5)R_8 + (a_3b_{93}e_1 + a_4b_{94}e_2 + a_5b_{95}e_3 + a_6b_{96}e_4 + a_7b_{97}e_5 + a_8b_{98}e_6)R_9 = \frac{1}{168}$$

$$[_2[_2\phi]_2[\phi]]_2 = c_1c_2b_{43}b_{54}R_5 + [c_1c_2b_{43}b_{64} + (c_1b_{53} + c_2b_{54})c_3b_{65}]R_6 + [c_1c_2b_{43}b_{74} + (c_1b_{53} + c_2b_{54})c_3b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4b_{76}]R_7 + [c_1c_2b_{43}b_{84} + (c_1b_{53} + c_2b_{54})c_3b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})c_5b_{87}]R_8 + [c_1c_2b_{43}b_{94} + (c_1b_{53} + c_2b_{54})c_3b_{95} + (c_1b_{63} + c_2b_{64} + c_3b_{65})c_4b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})c_5b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})c_6b_{98}]R_9 = \frac{1}{504}$$

$$[_2[\phi^2][\phi]]_2 = c_1d_1b_{43}R_4 + (c_1d_1b_{53} + c_2d_2b_{54})R_5 + (c_1d_1b_{63} + c_2d_2b_{64} + c_3d_3b_{65})R_6 + (c_1d_1b_{73} + c_2d_2b_{74} + c_3d_3b_{75} + c_4d_4b_{76})R_7 + (c_1d_1b_{83} + c_2d_2b_{84} + c_3d_3b_{85} + c_4d_4b_{86} + c_5d_5b_{87})R_8 + (c_1d_1b_{93} + c_2d_2b_{94} + c_3d_3b_{95} + c_4d_4b_{96} + c_5d_5b_{97} + c_6d_6b_{98})R_9 = \frac{1}{252}$$

$$[_2[_2\phi]_2\phi^2]_2 = a_4^2c_1b_{43}b_{54}R_5 + [a_4^2c_1b_{43}b_{64} + a_5^2(c_1b_{53} + c_2b_{54})b_{65}]R_6$$
$$+ [a_4^2c_1b_{43}b_{74} + a_5^2(c_1b_{53} + c_2b_{54})b_{75} + a_6^2(c_1b_{63} + c_2b_{64}$$
$$+ c_3b_{65})b_{76}]R_7 + [a_4^2c_1b_{43}b_{84} + a_5^2(c_1b_{53} + c_2b_{54})b_{85}$$
$$+ a_6^2(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + a_7^2(c_1b_{73} + c_2b_{74}$$
$$+ c_3b_{75} + c_4b_{76})b_{87}]R_8 + [a_4^2c_1b_{43}b_{94} + a_5^2(c_1b_{53} + c_2b_{54})b_{95}$$
$$+ a_6^2(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + a_7^2(c_1b_{73} + c_2b_{74} + c_3b_{75}$$
$$+ c_4b_{76})b_{97} + a_8^2(c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86}$$
$$+ c_5b_{87})b_{98}]R_9 = \frac{1}{252}$$

$$[_2[\phi^2]\phi^2]_2 = a_3^2b_{43}d_1R_4 + (a_3^2b_{53}d_1 + a_4^2b_{54}d_2)R_5 + (a_3^2b_{63}d_1 + a_4^2b_{64}d_2$$
$$+ a_5^2b_{65}d_3)R_6 + (a_3^2b_{73}d_1 + a_4^2b_{74}d_2 + a_5^2b_{75}d_3 + a_6^2b_{76}d_4)R_7$$
$$+ (a_3^2b_{83}d_1 + a_4^2b_{84}d_2 + a_5^2b_{85}d_3 + a_6^2b_{86}d_4 + a_7^2b_{87}d_5)R_8$$
$$+ (a_3^2b_{93}d_1 + a_4^2b_{94}d_2 + a_5^2b_{95}d_3 + a_6^2b_{96}d_4 + a_7^2b_{97}d_5$$
$$+ a_8^2b_{98}d_6)R_9 = \frac{1}{126}$$

$$[_2[\phi]^2\phi]_2 = a_3c_1^2b_{43}R_4 + (a_3c_1^2b_{53} + a_4c_2^2b_{54})R_5 + (a_3c_1^2b_{63} + a_4c_2^2b_{64}$$
$$+ a_5c_3^2b_{65})R_6 + (a_3c_1^2b_{73} + a_4c_2^2b_{74} + a_5c_3^2b_{75} + a_6c_4^2b_{76})R_7$$
$$+ (a_3c_1^2b_{83} + a_4c_2^2b_{84} + a_5c_3^2b_{85} + a_6c_4^2b_{86} + a_7c_5^2b_{87})R_8$$
$$+ (a_3c_1^2b_{93} + a_4c_2^2b_{94} + a_5c_3^2b_{95} + a_6c_4^2b_{96} + a_7c_5^2b_{97}$$
$$+ a_8c_6^2b_{98})R_9 = \frac{1}{168}$$

$$[_2[\phi]\phi^3]_2 = a_3{}^2b_{43}c_1R_4 + (a_3{}^3b_{53}c_1 + a_4{}^3b_{54}c_2)R_5 + (a_3{}^3b_{63}c_1 + a_4{}^3b_{64}c_2 + a_5{}^3b_{65}c_3)R_6 + (a_3{}^3b_{73}c_1 + a_4{}^3b_{74}c_2 + a_5{}^3b_{75}c_3 + a_6{}^3b_{76}c_4)R_7 + (a_3{}^3b_{83}c_1 + a_4{}^3b_{84}c_2 + a_5{}^3b_{85}c_3 + a_6{}^3b_{86}c_4 + a_7{}^3b_{87}c_5)R_8 + (a_3{}^3b_{93}c_1 + a_4{}^3b_{94}c_2 + a_5{}^3b_{95}c_3 + a_6{}^3b_{96}c_4 + a_7{}^3b_{97}c_5 + a_8{}^3b_{98}c_6)R_9 = \frac{1}{84}$$

$$[_2\phi^5]_2 = g_1R_3 + g_2R_4 + g_3R_5 + g_4R_6 + g_5R_7 + g_6R_8 + g_7R_9 = \frac{1}{42}$$

$$[[_4\phi]_4\phi] = a_6c_1b_{43}b_{54}b_{65}R_6 + \{c_1b_{43}b_{54}b_{75} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{76}\}a_7R_7 + \{c_1b_{43}b_{54}b_{85} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65} + (c_1b_{63} + c_2b_{64} + c_3b_{65})]b_{86} + [c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{87}\}a_8R_8 + \{c_1b_{43}b_{54}b_{95} + [c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]b_{96} + [c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]b_{97} + [c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]b_{98}\}a_9R_9 = \frac{1}{840}$$

$$[[{}_3\phi^2]_3\phi] = a_5d_1b_{43}b_{54}R_5 + a_6[d_1b_{43}b_{64} + (d_1b_{53} + d_2b_{54})b_{65}]R_6$$
$$+ a_7[d_1b_{43}b_{74} + (d_1b_{53} + d_2b_{54})b_{75} + (d_1b_{63} + d_2b_{64}$$
$$+ d_3b_{65})b_{76}]R_7 + a_8[d_1b_{43}b_{84} + (d_1b_{53} + d_2b_{54})b_{85}$$
$$+ (d_1b_{63} + d_2b_{64} + d_3b_{65})b_{86} + (d_1b_{73} + d_2b_{74} + d_3b_{75}$$
$$+ d_4b_{76})b_{87}]R_8 + a_9[d_1b_{43}b_{94} + (d_1b_{53} + d_2b_{54})b_{95} + (d_1b_{63}$$
$$+ d_2b_{64} + d_3b_{65})b_{96} + (d_1b_{73} + d_2b_{74} + d_3b_{75} + d_4b_{76})b_{97}$$
$$+ (d_1b_{83} + d_2b_{84} + d_3b_{85} + d_4b_{86} + d_5b_{87})b_{98}]R_9 = \frac{1}{420}$$

$$[[{}_2[\phi]\phi]_2\phi] = a_3a_5c_1b_{43}b_{54}R_5 + a_6[a_3c_1b_{43}b_{64} + (a_3c_1b_{53} + a_4c_2b_{54})b_{65}]R_6$$
$$+ a_7[a_3c_1b_{43}b_{74} + (a_3c_1b_{53} + a_4c_2b_{54})b_{75} + (a_3c_1b_{63}$$
$$+ a_4c_2b_{64} + a_5c_3b_{65})b_{76}]R_7 + a_8[a_3c_1b_{43}b_{84} + (a_3c_1b_{53}$$
$$+ a_4c_2b_{54})b_{85} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{86} + (a_3c_1b_{73}$$
$$+ a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{87}]R_8 + a_9[a_3c_1b_{43}b_{94}$$
$$+ (a_3c_1b_{53} + a_4c_2b_{54})b_{95} + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})b_{96}$$
$$+ (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})b_{97} + (a_3c_1b_{83}$$
$$+ a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} + a_7c_5b_{87})b_{98}]R_9 = \frac{1}{280}$$

$$[[{}_2\phi^3]_2\phi] = a_4e_1b_{43}R_4 + a_5(b_{53}e_1 + b_{54}e_2)R_5 + a_6(b_{63}e_1 + b_{64}e_2$$
$$+ b_{65}e_3)R_6 + a_7(b_{73}e_1 + b_{74}e_2 + b_{75}e_3 + b_{76}e_4)R_7 + a_8(b_{83}e_1$$
$$+ b_{84}e_2 + b_{85}e_3 + b_{86}e_4 + b_{87}e_5)R_8 + a_9(b_{93}e_1 + b_{94}e_2 + b_{95}e_3$$
$$+ b_{96}e_4 + b_{97}e_5 + b_{98}e_6)R_9 = \frac{1}{140}$$

$$[[[_2\phi]_2\phi]\phi] = a_4a_5c_1b_{43}b_{54}R_5 + a_6[a_4c_1b_{43}b_{64} + a_5(c_1b_{53} + c_2b_{54})b_{65}]R_6$$
$$+ a_7[a_4c_1b_{43}b_{74} + a_5(c_1b_{53} + c_2b_{54})b_{75} + a_6(c_1b_{63} + c_2b_{64}$$
$$+ c_3b_{65})b_{76}]R_7 + a_8[a_4c_1b_{43}b_{84} + a_5(c_1b_{53} + c_2b_{54})b_{85}$$
$$+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75}$$
$$+ c_4b_{76})b_{87}]R_8 + a_9[a_4c_1b_{43}b_{94} + a_5(c_1b_{53} + c_2b_{54})b_{95}$$
$$+ a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75}$$
$$+ c_4b_{76})b_{97} + a_8(c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86}$$
$$+ c_5b_{87})b_{98}]R_9 = \frac{1}{210}$$

$$[[[\phi^2]\phi]\phi] = a_3a_4d_1b_{43}R_4 + (a_3d_1b_{53} + a_4d_2b_{54})a_5R_5 + (a_3d_1b_{63} + a_4d_2b_{64}$$
$$+ a_5d_3b_{65})a_6R_6 + (a_3d_1b_{73} + a_4d_2b_{74} + a_5d_3b_{75} + a_6d_4b_{76})a_7R_7$$
$$+ (a_3d_1b_{83} + a_4d_2b_{84} + a_5d_3b_{85} + a_6d_4b_{86} + a_7d_5b_{87})a_8R_8$$
$$+ (a_3d_1b_{93} + a_4d_2b_{94} + a_5d_3b_{95} + a_6d_4b_{96} + a_7d_5b_{97}$$
$$+ a_8d_6b_{98})a_9R_9 = \frac{1}{105}$$

$$[[[\phi]^2]\phi] = a_4c_1^{\,2}b_{43}R_4 + (c_1^{\,2}b_{53} + c_2^{\,2}b_{54})a_5R_5 + (c_1^{\,2}b_{63} + c_2^{\,2}b_{64}$$
$$+ c_3^{\,2}b_{65})a_6R_6 + (c_1^{\,2}b_{73} + c_2^{\,2}b_{74} + c_3^{\,2}b_{75} + c_4^{\,2}b_{76})a_7R_7$$
$$+ (c_1^{\,2}b_{83} + c_2^{\,2}b_{84} + c_3^{\,2}b_{85} + c_4^{\,2}b_{86} + c_5^{\,2}b_{87})a_8R_8$$
$$+ (c_1^{\,2}b_{93} + c_2^{\,2}b_{94} + c_3^{\,2}b_{95} + c_4^{\,2}b_{96} + c_5^{\,2}b_{97}$$
$$+ c_6^{\,2}b_{98})a_9R_9 = \frac{1}{140}$$

$$[[[\phi]\phi^2]\phi] = a_4a_3^2b_{43}c_1R_4 + a_5(a_3^2b_{53}c_1 + a_4^2b_{54}c_2)R_5 + a_6(a_3^2b_{63}c_1 + a_4^2b_{64}c_2 + a_5^2b_{65}c_3)R_6 + a_7(a_3^2b_{73}c_1 + a_4^2b_{74}c_2 + a_5^2b_{75}c_3 + a_6^2b_{76}c_4)R_7 + a_8(a_3^2b_{83}c_1 + a_4^2b_{84}c_2 + a_5^2b_{85}c_3 + a_6^2b_{86}c_4 + a_7^2b_{87}c_5)R_8 + a_9(a_3^2b_{93}c_1 + a_4^2b_{94}c_2 + a_5^2b_{95}c_3 + a_6^2b_{96}c_4 + a_7^2b_{97}c_5 + a_8^2b_{98}c_6)R_9 = \frac{1}{70}$$

$$[[\phi^4]\phi] = a_3f_1R_3 + a_4f_2R_4 + a_5f_3R_5 + a_6f_4R_6 + a_7f_5R_7 + a_8f_6R_8 + a_9f_7R_9 = \frac{1}{35}$$

$$[[{}_3\phi]_3[\phi]] = c_1c_3b_{43}b_{54}R_5 + c_4[c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]R_6 + c_5[c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]R_7 + c_6[c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]R_8 + c_7[c_1b_{43}b_{94} + (c_1b_{53} + c_2b_{54})b_{95} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})b_{98}]R_9 = \frac{1}{336}$$

$$[[{}_2\phi^2]_2[\phi]] = d_1b_{43}c_2R_4 + (d_1b_{53} + d_2b_{54})c_3R_5 + (d_1b_{63} + d_2b_{64} + d_3b_{65})c_4R_6 + (d_1b_{73} + d_2b_{74} + d_3b_{75} + d_4b_{76})c_5R_7 + (d_1b_{83} + d_2b_{84} + d_3b_{85} + d_4b_{86} + d_5b_{87})c_6R_8 + (d_1b_{93} + d_2b_{94} + d_3b_{95} + d_4b_{96} + d_5b_{97} + d_6b_{98})c_7R_9 = \frac{1}{168}$$

$$[[[\phi]\phi][\phi]] = a_3c_1b_{43}c_2R_4 + (a_3c_1b_{53} + a_4c_2b_{54})c_3R_5 + (a_3c_1b_{63} + a_4c_2b_{64} + a_5c_3b_{65})c_4R_6 + (a_3c_1b_{73} + a_4c_2b_{74} + a_5c_3b_{75} + a_6c_4b_{76})c_5R_7 + (a_3c_1b_{83} + a_4c_2b_{84} + a_5c_3b_{85} + a_6c_4b_{86} + a_7c_5b_{87})c_6R_8 + (a_3c_1b_{93} + a_4c_2b_{94} + a_5c_3b_{95} + a_6c_4b_{96} + a_7c_5b_{97} + a_8c_6b_{98})c_7R_9 = \frac{1}{112}$$

$$[[\phi^3][\phi]] = c_1e_1R_3 + c_2e_2R_4 + c_3e_3R_5 + c_4e_4R_6 + c_5e_5R_7 + c_6e_6R_8 + c_7e_7R_9 = \frac{1}{56}$$

$$[[{}_3\phi]{}_3\phi^2] = a_5{}^2c_1b_{43}b_{54}R_5 + a_6{}^2[c_1b_{43}b_{64} + (c_1b_{53} + c_2b_{54})b_{65}]R_6 + a_7{}^2[c_1b_{43}b_{74} + (c_1b_{53} + c_2b_{54})b_{75} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{76}]R_7 + a_8{}^2[c_1b_{43}b_{84} + (c_1b_{53} + c_2b_{54})b_{85} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{86} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{87}]R_8 + a_9{}^2[c_1b_{43}b_{94} + (c_1b_{53} + c_2b_{54})b_{95} + (c_1b_{63} + c_2b_{64} + c_3b_{65})b_{96} + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})b_{97} + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})b_{98}]R_9 = \frac{1}{168}$$

$$[[_2\phi^2]_2\phi^2] = a_4^2 d_1 b_{43} R_4 + a_5^2 (d_1 b_{53} + d_2 b_{54}) R_5 + a_6^2 (d_1 b_{63} + d_2 b_{64} + d_3 b_{65}) R_6 + a_7^2 (d_1 b_{73} + d_2 b_{74} + d_3 b_{75} + d_4 b_{76}) R_7 + a_8^2 (d_1 b_{83} + d_2 b_{84} + d_3 b_{85} + d_4 b_{86} + d_5 b_{87}) R_8 + a_9^2 (d_1 b_{93} + d_2 b_{94} + d_3 b_{95} + d_4 b_{96} + d_5 b_{97} + d_6 b_{98}) R_9 = \frac{1}{84}$$

$$[[[\phi]\phi]\ \phi^2] = a_3 a_4^2 b_{43} c_1 R_4 + a_5^2 (a_3 b_{53} c_1 + a_4 b_{54} c_2) R_5 + a_6^2 (a_3 b_{63} c_1 + a_4 b_{64} c_2 + a_5 b_{65} c_3) R_6 + a_7^2 (a_3 b_{73} c_1 + a_4 b_{74} c_2 + a_5 b_{75} c_3 + a_6 b_{76} c_4) R_7 + a_8^2 (a_3 b_{83} c_1 + a_4 b_{84} c_2 + a_5 b_{85} c_3 + a_6 b_{86} c_4 + a_7 b_{87} c_5) R_8 + a_9^2 (a_3 b_{93} c_1 + a_4 b_{94} c_2 + a_5 b_{95} c_3 + a_6 b_{96} c_4 + a_7 b_{97} c_5 + a_8 b_{98} c_6) R_9 = \frac{1}{56}$$

$$[[\phi^3]\phi^2] = a_3^2 e_1 R_3 + a_4^2 e_2 R_4 + a_5^2 e_3 R_5 + a_6^2 e_4 R_6 + a_7^2 e_5 R_7 + a_8^2 e_6 R_8 + a_9^2 e_7 R_9 = \frac{1}{28}$$

$$[[_2\phi]_2^2] = (c_1 b_{43})^2 R_4 + (c_1 b_{53} + c_2 b_{54})^2 R_5 + (c_1 b_{63} + c_2 b_{64} + c_3 b_{65})^2 R_6 + (c_1 b_{73} + c_2 b_{74} + c_3 b_{75} + c_4 b_{76})^2 R_7 + (c_1 b_{83} + c_2 b_{84} + c_3 b_{85} + c_4 b_{86} + c_5 b_{87})^2 R_8 + (c_1 b_{93} + c_2 b_{94} + c_3 b_{95} + c_4 b_{96} + c_5 b_{97} + c_6 b_{98})^2 R_9 = \frac{1}{252}$$

$$[[_2\phi]_2[\phi^2]] = c_1d_2b_{43}R_4 + (c_1b_{53} + c_2b_{54})d_3R_5 + (c_1b_{63} + c_2b_{64} + c_3b_{65})d_4R_6 + (c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})d_5R_7 + (c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})d_6R_8 + (c_1b_{93} + c_2b_{94} + c_3b_{95} + c_4b_{96} + c_5b_{97} + c_6b_{98})d_7R_9 = \frac{1}{126}$$

$$[[\phi^2]^2] = d_1{}^2R_3 + d_2{}^2R_4 + d_3{}^2R_5 + d_4{}^2R_6 + d_5{}^2R_7 + d_6{}^2R_8 + d_7{}^2R_9 = \frac{1}{63}$$

$$[[_2\phi]_2[\phi]\phi] = a_4c_1c_2b_{43}R_4 + a_5(c_1b_{53} + c_2b_{54})c_3R_5 + a_6(c_1b_{63} + c_2b_{64} + c_3b_{65})c_4R_6 + a_7(c_1b_{73} + c_2b_{74} + c_3b_{75} + c_4b_{76})c_5R_7 + a_8(c_1b_{83} + c_2b_{84} + c_3b_{85} + c_4b_{86} + c_5b_{87})c_6R_8 + a_9(c_1b_{93} + c_2b_{94} + c_3b_{95} + c_4b_{96} + c_5b_{97} + c_6b_{98})c_7R_9 = \frac{1}{84}$$

$$[[\phi^2][\phi]\phi] = a_3c_1d_1R_3 + a_4c_2d_2R_4 + a_5c_3d_3R_5 + a_6c_4d_4R_6 + a_7c_5d_5R_7 + a_8c_6d_6R_8 + a_9c_7d_7R_9 = \frac{1}{42}$$

$$[[{}_2\phi]_2\phi^3] = a_4{}^3b_{43}c_1R_4 + a_5{}^3(b_{53}c_1 + b_{54}c_2)R_5 + a_6{}^3(b_{63}c_1 + b_{64}c_2 + b_{65}c_3)R_6 + a_7{}^3(b_{73}c_1 + b_{74}c_2 + b_{75}c_3 + b_{76}c_4)R_7 + a_8{}^3(b_{83}c_1 + b_{84}c_2 + b_{85}c_3 + b_{86}c_4 + b_{87}c_5)R_8 + a_9{}^3(b_{93}c_1 + b_{94}c_2 + b_{95}c_3 + b_{96}c_4 + b_{97}c_5 + b_{98}c_6)R_9 = \frac{1}{42}$$

$$[[\phi^2]\phi^3] = a_3{}^3d_1R_3 + a_4{}^3d_2R_4 + a_5{}^3d_3R_5 + a_6{}^3d_4R_6 + a_7{}^3d_5R_7 + a_8{}^3d_6R_8 + a_9{}^3d_7R_9 = \frac{1}{21}$$

$$[[\phi]^3] = c_1{}^3R_3 + c_2{}^3R_4 + c_3{}^3R_5 + c_4{}^3R_6 + c_5{}^3R_7 + c_6{}^3R_8 + c_7{}^3R_9 = \frac{1}{56}$$

$$[[\phi]^2\phi^2] = a_3{}^2c_1{}^2R_3 + a_4{}^2c_2{}^2R_4 + a_5{}^2c_3{}^2R_5 + a_6{}^2c_4{}^2R_6 + a_7{}^2c_5{}^2R_7 + a_8{}^2c_6{}^2R_8 + a_9{}^2c_7{}^2R_9 = \frac{1}{28}$$

$$[[\phi]\phi^4] = a_3{}^4c_1R_3 + a_4{}^4c_2R_4 + a_5{}^4c_3R_5 + a_6{}^4c_4R_6 + a_7{}^4c_5R_7 + a_8{}^4c_6R_8 + a_9{}^4c_7R_9 = \frac{1}{14}$$

$$[\phi^6] = a_2{}^6R_2 + a_3{}^6R_3 + a_4{}^6R_4 + a_5{}^6R_5 + a_6{}^6R_6 + a_7{}^6R_7 + a_8{}^6R_8 + a_9{}^6R_9 = \frac{1}{7}$$