

[ On the Limb Darkening of Planetary  
Atmospheres in the Thermal Infrared

Thermal maps of Venus (Murray, et al., 1963; Westphal, et al., 1965) and Jupiter (Murray, et al., 1964) have demonstrated that limb darkening in the 8-14 micron wavelength interval is pronounced for these two planets. Since both these planets are completely cloud-covered, it is possible that both anisotropic multiple scattering (Goody, 1964) and a temperature gradient (Pollack and Sagan, 1965) are responsible. It will be the purpose of this note to demonstrate how exact solutions to the standard problem--that of describing the angular distribution of outgoing radiation from a slab of material bounded on both sides by a vacuum (Chandrasekhar, 1950)--may be expressed in terms of Chandrasekhar's  $\psi$ - and  $\phi$ - functions of zero order when both scattering and temperature gradient effects are present. The additional problem of imposing the correct lower boundary conditions to account for surface effects in order to specify the angular distribution of outgoing thermal radiation (limb darkening) from the atmosphere is quite straight-forward (cf. Chandrasekhar, 1950, pp. 269-274) and will not be presented here.

Let  $I_p(\tau, \mu)$  be the monochromatic specific intensity of thermal radiation in a doubly infinite plane-parallel atmosphere. Consider an atmosphere of finite thickness  $\tau_1 = \tau_3 - \tau_2$  having scattering and thermal emission properties identical with those of the doubly infinite atmosphere has in the range  $(\tau_2 \leq \tau \leq \tau_3)$ . Let  $I(0, +\mu)$  and  $I(\tau_1, -\mu)$

HC # 3.00  
MF 1.65

FACILITY FORM 602

**N67 18280**  
(ACCESSION NUMBER)

12  
(PAGES)

TX-57359  
(NASA CR OR TMX OR AD NUMBER)

(THRU)

(CODE)

30  
(CATEGORY)

designate the outgoing radiation fields in the directions  $+\mu$  and  $-\mu$  ( $0 < \mu \leq 1$ ) at the levels  $\tau = 0$  and  $\tau = \tau_1$  respectively.

By developing an invariance principle known to Chandrasekhar (1950), Mullikin (1962) and many others, it is found that the solutions for the standard problem are given by

$$I(0, +\mu) = I_p(0, +\mu) - \frac{1}{2\mu} \int_0^1 S(\tau; \mu, \mu') I_p(0, -\mu') d\mu' - e^{-\tau/\mu} I_p(\tau_1, +\mu) - \frac{1}{2\mu} \int_0^1 T(\tau; \mu, \mu') I_p(\tau_1, +\mu') d\mu' \quad (1)$$

and

$$I(\tau_1, -\mu) = I_p(\tau_1, -\mu) - \frac{1}{2\mu} \int_0^1 S(\tau; \mu, \mu') I_p(\tau_1, +\mu') d\mu' - e^{-\tau/\mu} I_p(0, -\mu) - \frac{1}{2\mu} \int_0^1 T(\tau; \mu, \mu') I_p(0, -\mu') d\mu', \quad (2)$$

where

$$I(0, -\mu) = I(\tau_1, +\mu) \equiv 0 \quad (0 < \mu \leq 1), \quad (3)$$

and where  $S$  and  $T$  are the diffuse scattering and transmission functions in Chandrasekhar's (1950) notation for atmospheres of normal optical thickness  $\tau_1$ . Physically (1) and (2) arise from the fact that any mass element at a level  $\tau$  ( $0 \leq \tau \leq \tau_1$ ) is unable to distinguish between being imbedded in a doubly infinite atmosphere, and being imbedded in a finite atmosphere that is irradiated on the top and bottom by the radiation fields  $I_p(0, -\mu)$  and  $I_p(\tau_1, +\mu)$  respectively.

Consider the equation of transfer for a partially thermally emitting, partially anisotropically scattering plane-parallel atmosphere:

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{+1} p(\tau, \mu, \mu') I(\tau, \mu') d\mu' - (1 - \omega_0) B(\tau). \quad (4)$$

Let  $p(\tau; \mu, \mu') = p(\mu, \mu')$  be independent of  $\tau$ . Further let  $p(\mu, \mu')$  be the azimuth-independent part of the expansion in spherical harmonics of a finite Legendre polynomial series approximation of the phase function for single scattering, and let  $B(\tau)$  be approximated by a finite power series in  $\tau$ ; i.e.,

$$p(\mu, \mu') = \sum_{l=0}^N \omega_l P_l(\mu) P_l(\mu') \quad (5)$$

and

$$B(\tau) = \sum_{r=0}^M b_r \tau^r. \quad (6)$$

It can be shown (cf. Samuelson, 1965) that a particular integral satisfying equation (4) under conditions (5) - (6) is given by

$$I_p(\tau, \mu) = \sum_{r=0}^M \tau^r \sum_{s=r}^M c_{r,s} P_{s-r}(\mu), \quad (7)$$

where the constant coefficients  $c_{r,s}$  are uniquely generated by the recursion relations

$$\left. \begin{aligned} \left[1 - \frac{\bar{\omega}_{M-r}}{2(M-r)+1}\right] C_{r,M} &= \frac{(r+1)(M-r)}{2(M-r)-1} C_{r+1,M} & (0 \leq r \leq M-1) \\ \left[1 - \frac{\bar{\omega}_{M-r-1}}{2(M-r)-1}\right] C_{r,M-1} &= \frac{(r+1)(M-r-1)}{2(M-r)-3} C_{r+1,M-1} & (0 \leq r \leq M-2) \\ (1-\bar{\omega}_0)(C_{r,r}-b_r) &= \frac{1}{3}(r+1) C_{r+1,r+2} & (0 \leq r \leq M-2) \\ \left[1 - \frac{\bar{\omega}_{s-r}}{2(s-r)+1}\right] C_{r,s} &= (r+1) \left[ \frac{(s-r-1)}{2(s-r)+3} C_{r+1,s+2} + \frac{(s-r)}{2(s-r)-1} C_{r+1,s} \right] & \begin{matrix} (0 \leq r \leq M-3) \\ (r+1 \leq s \leq M-2) \end{matrix} \end{aligned} \right\} \quad (8)$$

where

$$\left. \begin{aligned} C_{M,M} &= b_M \\ C_{M-1,M-1} &= b_{M-1} \\ \bar{\omega}_\lambda &= 0 \quad (\lambda > N) \end{aligned} \right\} \quad (9)$$

That equation (7) is also the complete solution for a doubly infinite atmosphere is rendered plausible on physical grounds by noting that [cf. eq. (6)]

$$\lim_{\tau \rightarrow \infty} \left[ \frac{1}{2} \int_{-1}^{+1} I_p(\tau, \mu) d\mu \right] = \lim_{\tau \rightarrow \infty} [B(\tau)] \quad (10)$$

and

$$\lim_{\tau \rightarrow \infty} \left[ 2\pi \int_{-1}^{+1} \mu I_p(\tau, \mu) d\mu \right] = \frac{4\pi}{3-\bar{\omega}} \lim_{\tau \rightarrow \infty} \left[ \frac{dB(\tau)}{d\tau} \right]; \quad (11)$$

i.e. the limiting values of the average specific intensity and the net flux are respectively equal to and proportional to the limiting values of the Planck function and its divergence.

Now Chandrasekhar's  $\psi$ - and  $\phi$ - functions of order zero and degree are defined by (Chandrasekhar, 1950)

$$\psi_{\ell}(\tau; \mu) = P_{\ell}(\mu) + \frac{(-1)^{\ell}}{2} \int_0^1 S(\tau; \mu, \mu') P_{\ell}(\mu') \frac{d\mu'}{\mu'} \quad (12)$$

and

$$\phi_{\ell}(\tau; \mu) = e^{-\tau/\mu} P_{\ell}(\mu) + \frac{1}{2} \int_0^1 T(\tau; \mu, \mu') P_{\ell}(\mu') \frac{d\mu'}{\mu'} \quad (13)$$

Upon multiplying through both numerator and denominator in all the integrands of equations (1) - (2) by  $\mu'$ , it is readily demonstrated, with the aid of equations (7) and (12) - (13) and the relation

$$(2\ell+1)\mu P_{\ell}(\mu) = (\ell+1)P_{\ell+1}(\mu) + \ell P_{\ell-1}(\mu), \quad (14)$$

that (1) and (2) respectively reduce to

$$I(0, +\mu) = \frac{1}{\mu} \sum_{s=0}^M \frac{C_{0,s}}{2s+1} \left[ (s+1) \psi_{s+1}(\tau; \mu) + s \psi_{s-1}(\tau; \mu) \right] \\ - \frac{1}{\mu} \sum_{r=0}^M \tau_r^r \sum_{s=r}^M \frac{C_{r,s}}{2(s-r)+1} \left[ (s-r+1) \phi_{s-r+1}(\tau; \mu) + (s-r) \phi_{s-r-1}(\tau; \mu) \right] \quad (15)$$

and

$$I(\tau, -\mu) = \frac{1}{\mu} \sum_{r=0}^M \tau_r^r \sum_{s=r}^M \frac{(-1)^s C_{r,s}}{2(s-r)+1} \left[ (s-r+1) \psi_{s-r+1}(\tau; \mu) + (s-r) \psi_{s-r-1}(\tau; \mu) \right] \\ - \frac{1}{\mu} \sum_{s=0}^M \frac{(-1)^s C_{0,s}}{2s+1} \left[ (s+1) \phi_{s+1}(\tau; \mu) + s \phi_{s-1}(\tau; \mu) \right], \quad (16)$$

where all  $\psi$  and  $\phi$  of negative degree are defined to be bounded in the interval  $(0 \leq \mu \leq 1)$ .

Equations (15) - (16) are notable in two respects. First, the only non-elementary functions involved are the  $\psi$ - and  $\phi$ - functions of zero order. For a given  $\tau_1$  and set of  $\tilde{\omega}_\ell$   $\{\ell = 0, \dots, N[\text{cf. also eq. (9)}]\}$  these functions may be calculated once and for all with the aid of existing computer programs (cf. Churchill, et al., 1961). And secondly, the complete description of the thermal properties of the atmosphere are contained entirely in the constant coefficients  $c_{r,s}$  ( $r = 0, \dots, M$ ;  $s = r, \dots, M$ ). In view of the foregoing remarks and the simplicity of relations (8), the practical problem of solving (15) - (16) numerically should not be inordinately difficult. The additional problem of calculating the planetary limb darkening under restrictions (5) - (6) in accordance with Chandrasekhar's discussion of the "planetary problem" (loc. cit.) is quite straightforward.

R. E. Samuelson  
Goddard Space Flight Center  
Greenbelt, Maryland

## References

- Chandrasekhar, S. 1950, Radiative Transfer (Oxford: Clarendon Press), p. 92; pp. 177-180; Appendix III, pp. 383-385.
- Churchill, S., Chu, C., Evans, L., Tien, L., and Pang, S. 1961, U. of Mich. Rept. DASA-1257 (Ann Arbor: The University of Michigan).
- Goody, Richard. 1964, Icarus, 4, 98.
- Mullikin, T. W. 1962, Rand Mem. RM-3209-NASA (Santa Monica, Calif.: The Rand Corporation), p. 87.
- Murray, B. C., Wildey, R. L., and Westphal, J. A. 1963, J. Geophys. Res., 68, 4813.
- Murray, B. C., Wildey, R. L., and Westphal, J. A. 1964, Ap. J., 139, 986.
- Pollack, James B., and Sagan, Carl. 1965, J. Geophys. Res., 70, 4403.
- Samuelson, R. E. 1965, NASA Tech. Rept. R-215 (Washington, D. C.: National Aeronautics and Space Administration), Appendix B, pp. 61-67.
- Westphal, J. A., Wildey, R. L., and Murray, B. C. 1965, Ap. J., 142, 799.