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## ABSTRACT

The second term has been obtained in the asymptotic series for the second (longitudinal) adiabatic invariant of charged particle motion in a static magnetic field. This correction to the lowest order invariant has two sources: the correction to the lowest order magnetic moment and the integrated effect of the guiding-center drift across the field lines. The second term is found to vanish at the mirror points; therefore during its motion between mirror reflections, the guiding center deviates from the surface on which the lowest order invariant is constant and intersects this surface at reflection.

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## I. INTRODUCTION

This paper contains a systematic derivation of the next term beyond the lowest order for the second adiabatic invariant of a charged particle in a magnetic field. A charged particle in an electromagnetic field possesses up to three approximate invariants of its motion. The first is the Alfvén invariant[1] or magnetic moment $\mathfrak{p}_\perp^2/2m_0B$, where $\mathfrak{p}_\perp$ is the component of particle momentum $m\mathbf{v}$ perpendicular to the magnetic field $\mathbf{B}$. If there is an electric field, $\mathfrak{p}_\perp$ is the momentum in the reference frame in which the electric field vanishes. Magnetic moment invariance requires that fields vary slowly compared to the particle gyration period and gradually compared to the gyration radius. The second invariant[2,3] is an invariant of the guiding-center equations of motion, which are equations obtained by averaging over the particle gyration about the magnetic field line. The second invariant is therefore, also an invariant of the particle motion, from which the guiding-center equations are derived. This second invariant exists when the guiding-center motion along a field line is nearly periodic; the invariant is $\oint \mathfrak{p}_\parallel ds$, where $\mathfrak{p}_\parallel$ is the component of guiding-center momentum parallel to $\mathbf{B}$, and the integral extends over a period of the motion in s, which is distance along the line. The third invariant[3] is an invariant of equations of motion resulting when the guiding-center equations are averaged over the periodic motion along the field line (and therefore is an invariant of the particle motion also); it exists when these doubly averaged equations have nearly periodic solutions, and this occurs when the surfaces of constant second invariant are topologically cylinders. The third invariant $\Phi$ is the magnetic flux threading a second invariant cylinder. Its invariance is trivially true in static fields and becomes of significance only in time-dependent fields.

The three invariants described above are really only the lowest orders of three asymptotic series of the form

$$\begin{aligned} \text{constant} &= M_0 + \epsilon M_1 + \epsilon^2 M_2 + \text{--------}\,, \\ \text{constant} &= J_0 + \epsilon J_1 + \epsilon^2 J_2 + \text{--------}\,, \qquad (1) \\ \text{constant} &= \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \text{--------}\,, \end{aligned}$$

where $M_0$ is $\mathfrak{p}_\perp^2/2m_0B$, $J_0$ is $\oint \mathfrak{p}_\parallel ds$, and $\Phi_0$ is the magnetic flux threading through a surface of constant $J_0$. The expansion parameter $\epsilon$ is the mass-to-charge ratio of the particle. The invariance of $M_0$, $J_0$, and $\Phi_0$ can be surmised (and demonstrated) by rather physical methods, while to obtain higher order terms in each series may require deep insight or a systematic method. In the present paper we use a systematic method due to Kruskal[4] to obtain $J_1$ for static magnetic fields. Although the second invariant exists whenever the motion along a field line is nearly periodic, we confine ourselves to the case where the motion is oscillatory between two mirrors. The case where the particle traverses a closed field line always in the same sense must be treated separately in the systematic derivation. However, a direct derivation of $J_1$ to be given at the end of the paper shows that the result is the same as for the oscillatory case. The proof of the invariance of $J_0$ in reference 3 is valid for the closed field-line case as well as for the oscillatory case.

## II. THE SYSTEMATIC METHOD

The systematic method presupposes coupled equations of motion of the form:

$$d\underset{\sim}{x}/d\tau = \underset{\sim}{f}(\underset{\sim}{x}, \epsilon), \tag{2}$$

where $\underset{\sim}{x}$ has a finite number N of components and $\underset{\sim}{f}$ possesses a power series expansion in the small parameter $\epsilon$. Furthermore, it is required that all solutions of the system $d\underset{\sim}{x}/d\tau = \underset{\sim}{f}(\underset{\sim}{x}, 0)$ traverse closed trajectories in $\underset{\sim}{x}$ space. The equation of motion of a charged particle can be put into this form with $\epsilon$ equal to m/e. The method shows how to obtain a transformation from the N variables to another set $(\underset{\sim}{z}, \phi)$ which have the property that the equations of motion are

$$\begin{aligned} d\underset{\sim}{z}/d\tau &= \epsilon \underset{\sim}{h}(\underset{\sim}{z}, \epsilon), \\ d\phi/d\tau &= \omega(\underset{\sim}{z}, \epsilon), \end{aligned} \tag{3}$$

the important point being that $\phi$ does not appear on the right sides, and that $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{z}, \phi, \epsilon)$ is periodic in $\phi$. The vector $\underset{\sim}{z}$ has N-1 components. The transformation and the new functions $\underset{\sim}{h}$ and $\omega$ are obtained as series in $\epsilon$.

If in addition the equations (2) are canonical, with $\underset{\sim}{x}$ the vector $(\underset{\sim}{p}, \underset{\sim}{q})$, then there exists the adiabatic invariant

$$\text{constant} = \oint \underset{\sim}{p}(\underset{\sim}{z}, \phi, \epsilon) \cdot \frac{\partial \underset{\sim}{q}(\underset{\sim}{z}, \phi, \epsilon)}{\partial \phi}\, d\phi, \tag{4}$$

where $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are the canonical momentum and position, and where the integral is over a period of $\phi$. The invariant is obtained as a series in $\epsilon$ and, although a function of the $\mathbf{z}$ variables, may be rewritten in terms of the original $\mathbf{x}$ variables by inverting the transformation. To have an invariant, it is not really necessary that (2) themselves be canonical-only that they be transformable into canonical equations.

The transformation $\mathbf{x}$ to $(\mathbf{z}, \phi)$ is not made directly, but for convenience by way of intermediate variables $(\mathbf{y}, \upsilon)$, where $\mathbf{y}$ is any vector constant on the closed, lowest-order ($\epsilon = 0$) solutions of (2), and $\upsilon$ is an angle-like variable specifying the position around such closed curves.

The theory as described so far will produce only one adiabatic invariant series (the magnetic moment), but to any order in $\epsilon$ desired. The second (and third) adiabatic invariants are obtained from "reduced" equations of motion as follows: it can be shown that the Poisson bracket $[\phi, M]$ equals 1, which means that $\phi$ and $M$ can be used as conjugate variables in a canonical transformation from $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{p}', M; \mathbf{q}', \phi)$ where $\mathbf{p}'$ and $\mathbf{q}'$ have each one less component than $\mathbf{p}$ and $\mathbf{q}$. ($M$ is the sum of the magnetic moment series to the order in $\epsilon$ to which one is working). The new Hamiltonian $\mathcal{H}'(\mathbf{p}', \mathbf{q}', M, \epsilon)$ is independent of $\phi$ since $\dot{M}$ equals $-\partial\mathcal{H}'/\partial\phi$, and $\dot{M}$ is zero, being the first invariant. Thus the reduced system of equations is:

$$\dot{q}_i' = \frac{\partial \mathcal{H}'(\underset{\sim}{p}', \underset{\sim}{q}', M, \epsilon)}{\partial p_i'},$$

(5)

$$\dot{p}_i' = -\frac{\partial \mathcal{H}'(\underset{\sim}{p}', \underset{\sim}{q}', M, \epsilon)}{\partial q_i'}.$$

If these again have the property that all solutions are closed in $(\underset{\sim}{p}', \underset{\sim}{q}')$ space when $\epsilon$ is zero, then a second invariant exists in terms of new $\underset{\sim}{z}$ variables, which we will call $\underset{\sim}{z}'$:

$$J(\underset{\sim}{z}') = \oint \underset{\sim}{p}'(\underset{\sim}{z}', \phi') \cdot \frac{\partial \underset{\sim}{q}'(\underset{\sim}{z}', \phi')}{\partial \phi'} d\phi' \tag{6}$$

New intermediate variables $(\underset{\sim}{y}', \upsilon')$ will generally also be used. Finally a second reduction can be performed to obtain the third invariant. The variables and transformations are illustrated in Fig. 1. Each transformation is carried out as a series in the expansion parameter $\epsilon$.

Although $(\underset{\sim}{p}', \underset{\sim}{q}', M, \phi)$ are shown by the arrow in Fig. 1 as coming from $(\underset{\sim}{p}, \underset{\sim}{q})$, they really come from the entire first line, since $(\underset{\sim}{y}, \upsilon)$ and $(\underset{\sim}{z}, \phi)$ must be found in order to determine M. A similar statement holds from the 2nd invariant level to the 3rd.

$$(\underset{\sim}{p}, \underset{\sim}{q}) \longleftrightarrow (\underset{\sim}{y}, \upsilon) \longleftrightarrow (\underset{\sim}{z}, \phi) \dashrightarrow \text{1st invariant (the magnetic moment series M)}$$

$$\updownarrow$$

$$(\underset{\sim}{p}', \underset{\sim}{q}', M, \phi) \longleftrightarrow (\underset{\sim}{y}', \upsilon', M, \phi) \longleftrightarrow (\underset{\sim}{z}', \phi', M, \phi) \dashrightarrow$$

2nd invariant (the longitudinal invariant series J)

$$\updownarrow$$

$$(p'', q'', J, \phi', M, \phi) \longleftrightarrow (y'', \upsilon'', J, \phi', M, \phi) \longleftrightarrow (z'', \phi'', J, \phi', M, \phi) \dashrightarrow$$

3rd invariant (the flux series $\Phi$)

Firgure 1: The variables used at the level of first, second, and third adiabatic theory.

With so many transformations and variables, tricks for reducing the labor are very welcome. There are several such shortcuts. For one thing, $\phi$ is never needed, since it is simpler to use $\upsilon$ as the integration variable in (4). Likewise in (6), $\upsilon'$ is to be used. Secondly, if the $d\underset{\sim}{z}/d\tau$ equations of motion (3) have all solutions periodic when $\epsilon$ (or another parameter) is zero, as for Eq. (2), or if they can be transformed so that this is so, the theory following Eq. (2) may be repeated. For the charged particle, the equations of motion (3) for $\underset{\sim}{z}$ are themselves of the prescribed form (2) with the needed periodicity when $\epsilon = 0$, after a trivial rescaling of the independent variable. Thus $(\underset{\sim}{y}', \upsilon')$ and $(\underset{\sim}{z}', \phi')$ can be obtained from the $\underset{\sim}{z}$ variables rather than by the pathway of $(\underset{\sim}{p}', \underset{\sim}{q}')$. Furthermore, it is proved in reference (4) that:

$$\oint \underset{\sim}{p}'(\underset{\sim}{z}', \phi') \cdot \frac{\partial \underset{\sim}{q}'(\underset{\sim}{z}', \phi')}{\partial \phi'} d\phi' = \oint \underset{\sim}{p}(\underset{\sim}{z}', \phi', \phi) \cdot \frac{\partial \underset{\sim}{q}(\underset{\sim}{z}', \phi', \phi)}{\partial \phi'} d\phi, \qquad (7)$$

that is, the original canonical variables may be used. Thus the reduced canonical variables $(\underset{\sim}{p}', \underset{\sim}{q}')$, and $\mathcal{U}'$, need not be found. The pathway of transformations actually followed in the present calculation is illustrated in Fig. 2. The transformation is between $\underset{\sim}{z}$ and $(\underset{\sim}{y}', \upsilon')$ and not between $(\underset{\sim}{z}, \phi)$ and $(\underset{\sim}{y}', \upsilon')$.

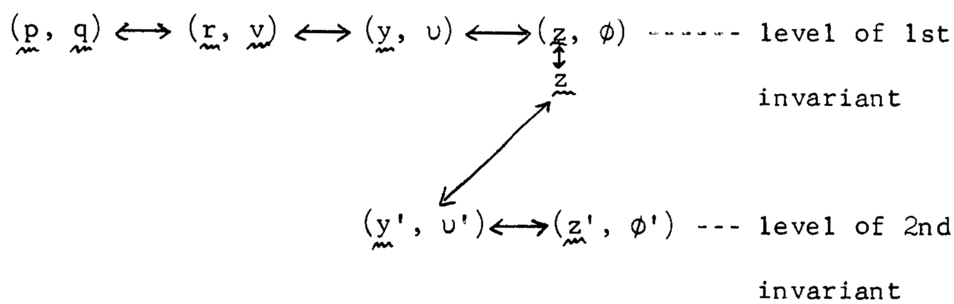


Figure 2: The transformations and variables actually used in this calculation.

Here $(\underset{\sim}{r}, \underset{\sim}{v})$ are the position and velocity of the particle. The $\underset{\sim}{x}$ variables can be regarded as either $(\underset{\sim}{p}, \underset{\sim}{q})$ or $(\underset{\sim}{r}, \underset{\sim}{v})$, since the particle equations of motion can be written in the form of (2) with the required properties using either set.

In (7) the right side appears to be a function of $\phi$ but really is not, since the left side is not; the $\phi$ dependence of the right side actually must disappear. This fact gives another shortcut -- namely, that any particular value of $\phi$ desired can be used in the functions $\underset{\sim}{p}(\underset{\sim}{z}', \phi', \phi)$ and $\underset{\sim}{q}(\underset{\sim}{z}', \phi', \phi)$. The choice $\phi = 0$ greatly reduces the algebra.

A fourth way to simplify the calculation is to choose the $\underset{\sim}{y}$ variables cleverly. The $\underset{\sim}{y}$ vector is required only to be a constant of the lowest-order motion, and the $(\underset{\sim}{z}, \phi)$ follow uniquely from the $(\underset{\sim}{y}, \upsilon)$. There clearly are infinitely many suitable choices for the $\underset{\sim}{y}$ variables, since any function of a given $\underset{\sim}{y}$ is also a constant. The guiding principle is to choose for the components of $\underset{\sim}{y}$ quantities which are both simple and constant to as high an order in $\epsilon$ as possible, even though the theory only requires $\underset{\sim}{y}$ to be a constant of the lowest-order motion. For example, one component of $\underset{\sim}{y}'$ is much better chosen as $M_0 + \epsilon M_1$ rather than just $M_0$ alone. Similarly, one component of $\underset{\sim}{y}$ is much better chosen as the approximate guiding-center position than as the particle position, since the guiding center does not move as rapidly as the particle.

Although this systematic method of calculating invariant series in principle requires only routine labor, in practice the amount of algebra gets out of hand rather rapidly unless the above short-cuts are used, along with physical intuition in the choice of $\underset{\sim}{y}$. After this long but necessary discussion of the method, we now begin the actual calculation.

## III. THE CHARGED PARTICLE

The particle equation of motion can be written as

$$d\underset{\sim}{r}/d\tau = \epsilon\underset{\sim}{v},$$
$$d\underset{\sim}{v}/d\tau = \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}), \qquad (8)$$

where $\underset{\sim}{x}$ is $(\underset{\sim}{r}, \underset{\sim}{v})$ and $\tau$ is $t/\epsilon$. When $\epsilon$ is zero, $\underset{\sim}{r}$ is constant and the second equation (8) is the equation of motion in a uniform magnetic field $\underset{\sim}{B}$. The solution $\underset{\sim}{v}$ has a constant projection along $\underset{\sim}{B}$ and is (harmonically) periodic in its two components perpendicular to $\underset{\sim}{B}$. Thus the trajectory is periodic in $\underset{\sim}{x}$ space. Let $\hat{\underset{\sim}{L}}$, $\hat{\underset{\sim}{M}}$, and $\hat{\underset{\sim}{N}}$ be three orthogonal unit vectors, with $\hat{\underset{\sim}{L}}$ parallel to $\underset{\sim}{B}$; $\hat{\underset{\sim}{M}}$ and $\hat{\underset{\sim}{N}}$ need not be specified further. The $\underset{\sim}{y}$ variables will be denoted by $(\underset{\sim}{\rho}, \sigma, \eta)$ and defined as

$$\underset{\sim}{\rho} = \underset{\sim}{r} + \epsilon\underset{\sim}{v} \times \hat{\underset{\sim}{L}}(\underset{\sim}{r})/B(\underset{\sim}{r}),$$
$$\sigma = |v_\perp|,$$
$$\eta = \hat{\underset{\sim}{L}}(\underset{\sim}{r}) : \underset{\sim}{v}. \qquad (9)$$

The choice of $\underset{\sim}{r}$ itself instead of the guiding center was tried for $\underset{\sim}{\rho}$ but led to a more difficult calculation. The $\underset{\sim}{y}$ components $\eta$ and $\sigma$

are the components of particle velocity parallel and perpendicular to the field line at the particle position. The $\upsilon$ variable is chosen as

$$\upsilon = \frac{1}{2\pi} \text{ arc tan } \frac{\underset{\sim}{v} \cdot \hat{\underset{\sim}{M}}(\underset{\sim}{r})}{\underset{\sim}{v} \cdot \hat{\underset{\sim}{N}}(\underset{\sim}{r})} \tag{10}$$

The inverse transformation can be obtained to any order desired from

$$\begin{aligned} \underset{\sim}{v} &= \eta\hat{\underset{\sim}{L}}(\underset{\sim}{r}) + \sigma[\hat{\underset{\sim}{M}}(\underset{\sim}{r}) \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}}(\underset{\sim}{r}) \cos\tilde{\upsilon}], \\ \underset{\sim}{r} &= \underset{\sim}{\rho} + \frac{\epsilon\sigma}{B(\underset{\sim}{r})}[\hat{\underset{\sim}{N}}(\underset{\sim}{r}) \sin\tilde{\upsilon} - \hat{\underset{\sim}{M}}(\underset{\sim}{r}) \cos\tilde{\upsilon}], \end{aligned} \tag{11}$$

where $\tilde{\upsilon} = 2\pi\upsilon$. The equations of motion for $\underset{\sim}{y}$ ($d\upsilon/d\tau$ is not needed) are:

$$\begin{aligned} d\sigma/d\tau &= -(\eta\epsilon/\sigma)\, \underset{\sim}{v}\underset{\sim}{v}:\nabla\hat{\underset{\sim}{L}}(\underset{\sim}{r}), \\ d\eta/d\tau &= \epsilon\underset{\sim}{v}\underset{\sim}{v}:\nabla\hat{\underset{\sim}{L}}(\underset{\sim}{r}), \\ d\underset{\sim}{\rho}/d\tau &= \epsilon\eta\hat{\underset{\sim}{L}}(\underset{\sim}{r}) + \frac{\epsilon^2}{B}\Big\{\eta^2\hat{\underset{\sim}{L}} \times (\hat{\underset{\sim}{L}}\cdot\nabla\hat{\underset{\sim}{L}}) + \sigma^2[(\hat{\underset{\sim}{M}} \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}} \cos\tilde{\upsilon}) \times \\ &\Big((\hat{\underset{\sim}{M}} \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}} \cos\tilde{\upsilon})\cdot\nabla\hat{\underset{\sim}{L}}\Big) + (\hat{\underset{\sim}{N}} \sin\tilde{\upsilon} - \hat{\underset{\sim}{M}} \cos\tilde{\upsilon})\,(\hat{\underset{\sim}{M}} \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}} \cos\tilde{\upsilon})\cdot\frac{\nabla B}{B}] \\ &+ \eta\sigma[\hat{\underset{\sim}{M}} \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}} \cos\tilde{\upsilon}) \times (\hat{\underset{\sim}{L}}\cdot\nabla\hat{\underset{\sim}{L}}) + \hat{\underset{\sim}{L}} \times \Big((\hat{\underset{\sim}{M}} \sin\tilde{\upsilon} + \hat{\underset{\sim}{N}} \cos\tilde{\upsilon})\cdot\nabla\hat{\underset{\sim}{L}}\Big)]\Big\} \end{aligned} \tag{12}$$

where the : notation means contraction first of the two inner vectors and then of the two outer ones. For example in (13b) $\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} : \nabla\hat{\underset{\sim}{L}}$ means $\hat{\underset{\sim}{M}} \cdot [(\hat{\underset{\sim}{N}}\cdot\nabla)\,\hat{\underset{\sim}{L}}]$.

The components of $\underset{\sim}{z}$ will be denoted by the Greek capitals of the corresponding $\underset{\sim}{y}$ components - i.e., by $(\underset{\sim}{P}, \Sigma, H)$. By use of the recursion and periodicity relations (B19 – B28) in reference (4), we obtain

$$\underset{\sim}{P} = \underset{\sim}{\rho} + O(\epsilon^2), \tag{13a}$$

$$\Sigma = \sigma + \frac{\epsilon}{B(\underset{\sim}{\rho})}\Big\{\eta^2(-\hat{\underset{\sim}{M}} \cos \tilde{\upsilon} + \hat{\underset{\sim}{N}} \sin \tilde{\upsilon} + \hat{\underset{\sim}{M}})\ \hat{\underset{\sim}{L}}:\nabla \hat{\underset{\sim}{L}} + \eta\sigma[-\tfrac{1}{2}\hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}} \sin \tilde{\upsilon} \cos \tilde{\upsilon} + \tfrac{1}{2}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} + \hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}}) \sin^2 \tilde{\upsilon} + \tfrac{1}{2}\hat{\underset{\sim}{N}}\hat{\underset{\sim}{N}} \sin \tilde{\upsilon} \cos \tilde{\upsilon}] : \nabla \hat{\underset{\sim}{L}}\Big\} + O(\epsilon^2), \tag{13b}$$

$$H = \eta + \frac{\epsilon}{B(\underset{\sim}{\rho})}\Big\{-\eta\sigma(-\hat{\underset{\sim}{M}} \cos \tilde{\upsilon} + \hat{\underset{\sim}{N}} \sin \tilde{\upsilon} + \hat{\underset{\sim}{M}})\ \hat{\underset{\sim}{L}} : \nabla \hat{\underset{\sim}{L}} - \sigma^2[-\tfrac{1}{2}\hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}} \sin \tilde{\upsilon} \cos \tilde{\upsilon} + \tfrac{1}{2}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} + \hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}}) \sin^2 \tilde{\upsilon} + \tfrac{1}{2}\hat{\underset{\sim}{N}}\hat{\underset{\sim}{N}} \sin \tilde{\upsilon} \cos \tilde{\upsilon}] : \nabla \hat{\underset{\sim}{L}}\Big\} + O(\epsilon^2), \tag{13c}$$

$$\phi = \int_0^{\upsilon} \text{-----------}. \tag{13d}$$

The expression for $\phi$ is not needed, since $\upsilon$ will be used as the integration variable; (13d) is included only to show that $\phi = 0$ when $\upsilon = 0$, a fact that will be used later. The $\epsilon^2$ terms of $(\underset{\sim}{P}, \Sigma, H)$ have also been calculated, but are too lengthy to include here. Note that choosing $\underset{\sim}{\rho}$ to be the guiding center has made the $\epsilon$ contribution to $\underset{\sim}{P}$ vanish, and this will afford much simplification later.

The equations of motion for the $\underset{\sim}{z}$ variables ($d\phi/d\tau$ is not needed) are:

$$\frac{1}{\epsilon}\frac{d\underset{\sim}{P}}{d\tau} = H\hat{\underset{\sim}{L}}(\underset{\sim}{P}) + \frac{\epsilon}{B(\underset{\sim}{P})} H^2\hat{\underset{\sim}{L}} \times (\hat{\underset{\sim}{L}} \cdot \nabla \hat{\underset{\sim}{L}}) + \frac{\epsilon\Sigma^2}{2B^2}\hat{\underset{\sim}{L}} \times \nabla B + \frac{\epsilon\Sigma^2}{2B}\hat{\underset{\sim}{L}}\ (\tfrac{3}{2}\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} - \tfrac{1}{2}\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}}) : \nabla \hat{\underset{\sim}{L}} + \frac{\epsilon}{B} H\Sigma\hat{\underset{\sim}{L}}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}} : \nabla \hat{\underset{\sim}{L}}) + O(\epsilon^2), \tag{14a}$$

$$\frac{1}{\epsilon}\frac{d\Sigma}{d\tau} = -\frac{H\Sigma}{2}\nabla\cdot\hat{\underset{\sim}{L}} + \frac{\epsilon H^3}{B}\left[B\nabla\cdot\frac{\hat{\underset{\sim}{L}}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}} : \nabla\hat{\underset{\sim}{L}})}{B} - \tfrac{1}{2}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}} : \nabla\hat{\underset{\sim}{L}})\nabla\cdot\hat{\underset{\sim}{L}}\right]$$

$$+ \frac{\epsilon}{B}\frac{H^2\Sigma}{2}\left\{\tfrac{1}{2}\nabla\cdot[\hat{\underset{\sim}{L}}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} + \hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}}) : \nabla\hat{\underset{\sim}{L}}] + B\nabla\cdot\frac{\hat{\underset{\sim}{M}}(\hat{\underset{\sim}{N}}\hat{\underset{\sim}{L}} : \nabla\hat{\underset{\sim}{L}})}{B}\right.$$

$$\left. - B\nabla\cdot\frac{\hat{\underset{\sim}{N}}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}} : \nabla\hat{\underset{\sim}{L}})}{B}\right\} + \frac{\epsilon}{B}\frac{H\Sigma^2}{2}(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}} : \nabla\hat{\underset{\sim}{L}})\nabla\cdot\hat{\underset{\sim}{L}} + O(\epsilon^2), \tag{14b}$$

$$\frac{1}{\epsilon}\frac{dH}{d\tau} = \frac{\Sigma^2}{2}\nabla\cdot\hat{\underset{\sim}{L}} - \frac{\Sigma}{H}\text{ times the order }\epsilon\text{ terms of }\frac{1}{\epsilon}\frac{d\Sigma}{d\tau} + O(\epsilon^2). \tag{14c}$$

Since $\epsilon\tau$ is time t, the left sides are the guiding-center velocity $d\underset{\sim}{P}/dt$, etc. The component of $d\underset{\sim}{P}/dt$ perpendicular to the field is the sum of the usual gradient-B and line-curvature drifts, while the parallel motion has both zero-order and first-order components. The equations for $d\Sigma/dt$ and $dH/dt$ give

$$\frac{d}{dt}\frac{\Sigma^2 + H^2}{2} = O(\epsilon^2) \tag{15}$$

which is energy conservation, while those for $d\Sigma/dt$ and $d\underset{\sim}{P}/dt$ give

$$\frac{d}{dt}\frac{\Sigma^2}{2B(\underset{\sim}{P})} = O(\epsilon) \tag{16}$$

which is conservation of the lowest-order magnetic moment $M_0$.

The so-called "guiding-center equations", which are usual in numerical integrations designed to follow the guiding center, are the set (see for example reference 5)

$$\frac{d\underset{\sim}{P}}{dt} = H\hat{\underset{\sim}{L}}(\underset{\sim}{P}) + \frac{\epsilon H^2}{B(\underset{\sim}{P})}\,\hat{\underset{\sim}{L}} \times (\hat{\underset{\sim}{L}}\cdot\nabla\hat{\underset{\sim}{L}}) + \frac{\epsilon\Sigma^2}{2B^2}\,\hat{\underset{\sim}{L}} \times\nabla B,$$

$$\tfrac{1}{2}(\Sigma^2 + H^2) = \text{constant}, \tag{17}$$

$$\frac{\Sigma^2}{2B(\underset{\sim}{P})} = \text{constant}.$$

This is a hybrid set, in that only some of the terms of order $\epsilon$ are retained in $d\underset{\sim}{P}/dt$ and none of them in $d\Sigma/dt$. It is simpler than the complete set (14) and its use has validity, as will be discussed at the end of the paper.

At this point the next term of the magnetic moment series can be calculated. We will not give details, but just the result.

$$M_0 + \epsilon M_1 = \frac{\Sigma^2}{2B(\underset{\sim}{P})} - \frac{\epsilon H\Sigma^2}{2B^2}\left[(\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}} - \hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}}):\nabla\hat{\underset{\sim}{L}} + \tfrac{1}{2}(\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}} + \hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}}):\nabla\hat{\underset{\sim}{L}}\right] - \frac{\epsilon H^2\Sigma}{B^2}\,\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}}:\nabla\hat{\underset{\sim}{L}} \tag{18}$$

Since the highest order term we will need in the magnetic moment series is $\epsilon M_1$, M will stand henceforth for $M_0 + \epsilon M_1$, so that $dM(\underset{\sim}{z})/dt$ is of order $\epsilon^2$. When expression (18) is written in terms of $(\underset{\sim}{r}, \underset{\sim}{v})$ it agrees with Eq. (28) of reference (6).

We now proceed to the reduced system and the second invariant. If we set $\epsilon = 0$ in (14) we have

$$d\underset{\sim}{P}/dt = H\hat{\underset{\sim}{L}}(\underset{\sim}{P})$$

$$d\Sigma/dt = -\tfrac{1}{2}H\Sigma\nabla\cdot\hat{\underset{\sim}{L}},$$

$$dH/dt = \tfrac{1}{2}\Sigma^2\nabla\cdot\hat{\underset{\sim}{L}}. \tag{19}$$

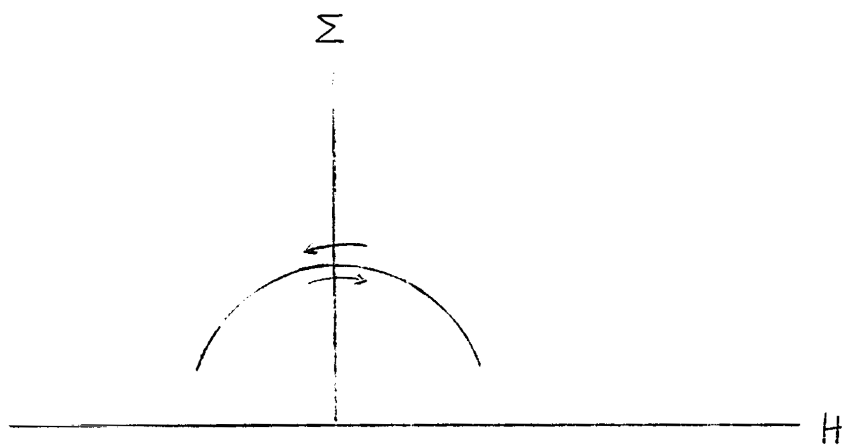


Figure 3: The projection of the lowest-order trajectory into the $\Sigma - H$ plane.

The first equation says that the guiding-center moves along a field line without deviating from it, the drifts having vanished for $\epsilon = 0$. The second and third equations show how the parallel and perpendicular energies interchange in the usual mirror manner. The motion is the five-dimensional space $(\underset{\sim}{P}, \Sigma, H)$ is a closed loop; the projection of the loop into the $\Sigma - H$ plane is double, as illustrated in Fig. 3.

The $\underset{\sim}{y}'$ variables are to be chosen as constants of the motion along this path. One component of $\underset{\sim}{y}'$ is naturally taken as the energy $K \equiv (\Sigma^2 + H^2)/2$. Another component is the magnetic moment, and in accord with the technique of choosing the $\underset{\sim}{y}$ variables as constant as possible, we use $M = \Sigma^2/2B(\underset{\sim}{P}) + \epsilon M_1$, where $M_1$ is given in (18). The final two components of $\underset{\sim}{y}'$ are $\alpha(\underset{\sim}{P})$ and $\beta(\underset{\sim}{P})$, where $\alpha$ and $\beta$ are two functions of position, constant on each field line, such that the vector potential $\underset{\sim}{A}$ is $\alpha\nabla\beta$ and $\underset{\sim}{B}$ is $\nabla\alpha \times \nabla\beta$. (See references 7 and 8). Since the lowest-order motion is strictly on a field line, $\alpha$ and $\beta$ are sutiable for $\underset{\sim}{y}'$ components. The angle variable $\upsilon'$ we define as the fraction of its total longitudinal oscillation period T the particle has completed:

$$\upsilon'(\Sigma, H, \underset{\sim}{P}) = \frac{1}{T(\Sigma,H,\underset{\sim}{P})} \int_{\underset{\sim}{P}_0(\Sigma,H,\underset{\sim}{P})}^{\underset{\sim}{P}} \frac{\hat{\underset{\sim}{L}}(\underset{\sim}{P}')\cdot d\underset{\sim}{P}'}{\pm 2^{\frac{1}{2}}\left\{\frac{\Sigma^2+H^2}{2} - \left[\frac{\Sigma^2}{2B(\underset{\sim}{P})} + \epsilon M_1(\Sigma,H,\underset{\sim}{P})\right] B(\underset{\sim}{P}')\right\}^{\frac{1}{2}}} \tag{20}$$

where the path of integration is along the field line on which $\underset{\sim}{P}$ is located, and $\underset{\sim}{P}_0$ is a zero of the denominator. For the oscillatory case we are considering the denominator will vanish at two points $\underset{\sim}{P}_0$ and $\underset{\sim}{P}_1$. For definiteness, choose $\underset{\sim}{P}_0$ as the one at which $\underset{\sim}{B}$ is directed towards rather than away from the other, as in Fig. 4. The period T is twice the integral from $\underset{\sim}{P}_0$ to $\underset{\sim}{P}_1$. The positive sign in the denominator is to be used when integrating from $\underset{\sim}{P}_0$ to $\underset{\sim}{P}_1$ and the negative sign on the return. A point on a field line has two values of $\upsilon'$ whose sum in unity. It should be noted that the denominator in the integral (20) is not exactly the parallel guiding-center velocity at $\underset{\sim}{P}'$; it is to lowest order only. From (14) the parallel guiding-center velocity is

$$v_{\parallel} \equiv \hat{\underset{\sim}{L}}\cdot\frac{d\underset{\sim}{P}}{dt} = H + \frac{\epsilon\Sigma^2}{2B}\left(\tfrac{3}{2}\hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}} - \tfrac{1}{2}\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}}\right):\nabla\hat{\underset{\sim}{L}} + \frac{\epsilon H\Sigma}{B}\left(\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}}:\nabla\hat{\underset{\sim}{L}}\right).$$

This can be solved for H, which can then be substituted into $2[K - (M_0 + \epsilon M_1)B]$. The result is

$$2[K - (M_0 + \epsilon M_1)B] = \left[v_{\parallel}^2 + \epsilon v_{\parallel}\frac{2\Sigma^2}{B}\left(\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}} - \hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}}\right):\nabla\hat{\underset{\sim}{L}}\right] + O(\epsilon^2). \tag{21}$$

Therefore the denominator of (20) is not exactly $v_{\parallel}$. When $v_{\parallel}$ is not too small, the square root of (21) may be expanded to give

$$2^{\frac{1}{2}}(K - MB)^{\frac{1}{2}} = v_{\parallel} + 2\epsilon M_0\left(\hat{\underset{\sim}{N}}\hat{\underset{\sim}{M}} - \hat{\underset{\sim}{M}}\hat{\underset{\sim}{N}}\right):\nabla\underset{\sim}{L} + O(\epsilon^2).$$

In the special case where $\underline{B}\cdot\nabla\times\underline{B} = 0$, the difference between the parallel guiding-center velocity and $\pm 2^{\frac{1}{2}}(K - MB)^{\frac{1}{2}}$ does vanish. The reason is that $(\hat{\underline{N}}\hat{\underline{M}} - \hat{\underline{M}}\hat{\underline{N}}):\nabla\hat{\underline{L}}$ equals $(\hat{\underline{L}}/B)\cdot\nabla\times\underline{B}$. The vanishing of $\underline{B}\cdot\nabla\times\underline{B}$ is the necessary and sufficient condition for the existence of a family of surfaces orthogonal to the $\underline{B}$ field. Simplification sometimes appears in adiabatic theory when this condition is met. For two other cases, see reference 7, pages 30 and 70.

In Fig. 4, the guiding center is shown reversing its parallel velocity at $\underline{P}_1$ (i.e. at $\upsilon' = \frac{1}{2}$) because by (21) the denominator of (20) vanishes up to terms of order $\epsilon^2$ when $v_{\parallel} = 0$, and $\underline{P}_1$ is defined as a zero of the denominator. The order $\epsilon^2$ difference is invisible to the order to which we are working.

The equations of motion (14) in terms of $(\underline{y}', \upsilon')$ are

$$\begin{aligned}
\dot{\alpha} &= \dot{\underline{P}}\cdot\nabla\alpha, \\
\dot{\beta} &= \dot{\underline{P}}\cdot\nabla\beta, \\
\dot{M} &= O(\epsilon^2), \\
\dot{K} &= O(\epsilon^2) \\
\dot{\upsilon}' &= \frac{1}{T},
\end{aligned} \tag{22}$$

where the dots mean d/dt and where $\dot{\underline{P}}$ is given by (14a); $\Sigma$, H, and $\underline{P}$ are to be expressed in terms of $(\alpha, \beta, M, K, \upsilon')$, a procedure that is in principle possible given the form of the field.

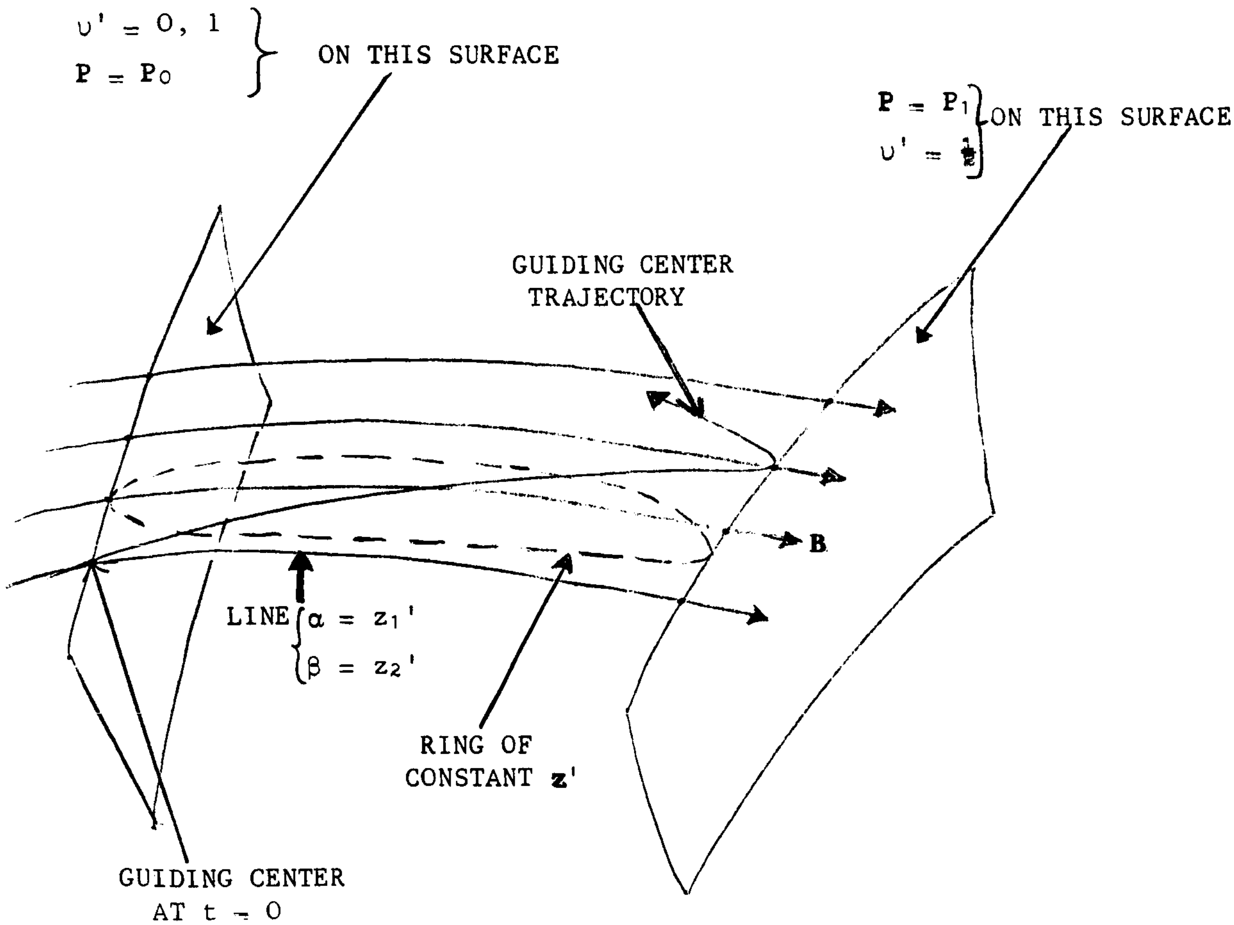


Figure 4: The geometry used in the calculation of the second adiabatic invariant.

By using the systematic method for finding the $\underset{\sim}{z}'$ variables, we obtain

$$z_1' = \alpha + T(\alpha,\beta,K,M)\int_0^{\upsilon'} d\upsilon\left[\int_0^1 d\upsilon''\dot{\underset{\sim}{P}}(\upsilon'')\cdot\nabla\alpha(\upsilon'') - \dot{\underset{\sim}{P}}(\upsilon)\cdot\nabla\alpha(\upsilon)\right] + O(\epsilon^2),$$

$$z_2' = \beta + T(\alpha,\beta,K,M)\int_0^{\upsilon'} d\upsilon\left[\int_0^1 d\upsilon''\,\dot{\underset{\sim}{P}}(\upsilon'')\cdot\nabla\beta(\upsilon'') - \dot{\underset{\sim}{P}}(\upsilon)\cdot\nabla\beta(\upsilon)\right] + O(\epsilon^2). \tag{23}$$

$$z_4' = K + O(\epsilon^2),$$

$$z_3' = M + O(\epsilon^2) = M_0 + \epsilon M_1 + O(\epsilon^2).$$

The integral $\int_0^1 d\upsilon''\dot{\underset{\sim}{P}}(\upsilon'')\cdot\nabla\alpha(\upsilon'')$ is (to the order needed) the average of $\alpha$ over the unperturbed path and will be denoted $\langle\dot{\alpha}\rangle$, and similarly for $\dot{\beta}$. The equations of motion for the $\underset{\sim}{z}'$ variables have not been obtained from (14) because they are not needed in calculating the second invariant.

The second invariant (since $\underset{\sim}{p} = m\underset{\sim}{v} + e\underset{\sim}{A}(\underset{\sim}{r})$) is

$$\frac{J(\underset{\sim}{z}')}{m} = \frac{1}{\epsilon}\int_0^1 d\upsilon\left\{\epsilon\underset{\sim}{v}(\underset{\sim}{z}', \upsilon', \phi) + \underset{\sim}{A}[\underset{\sim}{r}(\underset{\sim}{z}', \upsilon', \phi]\right\}\cdot\frac{\partial\underset{\sim}{r}(\underset{\sim}{z}', \upsilon', \phi)}{\partial\upsilon'}, \tag{24}$$

where (7) has been used and the integration variable switched to $\upsilon'$ from $\phi'$. The integral (24) is independent of $\phi$. The range of integration for either $\phi'$ or $\upsilon'$ is 0 to 1. The right side of (24) must be evaluated through order $\epsilon$ so as to get the $J_1$ term of the series $J_0 + \epsilon J_1 + \cdots\cdots$. The contribution to order $1/\epsilon$ in (24) vanishes because it is $1/\epsilon$ times the magnetic flux through the zero-order trajectory. But the zero-order trajectory is along a field line and back along the same path, thus enclosing no flux. The contribution to order 1 in (24) will be $J_0$, so we need up to and including order $\epsilon$.

To get $\underset{\sim}{r}(\underset{\sim}{z}', \upsilon', \phi)$ correct through order $\epsilon^2$ and $\underset{\sim}{v}(\underset{\sim}{z}', \upsilon', \phi)$ correct through order $\epsilon$ is straightforward and tedious. Great simplification occurs when we set $\phi = 0$. When $\phi = 0$, $\upsilon = 0$ also, as shown by (13d). From (13a-c) we then have that $\underset{\sim}{P} = \underset{\sim}{\rho}$, $\Sigma = \sigma$, and $H = \eta$. In fact according to the general theory these are exact relations (to all orders in $\epsilon$) when $\phi = 0$. Then from (11)

$$\underset{\sim}{r}\Big|_{\phi=0} = \underset{\sim}{P} - \frac{\epsilon\Sigma\hat{\underset{\sim}{M}}(\underset{\sim}{r})}{B(\underset{\sim}{r})} = \underset{\sim}{P} - \frac{\epsilon\Sigma\hat{\underset{\sim}{M}}(\underset{\sim}{P})}{B(\underset{\sim}{P})} + \frac{\epsilon^2\Sigma^2\hat{\underset{\sim}{M}}\cdot\nabla\hat{\underset{\sim}{M}}}{B^2} - \frac{\epsilon^2\Sigma^2\hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}}\cdot\nabla B}{B^3} + O(\epsilon^3), \qquad (25)$$

$$\underset{\sim}{v}\Big|_{\phi=0} = H\hat{\underset{\sim}{L}}(\underset{\sim}{r}) + \Sigma\hat{\underset{\sim}{N}}(\underset{\sim}{r}) = H\hat{\underset{\sim}{L}}(\underset{\sim}{P}) + \Sigma\hat{\underset{\sim}{N}}(\underset{\sim}{P}) - \frac{\epsilon\Sigma H\hat{\underset{\sim}{M}}\cdot\nabla\hat{\underset{\sim}{L}}}{B(\underset{\sim}{P})} - \frac{\epsilon\Sigma^2\hat{\underset{\sim}{M}}\cdot\nabla\hat{\underset{\sim}{N}}}{B(\underset{\sim}{P})} + O(\epsilon^2).$$

The next step is to express $(\underset{\sim}{P}, \Sigma, H)$ in terms of $(\underset{\sim}{z}', \upsilon')$. In principle it is possible to invert the definitions of the $(\underset{\sim}{y}', \upsilon')$ to get $(\underset{\sim}{P}, \Sigma, H)$ in terms of $(\underset{\sim}{y}', \upsilon')$. Let

$$\underset{\sim}{P} = \underset{\sim}{R}(\alpha, \beta, M, K, \upsilon') \qquad (26)$$

be the formula for $\underset{\sim}{P}$ obtained by inversion. Then $\underset{\sim}{P}(\underset{\sim}{z}', \upsilon')$ is obtained by substitution of (23) into the funtion $\underset{\sim}{R}$ and then Taylor-expanding:

$$\underset{\sim}{P}(\underset{\sim}{z}', \upsilon') = \underset{\sim}{R}(z_1', z_2', z_3', z_4', \upsilon') - T(\langle\dot{\alpha}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\alpha})$$

$$\frac{\partial \underset{\sim}{R}}{\partial z_1'}(z_1', z_2', z_3', z_4', \upsilon') - T(\langle\dot{\beta}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\beta})\frac{\partial \underset{\sim}{R}}{\partial z_2'}, \qquad (27)$$

where the T, $\dot{\alpha}$, and $\dot{\beta}$ are to be expressed as functions of $(\underset{\sim}{z}', \upsilon')$.

Consider the contribution of the vector potential to the integral in (24):

$$\int_0^1 d\upsilon' \underset{\sim}{A}\left[\underset{\sim}{P}(\underset{\sim}{z}', \upsilon') - \epsilon\Sigma\hat{\underset{\sim}{M}}(\underset{\sim}{P}) + \frac{\epsilon^2\Sigma^2}{B^2}\left(\hat{\underset{\sim}{M}}\cdot\nabla\hat{\underset{\sim}{M}} - \frac{\hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}}\cdot\nabla B}{B}\right)\right]\cdot\frac{\partial}{\partial\upsilon'}\left[\underset{\sim}{P} - \epsilon\Sigma\hat{\underset{\sim}{M}}\right.$$

$$\left.+\frac{\epsilon^2\Sigma^2}{B^2}\left(\hat{\underset{\sim}{M}}\cdot\nabla\hat{\underset{\sim}{M}} - \frac{\hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}}\cdot\nabla\hat{B}}{B}\right)\right], \qquad (28)$$

where $\underset{\sim}{P}(\underset{\sim}{z}', \upsilon')$ is to be replaced from (27). Since the integral (28) is needed through order $\epsilon^2$, it seems that the $\epsilon^2$ term of $\underset{\sim}{P}(\underset{\sim}{z}', \upsilon')$ would be needed. As the following analysis will show, neither it nor $(\epsilon^2\Sigma^2/B^2)$ $(\hat{\underset{\sim}{M}}\cdot\nabla\hat{M} - \hat{\underset{\sim}{M}}\hat{\underset{\sim}{M}}\cdot\nabla B/B)$ contributes and all of the order $\epsilon^2$ contribution to the integral arises from the products of first order terms. The path of integration is at constant $\underset{\sim}{z}'$. Varying $\upsilon'$ at constant $\underset{\sim}{z}'$ does not make $\underset{\sim}{r}(\underset{\sim}{z}', \upsilon')$ follow a field line unless $\alpha[\underset{\sim}{r}(\underset{\sim}{z}', \upsilon')]$ and $\beta[\underset{\sim}{r}(\underset{\sim}{z}', \upsilon')]$ are independent of $\upsilon'$; but they do depend on $\upsilon'$. In fact

$$\alpha[\underset{\sim}{r}(\underset{\sim}{z}', \upsilon')] = z_1' - T(\langle\dot{\alpha}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\alpha}) - \frac{\epsilon\Sigma\hat{\underset{\sim}{M}}\cdot\nabla\alpha(\underset{\sim}{R})}{B}, \qquad (29)$$

and

$$\beta[\underset{\sim}{r}(\underset{\sim}{z}', \upsilon')] = z_2' - T(\langle\dot{\beta}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\beta}) - \frac{\epsilon\Sigma\hat{\underset{\sim}{M}}\cdot\nabla\beta(\underset{\sim}{R})}{B}. \qquad (30)$$

The ring of constant $\underset{\sim}{z}'$ might look as in Fig. 4. The vector-potential integral in (28) is the magnetic flux passing through this ring, and that is the negative of the double integral $\iint d\alpha d\beta$ over the ring, because $\underset{\sim}{A}$ is $\alpha\nabla\beta$. In an $\alpha - \beta$ plane the ring might look as in Fig. 5. The double integral is the area of this ring, which by a little geometry is

$$\iint d\alpha d\beta = \tfrac{1}{2}\int_0^1 d\upsilon' \left\{[\beta(\underset{\sim}{z}', \upsilon') - \beta(\underset{\sim}{z}', 0)]\frac{\partial\alpha}{\partial\upsilon'} - [\alpha(\underset{\sim}{z}', \upsilon') - \alpha(\underset{\sim}{z}', 0)]\frac{\partial\beta}{\partial\upsilon'}\right\}, \tag{31}$$

where $\alpha$ and $\beta$ are given in (29) and (30) as functions of $(\underset{\sim}{z}', \upsilon')$. The difference $\alpha(\underset{\sim}{z}', \upsilon') - \alpha(\underset{\sim}{z}', 0)$ is order $\epsilon$, since the zero order of each is $z_1'$. Moreover $\partial\alpha(\underset{\sim}{z}', \upsilon')/\partial\upsilon'$ is order $\epsilon$. Similar statements hold for $\beta(\underset{\sim}{z}', \upsilon')$. These facts are also clear from Figs. 4 and 5, where the deviation $\alpha(\underset{\sim}{z}', \upsilon') - \alpha(\underset{\sim}{z}', 0)$ of the ring from the $\upsilon' = 0$ field line is due to the drifts, which are of order $\epsilon$. To summarize, (31) vanishes through order $\epsilon$ while its $\epsilon^2$ part comes only from the products of $\epsilon$ terms.

We will next show that only the $\epsilon\Sigma(\hat{\underset{\sim}{M}}\cdot\nabla\alpha)/B$ and $\epsilon\Sigma(\hat{\underset{\sim}{M}}\cdot\nabla\beta)/B$ terms of (29) and (30) contribute to (31); products of $\epsilon$ terms coming from the $\dot{\alpha}$ and $\dot{\beta}$ parts all cancel. Substitution of (29) and (30) into (31) gives

$$\iint d\alpha d\beta = \frac{T^2}{2}\int_0^1 d\upsilon' \, [(\langle\dot{\beta}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\beta})(\langle\dot{\alpha}\rangle - \dot{\alpha}) - (\langle\dot{\alpha}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\alpha})(\langle\dot{\beta}\rangle - \dot{\beta})]$$

$$+ \frac{\epsilon T}{2}\int_0^1 d\upsilon' \left[(\langle\dot{\beta}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\beta})\frac{\partial}{\partial\upsilon'}\frac{\Sigma\hat{\underset{\sim}{M}}\cdot\nabla\alpha}{B} - \left(\langle\dot{\alpha}\rangle\upsilon' - \int_0^{\upsilon'}\dot{\alpha}\right)\frac{\partial}{\partial\upsilon'}\frac{\Sigma\hat{\underset{\sim}{M}}\cdot\nabla\beta}{B}\right]$$

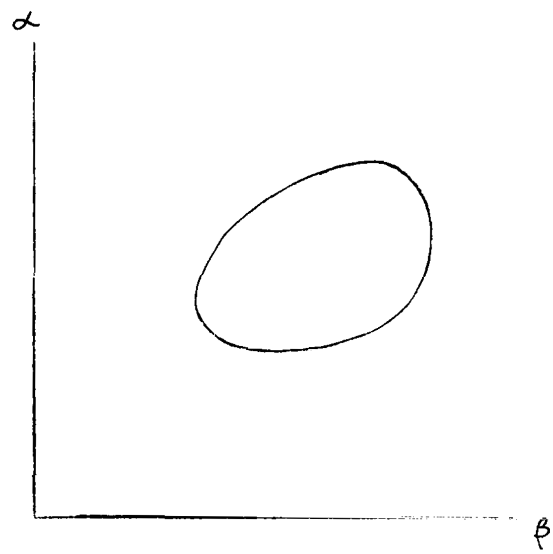


Figure 5: An integration ring in the $\alpha - \beta$ plane.

$$+ \frac{\epsilon T}{2} \int_0^1 d\upsilon' \left[ (<\dot{\alpha}> - \dot{\alpha}) \frac{\Sigma \hat{M} \cdot \nabla \beta}{B} - (<\dot{\beta}> - \dot{\beta}) \frac{\Sigma \hat{M} \cdot \nabla \alpha}{B} \right]$$

$$- \frac{\epsilon T}{2} \int_0^1 d\upsilon' \left[ (<\dot{\alpha}> - \dot{\alpha}) \left( \frac{\Sigma \hat{M} \cdot \nabla \beta}{B} \right)_{\upsilon'=0} - (<\dot{\beta}> - \dot{\beta}) \left( \frac{\Sigma \hat{M} \cdot \nabla \alpha}{B} \right)_{\upsilon'=0} \right] \quad (32)$$

The last integral vanishes because $\int_0^1 (<\dot{\alpha}> - \dot{\alpha}) = \int_0^1 (<\dot{\beta}> - \dot{\beta}) = 0$. The second integral in (32) can be integrated by parts, and after some cancellation one finds

$$\int_0^1 \underset{\sim}{A} \cdot \frac{\partial \underset{\sim}{r}}{\partial \upsilon'} d\upsilon' = - \iint d\alpha d\beta = \epsilon T \int_0^1 d\upsilon' \frac{\Sigma}{B} \left[ (<\dot{\beta}> - \dot{\beta}) \hat{\underset{\sim}{M}} \cdot \nabla \alpha - (<\dot{\alpha}> - \dot{\alpha}) \hat{M} \cdot \nabla \beta \right]$$

$$+ \frac{T^2}{2} (<\dot{\beta}><\dot{\alpha}\upsilon'> - <\dot{\alpha}><\dot{\beta}\upsilon'>)$$

$$+ \frac{T^2}{2} \int_0^1 d\upsilon' [<\dot{\alpha}> \int_0^{\upsilon'} d\upsilon'' \dot{\beta}(\upsilon'') - <\dot{\beta}> \int_0^{\upsilon'} d\upsilon'' \dot{\alpha}(\upsilon'')]$$

$$+ \frac{T^2}{2} \int_0^1 d\upsilon' [\dot{\beta}(\upsilon') \int_0^{\upsilon'} d\upsilon'' \dot{\alpha}(\upsilon'') - \dot{\alpha}(\upsilon') \int_0^{\upsilon'} d\upsilon'' \dot{\beta}(\upsilon'')]. \quad (33)$$

We proceed to show that the terms with $T^2$ give zero. In the second $T^2$ integral interchange the order of the $\upsilon'$ and $\upsilon''$ integration to get

$$\int_0^1 d\upsilon' <\dot{\alpha}> \int_0^{\upsilon'} d\upsilon'' \dot{\beta}(\upsilon'') = <\dot{\alpha}> \int_0^1 d\upsilon'' \dot{\beta}(\upsilon'')(1 - \upsilon'') = <\dot{\alpha}><\dot{\beta}> - <\dot{\alpha}><\dot{\beta}\upsilon'> \quad (34)$$

and similarly for the other part of the integral. Thus the $T^2$ terms in (33) become

$$T^2\Big\{\langle\dot\beta\rangle\langle\dot\alpha\upsilon'\rangle - \langle\dot\alpha\rangle\langle\dot\beta\upsilon'\rangle - \tfrac{1}{2}\int_0^1 d\upsilon'\int_0^{\upsilon'} d\upsilon''[\dot\alpha(\upsilon')\dot\beta(\upsilon'') - \dot\beta(\upsilon')\dot\alpha(\upsilon'')]\Big\}. \tag{35}$$

Further progress depends on a theorem: Given two functions $f(\upsilon)$ and $g(\upsilon)$ such that $f(1-\upsilon)$ equals $f(\upsilon)$ and $g(1-\upsilon)$ equals $g(1) - g(\upsilon)$. Then $\int_0^1 d\upsilon g(\upsilon)f(\upsilon) = \frac{1}{2}g(1)\int_0^1 d\upsilon f(\upsilon)$. In (35) the first term is $\int_0^1\dot\beta\int_0^1\dot\alpha\upsilon'$. Clearly, $\upsilon'$ is a g function. Also, $\dot\alpha$ is an f function; this is because $1-\upsilon'$ is the same point on the zero-order field line as $\upsilon'$, and the drift velocity, which produces $\dot\alpha$ and $\dot\beta$, is independent of the sign of the parallel guiding-center velocity. Application of the theorem now gives $\int_0^1\dot\alpha\upsilon' \equiv \langle\dot\alpha\upsilon'\rangle = \frac{1}{2}\langle\dot\alpha\rangle$, and similarly $\langle\dot\beta\upsilon'\rangle = \frac{1}{2}\langle\dot\beta\rangle$. And so the first two terms of (35) cancel. In the last term we have $\int_0^1 d\upsilon'\dot\beta(\upsilon')\int_0^{\upsilon'}d\upsilon''\dot\alpha(\upsilon'')$, where $\dot\beta(\upsilon')$ is an f function, and $\int_0^{\upsilon'}d\upsilon''\dot\alpha(\upsilon'')$ is a g function of its upper limit. Application of the theorem yields $\int_0^1 d\upsilon'\dot\beta(\upsilon')\int_0^{\upsilon'}d\upsilon''\dot\alpha(\upsilon'') = \frac{1}{2}\langle\dot\alpha\rangle\langle\dot\beta\rangle$ and similarly $\int_0^1 d\upsilon'\dot\alpha(\upsilon')\int_0^{\upsilon'}d\upsilon''\dot\beta(\upsilon'') = \frac{1}{2}\langle\dot\beta\rangle\langle\dot\alpha\rangle$. Thus all the $T^2$ terms disappear and the $\epsilon$ contribution of the vector potential to $J(\underset{\sim}{z}')/m$ is

$$\frac{1}{\epsilon}\int_0^1 \underset{\sim}{A}[\underset{\sim}{r}(\underset{\sim}{z}',\upsilon')]\cdot\frac{\partial\underset{\sim}{r}(\underset{\sim}{z}',\upsilon')}{\partial\upsilon'}d\upsilon' = T\int_0^1 d\upsilon'\frac{\Sigma}{B(\underset{\sim}{R})}\,[(\langle\dot\beta\rangle - \dot\beta)\hat{\underset{\sim}{M}}(\underset{\sim}{R})\cdot\nabla\alpha(\underset{\sim}{R}) -$$

$$(\langle\dot\alpha\rangle - \dot\alpha)\hat{\underset{\sim}{M}}(\underset{\sim}{R})\cdot\nabla\beta(\underset{\sim}{R})]. \tag{36}$$

In this expression $\underset{\sim}{R}$ means $\underset{\sim}{R}(z_1', z_2', z_3', z_4', \upsilon')$, and $\Sigma$ is $[2z_3'\,B(\underset{\sim}{R})]^{\frac{1}{2}}$ to the order needed. The $\dot\alpha$ and $\dot\beta$ are to be expressed as functions of $\underset{\sim}{z}'$ and $\upsilon'$.

We now turn to the evaluation of the integral in (24) containing the velocity. Substitute $\underset{\sim}{r}$ and $\underset{\sim}{v}$ from (25) into (24):

$$\int_0^1 d\upsilon' \; \underset{\sim}{v}(\underset{\sim}{z}', \upsilon') \cdot \frac{\partial \underset{\sim}{r}(\underset{\sim}{z}', \upsilon')}{\partial \upsilon'} \, d\upsilon' = \int_0^1 d\upsilon' \Big[ H\hat{\underset{\sim}{L}}(\underset{\sim}{P}) + \Sigma \hat{\underset{\sim}{N}}(\underset{\sim}{P}) - \frac{\epsilon \Sigma H \hat{\underset{\sim}{M}} \cdot \nabla \hat{\underset{\sim}{L}}}{B}$$

$$- \frac{\epsilon \Sigma^2 \underset{\sim}{M} \cdot \nabla \hat{\underset{\sim}{N}}}{B} \Big] \cdot \frac{\partial}{\partial \upsilon'} \Big[ \underset{\sim}{P}(\underset{\sim}{z}', \upsilon') - \frac{\epsilon \Sigma \hat{\underset{\sim}{M}}(\underset{\sim}{P})}{B} \Big], \tag{37}$$

where $(\underset{\sim}{P}, H, \Sigma)$ are to be expressed in terms of $(\underset{\sim}{z}', \upsilon')$. Equation (27) gives $\underset{\sim}{P}(\underset{\sim}{z}', \upsilon')$; $H(\underset{\sim}{z}', \upsilon')$ and $\Sigma(\underset{\sim}{z}', \upsilon')$ can be obtained by inverting the definitions of the $\underset{\sim}{y}'$ variables and then expressing $\underset{\sim}{y}'$ in terms of $(\underset{\sim}{z}', \upsilon')$.

The lowest order of (37) is found to be the usual longitudinal invariant

$$\frac{J_0(\underset{\sim}{z}')}{m} = \int_0^1 \pm\, 2^{\frac{1}{2}} \left[ z_4' - z_3' \, B(\underset{\sim}{R}) \right]^{\frac{1}{2}} \hat{\underset{\sim}{L}}(\underset{\sim}{R}) \cdot \frac{\partial \underset{\sim}{R}}{\partial \upsilon'} \, d\upsilon' \tag{38}$$

where the plus sign is to be used for $0 \leq \upsilon' \leq \frac{1}{2}$ and the minus sign for $\frac{1}{2} \leq \upsilon' \leq 1$, because the parallel velocity, which to lowest order is H, changes sign at the turning point where $\upsilon'$ is $\frac{1}{2}$. The plus-or-minus sign is essential; without it $J_0$ would vanish because $\hat{\underset{\sim}{L}} \cdot \partial \underset{\sim}{R} / \partial \upsilon'$ has opposite signs but the same magnitude at $\upsilon'$ and $1-\upsilon'$. Because of it, many terms in $J_1$ will vanish due to the symmetry properties of their integrands.

The $\epsilon$ part of (37) is not difficult to calculate. Many terms vanish due to the symmetry of the integrand. A sample term is

$\epsilon\int_0^1 2z_3' \frac{\partial \mathbf{R}}{\partial \upsilon'} \hat{\mathbf{M}}:\nabla\hat{\mathbf{N}}$ which vanishes because $\hat{\mathbf{M}}\cdot\nabla\hat{\mathbf{N}}$ is the same at $\upsilon'$ and $1-\upsilon'$, but $\partial\mathbf{R}/\partial\upsilon'$ has opposite signs. The result we find for the contribution of the velocity to J/m is

$$\int_0^1 d\upsilon' \mathbf{v}(\mathbf{z}', \upsilon') \cdot \frac{\partial \mathbf{r}(\mathbf{z}', \upsilon')}{\partial \upsilon'} d\upsilon' = \int_0^1 d\upsilon' (\pm) 2^{\frac{1}{2}} [z_4' - z_3' B(\mathbf{R})]^{\frac{1}{2}} \hat{\mathbf{L}}(\mathbf{R}) \cdot \frac{\partial \mathbf{R}}{\partial \upsilon'}$$

$$- T\int_0^1 d\upsilon' \Big\{ (\langle\dot{\alpha}\rangle\upsilon' - \int_0^{\upsilon'} \dot{\alpha}) \frac{\partial}{\partial z_1'} [\pm 2^{\frac{1}{2}} (z_4' - z_3' B)^{\frac{1}{2}} \hat{\mathbf{L}} \cdot \frac{\partial \mathbf{R}}{\partial \upsilon'}]$$

$$+ (\langle\dot{\beta}\rangle\upsilon' - \int_0^{\upsilon'} \dot{\beta}) \frac{\partial}{\partial z_2'} [\pm 2^{\frac{1}{2}} (z_4' - z_3' B)^{\frac{1}{2}} \hat{\mathbf{L}} \cdot \frac{\partial \mathbf{R}}{\partial \upsilon'}]$$

$$+ (\langle\dot{\alpha}\rangle - \dot{\alpha}) \hat{\mathbf{N}}(\mathbf{R}) \cdot \frac{\partial \mathbf{R}}{\partial z_1'} (2z_3' B)^{\frac{1}{2}}$$

$$+ (\langle\dot{\beta}\rangle - \dot{\beta}) \hat{\mathbf{N}}(\mathbf{R}) \cdot \frac{\partial \mathbf{R}}{\partial z_2'} (2z_3' B)^{\frac{1}{2}}$$

$$- \epsilon\int_0^1 d\upsilon' (\pm) 2^{\frac{1}{2}} (z_4' - z_3' B)^{\frac{1}{2}} \left(\frac{2z_3'}{B}\right)^{\frac{1}{2}} [\hat{\mathbf{L}} \frac{\partial \mathbf{R}}{\partial \upsilon'} : \nabla\hat{\mathbf{M}} + \hat{\mathbf{L}} \cdot \frac{\partial \mathbf{R}}{\partial \upsilon'} \hat{\mathbf{M}}\hat{\mathbf{L}} : \nabla\hat{\mathbf{L}}]. \qquad (39)$$

The last integral vanishes because $\partial\mathbf{R}/\partial\upsilon'$ is parallel (or anti-parallel to $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}\hat{\mathbf{L}}:\nabla\hat{\mathbf{M}} + \hat{\mathbf{M}}\hat{\mathbf{L}}:\nabla\hat{\mathbf{L}} = \hat{\mathbf{L}}\cdot\nabla(\hat{\mathbf{L}}\cdot\hat{\mathbf{M}}) = 0$.

When (36) and (39) are added there is a term containing the factor $[(\hat{\mathbf{N}}\cdot\partial\mathbf{R}/\partial z_1') + (\hat{\mathbf{M}}\cdot\nabla\beta/B)]$ which vanishes because of the relation (see reference 7, page 51) $\nabla\beta/B = \hat{\mathbf{L}} \times \partial\mathbf{R}/\partial z_1'$ which holds for these $(\alpha, \beta)$ systems. The result for $J(z')$ is

$$\frac{J(\underset{\sim}{z}')}{m} = \int_0^1 d\upsilon'(\pm)2^{\frac{1}{2}}[z_4' - z_3'B(\underset{\sim}{R})]^{\frac{1}{2}}\hat{\underset{\sim}{L}}(\underset{\sim}{R})\cdot\frac{\partial \underset{\sim}{R}}{\partial \upsilon'} - T\int_0^1 d\upsilon'\Big\{[<\dot{\alpha}>\upsilon'$$

$$-\int_0^{\upsilon'}\dot{\alpha}(\underset{\sim}{z}', \upsilon'')d\upsilon''] \frac{\partial}{\partial z_1'} + [<\dot{\beta}>\upsilon' - \int_0^{\upsilon'}\dot{\beta}(\underset{\sim}{z}', \upsilon'')d\upsilon''] \frac{\partial}{\partial z_2'}\Big\} \tag{40}$$

$$[(\pm)2^{\frac{1}{2}}(z_4' - z_3'B)^{\frac{1}{2}}\hat{\underset{\sim}{L}}\cdot\frac{\partial \underset{\sim}{R}}{\partial \upsilon'}] + O(\epsilon^2).$$

$J(\underset{\sim}{z}')$ can be expressed in terms of the $(\underset{\sim}{y}', \upsilon')$ variables, which have more physical appeal than the $\underset{\sim}{z}'$ ones. The reversion is not too difficult if liberal use is made of the theorem regarding f and g type functions. We give only the result:

$$\frac{J(\underset{\sim}{y}', \upsilon')}{m} = \int_0^1 d\upsilon''(\pm)2^{\frac{1}{2}}\Big\{K - MB[\underset{\sim}{R}(\alpha, \beta, M, K, \upsilon'')]\Big\}^{\frac{1}{2}}\hat{\underset{\sim}{L}}(\underset{\sim}{R})\cdot\frac{\partial \underset{\sim}{R}}{\partial \upsilon''}$$

$$+ \frac{T^2(\alpha, \beta, M, K)}{\epsilon}\int_0^1 d\upsilon''[<\dot{\beta}>\dot{\alpha}(\underset{\sim}{y}', \upsilon'') - <\dot{\alpha}>\dot{\beta}(\underset{\sim}{y}', \upsilon'')] + O(\epsilon^2), \tag{41}$$

Definitions and equations needed to interpret Eq. (41) are:

$\underset{\sim}{y}'$: a vector with components $\alpha$, $\beta$, M, and K.

$\upsilon'$: defined in Eq. (20).

$\alpha$ and $\beta$: functions of position, constant on a field line, such that

$$\underset{\sim}{A} = \alpha\nabla\beta.$$

M: the magnetic moment constant through order $\epsilon$, given by Eq. (18).

K: the particle energy, $(\Sigma^2 + H^2)/2$.

T: the period of oscillation, defined below Eq. (20).

$\dot{\alpha}(\underset{\sim}{y}', \upsilon')$ and $\dot{\beta}(\underset{\sim}{y}', \upsilon')$: the rate of change of $\alpha$ and $\beta$ due to the drifts, and given by Eq. (22) and (14a).

$<\dot{\alpha}>$: the rate of change of $\alpha$ averaged over a longitudinal oscillation and given by $\int_0^1 d\upsilon' \dot{\alpha}(\underset{\sim}{y}', \upsilon')$.

$<\dot{\beta}>$: similar to $<\dot{\alpha}>$. Thus $<\dot{\alpha}>$ and $<\dot{\beta}>$ are functions of $\underset{\sim}{y}'$ in Eq. (41).

$\Sigma$, H, $\underset{\sim}{P}$: the guiding-center variables, defined by Eq. (13).

$\underset{\sim}{R}(\alpha, \beta, M, K, \upsilon')$: defined by Eq. (26).

$\hat{\underset{\sim}{L}}$: the unit vector $\underset{\sim}{B}/B$.

$\epsilon$: mass to charge ratio m/e.

The first integral (41) contains $\epsilon$ terms in M, which is $M_0 + \epsilon M_1$; for a given M and K the integral is a function of only $\alpha$ and $\beta$ -- i.e., of the field line on which the guiding-center is located, and not of $\upsilon'$, which gives the position along the field line. The integral is taken between zeros of $K - MB(\underset{\sim}{R})$, even though the guiding-center never actually moves along the line $(\alpha, \beta)$ between these "virtual" mirror points, C and D in Fig. 6. The guiding-center will eventually be reflected on some field line, but it generally will not be the line CD.

The term of J proportional to $T^2$ is of order $\epsilon$, since $\dot{\alpha}$ and $\dot{\beta}$ are of order $\epsilon$. It depends on $\upsilon'$ through the upper limit of the integral, and vanishes at $\upsilon' = 0$, $\frac{1}{2}$, and 1. In fact because $\epsilon M_1$ vanishes when $v_{\parallel} = 0$, the entire $\epsilon J_1$ vanishes at the mirror points. Thus the guiding-center trajectory will look somewhat as in Fig. 6, oscillating about the surface of constant $J_0$, but always coinciding with it at the mirror points. In the figure, the dotted part of the

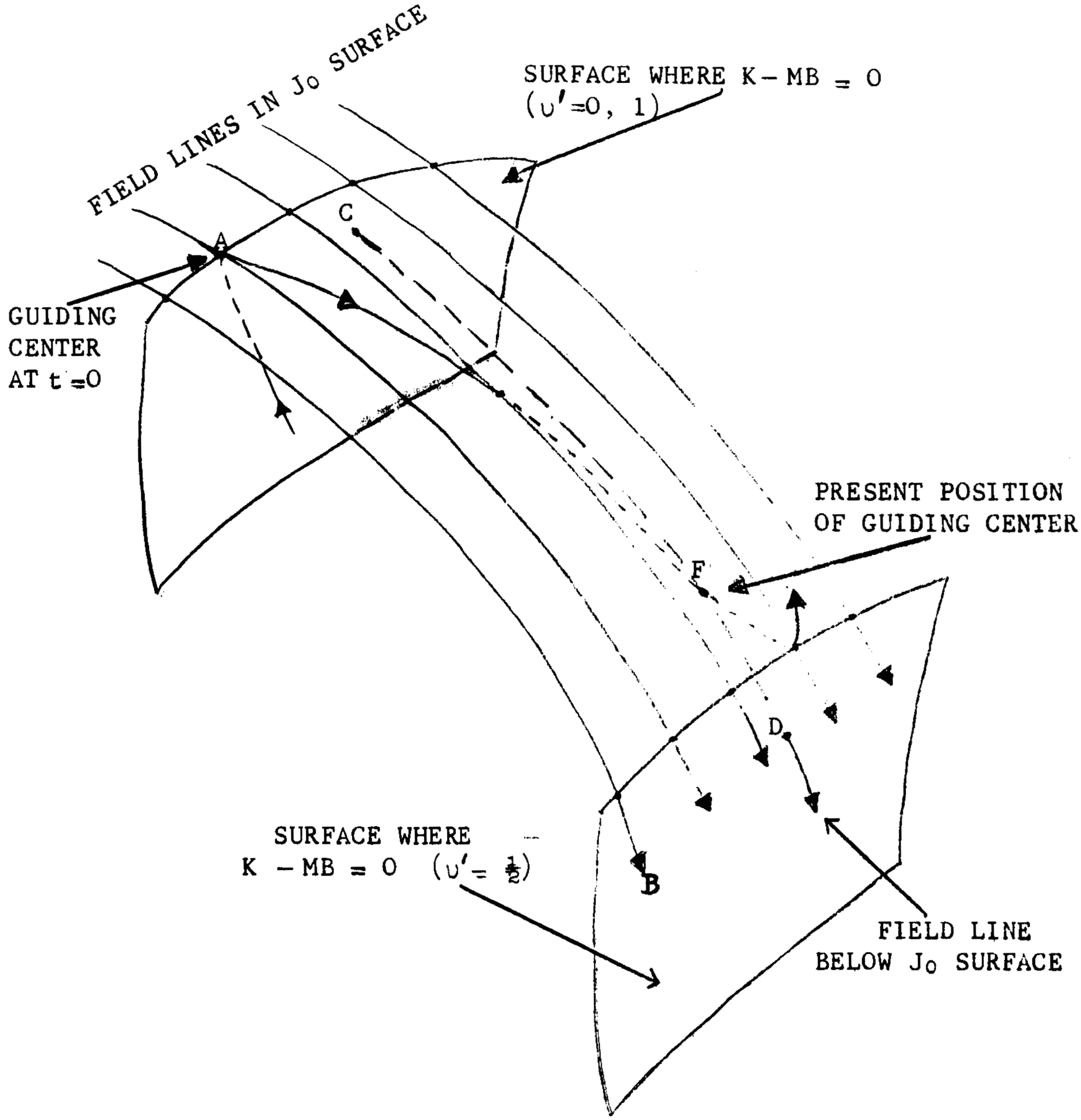


Figure 6: The guiding center oscillates about the surface of constant $J_0$.

trajectory lies below the $J_0$ surface.

As shown by Eq. (21) the radical $\pm 2^{\frac{1}{2}}(K - MB)^{\frac{1}{2}}$ in the first term of J/m is not the parallel guiding-center velocity through order $\epsilon$. However the integral over a complete period of the difference does vanish; the order $\epsilon$ term on the right side of (21) is proportional $v_{\parallel}$ and therefore has opposite signs in the ranges $\upsilon' = 0$ to $\frac{1}{2}$ and $\frac{1}{2}$ to 1, whereas $\pm\hat{\underset{\sim}{L}}\cdot\partial\underset{\sim}{R}/\partial\upsilon'$ has the same sign. Thus the integral of the order $\epsilon$ difference vanishes by symmetry. For a similar reason, even the term of $M_1$ that is porportional to H would contribute nothing to the integral, leaving only the $\epsilon\hat{\underset{\sim}{M}}\hat{\underset{\sim}{L}}:\nabla\hat{\underset{\sim}{L}}H^2\Sigma/B^2$ term.

## IV: DIRECT DERIVATION

Verification that $d(J/m)/dt = O(\epsilon^2)$ will serve as a check on the result and at the same time suggest a direct derivation of J that shortcuts much of the present work. We have

$$\frac{d}{dt}\frac{J}{m} = \dot{\alpha}\frac{\partial}{\partial\alpha}\left(\frac{J}{m}\right) + \dot{\beta}\frac{\partial}{\partial\beta}\left(\frac{J}{m}\right) + \dot{\upsilon}'\frac{\partial}{\partial\upsilon'}\left(\frac{J}{m}\right) + O(\epsilon^2) \tag{42}$$

So we need $\partial(J/m)/\partial\alpha$ and $\partial(J/m)/\partial\beta$ correct only to zero order, since $\dot{\alpha}$ and $\dot{\beta}$ are of order $\epsilon$. In reference (3) it is shown that

$$\frac{\partial}{\partial\alpha}\frac{J_0}{m} = -\frac{T}{\epsilon}\langle\dot{\beta}\rangle,$$

$$\frac{\partial}{\partial\beta}\frac{J_0}{m} = \frac{T}{\epsilon}\langle\dot{\alpha}\rangle. \tag{43}$$

Furthermore, $\dot{\upsilon}'$ is 1/T and $\partial(J/m)/\partial\upsilon'$ equals $(T^2/\epsilon)(\langle\beta\rangle\dot{\alpha} - \langle\dot{\alpha}\rangle\dot{\beta})$. Putting it all together gives what we want: $d(J/m)/dt = O(\epsilon^2)$.

The direct derivation goes as follows: start with the time derivative of the lowest order J,

$$\frac{d}{dt}\frac{J_0}{m} = \dot{\alpha}\frac{\partial}{\partial\alpha}\frac{J_0}{m} + \dot{\beta}\frac{\partial}{\partial\beta}\frac{J_0}{m} + \dot{M}_0\frac{\partial}{\partial M_0}\frac{J_0}{m} + O(\epsilon^2), \tag{44}$$

and by (43) convert this to

$$\frac{d}{dt}\frac{J_0}{m} = \frac{T}{\epsilon}[\dot{\beta}\langle\dot{\alpha}\rangle - \dot{\alpha}\langle\dot{\beta}\rangle] + \dot{M}_0\frac{\partial}{\partial M_0}\frac{J_0}{m} + O(\epsilon^2)$$

$$= \frac{d}{dt}\frac{T}{\epsilon}\int_0^t dt[\dot{\beta}\langle\dot{\alpha}\rangle - \dot{\alpha}\langle\dot{\beta}\rangle] - \epsilon\dot{M}_1\frac{\partial}{\partial M_0}\frac{J_0}{m} + O(\epsilon^2)$$

where $\dot{M}_0$ has been replaced by $-\epsilon\dot{M}_1$. The period T, which depends on time has been placed inside the d/dt because its time derivative is proportional to the order $\epsilon$ drifts, making the error of order $\epsilon^2$. The term containing $\dot{M}_1$ may be written as

$$\epsilon\dot{M}_1\frac{\partial}{\partial M_0}\frac{J_0}{m} = \frac{d}{dt}\left[\epsilon M_1\frac{\partial(J_0/m)}{\partial M_0}\right] - \epsilon M_1\frac{d}{dt}\frac{\partial(J_0/m)}{\partial M_0}.$$

Because $J_0$ is an integral along a field line, so is $\partial(J_0/m)/\partial M_0$, which is proportional to $\langle B\rangle$, the average magnetic field over an oscillation. Therefore $\partial(J_0/m)/\partial M_0$ is changed only by the order $\epsilon$

drifts so that $\epsilon M_1 \frac{d}{dt} \frac{\partial(J_0/m)}{\partial M_0}$ is of $O(\epsilon^2)$ and can be dropped.

Transposing all terms of (45) to the left side and integrating over time gives

$$\frac{J_0}{m} + \epsilon M_1 \frac{\partial(J_0/m)}{\partial M_0} + \frac{T}{\epsilon}\int_0^t dt[\langle\dot{\beta}\rangle\dot{\alpha} - \langle\dot{\alpha}\rangle\dot{\beta}] = \text{constant}$$

or

$$\int_0^1 d\upsilon'(\pm)2^{\frac{1}{2}}[K - (M_0 + \epsilon M_1)B]^{\frac{1}{2}}\hat{\underset{\sim}{L}}\cdot\frac{\partial \underset{\sim}{R}}{\partial\upsilon'} + \frac{T}{\epsilon}\int_0^t dt[\langle\dot{\beta}\rangle\dot{\alpha} - \langle\dot{\alpha}\rangle\dot{\beta}] = \text{constant} \tag{46}$$

At this point we would have the result, except for the fact that the integral is over time in (46) and so is not a local quantity. This objection can be circumvented by realizing that the correction $\epsilon J_1$ to $J_0$ should be small, which means that the guiding center should be distant only order $\epsilon$ from the surface on which $J_0$ is constant. The time integral from A to F in Fig. 6 can be replaced by the integral from C to F along the instantaneous field line, and finally we have (41). The $\upsilon'$-dependent term is just the effect of the drifts, and the total $\epsilon J_1$ arises from this and from the correction of the lowest-order magnetic moment.

The direct derivation also reveals that the closed field-line case must yield the same expression for $J_1$ as the oscillatory case, which has been considered to this point. In the closed field-line case, the guiding center traverses the line always in the same sense, parallel or antiparallel to $\underset{\sim}{B}$, but with varying speed, depending on the magnitude of $\underset{\sim}{B}$. At the same time it drifts slowly at right angles to the line. There is nothing in the direct derivation that relies on the oscillatory motion, thus we may conclude that the form of the expression (41) for J will be unchanged; $\upsilon'$ may be taken as 0 and 1 at an arbitrary point on the field line, and only one of the plus-or-minus signs will be needed.

There is however a difference between the oscillatory and closed field line cases - namely, that the order $\epsilon$ difference between the integral of $v_{\parallel}$ and of $\pm\, 2^{\frac{1}{2}}(K - MB)^{\frac{1}{2}}$ no longer vanishes, and the first integral on the right side of (41) may not be replaced by $\oint v_{\parallel} ds$, nor may the part of $\epsilon M_1$ that is proportional to H be omitted.

The direct derivation also raises the question of why the time derivative of the magnetic moment (as in Eq. 44) did not need to be considered in earlier proofs (references 3 and 7) of the conservation of $J_0$. The answer depends on whether $J_0$ is defined with $(K - M_0 B)^{\frac{1}{2}}$ or with $(K - MB)^{\frac{1}{2}}$, which includes some terms of order $\epsilon$. If the former, the proofs should indeed consider the effect of $\dot{M}_0$. Since the proofs show only that $\langle d(J_0/m)/dt \rangle = O(\epsilon^2)$ -- i.e., that $J_0$ is only conserved on the average, it is only necessary to show that $\langle \dot{M}_0 \rangle$ vanishes. This is easy to do: $\langle \dot{M}_0 \rangle = -\langle \epsilon \dot{M}_1 \rangle = \langle \epsilon/T) \int_0^T dt \dot{M}_1 = (\epsilon/T)\Delta M_1$, where $\Delta M_1$ is the change in $M_1$ between the time the guiding center leaves one mirror and (to lowest order) returns to it again. But from (18) this change is zero because $M_1$ is zero when H is zero. If on the other hand $J_0$ is defined with M instead of $M_0$, previous proofs are valid. In practice, defining $J_0$ with M is preferable because of the use to which $J_0$ is usually put. The invariance of $J_0$ is often used to determine the surface on which the guiding-center moves on the average. In a numerical calculation to find the surface one picks a number representing the magnetic moment and holds this number constant, so that in effect one is using $M_0 + \epsilon M_1$.

The above discussion leads to the final subject to be treated—namely, the order to which the second invariant is conserved. Since $J_0$ is an integral along a field line, the parallel velocity does not change it, and therefore $d(J_0/m)dt = O(\epsilon)$: $J_0$ is trivially conserved to lowest order and is affected by the drifts, which are of order $\epsilon$. The usual proof of the invariance of $J_0$ (ref. 3) does not show that the $\epsilon$ term is zero, but only that its average vanishes: $\langle d(J_0/m)/dt \rangle = O(\epsilon^2)$. So far it would not matter whether the complete set of equations (14) were used or the hybrid set (17). The drifts are the same for each, and these are what are involved in the proofs that $J_0$ is conserved on the average. In the present paper we have shown that $(1/m)\ d(J_0 + \epsilon J_1)/dt = O(\epsilon^2)$ without any averaging involved, but the full set of equations (14) must be used in order that $dM/dt = O(\epsilon^2)$.

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## REFERENCES

[1]H. Alfvén, Cosmical Electrodynamics, Clarendon Press Oxford, 1950.

[2]G. F. Chew, M. Goldberger, and F. Low, "The Individual Particle Equations of Motion in the Adiabatic Approximation", Los Alamos Scientific Laboratory Report LA-2055, Chapter T-759, p. 9, 1955.

[3]T. Northrop and E. Teller, Phys. Rev. 117, 215 (1960).

[4]M. Kruskal, J. Math. Phys. 3, 806 (1962).

[5]J. Siambis and A. W. Trivelpiece, Phys. Fluids 8, 2047 (1965).

[6]M. Kruskal, Princeton University, Project Matterhorn Report PM-S-33 (NYO-7903), March 1958.

[7]T. Northrop, The Adiabatic Motion of Charged Particles, John Wiley & Sons, Inc., 1963.

[8]E. Ray, Ann. Phys. (N. Y.), 24, 1 (1963).