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MATHEMATICS-SCIENCE-EXPERIENCE

BY

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In this paper I propose to examine critically some fundamental things. I hope not to make platitudinous remarks. There will be quotations but only from persons of the highest attainment.

If we are to discuss mathematics we must first decide in as exact a way as we can what it is that we are to discuss. A few years ago Courant and Robbins published a book of some four hundred pages entitled "What is Mathematics." The book was apparently written under the assumption that it takes at least four hundred pages to show, not to tell, what mathematics does and how. I believe that some mathematicians would contend that it takes at least three years of graduate study. However, we bear in mind that the professional mathematician may be "unable to see the forest for the trees." It is good for him to study some philosophy and to stand and contemplate his subject as a whole and to critically examine the value to mankind of what he is doing.

If we view what actually is done by those who call themselves mathematicians and those who call themselves physicists or astronomers or for that matter almost any type of scientist we are impressed by much overlapping and a confused and ill-marked boundary. Once at a meeting of the Society for Engineering Education there was a joint meeting of the section on mechanics and the section on mathematics. I rose to the occasion, saying that, as mechanics really was mathematics, I thought the joint meeting splendid and that such meetings should be held frequently. On being pushed, I gave a few reasons, satisfactory to me, as to why mechanics was mathematics. Whereupon, one of the members present, a distinguished electrical engineer, arose and said that he agreed with everything that I had said but that the arguments which I had advanced classified the whole science of electricity as mathematics quite as much as it did mechanics. Then there is geometry. What subject is more universally classified as mathematics than geometry. Although strict formalists may loudly protest, geometry may be regarded as the study of the space of experience. In passing I quote the late Maxime Bôcher (1), professor of mathematics at Harvard and one of America's most distinguished mathematicians:

> Thus, geometry becomes the simplest of the natural sciences and its axioms are of the nature of physical laws, to be tested by experience.

It is hard for me to believe that Bôcher, who was a man of profound scholarship, had only elementary geometry in mind. I shall have more to say on this topic later.

Many non-mathematicians attempt to separate their subject from mathematics by aim. Mechanics is taught both by physicists and mathematicians. Physicists usually make a great point of aim and on the basis of aim contend loudly that mechanics is physics. Some even refer to what they call "pure mathematics" as a "game." This seems an attempt to disclaim that mathematics with which they themselves are not familiar. Again I become lost in confusion: Fourier wished to study the flow of heat (physics) but he developed the theory of trigonometric series ("pure" mathematics).

Let us proceed and let us realize that we are not remaking the language and that any meaning which we give to a term must not depart too far from common usage. As a matter of fact, at this juncture instead of laying down a definition of mathematics I shall quote some definitions that have been given and examine them briefly but critically.

I have heard it said that Charles Darwin gave the following (He probably never did):

> A mathematician is a blind man in a dark room looking for a black hat that isn't there.

We may also give the frequently made remark: "Mathematics is what mathematicians do."

In a recent issue of the magazine Science, Bernard Friedman (2) says: "What are mathematicians doing? They are trying to make precise the intuition of poets."

The first two statements are of course jocular. The other two only partly so. To be properly appreciated they must be pondered in the context in which they appear. It is hoped that they will be appreciated by those who read this paper to the end.

Not long ago I was chatting with a friend when I was a bit startled by his telling me, in order to put me in my place, in an argument that mathematics was the science of measurement and that alone. I also recently encountered this - "Mathematics is the science of quantity or magnitude." Heaven knows where this type of definition originated but like Shakespeare's Cleopatra, "Age cannot wither her nor custom stale her infinite variety."

Turning to what is to us a more serious approach: The first American mathematician of international reputation was Benjamin Peirce of Harvard, popularly known as the Astronomer, in that charming book The Autocrat of the Breakfast Table. In 1870 Peirce (3) wrote:

> Mathematics is the science which draws necessary conclusions.

I call this a great definition in as much as for now nearly a hundred years it has been the starting point for many discussions on the nature of mathematics. It is to be noticed that its emphasis is on method rather than on subject matter or aim. The word "necessary" is not sufficiently precise, that is we are not told what methods of drawing conclusions are allowable. Finally, many of us regard the whole definition as too broad.

Whitehead (4) says:

> Mathematics in its widest significance is the development of all types of formal necessary deductive reasoning.

This is an attempt to be more precise than Peirce. The words all types are satisfying. Deductive was probably intended by Peirce but its specific mention is again satisfying. Yet, there is with some a wishful thinking to include as a part of mathematics the intuition or induction which precedes formal proof. There is also again the possible criticism that the definition is too broad and will include more than is commonly understood as mathematics.

Numerous writers have compared mathematics and logic. Morris R. Cohen (5) at one time professor of philosophy at the University of Chicago says: "Mathematics as a purely formal subject is indeed identical with logic." We add nothing to Nature by calling it God.

Percy Bridgman, Nobel Laureate in physics, (6) says: "Mathematics which is usually recognized to be properly a branch of logic."

Thus, according to Bridgman, mathematics is a branch of logic and not coextensive with logic as stated by Cohen. However, there has been a tremendous development of what was called symbolic logic and mostly by men trained in mathematics and who consider themselves mathematicians. There is now a tendency to discard the term "symbolic" under the assumption that (continued on page 5.)

there is no other type of logic. With this point of view logic becomes a part of mathematics, not mathematics a part of logic. Bearing in some way on this point, almost all present-day students of mathematics study some logic. Why? For a better understanding of their subject, to aid in its study, for general education or because it is a part of their subject?

The definitions, which I have quoted put the emphasis on method. I next quote the now classical definition of Bertrand Russell (7):

> Pure mathematics is the class of all propositions of the form -- p imlies q - where p and q are propositions containing one or more variables the same in the two propositions and neither p nor q contains any constants except logical constants. And logical constants are all notions defined in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

I wish I knew what Russell meant by "truth" Whitehead and Peirce say mathematics implies. At the time that he wrote this Russell must have been something of a fundamentalist. It was early in his career. Russell seems to have regarded mathematics as a body of propositions (theorems), not as a process but as the results of that process. As a definition the word implies may be construed inductively although this would be contrary to Russell's mathematical procedure. At a later time (8) he remarks:

> We shall assume that all mathematics is deductive.

In Russell's jocular definition quoted at the beginning of this paper the word "true" occurs again.

I started this discussion with some remarks on defining mathematics by means of its subject matter. I now return to this and give a very interesting definition. It comes from Study, formerly professor of mathematics at Bonn, and generally recognized as one of the leading geometers of this century. Study (9) says:

> Mathematics includes computation with natural numbers and everything that can be founded upon it, but nothing else.

This is a subject matter definition. At first glance it seems much too narrow. However, the more I ponder it the less sure I am of this. In support of Study, we have many people. Minkowski was professor of mathematics at Göttingen. His work seems to have inspired a good deal of Einstein's early work on relativity. He was certainly one of the great minds of the early twentieth century. Minkowski (10) says: "Integral numbers are the fountainhead of all mathematics."

The following is from A. Comtè (12): "There is no inquiry which is not finally reducible to a question of numbers."

Kronecker (13) says: "The whole numbers are from the Dear God all else is made by man."

A colleague of mine recently remarked when talking informally, that mathematics is largely a study of ordered pairs.

We now pass to a point of view that is different. David Hilbert, one of the leading mathematicians of the early twentieth century and a real founder of the present school of mathematical logic held that - Mathematics is essentially the manipulation of symbols. - I do not remember seeing a concise definition but this idea pervades all of his writings on the nature of mathematics:

I now give Poincare's (14) oft quoted remark:

> Mathematicians do not study objects but the relations between objects. Matter does not engage their attention. They are interested in form alone.

Here I again call to mind Russell's jocular remark quoted at the beginning of this paper.

> Mathematics is that subject in which one never knows what he is talking about or if what he says is true.

The most detailed and clearest definition along this line that I have seen was given by C. I. Lewis (15) at that time professor at Harvard.

> A mathematical system is any set of strings of recognizable marks in which some of the strings are taken initially and the remainder derived from these by operations performed according to rules which are independent of any meaning assigned to the marks.

I shall take the liberty of elaborating Lewis a little. As commonly held by mathematicians: A mathematical structure consists of certain undefined terms (symbols), certain axioms that is relations between them, an agreed logic, that is method of drawing conclusions, and strings of theorems derived by means of the logic. New terms or symbols may be introduced at any time for convenience provided of course, nothing outside the system with which we began is introduced. This structure is built rather than discovered.

Much has been written about the axioms of the system. The question of language in the broad sense is not usually discussed.

It seems to me that formalists in mathematics are satisfied with Lewis, and certainly the majority of mathematicians at the present time should be called formalists - follow the strings where-ever they lead.

The terms formalist and intuitionist are not precise. However, there is a usage. I think to understand these terms we must consider axioms and definitions for such difference as there is seems to me to stem from these two things.

About the turn of the century there was an awakening among physicists for more rigor in their field. This brought about among other things a demand for operational definitions - Physicists define things the way they do them - unless the operational definition is clear, an appreciation of the Einstein theory of relativity is wholly impossible. In the classical treatment of mechanics "time" is an undefined term. However, in the relativity theory time is defined by the way it is measured. Those of us who are old enough may recall sneers made by some at the statement: - Time is defined by a clock. - If this is understood we can understand Eddington (16) when he says that physics consists of pointer readings.

What is said applies to all sciences if properly studied or so it seems to me. Their primary purpose is to describe secondarily to forecast. I do not call mathematics a science preferring a separate classification. Of course, if I were addressing the National Science Foundation, I might feel differently.

I now digress briefly to discuss isomorphism. Two mathematical structures are isomorphic if they are mathematically identical the difference between them being one of language only. An example is the common dualism of projective geometry in which, for example, the system of lines and points in a plane is isomorphic to the system of points and lines in a way familiar to every student of geometry a generation ago. The mathematical structures are identical hence to a formalist the term isomorphic is meaningless. We have but one mathematical structure.

Now to physics - I shall not discard the term isomorphic. If a set of objects of physical or other significance, not the bare bones of mathematics can be put into one to one correspondence with the undefined terms of a mathematical structure and if laws can be stated about them corresponding axiom to law, law to axiom with the mathematical system, we may say that the two systems are isomorphic and applications of mathematics step into the picture. Theorems say: "If you do this you will get that."

I now ask the question: Is it really possible to have isomorphism between a physical system and a mathematical structure? I make one more quotation made almost a hundred years ago by Tait and Steele, in their Natural Philosophy which for many years was a standard English textbook.

> It has been long understood that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of abstractions or rather limitation of the data such as enables us easily to solve the modified form of the question while we are well assured that the circumstances (so modified) affect the result only in a superficial manner.

This was apparently the point of view of all the older physicists. It was the custom to construct in imagination a simple mechanical model which was assumed to have the same pointer readings as the phenomina being studied. This was called understanding the phenomina. Certain laws governing this model were observed and the model was assumed isomorphic to a mathematical structure and conclusions drawn. These results were regarded as an approximation to a fundamental "reality." This is still the point of view of many people. Confidence in the conclusions drawn was possibly greater than warranted but has usually been borne out by experience.

The modern tendency is to dispense with the intermediate model and to make a mathematical model directly from the pointer readings. I am one who doubts if this is ever really done. The action of symbols on paper ultimately goes back to a more primitive experience. Modern Atomic theory seems to be an elaborate model construction. The French mathematician and astronomer Painleve remarked (I cannot give exact reference): We say that the Earth revolves around the Sun for the benefit of women and children - Einstein remarked (again I can not give reference): - Ptolemy and Copernicus were equally right or equally wrong. - Painleve evokes the mechanical model for "women and children." Einstein apparently evokes that mechanical model that he needs. Students of relativity admit a model as "real" if it describes best the phenomina being studied at the time. In such work there is no other reality. Thus reality may change. It seems hard to lose Copernicus but it is conceivable. Times and men change. We might construct many models isomorphic to a set of pointer readings made on the stars. Subsequent pointer readings may be different and lead to different mathematical models even to the invention of new mathematics. Is space of experience Euclidean or non-Euclidean? This question is without meaning other than what I have tried to describe. For that matter why go as far away as the stars to illustrate an interpretation of experience?

The reader may remember the fable of the four blind men and the elephant. One man felt only the tail another the trunk, the third a leg and the fourth only a side. Well! The biologist says - But there really is an elephant to be studied. So answer fundamentalists the world over. But we are not blind men studying something that "exists." This word and the little word "is" have meaning only when operationally used. What are these operations? - Something that each of the two people who talk to each other do (can do).

A person's experience consists of discrete sensations. When a mathematician defines continuity it is to be noticed that the definition is in terms of the discrete. A simple physical law like pv = c must be construed in terms of the discrete, that is really in terms of the integers. This, of course, includes the rational numbers. Their use amounts only to a change of unit in a physical measurement. Isomorphism between a mathematical formula and a physical system involving the irrational of the infinite in any way is outside the pointer readings and meaningless.

I think now of Study and his definition of mathematics given earlier in this paper. What does Russell mean by "truth" in his two definitions?

Where do the axioms of a mathematical structure come from? Does the formalist simply make anything that he likes. If so, how? I ask this question,does the formalist dream human dreams, or the dreams of some other creature? To the intuitionist there is nowhere for the axioms to come from but from the experience of the man who makes them. What about definitions? If not operational, what? The intuitionist admists the infinite for example the/ infinite sequence. This is operationally defined. The Topologist must scrutinize his definitions with the greatest care.

What of this logic which mathematicians use? Is it a fixed thing? Some of us may remember the efforts made by Brouwer (17) and a few others to exclude proof by contradiction. There was a not-so-vocal effort, to exclude proof by mathematical induction. This last seems to me to have been fundamentally based on a feeling about the nature of the positive integers. Are they made with the induction axiom or have they a more fundamental origin? Grant that they are manmade like the rest of our language, but how and when was language made?

Much is said about mathematics as an art. Mathematics is a created structure not too different from a painting. Certainly there are resemblances between modern abstracts art and modern abstract mathematics. What do they stand for and how many people "understand" them? There are some advantages in art. More people criticize it and more people pay real money for it! Mathematics has the advantage that in the past it has helped with the bread and butter of mankind: We have hopes for the future. I shall never be surprised when some well-constructed mathematical system proves isomorphic to something physical. It has happened many times in the past. Remember that the axioms are from our experience. I am frequently annoyed by thoughtless people speaking of mathematics as eternal, immutable, etc. Mathematics is human and like all human structures subject to error and change. Those who think that the ultimate has been reached in any human endeavor are very young.

To say that the universe was made following a mathematical pattern as has been done in "high places" seems to me to be wholly wrong almost silly. Man has made his mathematics just as he has made the rest of his language to describe his experience.

Am I a formalist or an intuitionist? It depends upon the time of day.

(1) Bull, Amer. Math. Soc. Vol. 2, 1904, p. 124.

(2) Oct. 19, 1966

(3) Amer. Jour. of Math. Vol. 4, 1881, p. 97.

(4) Universal Algebra, Preface.

(5) Reason and Nature, p. 173.

(6) Nature of Physical Theory, p. 47.

(7) Principles of Mathematics, p. 3.

(8) Introduction to Mathematical Philosophy, p. 145.

(9) Mathematik and Physik, Braunschweig, 1923.

(10) Diopantische Approximationen, Leipzig 1907, Vorrede.

(11) Sartorius von Walterhausen: Gauss zum Gedachtniss, Leipzig 1856, p. 79.

(12) Positive Philosophy (Matineau) Bk. 1. Chap. 1.

(13) Jahresberichte der Deutschen Mathematiker Vereinigung Bd. 2, p. 19.

(14) Science and Hypothesis (Halsted)

(15) Survey of Symbolic Logic, p. 355.

(16) The Nature of the Physical World, p. 251.

(17) Bull. of Amer. Math. Soc. Vol. 30, p. 31.