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VIBRATION AND BUCKLING OF PRESTRESSED SHELLS OF REVOLUTION[1]

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SUMMARY

A linearized set of equations is developed for the infinitesimal vibration and buckling of an axisymmetrically prestressed thin shell with an arbitrary meridional configuration. A finite-difference numerical procedure is given for finding the natural frequencies, buckling loads, and associated mode shapes, and the procedure is applied to calculation of natural frequencies of an unstressed, simply supported cylinder. A closed-form solution is obtained for the simply supported cylinder and is used to verify the numerical procedure, which is then used to solve some other example problems including the vibrations of particular shells of zero and positive Gaussian curvature.

INTRODUCTION

In the design of shell structures for launch vehicles, planetary atmospheric entry probes, or similar structures, knowledge of the natural frequencies and mode shapes of the systems is of fundamental importance in determining their dynamic behavior. For strength considerations of such shells, knowledge of the structural stability of the configurations under varying aerodynamic pressure distributions is important. These shell configurations are often too complex to allow solutions in closed form, and numerical techniques are appropriate for investigating their dynamic and static behavior.

Several numerical methods have recently been developed for static stress analysis and non-prestressed free vibration analysis of general shells of revolution. Static stress analysis is considered in references 1 to 4 and non-prestressed vibrations are considered in references 5 and 6. Membrane and flexural vibrations of toroidal shells are treated in references 7 and 8 on the basis of the numerical approach of reference 1.

The investigation of prestressed vibrations and buckling of shells requires a consideration of the nonlinear shell equations. The object of this paper is to develop

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[1]The information presented herein is to be included with additional material to be offered in partial fulfillment of the thesis requirements for the degree of Doctor of Philosophy in Engineering Mechanics, Virginia Polytechnic Institute, Blacksburg, Virginia.

differential and difference equations that govern the asymmetric vibration and buckling of a class of shapes of prestressed shells of revolution. These equations are based on the nonlinear theory of reference 9. A finite-difference procedure similar to that of reference 1 (utilizing and extending the ideas of refs. 7 and 8) is then formulated to obtain solutions of the equations. A comparison with known solutions (refs. 10 to 13) for free vibrations and buckling of a simply supported cylinder indicates the validity of the equations derived in the present paper. In addition, consideration is given to some problems which have not been previously treated in the literature. Natural frequencies are calculated for a cylinder with one end simply supported (with in-plane displacements free) and the other end clamped. Also, a comparison is made between natural frequencies for a cylindrical shell and a similar shell with constant positive Gaussian curvature.

SYMBOLS

a	reference length
B	extensional stiffness (eq. (16))
D	bending stiffness (eq. (16))
$\bar{e}_x,\bar{e}_\theta,\bar{E}_x,\bar{E}_\theta$	nondimensional prestress parameters (see eqs. (19) and (52))
E	Young's modulus of elasticity
h	shell thickness
i	station number, $i = 0, 1, 2, \ldots N$
k_x,k_θ	nondimensional curvatures (eqs. (12))
K	critical buckling load parameter
m	number of axial half-waves for cylinder (see eqs. (26))
m_x	moment variable (see eqs. (20))
$M_\xi,M_\theta,M_{\xi\theta},M_{\theta\xi}$	modified moment resultants associated with perturbed state
n	number of circumferential waves

N	total number of intervals along the meridian
$N_\xi, N_\theta, N_{\xi\theta}, N_{\theta\xi}, Q_\xi, Q_\theta$	modified stress resultants associated with perturbed state
$\overline{N}_\xi, \overline{N}_\theta$	modified prestress stress resultants
P, P_ξ, P_θ	surface loading
r	nondimensional radius of cross section, ρ/a
R_ξ, R_θ	principal radii of curvature (see fig. 1)
s	total meridional arc length
S	total nondimensional meridional arc length, s/a
t	time
u,v,w	displacement variables defining perturbed state (see eqs. (20))
U,V,W	displacements in meridional (ξ), circumferential (θ), and normal directions, respectively, of undeformed middle surface defining perturbed state (see eqs. (20))
x	nondimensional meridional coordinate, ξ/a
$\gamma = \frac{1}{r}\frac{dr}{dx}$	
Δ	length of interval between stations, S/N
$\epsilon_\xi, \epsilon_\theta, \epsilon_{\xi\theta}$	middle-surface strains associated with perturbed state
θ	circumferential coordinate in undeformed shell
$\kappa_\xi, \kappa_\theta, \kappa_{\xi\theta}$	middle-surface bending strains associated with the perturbed state
λ	thickness parameter, $\frac{h}{a}$
μ	Poisson's ratio
ν	mass density

ξ	meridional coordinate in undeformed shell
ρ	cross-sectional radius (fig. 1)
$\varphi_\xi, \varphi_\theta, \varphi$	middle-surface rotations associated with perturbed state
$\bar{\varphi}_\xi, \bar{\Phi}_\xi$	prestress meridional rotation
ω	natural frequency
$\Omega^2 = \dfrac{a^2\omega^2\nu\left(1 - \mu^2\right)}{E}$	frequency parameter

Notations Used to Identify Load and Deformation Variables:

Unmarked variables indicate variables associated with perturbed state only.

$\tilde{(\;)}$	indicates modified variables associated with the total deformation
$\bar{(\;)}$	indicates modified variables associated with the prestress state only
$(\;)^o$	indicates physical stress resultant quantities

The comma before a subscript denotes differentiation with respect to the following subscripted variable. A dot over a symbol indicates differentiation of the quantities with respect to time.

1×4 column matrices:

Z_i	dependent variable

4×4 matrices:

$A_i, B_i, C_i, D_0, D_N, E_0, E_N$	difference-equation coefficients
e_{jk}, f_{jk}	boundary-equation coefficients
F_{jk}, G_{jk}, H_{jk}	equilibrium-equation coefficients
P_i	recursion coefficients
α, β	boundary-condition-selection matrices

DEVELOPMENT OF GOVERNING EQUATIONS

The shell geometry is illustrated in figure 1. The location of points on the middle surface of the shell is described by the principal coordinates (ξ, θ), where ξ is the meridional distance measured on the middle surface from one boundary, and θ is the circumferential angle. Since the shell is axisymmetric, it is completely described by the meridional shape parameter $\rho(\xi)$ which is the radial distance from the axis of revolution to the middle surface of the shell.

The principal radii of curvature of the middle surface, $R_\xi(\xi)$ and $R_\theta(\xi)$, are given by:

$$\left.\begin{aligned} R_\xi &= -\frac{\sqrt{1 - \left(\frac{d\rho}{d\xi}\right)^2}}{\frac{d^2\rho}{d\xi^2}} \\ &\text{and} \\ R_\theta &= \frac{\rho}{\sqrt{1 - \left(\frac{d\rho}{d\xi}\right)^2}} \end{aligned}\right\} \tag{1}$$

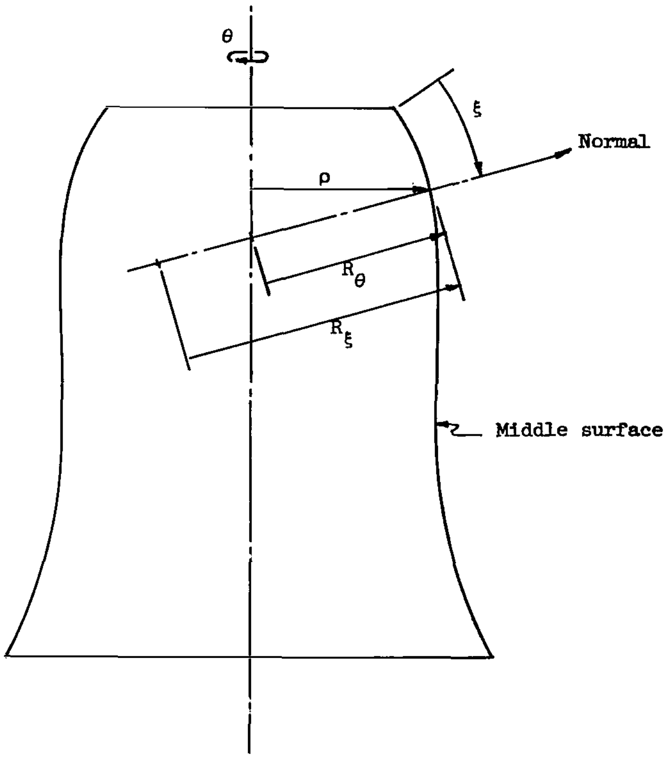


Figure 1.- Shell middle-surface geometry.

The shell is assumed to have a constant thickness h measured along the normal to the middle surface and boundaries at $\xi = 0$ and $\xi = s$, where s is the total meridional arc length. The material is assumed homogeneous and isotropic with mass density ν, Young's modulus of elasticity E, and Poisson's ratio μ.

Governing Nonlinear Equations

General nonlinear shell equations in which strains are assumed to be small and rotations, moderately small, are given in reference 9. For a shell of revolution, these equations become, when inertia terms are added,

$$\left(\rho\tilde{N}_{\xi}\right)_{,\xi} + \tilde{N}_{\xi\theta,\theta} - \frac{d\rho}{d\xi}\tilde{N}_{\theta} + \frac{\rho}{R_{\xi}}\tilde{Q}_{\xi} + \frac{1}{2}\left(\frac{1}{R_{\xi}} - \frac{1}{R_{\theta}}\right)\tilde{M}_{\xi\theta,\theta}$$

$$- \frac{\rho}{R_{\xi}}\left(\tilde{\varphi}_{\xi}\tilde{N}_{\xi} + \tilde{\varphi}_{\theta}\tilde{N}_{\xi\theta}\right) - \frac{1}{2}\left[\tilde{\varphi}\left(\tilde{N}_{\xi} + \tilde{N}_{\theta}\right)\right]_{,\theta} + \rho\tilde{P}_{\xi} = \rho\nu h\ddot{U} \tag{2a}$$

$$\left(\rho\tilde{N}_{\xi\theta}\right)_{,\xi} + \tilde{N}_{\theta,\theta} + \frac{d\rho}{d\xi}\tilde{N}_{\xi\theta} + \frac{\rho}{R_{\theta}}\tilde{Q}_{\theta} + \frac{\rho}{2}\left[\left(\frac{1}{R_{\theta}} - \frac{1}{R_{\xi}}\right)\tilde{M}_{\xi\theta}\right]_{,\xi}$$

$$- \frac{\rho}{R_{\theta}}\left(\tilde{\varphi}_{\xi}\tilde{N}_{\xi\theta} + \tilde{\varphi}_{\theta}\tilde{N}_{\theta}\right) + \frac{\rho}{2}\left[\tilde{\varphi}\left(\tilde{N}_{\xi} + \tilde{N}_{\theta}\right)\right]_{,\xi} + \rho\tilde{P}_{\theta} = \rho\nu h\ddot{V} \tag{2b}$$

$$\left(\rho\tilde{Q}_{\xi}\right)_{,\xi} + \tilde{Q}_{\theta,\theta} - \rho\left(\frac{\tilde{N}_{\xi}}{R_{\xi}} + \frac{\tilde{N}_{\theta}}{R_{\theta}}\right) - \left(\rho\tilde{\varphi}_{\xi}\tilde{N}_{\xi}\right)_{,\xi} - \left(\rho\tilde{\varphi}_{\theta}\tilde{N}_{\xi\theta}\right)_{,\xi}$$

$$- \left(\tilde{\varphi}_{\xi}\tilde{N}_{\xi\theta}\right)_{,\theta} - \left(\tilde{\varphi}_{\theta}\tilde{N}_{\theta}\right)_{,\theta} + \rho\tilde{P} = \rho h\nu\ddot{W} \tag{2c}$$

$$\left(\rho\tilde{M}_{\xi}\right)_{,\xi} + \tilde{M}_{\xi\theta,\theta} - \frac{d\rho}{d\xi}\tilde{M}_{\theta} - \rho\tilde{Q}_{\xi} = 0 \tag{2d}$$

$$\left(\rho\tilde{M}_{\xi\theta}\right)_{,\xi} + \tilde{M}_{\theta,\theta} + \frac{d\rho}{d\xi}\tilde{M}_{\xi\theta} - \rho\tilde{Q}_{\theta} = 0 \tag{2e}$$

where the comma before a subscript denotes partial differentiation with respect to the succeeding subscripted independent variables (ξ or θ) and dots over a quantity denote differentiation with respect to time. The equilibrium equations (2a), (2b), and (2c) represent the sum of forces along coordinates of the undeformed surface, and the perturbed displacements of the middle surface U, V, and W are measured in the direction of the coordinates of the undeformed surface with W measured positive along the outward normal.

The boundary conditions considered on the edges $\xi = 0$ and $\xi = s$ may be chosen from any combination of the following four pairs of quantities in which either quantity (but not both) of each pair is prescribed:

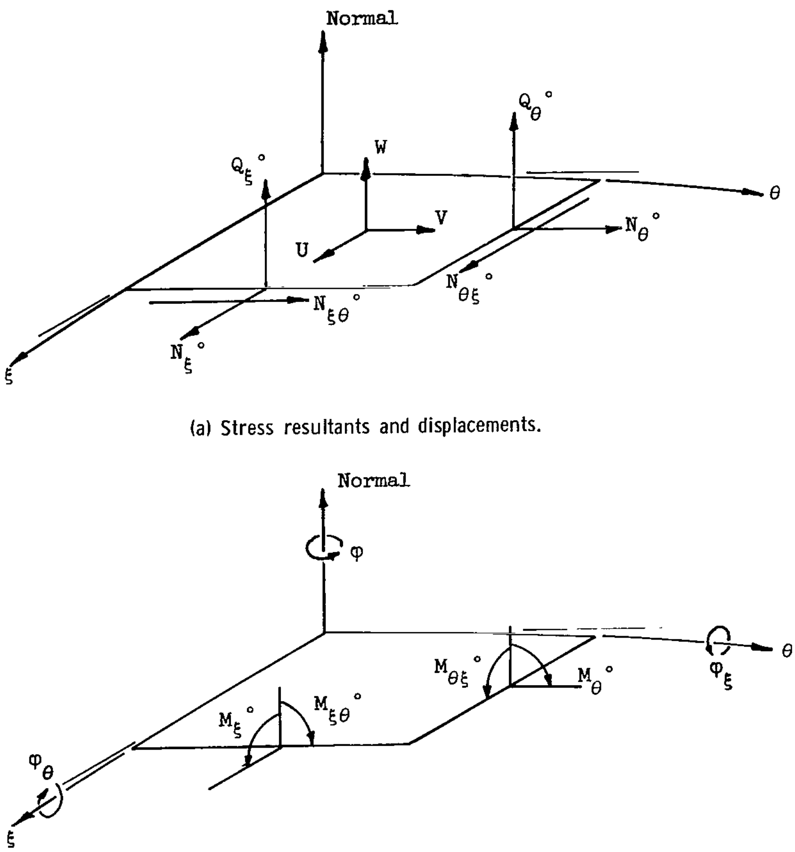
$$
\left.
\begin{array}{rl}
\tilde{N}_{\xi} & \text{or} \quad \tilde{U} \\[2ex]
\tilde{N}_{\xi\theta} + \left(\dfrac{3}{2R_{\theta}} - \dfrac{1}{2R_{\xi}}\right)\tilde{M}_{\xi\theta} + \dfrac{1}{2}\left(\tilde{N}_{\xi} + \tilde{N}_{\theta}\right)\tilde{\varphi} & \text{or} \quad \tilde{V} \\[2ex]
\tilde{Q}_{\xi} + \dfrac{1}{\rho}\tilde{M}_{\xi\theta,\theta} - \tilde{\varphi}_{\xi}\tilde{N}_{\xi} - \tilde{\varphi}_{\theta}\tilde{N}_{\xi\theta} & \text{or} \quad \tilde{W} \\[2ex]
\tilde{\varphi}_{\xi} & \text{or} \quad \tilde{M}_{\xi}
\end{array}
\right\} \tag{3}
$$

The equations have been derived by use of the Kirchhoff-Love assumptions; that is, normals to the undeformed middle surface remain normal to the deformed middle surface, normal strain is zero, and the normal stress is negligible. Rotary inertia terms have been neglected in the moment equations. Modified stress and moment resultants have been used in the development of equations (2) and (3) and are defined as follows:

$$
\left.
\begin{aligned}
\tilde{N}_{\xi} &= N_{\xi}^{o} - \frac{M_{\xi}^{o}}{R_{\xi}} \\
\tilde{N}_{\theta} &= N_{\theta}^{o} - \frac{M_{\theta}^{o}}{R_{\theta}} \\
\tilde{N}_{\xi\theta} &= N_{\xi\theta}^{o} - \frac{M_{\theta\xi}^{o}}{R_{\theta}} \\
\tilde{N}_{\theta\xi} &= N_{\theta\xi}^{o} - \frac{M_{\xi\theta}^{o}}{R_{\xi}} \\
\tilde{M}_{\xi} &= M_{\xi}^{o} \\
\tilde{M}_{\theta} &= M_{\theta}^{o} \\
\tilde{M}_{\xi\theta} &= \tilde{M}_{\theta\xi} = \frac{1}{2}\left(M_{\xi\theta}^{o} + M_{\theta\xi}^{o}\right)
\end{aligned}
\right\} \tag{4}
$$

The modified transverse shear stress resultants $\tilde{Q}_{\xi}$ and $\tilde{Q}_{\theta}$ may be found by applying the definitions in equations (4) to the moment equilibrium equations (eqs. (2d) and (2e)).

The quantities N_ξ^o, N_θ^o, $N_{\xi\theta}^o$, $N_{\theta\xi}^o$, Q_ξ^o, and Q_θ^o represent the total middle-surface stress resultants (fig. 2(a)), and the quantities M_ξ^o, M_θ^o, $M_{\xi\theta}^o$, and $M_{\theta\xi}^o$ represent the total middle-surface moment resultants introduced by the same combined effect (fig. 2(b)). No attempt is made to relate these stress and moment resultants to the distribution of stress through the thickness of the shell. The equations of reference 9 have been derived without dependence on such a relationship; thus, any formulation consistent with thin-shell theory is acceptable. According to reference 9, the addition of terms like M^o/R to the N^o quantities in the stress-strain relations does not introduce errors any greater than those already introduced by neglecting transverse shear flexibility in the Kirchhoff-Love hypothesis. Consequently, the $\tilde{N}$ quantities may be treated as stress resultants without introducing an inconsistency in the thin-shell analysis.



(a) Stress resultants and displacements.

(b) Moment resultants and rotations.

Figure 2.- Middle-surface quantities.

The sum of the moments about the normal direction is

$$N_{\xi\theta}^o - \frac{M_{\theta\xi}^o}{R_\theta} - N_{\theta\xi}^o + \frac{M_{\xi\theta}^o}{R_\xi} = 0$$

Hence, from the definition of the modified stress resultants

$$\tilde{N}_{\theta\xi} = \tilde{N}_{\xi\theta} \tag{5}$$

Therefore, the sixth equilibrium equation, that of equilibrium of moments about the normal, is identically satisfied by symmetric modified stress resultants.

Vibration and Buckling Equations

In the derivation of the vibration and buckling equations, the total rotations and total stress and moment resultants are separated into the parts associated with the initial axisymmetric prestress and the parts associated with the infinitesimal perturbed displacements about the prestressed state. The perturbed quantities can be time dependent for infinitesimal vibration investigations or static for buckling investigations. Symbols with bars represent those quantities associated with the prestress conditions and symbols without bars represent those associated with the perturbed state. Thus, the total stress state may be completely described by these quantities from the relations:

$$\left.\begin{aligned}
\tilde{N}_\xi &= \bar{N}_\xi + N_\xi \\
\tilde{N}_\theta &= \bar{N}_\theta + N_\theta \\
\tilde{N}_{\xi\theta} &= N_{\xi\theta} \\
\tilde{Q}_\xi &= \bar{Q}_\xi + Q_\xi \\
\tilde{Q}_\theta &= Q_\theta \\
\tilde{M}_\xi &= \bar{M}_\xi + M_\xi \\
\tilde{M}_\theta &= \bar{M}_\theta + M_\theta \\
\tilde{M}_{\xi\theta} &= M_{\xi\theta}
\end{aligned}\right\} \tag{6}$$

The total rotations are

$$\left.\begin{aligned}
\tilde{\varphi}_\xi &= \bar{\varphi}_\xi + \varphi_\xi \\
\tilde{\varphi}_\theta &= \varphi_\theta \\
\tilde{\varphi} &= \varphi
\end{aligned}\right\} \tag{7}$$

and the total surface loading is described by

$$\left.\begin{aligned}\tilde{P}_\xi &= \bar{P}_\xi \\ \tilde{P} &= \bar{P}\end{aligned}\right\} \tag{8}$$

since no additional surface loading is assumed to be associated with the perturbed state.

Substitution of equations (6), (7), and (8) in equation (2a) yields

$$\begin{aligned}&\left\{\frac{d\rho}{d\xi}\bar{N}_\xi + \rho\frac{d\bar{N}_\xi}{d\xi} - \frac{d\rho}{d\xi}\bar{N}_\theta + \frac{\rho}{R_\xi}\bar{Q}_\xi - \frac{\rho}{R_\xi}\bar{\varphi}_\xi\bar{N}_\xi + \rho\bar{P}_\xi\right\} \\ &+ \left\{\frac{d\rho}{d\xi}N_\xi + \rho N_{\xi,\xi} + N_{\xi\theta,\theta} - \frac{d\rho}{d\xi}N_\theta + \frac{\rho}{R_\xi}Q_\xi + \frac{1}{2}\left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right)M_{\xi\theta,\theta}\right. \\ &\left. - \frac{\rho}{R_\xi}\varphi_\xi N_\xi - \frac{\rho}{R_\xi}\varphi_\theta N_{\xi\theta} - \frac{1}{2}\left[\varphi\left(N_\xi + N_\theta\right)\right]_{,\theta}\right\} + \left\{-\frac{\rho}{R_\xi}\bar{\varphi}_\xi N_\xi\right\} \\ &+ \left\{-\frac{\rho}{R_\xi}\varphi_\xi\bar{N}_\xi - \frac{1}{2}\varphi_{,\theta}\left(\bar{N}_\xi + \bar{N}_\theta\right)\right\} = \rho\nu h\ddot{U}\end{aligned} \tag{9}$$

where θ-derivatives of barred quantities vanish as a result of axisymmetry of the prestressed state. The prestressed shell is in equilibrium; thus, the sum of the terms enclosed by the first set of braces in equation (9) vanishes identically. Furthermore, the perturbation of the shell away from the prestressed configuration is governed by linear theory. Therefore, the nonlinear terms enclosed by the second pair of braces are neglected.

The term in equation (9) enclosed by the third pair of braces (i.e., the interaction between the prestress deformation and the perturbation stress resultants) is usually neglected in the procedure followed in the classical linearization process for a cylinder. If this term is neglected, the general assumption is made that the prestress rotation is uniformly zero throughout the shell. The error introduced is usually negligible, but for certain boundary conditions or for sharply varying surface loads, this term may be significant and is consequently retained in this analysis. If $\bar{\varphi}_\xi$ and $\bar{\varphi}_{\xi,\xi}$ are neglected, the problem may be reinterpreted as that of a prestressed but undeformed shell of revolution perturbed about the undeformed state. With this term retained, the equilibrium equation in the meridional direction (eq. (9)) reduces to

$$\left(\rho N_\xi\right)_{,\xi} + N_{\xi\theta,\theta} - \frac{d\rho}{d\xi} N_\theta + \frac{\rho}{R_\xi} Q_\xi + \frac{1}{2}\left[\left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) M_{\xi\theta}\right]_{,\theta}$$

$$- \frac{\rho}{R_\xi} \bar{N}_\xi \varphi_\xi - \frac{1}{2}\left(\bar{N}_\xi + \bar{N}_\theta\right)\varphi_{,\theta} - \frac{\rho}{R_\xi} \bar{\varphi}_\xi N_\xi = \rho h \nu \ddot{U} \qquad (10)$$

By the same procedure, the remaining equilibrium equations (eqs. (2b) to (2e)) are linearized. Solving equations (2d) and (2e) for Q_ξ and Q_θ and substituting for them into equations (2a), (2b), and (2c), eliminates these quantities from the system. The parameters defining the geometry of the middle surface can be nondimensionalized by using a reference length a, as follows:

$$\left.\begin{aligned} x &= \frac{\xi}{a} \\ r &= \frac{\rho}{a} \end{aligned}\right\} \qquad (11)$$

and nondimensional curvatures can be defined as

$$\left.\begin{aligned} k_x &= \frac{a}{R_\xi} \\ k_\theta &= \frac{a}{R_\theta} \end{aligned}\right\} \qquad (12)$$

Upon completion of these manipulations, the following equilibrium equations result:

$$a\left(\frac{dr}{dx} N_\xi + rN_{\xi,x} + N_{\xi\theta,\theta} - \frac{dr}{dx} N_\theta\right) + k_x \frac{dr}{dx} M_\xi + rk_x M_{\xi,x} - k_x \frac{dr}{dx} M_\theta + \frac{1}{2}\left(3k_x - k_\theta\right) M_{\xi\theta,\theta}$$

$$- a\left[rk_x \bar{N}_\xi \varphi_\xi + \frac{1}{2}\left(\bar{N}_\xi + \bar{N}_\theta\right)\varphi_{,\theta} + rk_x \bar{\varphi}_\xi N_\xi\right] = ra^2 h\nu\ddot{U} \qquad (13a)$$

$$a\left(2 \frac{dr}{dx} N_{\xi\theta} + rN_{\xi\theta,x} + N_{\theta,\theta}\right) + \frac{1}{2}\left[\left(k_x + 3k_\theta\right)\frac{dr}{dx} - \frac{dk_x}{dx} r\right] M_{\xi\theta} + \frac{r}{2}\left(3k_\theta - k_x\right) M_{\xi\theta,x} + k_\theta M_{\theta,\theta}$$

$$+ a\left[-rk_\theta \bar{N}_\theta \varphi_\theta + \frac{r}{2} \frac{d}{dx}\left(\bar{N}_\xi + \bar{N}_\theta\right)\varphi + \frac{r}{2}\left(\bar{N}_\xi + \bar{N}_\theta\right)\varphi_{,x} - rk_\theta \bar{\varphi}_\xi N_{\xi\theta}\right] = ra^2 h\nu\ddot{V} \qquad (13b)$$

$$
\begin{aligned}
&-ar\left(k_x N_\xi + k_\theta N_\theta\right) + \frac{d^2r}{dx^2} M_\xi + 2\,\frac{dr}{dx} M_{\xi,x} + rM_{\xi,xx} \\
&- \frac{d^2r}{dx^2} M_\theta - \frac{dr}{dx} M_{\theta,x} + \frac{2}{r}\frac{dr}{dx} M_{\xi\theta,\theta} + 2M_{\xi\theta,x\theta} + \frac{1}{r} M_{\theta,\theta\theta} \\
&+ a\Bigg(- \frac{dr}{dx} \bar{N}_\xi \varphi_\xi - r\bar{N}_\xi \varphi_{\xi,x} - r\,\frac{d\bar{N}_\xi}{dx} \varphi_\xi - \bar{N}_\theta \varphi_{\theta,\theta} \\
&- \frac{dr}{dx} \bar{\varphi}_\xi N_\xi - r\,\frac{d\bar{\varphi}_\xi}{dx} N_\xi - r\bar{\varphi}_\xi N_{\xi,x} - \bar{\varphi}_\xi N_{\xi\theta,\theta}\Bigg) = ra^2 h\nu \ddot{W}
\end{aligned}
\tag{13c}
$$

Similarly, the boundary conditions (eqs. (3)) are given as

$$
\left.
\begin{aligned}
N_\xi = 0 \quad &\text{or} \quad U = 0 \\
N_{\xi\theta} + \frac{1}{2a}\left(3k_\theta - k_x\right)M_{\xi\theta} + \frac{1}{2}\left(\bar{N}_\xi + \bar{N}_\theta\right)\varphi = 0 \quad &\text{or} \quad V = 0 \\
\frac{1}{a}\left[M_{\xi,x} + \frac{1}{r}\frac{dr}{dx}\left(M_\xi - M_\theta\right) + \frac{2M_{\xi\theta,\theta}}{r}\right] - \bar{N}_\xi \varphi_\xi - \bar{\varphi}_\xi N_\xi = 0 \quad &\text{or} \quad W = 0 \\
\varphi_\xi = 0 \quad &\text{or} \quad M_\xi = 0
\end{aligned}
\right\}
\tag{14}
$$

The modified stress-resultant—strain relationships, if physical linearity is assumed, are

$$
\left.
\begin{aligned}
&N_\xi = B\left(\epsilon_\xi + \mu\epsilon_\theta\right) \\
&N_\theta = B\left(\epsilon_\theta + \mu\epsilon_\xi\right) \\
&N_{\xi\theta} = B(1 - \mu)\epsilon_{\xi\theta} \\
&M_\xi = D\left(\kappa_\xi + \mu\kappa_\theta\right) \\
&M_\theta = D\left(\kappa_\theta + \mu\kappa_\xi\right) = \mu M_\xi + D\left(1 - \mu^2\right)\kappa_\theta \\
&M_{\xi\theta} = D(1 - \mu)\kappa_{\xi\theta}
\end{aligned}
\right\}
\tag{15}
$$

where

$$\left.\begin{aligned} B &= \frac{Eh}{1 - \mu^2} \\ D &= \frac{Eh^3}{12\left(1 - \mu^2\right)} \end{aligned}\right\} \tag{16}$$

The linearized strain-displacement relationships, from reference 9, reduce to

$$\left.\begin{aligned} \epsilon_{\xi} &= \frac{1}{a}\left(U_{,x} + k_x W + \bar{\varphi}_{\xi} k_x U - \bar{\varphi}_{\xi} W_{,x}\right) \\ \epsilon_{\theta} &= \frac{1}{a}\left(\frac{dr}{dx}\frac{U}{r} + \frac{V_{,\theta}}{r} + k_{\theta} W\right) \\ \epsilon_{\xi\theta} &= \frac{1}{a}\left(\frac{U_{,\theta}}{2r} + \frac{V_{,x}}{2} - \frac{dr}{dx}\frac{V}{2r} + \frac{\bar{\varphi}_{\xi}}{2} k_{\theta} V - \frac{\bar{\varphi}_{\xi}}{2r} W_{,\theta}\right) \\ \kappa_{\xi} &= \frac{1}{a^2}\left(k_x U_{,x} + \frac{dk_x}{dx} U - W_{,xx}\right) \\ \kappa_{\theta} &= \frac{1}{a^2}\left(\frac{dr}{dx}\frac{k_x}{r} U + \frac{k_{\theta}}{r} V_{,\theta} - \frac{W_{,\theta\theta}}{r^2} - \frac{dr}{dx}\frac{W_{,x}}{r}\right) \\ \kappa_{\xi\theta} &= \frac{1}{a^2}\left[\frac{\left(3k_x - k_{\theta}\right)}{4r} U_{,\theta} + \frac{\left(3k_{\theta} - k_x\right)}{4} V_{,x} + \frac{\left(k_x - 3k_{\theta}\right)}{4r}\frac{dr}{dx} V - \frac{W_{,x\theta}}{r} + \frac{dr}{dx}\frac{W_{,\theta}}{r^2}\right] \end{aligned}\right\} \tag{17}$$

The middle-surface rotations are given in terms of displacements as

$$\left.\begin{aligned} \varphi_{\xi} &= \frac{1}{a}\left(k_x U - W_{,x}\right) \\ \varphi_{\theta} &= \frac{1}{a}\left(k_{\theta} V - \frac{W_{,\theta}}{r}\right) \\ \varphi &= \frac{1}{2a}\left(V_{,x} + \frac{dr}{dx}\frac{V}{r} - \frac{U_{,\theta}}{r}\right) \end{aligned}\right\} \tag{18}$$

The prestress terms are given nondimensionally as

$$\bar{N}_{\xi} = B\bar{e}_x(x) \qquad \bar{N}_{\theta} = B\bar{e}_{\theta}(x) \tag{19}$$

Reduction to Ordinary Differential Equations

With equations (13) and (15) to (19), the equilibrium equations can be reduced to three partial differential equations with the displacements as the unknown dependent variables where the highest order derivative in x is a fourth-order derivative. However, since the solution, in general, can only be achieved by numerical techniques, the procedure of reference 1 is followed, where dependence on θ is removed by assuming a solution of the separable type and introducing M_ξ as an additional unknown. This procedure yields a set of four second-order ordinary differential equations with variable coefficients. The fourth equation is simply the equation for M_ξ in terms of the displacements. This reduction in order is essential for the numerical treatment that follows.

A solution is assumed of the form

$$\left.\begin{aligned} U &= u(x)(\cos n\theta)e^{i\omega t} \\ V &= v(x)(\sin n\theta)e^{i\omega t} \\ W &= w(x)(\cos n\theta)e^{i\omega t} \\ M_\xi &= \frac{Eh^3}{a^2} m_x(x)(\cos n\theta)e^{i\omega t} \end{aligned}\right\} \qquad (20)$$

Defining the perturbation displacements in this manner assures compatibility in the θ-coordinate. The special case of axisymmetric torsional vibration or torsional buckling is precluded in this investigation by the introduction of this form of θ-variation. This vibration mode uncouples from the extensional and bending modes (see ref. 14) and may be treated directly by interchanging the sines and cosines in equations (20).

For the buckling problem, the time dependence of the perturbed displacements is removed by allowing ω (eqs. (20)) to vanish. The perturbed displacements then represent a possible equilibrium state. Any loading that would maintain this equilibrium state as well as the prestressed equilibrium state is a critical loading for buckling.

Performing the operations indicated and utilizing the following geometric relationships

$$\left.\begin{aligned} \frac{dk_\theta}{dx} &= \frac{1}{r}\frac{dr}{dx}\left(k_x - k_\theta\right) \\ \frac{d^2r}{dx^2} &= -rk_xk_\theta \end{aligned}\right\} \qquad (21)$$

which are the Codazzi and Gauss equations, respectively, yields the governing equations, as follows:

$$F_{11}u'' + G_{11}u' + H_{11}u + G_{12}v' + H_{12}v + F_{13}w'' + G_{13}w' + H_{13}w + G_{14}m_x' + H_{14}m_x = 0 \tag{22a}$$

$$G_{21}u' + H_{21}u + F_{22}v'' + G_{22}v' + H_{22}v + F_{23}w'' + G_{23}w' + H_{23}w + H_{24}m_x = 0 \tag{22b}$$

$$\begin{aligned} &F_{31}u'' + G_{31}u' + H_{31}u + F_{32}v'' + G_{32}v' + H_{32}v \\ &+ F_{33}w'' + G_{33}w' + H_{33}w + F_{34}m_x'' + G_{34}m_x' + H_{34}m_x = 0 \end{aligned} \tag{22c}$$

$$G_{41}u' + H_{41}u + H_{42}v + F_{43}w'' + G_{43}w' + H_{43}w + H_{44}m_x = 0 \tag{22d}$$

The same procedure yields the boundary conditions, as follows:

$$e_{11}u' + f_{11}u + f_{12}v + e_{13}w' + f_{13}w = 0 \quad \text{or} \quad u = 0 \tag{23a}$$

$$f_{21}u + e_{22}v' + f_{22}v + e_{23}w' + f_{23}w = 0 \quad \text{or} \quad v = 0 \tag{23b}$$

$$e_{31}u' + f_{31}u + e_{32}v' + f_{32}v + e_{33}w' + f_{33}w + e_{34}m_x' + f_{34}m_x = 0 \quad \text{or} \quad w = 0 \tag{23c}$$

$$f_{41}u + e_{43}w' = 0 \quad \text{or} \quad m_x = 0 \tag{23d}$$

Primes denote total differentiation with respect to the nondimensional variable x, and the coefficients are subscripted for convenience in subsequent matrix manipulations. The coefficients F_{jk}, G_{jk}, H_{jk} are given in appendix A in terms of the parameters γ and λ, where

$$\left.\begin{aligned} \gamma &= \frac{1}{r}\frac{dr}{dx} \\ \lambda &= \frac{h}{a} \end{aligned}\right\} \tag{24}$$

For the vibration problem, the frequency parameter Ω^2 occurs in H_{11}, H_{22}, and H_{33}, where

$$\Omega^2 = \frac{a^2\omega^2\nu\left(1 - \mu^2\right)}{E} \tag{25}$$

CLOSED-FORM SOLUTION FOR CYLINDER VIBRATIONS

The vibratory characteristics of a "freely supported" cylinder (simply supported but unrestrained in the axial direction) with prestress deformations neglected are well known. (See refs. 10, 11, and 12.) Thus, these known results can be used as a check of the validity of the governing equations and of the accuracy of the numerical techniques to be suggested subsequently in this report.

When prestress deformations are neglected and the in-plane stresses are constant, the equations (22) reduce to ordinary differential equations with constant coefficients which, for freely supported boundary conditions, have a solution of the form

$$\left.\begin{aligned} u(x) &= A_m \cos \frac{m\pi x}{S} \\ v(x) &= B_m \cos \frac{m\pi x}{S} \\ w(x) &= C_m \sin \frac{m\pi x}{S} \\ m_x(x) &= D_m \sin \frac{m\pi x}{S} \end{aligned}\right\} \quad m = 1,2,\ .\ .\ . \tag{26}$$

where S is the length-radius ratio of the cylinder (a = cylinder radius). The classical procedure of neglecting prestress deformations to ensure constant coefficients in the field equations implies that the cylinder is initially prestressed as a shell with free edges and then subsequently supported for vibration.

Equations (26) are substituted into equations (22) to yield a set of linear homogeneous algebraic equations. For a nontrivial solution to exist, the determinant of the coefficient matrix of the resultant set of equations must vanish. This procedure leads to the characteristic equation

$$\Lambda_3\left(\Omega^6\right) + \Lambda_2\left(\Omega^4\right) + \Lambda_1\left(\Omega^2\right) + \Lambda_0 = 0 \tag{27}$$

where the coefficients Λ_i are given in appendix B.

Equation (27) has been solved for the frequency parameter for an unstressed circular cylindrical shell with $\mu = 0.3$, $\lambda = 0.001$, and $S = 3$; and the results for the axial mode $m = 1$ are given in figure 3. For the purpose of comparing the results of equation (27) with those given in references 10, 11, and 12, the in-plane inertia terms are dropped from equations (22a), (22b), and (22c), so that equation (27) is reduced from a cubic to a linear equation in Ω^2. In all studies performed, the frequencies calculated, when in-plane inertias were neglected, agreed closely with those given in references 10, 11, and 12, and all trends observed in these references were similarly verified.

The effect on the natural frequencies due to the neglect of in-plane inertia is illustrated in figure 3. The error introduced by this approximation decreases as n increases, the largest error occurring at $n = 2$. For n larger than 5 the error is negligible. For $n = 0$, the fundamental frequency, which corresponds to pure torsional oscillations, is excluded when the assumption of negligible in-plane inertias is made.

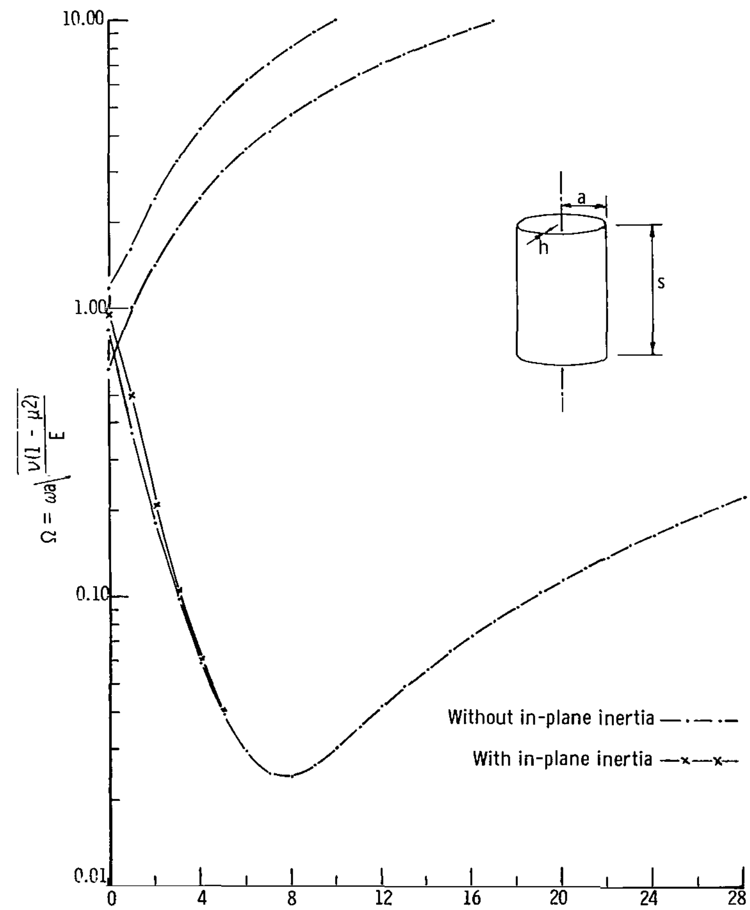


Figure 3.- Natural frequencies of freely supported circular cylindrical shell. $\lambda = \frac{h}{a} = 0.001$; $S = \frac{s}{a} = 3$; $m = 1$.

DEVELOPMENT OF NUMERICAL PROCEDURE

Development of Difference Equations

A numerical procedure is needed for those shells of revolution and loading conditions which do not admit a solution in closed form. The meridian of the shell is divided into increments and a three-point central difference method is used to reduce the differential equations to algebraic form. The distance measured along the meridian between adjacent stations is constant and is represented nondimensionally by Δ where

$$\Delta = x_i - x_{i-1} = \frac{S}{N} \tag{28}$$

and where the subscript i on symbols and matrices indicates the evaluation of the subscripted variable or matrix at the ith station, $i = 0, 1, 2, \ldots, N$ and where

S total meridional arc length of the nondimensional shell, s/a

N total number of intervals

The three-point difference formulas, when applied at the ith station for some function $z(x)$, are

$$\left.\begin{aligned} z_i'' &\approx \frac{1}{\Delta^2}\left(z_{i-1} - 2z_i + z_{i+1}\right) \\ z_i' &\approx \frac{1}{2\Delta}\left(-z_{i-1} + z_{i+1}\right) \end{aligned}\right\} \tag{29}$$

Reference 2 indicates that this simple approximation leads to sufficiently accurate results.

The governing equations (22) may be written in matrix form at station i as

$$F_iZ_i'' + G_iZ_i' + H_iZ_i = 0 \tag{30}$$

where

$$F_i = \begin{bmatrix} F_{11} & 0 & F_{13} & 0 \\ 0 & F_{22} & F_{23} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ 0 & 0 & F_{43} & 0 \end{bmatrix}_i \tag{31a}$$

$$G_i = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & 0 \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & 0 & G_{43} & 0 \end{bmatrix}_i \tag{31b}$$

$$H_i = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}_i \tag{31c}$$

and

$$Z_i = \begin{Bmatrix} u \\ v \\ w \\ m_x \end{Bmatrix}_i \tag{31d}$$

Similarly, all the boundary equations (23) may be written in matrix form at the boundaries, $i = 0$ and $i = N$, as

$$\left.\begin{aligned} \alpha_0 e_0 Z_0' + \left(\alpha_0 f_0 + \beta_0\right) Z_0 = 0 \\ \alpha_N e_N Z_N' + \left(\alpha_N f_N + \beta_N\right) Z_N = 0 \end{aligned}\right\} \tag{32}$$

where

$$e_{0,N} = \begin{bmatrix} e_{11} & 0 & e_{13} & 0 \\ 0 & e_{22} & e_{23} & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} \\ 0 & 0 & e_{43} & 0 \end{bmatrix}_{0,N} \tag{33a}$$

$$f_{0,N} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 \\ f_{21} & f_{22} & f_{23} & 0 \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & 0 & 0 & 0 \end{bmatrix}_{0,N} \tag{33b}$$

and where

$$\left.\begin{aligned} \alpha_{0,N} &= \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{bmatrix}_{0,N} \\ \beta_{0,N} &= \begin{bmatrix} (1 - \alpha_{11}) & 0 & 0 & 0 \\ 0 & (1 - \alpha_{22}) & 0 & 0 \\ 0 & 0 & (1 - \alpha_{33}) & 0 \\ 0 & 0 & 0 & (1 - \alpha_{44}) \end{bmatrix}_{0,N} \end{aligned}\right\} \tag{34}$$

The elements α_{jj} take on the value 1 or 0 depending on the prescribed conditions.

The α- and β-matrices (eqs. (34)) are used to select the prescribed boundary conditions. If, for example, $u = 0$ is prescribed at $i = 0$ then $(\alpha_{11})_0 = 0$ and if u is not prescribed at $i = N$ (i.e., if the u displacement is unrestrained in the meridional direction of the undeformed shell), then $(\alpha_{11})_N = 1$. If desired, the present theory can be extended to allow for elastic and directional supports in the boundary conditions by appropriate redefinition of the α- and β-matrices.

When equations (29) are applied to equations (30) and (32), the governing equations become

$$\left(\frac{F_i}{\Delta^2} - \frac{G_i}{2\Delta}\right)Z_{i-1} + \left(H_i - \frac{2F_i}{\Delta^2}\right)Z_i + \left(\frac{F_i}{\Delta^2} + \frac{G_i}{2\Delta}\right)Z_{i+1} = 0 \qquad (i = 0, 1, 2, \ldots, N - 1, N) \tag{35}$$

and the boundary equations become

$$\left.\begin{aligned}
&-\frac{\alpha_0 e_0}{2\Delta} Z_{-1} + \left(\alpha_0 f_0 + \beta_0\right) Z_0 + \frac{\alpha_0 e_0}{2\Delta} Z_1 = 0 \\
&-\frac{\alpha_N e_N}{2\Delta} Z_{N-1} + \left(\alpha_N f_N + \beta_N\right) Z_N + \frac{\alpha_N e_N}{2\Delta} Z_{N+1} = 0
\end{aligned}\right\} \qquad (36)$$

Equations (36) can be solved for Z_{-1} and Z_{N+1} and the results can be substituted into equation (35) to yield, at $i = 0$,

$$\left\{\alpha_0\left[\frac{e_0}{2\Delta}\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(H_0 - \frac{2F_0}{\Delta^2}\right) + f_0\right] + \beta_0\right\} Z_0 + \frac{\alpha_0 e_0}{2\Delta}\left[\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(\frac{F_0}{\Delta^2} + \frac{G_0}{2\Delta}\right) + I\right] Z_1 = 0 \qquad (37)$$

and, at $i = N$,

$$\left\{\alpha_N\left[-\frac{e_N}{2\Delta}\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(H_N - \frac{2F_N}{\Delta^2}\right) + f_N\right] + \beta_N\right\} Z_N$$

$$-\frac{\alpha_N e_N}{2\Delta}\left[\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(\frac{F_N}{\Delta^2} - \frac{G_N}{2\Delta}\right) + I\right] Z_{N-1} = 0 \qquad (38)$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The difference equations (35), (37), and (38) constitute a complete set of field equations governing the behavior of the perturbed state.

Numerical Solution

The problem is now one of solving a set of homogeneous equations (eqs. (35), (37), and (38)). This set constitutes an eigenvalue problem such that the mode shape Z_i is the eigenvector and the frequency parameter Ω^2 is the corresponding eigenvalue for the vibration problem. The eigenvalue for the buckling problem is contained in all barred terms. The fourth equation, being simply the definition of m_x in terms of displacements, will not contain an eigenvalue for either problem. For a nontrivial solution to exist, the determinant of the coefficient matrix must vanish.

The coefficient matrix will be a 12-element-wide band matrix. A convenient technique for solution of this problem can be formulated by modifying the method of references 7 and 8. Such a modification is presented herein to handle the free vibrations and buckling of a prestressed shell governed by four second-order difference equations with two-point boundary conditions.

Define the following (4×4) matrices:

$$\left.\begin{aligned}
A_i &= \frac{F_i}{\Delta^2} - \frac{G_i}{2\Delta} \\
B_i &= H_i - \frac{2F_i}{\Delta^2} \\
C_i &= \frac{F_i}{\Delta^2} + \frac{G_i}{2\Delta} \\
D_0 &= \alpha_0 \left[\frac{e_0}{2\Delta}\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(H_0 - \frac{2F_0}{\Delta^2}\right) + f_0\right] + \beta_0 \\
D_N &= \alpha_N \left[-\frac{e_N}{2\Delta}\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(H_N - \frac{2F_N}{\Delta^2}\right) + f_N\right] + \beta_N \\
E_0 &= \alpha_0 \frac{e_0}{2\Delta}\left[\left(\frac{F_0}{\Delta^2} - \frac{G_0}{2\Delta}\right)^{-1}\left(\frac{F_0}{\Delta^2} + \frac{G_0}{2\Delta}\right) + I\right] \\
E_N &= -\alpha_N \frac{e_N}{2\Delta}\left[\left(\frac{F_N}{\Delta^2} + \frac{G_N}{2\Delta}\right)^{-1}\left(\frac{F_N}{\Delta^2} - \frac{G_N}{2\Delta}\right) + I\right]
\end{aligned}\right\} \tag{39}$$

Equations (35), (37), and (38) may now be written as

$$A_i Z_{i-1} + B_i Z_i + C_i Z_{i+1} = 0 \qquad (i = 1, 2, \ldots, N - 1) \tag{40}$$

$$D_0 Z_0 + E_0 Z_1 = 0 \tag{41}$$

$$E_N Z_{N-1} + D_N Z_N = 0 \tag{42}$$

For such a set of homogeneous equations, a recursion formula for Z_i may be written as

$$Z_i + P_i Z_{i+1} = 0 \qquad \begin{cases} (i = 1, 2, \ldots, N-2) \\ \text{and} \\ \left(i = 0 \quad \text{if} \quad Z_0 \neq 0\right) \\ \left(i = N-1 \quad \text{if} \quad Z_N \neq 0\right) \end{cases} \tag{43}$$

where P_i is a (4×4) recursion matrix. To find P_i, combine equation (43) and equation (40) to obtain

$$Z_i + \left(B_i - A_i P_{i-1}\right)^{-1} C_i Z_{i+1} = 0 \qquad (i = 1, 2, \ldots, N-1) \tag{44}$$

Comparison of equation (44) with equation (43) shows that

$$P_i = \left(B_i - A_i P_{i-1}\right)^{-1} C_i \qquad (i = 1, 2, \ldots, N-1) \tag{45}$$

Comparison of equation (41) with equation (43), the latter written at $i = 0$, shows that

$$P_0 = D_0^{-1} E_0 \tag{46}$$

From equations (45) and (46), P_i may be found at all points with the exception of the point $i = N$. This process of determining all required values of P_i in terms of P_0 is in essence a Gaussian elimination process.

Equation (43) written at $i = N - 1$, in combination with equation (42), yields

$$\left(D_N - E_N P_{N-1}\right) Z_N = 0 \tag{47}$$

If $Z_N \neq 0$, then for a solution to exist,

$$\left| D_N - E_N P_{N-1} \right| = 0 \tag{48}$$

Therefore, any frequency parameter Ω^2 (or buckling parameter in the corresponding stability problem) which satisfies equation (48) contains a natural frequency (or critical load) of the system. The natural frequencies can be found by trial and error by selecting successive values for Ω^2, calculating the matrices of equations (39), and using equations (45) and (46) to evaluate the determinant in equation (48). This procedure is continued until the desired zeroes of the determinant are found.

The method must be slightly modified for the case $Z_N = 0$. Substituting equation (43) written at $N - 2$ into equation (40) written at $N - 1$ yields

$$\left(B_{N-1} - A_{N-1}P_{N-2}\right)Z_{N-1} = 0 \tag{49}$$

If $Z_{N-1} = 0$, then $Z_i = 0$ from equation (43), and the solution is trivial. Therefore

$$\left|B_{N-1} - A_{N-1}P_{N-2}\right| = 0 \tag{50}$$

Consequently, for the case $Z_N = 0$, equation (50) is used in place of equation (48) in the elimination process.

After the natural frequencies have been found, the mode shapes are determined by first solving equation (47) for Z_N in terms of an appropriate normalizing factor. For the case where $Z_N = 0$, equation (49) is used to solve for Z_{N-1}. The remaining Z_i's are then determined by using the recursion formula, equation (43).

This numerical procedure is particularly well suited for use with a large number of stations since only the band elements need be retained during the computation process. References 1, 2, and 4 give further advantages in using this general method of solution.

In an investigation of buckling, the procedure must be slightly altered. The external axisymmetric loading $\bar{P}_\xi$ and $\bar{P}$ can be given by

$$\left.\begin{aligned} \bar{P}_\xi(x) &= K\bar{p}_\xi(x) \\ \bar{P}(x) &= K\bar{p}(x) \end{aligned}\right\} \tag{51}$$

where $\bar{p}_\xi$ and $\bar{p}$ characterize the form of the external applied load and K is a constant governing its magnitude. If linearity between the external applied loads and the prebuckling in-plane stresses and deformations is tacitly assumed, the prebuckling quantities may be written as

$$\left.\begin{aligned} \bar{e}_x &= K\bar{E}_x \\ \bar{e}_\theta &= K\bar{E}_\theta \\ \bar{\varphi}_\xi &= K\bar{\Phi}_\xi \end{aligned}\right\} \tag{52a}$$

and it follows that

$$\left.\begin{aligned} \frac{d\bar{e}_x}{dx} &= K\frac{d\bar{E}_x}{dx} \\ \frac{d\bar{e}_\theta}{dx} &= K\frac{d\bar{E}_\theta}{dx} \\ \frac{d\bar{\varphi}_\xi}{dx} &= K\frac{d\bar{\Phi}_\xi}{dx} \end{aligned}\right\} \tag{52b}$$

where $\bar{E}_x$, $\bar{E}_\theta$, $\bar{\Phi}_\xi$ characterize the in-plane stress and deformation due to $\bar{p}_\xi$ and $\bar{p}$. These quantities can be found, for example, from the stress program of reference 1. Thus, K becomes the buckling load parameter, and any value of K which allows two (or more) equilibrium states to exist simultaneously defines a critical buckling load for the system. The numerical procedure is followed directly, as before, with trial values of K rather than Ω^2 selected to satisfy equation (48) or equation (50). With K known, equation (51) yields the critical loading for the particular harmonic wave number n investigated.

Although equation (48) (or eq. (50)) contains all the roots of the system of equations (40), (41), and (42), this method of elimination introduces spurious singularities in the determinants of equation (48) or (50). For some shell configurations and boundary conditions, it is found that the roots and singularities very nearly coincide, and the usual predictor-corrector methods fail to indicate a root if the increments given to the frequency parameter or buckling load parameter are too large. Moreover, some of these singularities are associated with a change in sign in the value of the determinant even though no zero exists at that value of the frequency or buckling parameter. A technique for avoiding this difficulty is presented in reference 15.

EXAMPLES OF APPLICATION OF NUMERICAL PROCEDURE

Cylinder Vibrations and Buckling

An indication of the accuracy of the numerical procedure is obtained by investigating the free vibration and buckling of an unstressed, freely supported, cylindrical shell. The freely supported boundary conditions are introduced by defining

$$\alpha_{0,N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The particular shell investigated has the parameters $\mu = 0.3$, $\lambda = 0.001$, and $S = 3$, where the reference length is taken to be the cylinder radius, so that λ is the ratio of the thickness to the radius and S is the ratio of the length to the radius. The calculations are based on 200 intervals.

The lowest frequency for each circumferential harmonic mode number is found for an unstressed shell and compared with the corresponding results calculated using the closed-form solution of equation (27) with m taken as 1. If S is held constant and m is varied for any n, it can be shown that the lowest frequency will result for $m = 1$. The numerical and exact results agree to three significant figures or better for all values of n from 1 to 15.

The same cylinder is investigated with one edge clamped and one edge freely supported. The clamped edge is introduced by defining

$$\alpha_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A comparison between the two edge conditions for the lowest frequency for each circumferential harmonic mode number is presented in figure 4. As expected, introducing the clamped edge increases the natural frequencies of the system, the lowest natural frequency having a relative increase based on the freely supported cylinder of 26 percent. As n increases, the effect of the edge conditions appears to diminish rapidly.

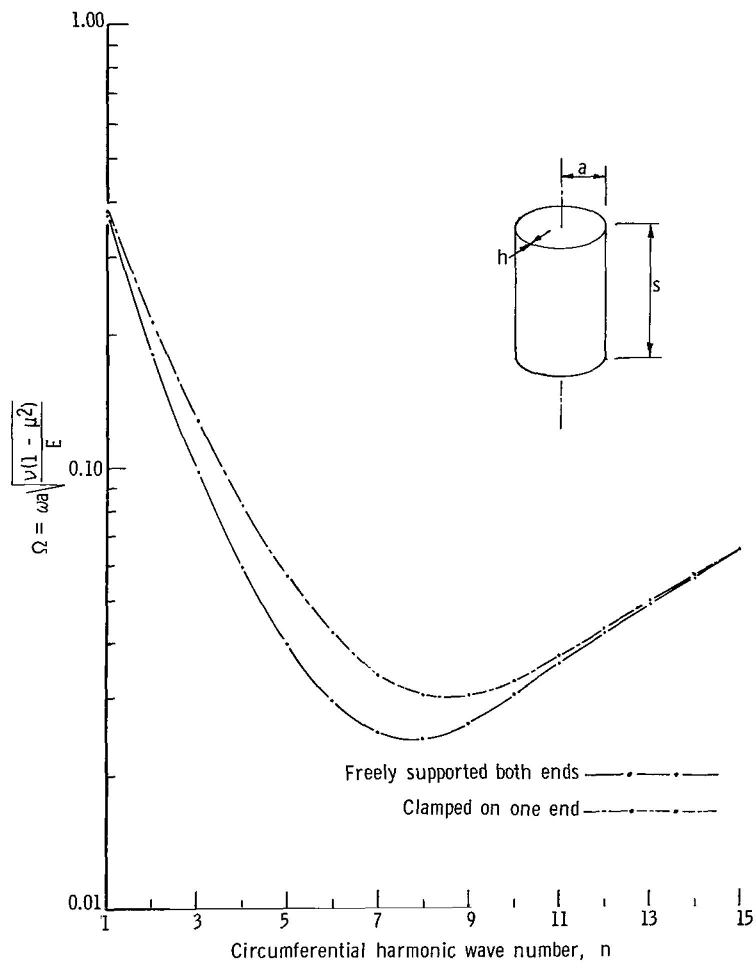


Figure 4.- Comparison between the natural frequencies of a circular cylinder freely supported at both ends and freely supported at one end and clamped at the other end. $\lambda = 0.001$; $S = \frac{s}{a} = 3$; $\mu = 0.3$.

The buckling load is found for a closed circular cylinder subjected to uniform hydrostatic pressure. The results from the numerical procedure are compared with the closed-form solutions based on Donnell theory given in reference 13. Prebuckling deformations are neglected in the derivation of reference 13; consequently, to allow for consistent comparisons, the assumption is made that $\overline{\varphi}_\xi = 0$.

For a closed cylinder under external pressure, $\overline{N}_\xi = \frac{\overline{N}_\theta}{2}$. Thus, unit stress conditions characterizing this loading are introduced by letting $\overline{E}_\theta = -1$ and $\overline{E}_x = -\frac{1}{2}$. Hence, from equations (19) and (52)

$$\left.\begin{aligned} \overline{e}_x &= -\frac{K}{2} = \frac{\overline{N}_\theta}{2B} \\ \overline{e}_\theta &= -K = \frac{\overline{N}_\theta}{B} \end{aligned}\right\} \tag{53}$$

where $\overline{N}_\theta$ may be taken to be $P\rho$, P is the uniform external pressure (P is a negative quantity), and ρ is the radius of the cylinder measured to the middle surface. Thus, the buckling load parameter is

$$K = \frac{-\left(1 - \mu^2\right)P}{E\lambda} \tag{54}$$

The corresponding K from reference 13 in terms of the nondimensional parameters used in this development is given by

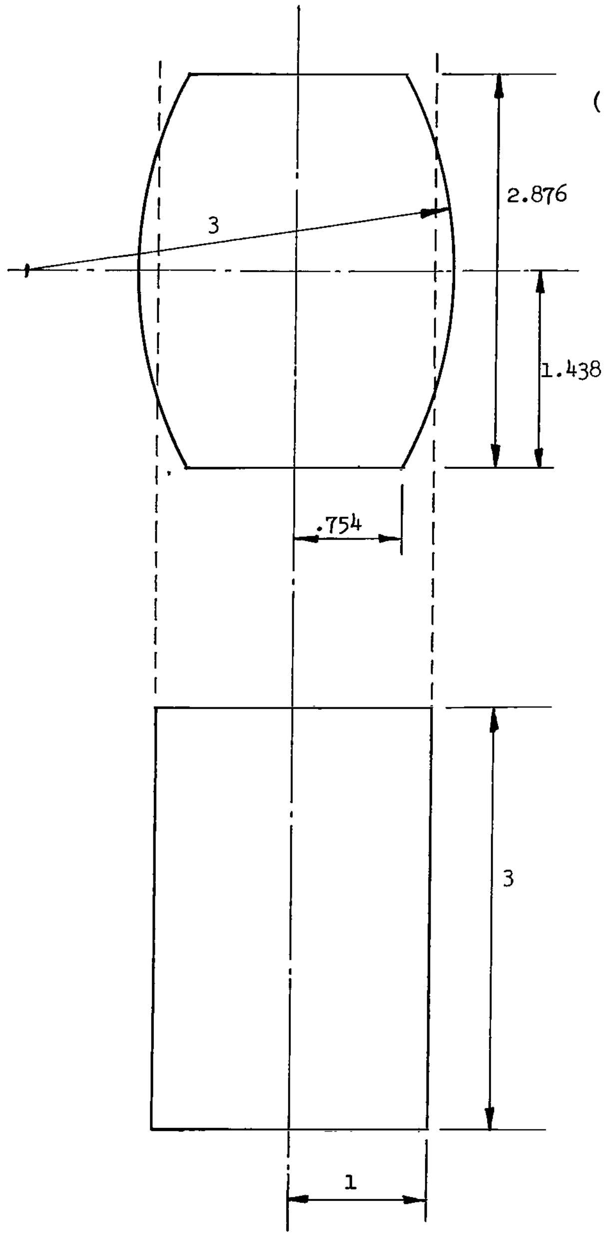
$$K = \frac{\pi^2\lambda^2\left[1 + \left(\frac{Sn}{\pi}\right)^2\right]^2}{12S^2\left[\frac{1}{2} + \left(\frac{Sn}{\pi}\right)^2\right]} + \frac{S\left(1 - \mu^2\right)}{\pi^2\left[1 + \left(\frac{Sn}{\pi}\right)^2\right]^2\left[\frac{1}{2} + \left(\frac{Sn}{\pi}\right)^2\right]} \tag{55}$$

The lowest buckling load occurs for this shell at $n = 9$. The closed-form solution is generally accepted as giving accurate results for n greater than 4; thus, the closed-form value for $K = 0.888 \times 10^{-5}$ at $n = 9$ is a valid basis for comparison. By use of the numerical procedures outlined herein, the critical buckling load was similarly found to be at $n = 9$ and gave a relative error of 0.68 percent. For $n = 1$ and $n = 2$ it is known that the lowest buckling load calculated, when the closed-form solution is used, will be in error on the high side. The numerical procedure gave relative errors based on the closed-form solution of -18.13 percent for $n = 1$ and -14.23 percent for $n = 2$. These errors indicate that for these cases the use of the numerical procedure coupled with the more consistent shell theory yields more accurate results.

Vibrations of a Shell of Positive Gaussian Curvature

The previous examples are applications of the numerical procedure for both buckling and vibration of systems governed by equations with constant coefficients. The capabilities of the numerical procedure are further demonstrated by investigating the vibration characteristics of a shell of positive Gaussian curvature with a constant positive meridional curvature and comparing them with those of a corresponding shell of zero Gaussian curvature (cylinder). The correspondence is introduced by requiring the total surface area (and thus the mass) of both shells to be equal. Furthermore, the meridional lengths of both shells are selected to be equal and thus the perpendicular distance from the axis of revolution to the centroid of the meridional curve is the same for both shells. This distance is selected as the reference length for each shell.

The shapes of both shells investigated are illustrated in figure 5 along with a tabulation of the equations used in the numerical program to define the geometries. The



Shell of positive Gaussian curvature
(constant positive meridional curvature)

$$r = 3 \cos\left(\frac{1}{2} - \frac{x}{3}\right) - 1.879$$

$$k_x = \frac{1}{3}$$

$$k_\theta = \frac{\cos\left(\frac{1}{2} - \frac{x}{3}\right)}{3 \cos\left(\frac{1}{2} - \frac{x}{3}\right) - 1.879}$$

$$S = 3$$

$$\gamma = \frac{\sin\left(\frac{1}{2} - \frac{x}{3}\right)}{3 \cos\left(\frac{1}{2} - \frac{x}{3}\right) - 1.879}$$

Corresponding cylindrical shell
(zero Gaussian curvature)

$$r = 1$$

$$k_x = 0$$

$$k_\theta = 1$$

$$S = 3$$

$$\gamma = 0$$

Figure 5.- Geometry of shells investigated.

lowest natural frequencies for successive circumferential harmonic mode numbers, based on freely supported edge conditions, have been found for each unstressed shell. The results are presented in figure 6.

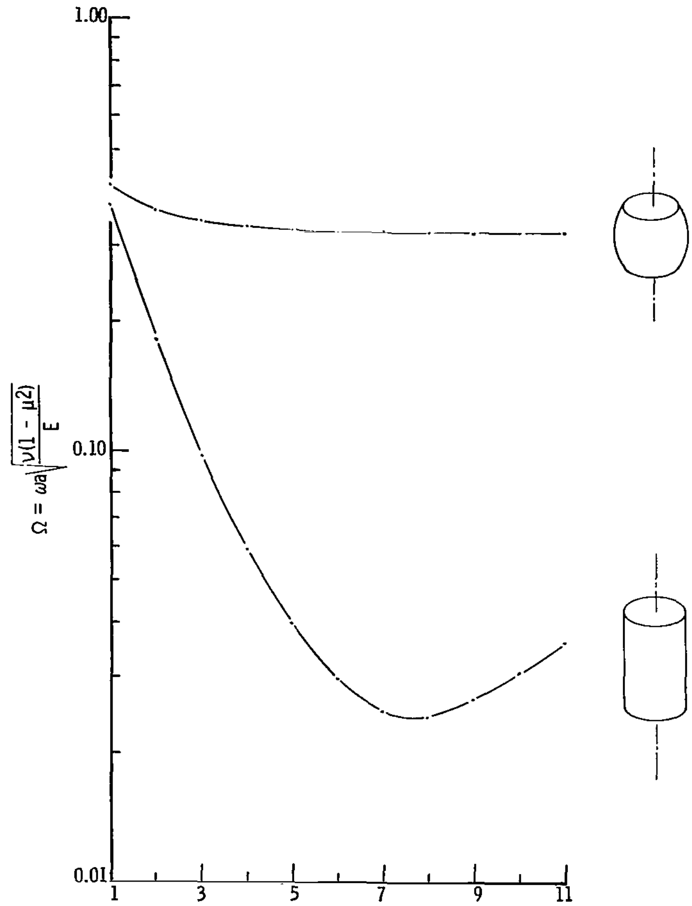


Figure 6.- Comparison between frequencies of shell of positive Gaussian curvature and a corresponding cylinder

The lowest frequency for the positive-curvature shell was considerably higher than the corresponding frequency for the zero curvature shell for each circumferential mode number greater than 2. The maximum increase tabulated was in excess of 1,000 percent indicating a large increase in stiffness due to the curvature of the meridian.

CONCLUDING REMARKS

A set of linear equations governing the infinitesimal vibrations and buckling of axisymmetrically prestressed shells has been developed and both the in-plane inertia and prestress deformation effects have been retained in the analysis. The equations derived are consistent with first-order thin-shell theory and can be used to describe the behavior of shells with arbitrary meridional configuration having moderately small prestress rotation.

A numerical procedure has been given for solving the governing equations for the natural frequencies, buckling loads, and associated mode shapes for a general shell of revolution with homogeneous boundary conditions. The numerical procedure uses matrix methods in finite-difference form coupled with a Gaussian elimination to solve the governing eigenvalue problem.

Examples of applications of the numerical procedure have been presented. Results for natural vibration frequencies and buckling under hydrostatic pressure of a simply supported cylinder were found to be in good agreement with previously published results. A brief comparison between the vibration characteristics of two geometrically similar shells having positive and zero Gaussian curvature displayed the markedly greater stiffness and resulting higher frequencies of the doubly curved shell.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 14, 1966,
124-08-06-11-23.

APPENDIX A

COEFFICIENTS OF EQUATIONS (22) AND (23)

Coefficients of Equations (22)

The coefficients of terms in the governing equations (22) are defined as follows:

$$F_{11} = 1$$

$$F_{13} = -\overline{\varphi}_\xi$$

$$F_{22} = \frac{(1 - \mu)}{2} + \frac{\lambda^2(1 - \mu)}{96}\left(3k_\theta - k_x\right)^2 + \frac{1}{4}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$F_{23} = \frac{\lambda^2(1 - \mu)n}{24r}\left(3k_\theta - k_x\right)$$

$$F_{31} = F_{13}$$

$$F_{32} = F_{23}$$

$$F_{33} = \frac{\lambda^2(1 - \mu)}{12}\left[\frac{2n^2}{r^2} + (1 + \mu)\gamma^2\right] + \overline{e}_x + \overline{\varphi}_\xi{}^2$$

$$F_{34} = \lambda^2\left(1 - \mu^2\right)$$

$$F_{43} = F_{34}$$

$$G_{11} = \gamma$$

$$G_{12} = \frac{(1 + \mu)n}{2r} + \frac{\lambda^2 n(1 - \mu)}{96r}\left(3k_x - k_\theta\right)\left(3k_\theta - k_x\right) - \frac{n}{4r}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$G_{13} = k_x + \mu k_\theta + \frac{\lambda^2(1 - \mu)}{12}\left[(1 + \mu)\gamma^2 k_x + \frac{n^2\left(3k_x - k_\theta\right)}{2r^2}\right] + k_x\overline{e}_x - (1 - \mu)\gamma\overline{\varphi}_\xi + k_x\overline{\varphi}_\xi{}^2 - \frac{d\overline{\varphi}_\xi}{dx}$$

$$G_{14} = \lambda^2\left(1 - \mu^2\right)k_x$$

$$G_{21} = -G_{12}$$

$$G_{22} = \frac{(1 - \mu)}{2}\gamma - \frac{\lambda^2(1 - \mu)}{96}\left(3k_\theta - k_x\right)\left[2\frac{dk_x}{dx} - \gamma\left(5k_x - 3k_\theta\right)\right] + \frac{1}{4}\frac{d}{dx}\left(\overline{e}_x + \overline{e}_\theta\right) + \frac{\gamma}{4}\left(\overline{e}_x + \overline{e}_\theta\right)$$

$$G_{23} = \frac{\lambda^2(1-\mu)n}{24r}\left[2(1+\mu)\gamma k_\theta - \frac{dk_x}{dx} + 3\gamma\left(k_x - k_\theta\right)\right] + \frac{(1+\mu)n}{2r}\,\bar{\varphi}_\xi$$

$$G_{31} = -G_{13} - 2\gamma\bar{\varphi}_\xi - 2\,\frac{d\bar{\varphi}_\xi}{dx}$$

$$G_{32} = \frac{\lambda^2(1-\mu)n}{24r}\left[3\gamma k_x - \gamma k_\theta(5+2\mu) - \frac{dk_x}{dx}\right] - \frac{n}{2r}(1+\mu)\bar{\varphi}_\xi$$

$$G_{33} = -\frac{\lambda^2(1-\mu)}{12}\left[(1+\mu)\left(2\gamma k_x k_\theta + \gamma^3\right) + \frac{2n^2\gamma}{r^2}\right] + \gamma\bar{e}_x + \frac{d\bar{e}_x}{dx} + \gamma\bar{\varphi}_\xi^{\,2} + 2\bar{\varphi}_\xi\,\frac{d\bar{\varphi}_\xi}{dx}$$

$$G_{34} = \lambda^2(2-\mu)\left(1-\mu^2\right)\gamma$$

$$G_{41} = -G_{14}$$

$$G_{43} = \lambda^2\left(1-\mu^2\right)\mu\gamma$$

$$H_{11} = -\mu k_x k_\theta - \gamma^2 - \frac{(1-\mu)n^2}{2r^2} - \frac{\lambda^2(1-\mu)}{12}\left[(1+\mu)\gamma^2 k_x^{\,2} + \frac{n^2\left(3k_x - k_\theta\right)^2}{8r^2}\right] - k_x^{\,2}\bar{e}_x$$
$$- \frac{n^2}{4r^2}\left(\bar{e}_x + \bar{e}_\theta\right) + \left[(1-2\mu)\gamma k_x + \frac{dk_x}{dx}\right]\bar{\varphi}_\xi - k_x^{\,2}\bar{\varphi}_\xi^{\,2} + k_x\,\frac{d\bar{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{12} = -\frac{(3-\mu)n\gamma}{2r} - \frac{\lambda^2(1-\mu)n\gamma}{12r}\left[\frac{\left(3k_x - k_\theta\right)\left(3k_\theta - k_x\right)}{8} + (1+\mu)k_x k_\theta\right] - \frac{n\gamma}{4r}\left(\bar{e}_x + \bar{e}_\theta\right)$$
$$- \frac{\mu n}{r}\,k_x\bar{\varphi}_\xi + \frac{(1-\mu)n}{2r}\,k_\theta\bar{\varphi}_\xi$$

$$H_{13} = \frac{dk_x}{dx} + \gamma\left(k_x - k_\theta\right) - \frac{\lambda^2(1-\mu)n^2\gamma}{12r^2}\left[\frac{\left(3k_x - k_\theta\right)}{2} + (1+\mu)k_x\right]$$
$$- k_x\left(k_x + \mu k_\theta\right)\bar{\varphi}_\xi + \frac{(1-\mu)n^2}{2r^2}\,\bar{\varphi}_\xi$$

$$H_{14} = \lambda^2\left(1-\mu^2\right)(1-\mu)\gamma k_x$$

$$H_{21} = -\frac{(3-\mu)n\gamma}{2r} + \frac{\lambda^2(1-\mu)n}{12r}\left[-(1+\mu)\gamma k_x k_\theta + \frac{\gamma}{8}\left(6k_x k_\theta - 7k_x^2 - 3k_\theta^2\right) - \frac{1}{4}\frac{dk_x}{dx}\left(5k_\theta - 3k_x\right)\right]$$

$$+ \frac{n}{4r}\frac{d}{dx}\left(\bar{e}_x + \bar{e}_\theta\right) - \frac{n\gamma}{4r}\left(\bar{e}_x + \bar{e}_\theta\right) - \frac{\mu n}{r} k_x \bar{\varphi}_\xi + \frac{(1-\mu)n}{2r} k_\theta \bar{\varphi}_\xi$$

$$H_{22} = -\gamma G_{22} + \frac{(1-\mu)}{2} k_x k_\theta - \frac{n^2}{r^2} - \frac{\lambda^2(1-\mu)}{12}\left[\frac{(1+\mu)n^2}{r^2} k_\theta^2 - \frac{k_x k_\theta}{8}\left(3k_\theta - k_x\right)^2\right]$$

$$+ \frac{\gamma}{2}\frac{d}{dx}\left(\bar{e}_x + \bar{e}_\theta\right) - k_\theta^2 \bar{e}_\theta - \frac{1}{4} k_x k_\theta\left(\bar{e}_x + \bar{e}_\theta\right) + \frac{(1-\mu)\gamma}{2}\left(k_x + 2k_\theta\right)\bar{\varphi}_\xi$$

$$- \frac{(1-\mu)}{2} k_\theta^2 \bar{\varphi}_\xi^2 + \frac{(1-\mu)}{2} k_\theta \frac{d\bar{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{23} = -\frac{n}{r}\left(k_\theta + \mu k_x\right) + \frac{\lambda^2(1-\mu)n}{24r}\left[\gamma\frac{dk_x}{dx} - 2\gamma^2 k_x - \frac{2(1+\mu)n^2}{r^2} k_\theta + \left(3k_\theta - k_x\right)\left(\gamma^2 + k_x k_\theta\right)\right]$$

$$- \frac{n}{r} k_\theta \bar{e}_\theta + \frac{(1-\mu)n\gamma}{2r}\bar{\varphi}_\xi - \frac{(1-\mu)n}{2r} k_\theta \bar{\varphi}_\xi^2 + \frac{(1-\mu)n}{2r}\frac{d\bar{\varphi}_\xi}{dx}$$

$$H_{24} = -\frac{\lambda^2\mu\left(1-\mu^2\right)n}{r} k_\theta$$

$$H_{31} = -\gamma\left(k_\theta + \mu k_x\right) + \frac{\lambda^2(1-\mu)}{12}\left[(1+\mu)\gamma\left(\gamma^2 k_x - \gamma\frac{dk_x}{dx} - \frac{n^2}{r^2} k_x + 2k_x^2 k_\theta\right)\right.$$

$$\left. + \frac{n^2}{2r^2}\left(\gamma k_x - \gamma k_\theta - 3\frac{dk_x}{dx}\right)\right] - \left(\gamma k_x + \frac{dk_x}{dx}\right)\bar{e}_x - k_x\frac{d\bar{e}_x}{dx} + \left[\frac{(1-\mu)n^2}{2r^2} - k_x^2\right]\bar{\varphi}_\xi$$

$$- \left(\gamma k_x + \frac{dk_x}{dx}\right)\bar{\varphi}_\xi^2 - \left(2k_x\bar{\varphi}_\xi + \mu\gamma\right)\frac{d\bar{\varphi}_\xi}{dx}$$

$$H_{32} = -\frac{n}{r}\left(k_\theta + \mu k_x\right) + \frac{\lambda^2(1-\mu)n}{24r}\left[2(1+\mu)\left(k_x k_\theta^2 - \gamma^2 k_x + 2\gamma^2 k_\theta - \frac{n^2}{r^2} k_\theta\right) + \gamma\frac{dk_x}{dx}\right.$$

$$\left. + 3\gamma^2\left(k_\theta - k_x\right) + k_x k_\theta\left(3k_\theta - k_x\right)\right] - \frac{n}{r} k_\theta \bar{e}_\theta + \frac{(1-\mu)n\gamma}{2r}\bar{\varphi}_\xi - \frac{(1-\mu)n}{2r} k_\theta \bar{\varphi}_\xi^2 - \frac{\mu n}{r}\frac{d\bar{\varphi}_\xi}{dx}$$

$$H_{33} = -k_x^2 - 2\mu k_x k_\theta - k_\theta^2 + \frac{\lambda^2(1 - \mu)n^2}{12r^2}\left[(1 + \mu)\left(k_x k_\theta - \frac{n^2}{r^2} + 2\gamma^2\right) + 2\left(\gamma^2 + k_x k_\theta\right)\right]$$

$$- \frac{n^2}{r^2}\bar{e}_\theta - \left[\gamma k_x(1 + \mu) + \frac{dk_x}{dx}\right]\bar{\varphi}_\xi - \frac{(1 - \mu)n^2}{2r^2}\bar{\varphi}_\xi^{\,2} - \left(k_x + \mu k_\theta\right)\frac{d\bar{\varphi}_\xi}{dx} + \Omega^2$$

$$H_{34} = -\lambda^2\left(1 - \mu^2\right)\left[(1 - \mu)k_x k_\theta + \frac{\mu n^2}{r^2}\right]$$

$$H_{41} = -\lambda^2\left(1 - \mu^2\right)\left(\frac{dk_x}{dx} + \mu\gamma k_x\right)$$

$$H_{42} = H_{24}$$

$$H_{43} = -\frac{\lambda^2\left(1 - \mu^2\right)\mu n^2}{r^2}$$

$$H_{44} = 12\lambda^2\left(1 - \mu^2\right)^2$$

Coefficients of Equations (23)

The coefficients of the terms associated with the boundary conditions (eqs. (23)) are defined in the following equations:

$$e_{11} = 1$$

$$e_{13} = -\bar{\varphi}_\xi$$

$$e_{22} = \frac{(1 - \mu)}{2} + \frac{\lambda^2(1 - \mu)}{96}\left(3k_\theta - k_x\right)^2 + \frac{1}{4}\left(\bar{e}_x + \bar{e}_\theta\right)$$

$$e_{23} = \frac{\lambda^2(1 - \mu)n}{24r}\left(3k_\theta - k_x\right)$$

$$e_{31} = e_{13}$$

$$e_{32} = e_{23}$$

$$e_{33} = \frac{\lambda^2(1 - \mu)}{12}\left[\frac{2n^2}{r^2} + (1 + \mu)\gamma^2\right] + \bar{e}_x + \bar{\varphi}_\xi{}^2$$

$$e_{34} = \lambda^2\left(1 - \mu^2\right)$$

$$e_{43} = e_{34}$$

$$f_{11} = \mu\gamma + k_x\bar{\varphi}_\xi$$

$$f_{12} = \frac{\mu n}{r}$$

$$f_{13} = k_x + \mu k_\theta$$

$$f_{21} = -\frac{(1 - \mu)n}{2r} - \frac{\lambda^2(1 - \mu)n}{96r}\left(3k_x - k_\theta\right)\left(3k_\theta - k_x\right) + \frac{n}{4r}\left(\bar{e}_x + \bar{e}_\theta\right)$$

$$f_{22} = -\frac{(1 - \mu)\gamma}{2} - \frac{\lambda^2\gamma(1 - \mu)}{96}\left(3k_\theta - k_x\right)^2 + \frac{\gamma}{4}\left(\bar{e}_x + \bar{e}_\theta\right) + \frac{(1 - \mu)}{2}k_\theta\bar{\varphi}_\xi$$

$$f_{23} = -\frac{\lambda^2\gamma(1 - \mu)n}{24r}\left(3k_\theta - k_x\right) + \frac{(1 - \mu)n}{2r}\bar{\varphi}_\xi$$

$$f_{31} = -\frac{\lambda^2(1 - \mu)}{12}\left[(1 + \mu)\gamma^2 k_x + \frac{n^2}{2r^2}\left(3k_x - k_\theta\right)\right] - k_x\bar{e}_x - \mu\gamma\bar{\varphi}_\xi - k_x\bar{\varphi}_\xi{}^2$$

$$f_{32} = -\frac{\lambda^2(1 - \mu)n\gamma}{24r}\left[3k_\theta - k_x + 2(1 + \mu)k_\theta\right] - \frac{\mu n}{r}\bar{\varphi}_\xi$$

$$f_{33} = -\frac{\lambda^2(1 - \mu)}{12}(3 + \mu)\frac{n^2\gamma}{r^2} - \left(k_x + \mu k_\theta\right)\bar{\varphi}_\xi$$

$$f_{34} = \lambda^2(1 - \mu)\left(1 - \mu^2\right)\gamma$$

$$f_{41} = -\lambda^2\left(1 - \mu^2\right)k_x$$

APPENDIX B

COEFFICIENTS OF EQUATION (27)

The coefficients of equation (27) are defined as follows:

$$\Lambda_0 = \left[a_{11}a_{22}a_{33} - a_{11}\left(a_{23}\right)^2 - a_{22}\left(a_{13}\right)^2 - a_{33}\left(a_{12}\right)^2 + 2a_{12}a_{23}a_{13}\right]a_{44}$$

$$+ \left[-a_{11}a_{22}a_{34} + 2a_{11}a_{23}a_{24} + \left(a_{12}\right)^2 a_{34} - 2a_{12}a_{24}a_{13}\right]a_{34} + \left[\left(a_{13}\right)^2 - a_{11}a_{33}\right]\left(a_{24}\right)^2$$

$$\Lambda_1 = \left[a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - \left(a_{12}\right)^2 - \left(a_{23}\right)^2 - \left(a_{13}\right)^2\right]a_{44}$$

$$- \left(a_{24}\right)^2\left(a_{11} + a_{33}\right) - \left(a_{34}\right)^2\left(a_{11} + a_{22}\right) + 2a_{23}a_{24}a_{34}$$

$$\Lambda_2 = \left(a_{11} + a_{22} + a_{33}\right)a_{44} - \left(a_{34}\right)^2 - \left(a_{24}\right)^2$$

$$\Lambda_3 = a_{44}$$

where

$$a_{11} = H_{11} - F_{11}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{12} = G_{12}\left(\frac{m\pi}{S}\right)$$

$$a_{13} = G_{13}\left(\frac{m\pi}{S}\right)$$

$$a_{22} = H_{22} - F_{22}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{23} = H_{23} - F_{23}\left(\frac{m\pi}{S}\right)^2$$

$$a_{24} = H_{24}$$

$$a_{33} = H_{33} - F_{33}\left(\frac{m\pi}{S}\right)^2 - \Omega^2$$

$$a_{34} = H_{34} - F_{34}\left(\frac{m\pi}{S}\right)^2$$

$$a_{44} = H_{44}$$

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