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CLASSIFICATION OF LUNAR SATELLITE ORBITS

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BY

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GODDARD SPACE FLIGHT CENTER

GREENBELT, MARYLAND

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SUMMARY

A classification is presented of lunar satellite orbits perturbed by the earth and by the non-sphericity of the lunar gravitational field (the second zonal harmonic only). The orbits are arranged in three categories — those in which the argument of perilune circulates, those in which it "librates" about some odd multiple value of $\pi/2$, and a class of "transition" orbits which belong to neither of the other two classes.

The discussion is kept as general as possible by holding the semi-major axis arbitrary. In addition, two examples are presented to clarify the analysis. Graphs for these cases show the permissible ranges of the two parameters involved (integrals of the satellite motion) for the orbit classes.

CLASSIFICATION OF LUNAR SATELLITE ORBITS

INTRODUCTION

Several studies have been made of the long-term effects on artificial satellite orbits disturbed by a third body (references 1, 2, 3 and 4). In addition, Lidov (4) has extended the analysis to include effects caused by the non-sphericity of the gravitational field of the central body (the second zonal harmonic only), in the particular case when the inclination of the disturbing body's orbit plane to the equatorial plane of the central body is 90°. However, this simplification is not applicable when the central body is the moon, the angle in question being about 6°41'.

The orbital motion of an artificial lunar satellite has already been studied by several people, among whom are Kozai (5) and Giacaglia, et al. (6). However, adequate analytical representations for the long-period and secular effects caused by the earth have proved particularly troublesome to obtain — for instance, these two papers present solutions involving elliptic integrals. Later solutions by Frost (7) and Fisher and Felsentreger (8) (which do not involve elliptic integrals) depend upon whether the motion of the argument of perilune is circulatory (secular) or libratory (periodic). It was therefore deemed appropriate to define the regions of circulation and libration.

The disturbing forces which will be considered here, then, give rise to long-period and secular effects caused by the earth and secular perturbations due to the second zonal harmonic of the lunar gravitational field.

Orbits will be divided essentially into two classes — those in which the argument of perilune either circulates or librates. The discussion will be limited to a presentation of the ranges of permissible values for two parameters (integral constants), which of course depend upon initial conditions. Values of the parameters for orbits which fall into neither category ("transition" orbits) lie on the boundaries of the ranges.

A list of symbols appears in Appendix A.

EQUATIONS OF MOTION AND CONSTANTS OF THE MOTION

The equations of motion for $\eta = \sqrt{1 - e^2}$ and the argument of perilune g are (6)

$$\frac{d\eta}{dt} = \frac{-15}{4} \frac{\alpha_1}{q} n (1 - \eta^2)(1 - \cos^2 i) \sin 2g$$

$$\frac{dg}{dt} = -\frac{3}{2} \frac{\alpha_1}{q} \frac{n}{\eta} \left[-2\eta^2 + 5(\eta^2 - \cos^2 i) \sin^2 g\right] - \frac{3}{4} \alpha_2 \frac{n}{\eta^4} (1 - 5\cos^2 i) . \tag{1}$$

The two integral constants of the motion are readily found to be

$$\alpha = \eta^2 \cos^2 i$$

$$c = (1 - \eta^2)(1 - \frac{5}{2} \sin^2 i \sin^2 g) - \frac{1}{6} A \frac{1 - 3\cos^2 i}{\eta^3} , \tag{2}$$

where

$$A = \frac{\alpha_2}{\alpha_1/q} .$$

In the case where the moon's second zonal harmonic J_2 is neglected (i.e., $A = 0$), the parameters become the α and β of Lorell (1), who has presented a classification of orbits disturbed only by a third body. However, the analysis is considerably more complicated when J_2 is not considered negligible.

To begin the study, it will be assumed that for all orbits the argument of perilune must reach some odd multiple of $\pi/2$. Therefore, the maximum c, α region compatible with elliptic orbits is that for which $\sin^2 g = 1$ is allowable. If the argument of perilune is to circulate, then g must eventually become 0. Hence, the region in the c, α plane describing "circulatory" orbits is that for which both $\sin^2 g = 0$ and $\sin^2 g = 1$ are allowable. The non-intersection of the two regions gives "libratory" orbits.

VALUES OF α AND c FOR WHICH $\mathrm{SIN}^2\ g = 1$

For $\sin^2 g = 1$, c can be rewritten as

$$c = \frac{3\eta^7 - (3 + 5\alpha)\,\eta^5 + 5\alpha\eta^3 - \frac{1}{3}A\eta^2 + A\alpha}{2\eta^5} \tag{3}$$

which, for $\eta = 1$ (i.e., $e = 0$), describes the line

$$c = -\frac{1}{6}A\,(1 - 3\alpha)\,. \tag{4}$$

For a particular value of η, α achieves its greatest value when $\cos^2 i = 1$. Replacing η^2 in Equation (3) by α, one obtains

$$c = 1 - \alpha + \frac{\frac{1}{3}A}{\alpha^{3/2}}\ , \tag{5}$$

which can be rewritten as

$$c = -\frac{1}{6}A(1-3\alpha) - \frac{(\alpha^{1/2} - 1)\left[\left(1+\frac{1}{2}A\right)\alpha^2 + \left(1+\frac{1}{2}A\right)\alpha^{3/2} + \frac{1}{3}A\alpha + \frac{1}{3}A\alpha^{1/2} + \frac{1}{3}A\right]}{\alpha^{3/2}}\ . \tag{6}$$

The last term in Equation (6) is always ≤ 0, indicating that Equation (5) describes a curve in the c, α plane to the right of the line $c = -(1/6)\,A(1 - 3\alpha)$, and which asymptotically approaches the axis $\alpha = 0$. Equations (4) and (5) intersect at $\alpha = 1, c = (1/3)A$. Therefore, the region to the right of the line of Equation (4) and bounded by Equation (5) and $\alpha = 0$, gives permissible values of α and c .

There remains the region to the left of the line to be explored. Equation (3) can be rewritten

$$\alpha = \frac{\eta^2\left[(2c + 3)\eta^3 - 3\eta^5 + \frac{1}{3}A\right]}{5(\eta^3 - \eta^5) + A}; \tag{7}$$

holding c constant and taking the derivative with respect to η yields

$$\frac{d\alpha}{d\eta} = \frac{2\eta\left[15\eta^{10} - 30\eta^{8} + 5(2c+3)\eta^{6} - 8A\eta^{5} + 5A\left(c+\frac{4}{3}\right)\eta^{3} + \frac{1}{3}A^{2}\right]}{[5(\eta^{3} - \eta^{5}) + A]^{2}} . \tag{8}$$

It is desirable to describe the roots of

$$f(\eta) = 15\eta^{10} - 30\eta^{8} + 5(2c+3)\eta^{6} - 8A\eta^{5} + 5A\left(c+\frac{4}{3}\right)\eta^{3} + \frac{1}{3}A^{2} \tag{9}$$

in order to determine possible maximum values of α.

Now, $\alpha = 0$ at $\eta = 0$. Since $f(0) = (1/3)A^2 > 0$, α increases from 0 until η assumes the value of a root of $f(\eta)$. Suppose, for the moment, that η_1 is a root of $f(\eta)$. If it can be shown that $\alpha(1) \leq \alpha(\eta_1)$, then one can conclude that α attains a maximum value either at η_1 or at some other root of $f(\eta)$ in $0 < \eta \leq 1$.

From Equation (9), then,

$$c = \frac{-15\eta_1^{10} + 30\eta_1^{8} - 15\eta_1^{6} + 8A\eta_1^{5} - \frac{20}{3}A\eta_1^{3} - \frac{1}{3}A^{2}}{5\eta_1^{3}(2\eta_1^{3} + A)} . \tag{10}$$

Substitution of Equation (10) into Equation (7) yields (for $\eta = \eta_1$)

$$\alpha(\eta_1) = \frac{\eta_1^{2}[-30\eta_1^{10} + 30\eta_1^{8} + A\eta_1^{5} + 5A\eta_1^{3} + A^{2}]}{5[5(\eta_1^{3} - \eta_1^{5}) + A](2\eta_1^{3} + A)} , \tag{11}$$

and, for $\eta = 1$,

$$\alpha(1) = \frac{-30\eta_1^{10} + 60\eta_1^{8} - \frac{10}{3}(9-A)\eta_1^{6} + 16A\eta_1^{5} - \frac{5}{3}A(8-A)\eta_1^{3} - \frac{2}{3}A^{2}}{5A\eta_1^{3}(2\eta_1^{3} + A)} . \tag{12}$$

Then,

$$\alpha(1) - \alpha(\eta_1) = \frac{(\eta_1 - 1)^2 \, g(\eta_1)}{15A\eta_1^3 \, (2\eta_1^3 + A) \, [5(\eta_1^3 - \eta_1^5) + A]} , \tag{13}$$

where

$$\begin{aligned} g(\eta_1) = {} & 90(5+A)\,\eta_1^{13} + 180(5+A)\,\eta_1^{12} + 180A\eta_1^{11} - 180(5-A)\,\eta_1^{10} - 10(45-13A)\eta_1^9 \\ & -A(250+3A)\eta_1^8 - 2A(290+3A)\eta_1^7 - A(290+49A)\eta_1^6 - 92A^2\eta_1^5 - 100A^2\eta_1^4 \\ & - A^2(50+3A)\eta_1^3 - 6A^3\eta_1^2 - 4A^3\eta_1 - 2A^3. \end{aligned} \tag{14}$$

The Theorem of Vincent and Descartes' Rule of Signs establishes that $g(\eta_1)$ has no root in $0 < \eta_1 < 1$ (see Appendix B). Since $g(0) = -2A^3 < 0$ and $g(1) = -15A(A^2 + 20A + 24) < 0$, one may conclude that

$$\alpha(1) - \alpha(\eta_1) \leq 0$$

for $0 < \eta_1 \leq 1$; equality occurs only when $\eta_1 = 1$.

It must now be shown that $\alpha(\eta_1) \leq 1$ for any root η_1 of $f(\eta)$. From Equation (11),

$$\alpha(\eta_1) \leq 1 \Longleftrightarrow h(\eta_1) = -30\eta_1^{12} + 30\eta_1^{10} + 50\eta_1^8 + A\eta_1^7 - 50\eta_1^6$$

$$+ 30A\eta_1^5 - 35A\eta_1^3 + A^2\eta_1^2 - 5A^2 \leq 0.$$

Again, it can be shown that $h(\eta_1)$ has no root in $0 \leq \eta_1 \leq 1$ (see Appendix B). Since $h(0) = -5A^2 < 0$, it has been established that $\alpha(\eta_1) < 1$ for any root η_1 of $f(\eta)$.

Finally, it must be demonstrated that, for any value of c less than $-(1/6)A(1-3\alpha)$, $f(\eta)$ has at least one root. From Equation (9), $f(0) = (1/3)A^2 > 0$. Existence of a root between 0 and 1 is assured if $f(1) < 0$ - hence

$$f(1) < 0 \Longleftrightarrow c < \frac{A(4-A)}{15(2+A)}.$$

However, when $c = A(4-A)/15(2+A)$, $\eta = 1$ is a root of $f(\eta)$, and the point $\alpha = (6+A)/5(2+A)$, $c = A(4-A)/15(2+A)$ lies on the line $c = -(1/6)A(1-3\alpha)$. Thus, $c < A(4-A)/15(2+A) \Longrightarrow f(\eta)$ has at least one root η_1 in $0 < \eta_1 < 1$. In addition, since $\alpha(1) < \alpha(\eta_1)$ for $\eta_1 \neq 1$, the point $\alpha(\eta_1)$, c lies above the line $c = -(1/6)A(1-3\alpha)$.

Hence, the curve whose parametric equations are (for $0 < \eta_1 \leq 1$)

$$c(\eta_1) = \frac{-15\eta_1^{10} + 30\eta_1^8 - 15\eta_1^6 + 8A\eta_1^5 - \frac{20}{3}A\eta_1^3 - \frac{1}{3}A^2}{5\eta_1^3(2\eta_1^3 + A)}$$

$$\alpha(\eta_1) = \frac{\eta_1^2(-30\eta_1^{10} + 30\eta_1^8 + A\eta_1^5 + 5A\eta_1^3 + A^2)}{5[5(\eta_1^3 - \eta_1^5) + A](2\eta_1^3 + A)} \qquad (15)$$

represents the upper boundary to the permissible c, α region lying to the left of the line $c = -(1/6)A(1-3\alpha)$. It intersects this line at the point $c = A(4-A)/15(2+A)$, $\alpha = (6+A)/5(2+A)$, and approaches the $\alpha = 0$ axis asymptotically.

VALUES OF α AND c FOR WHICH $\mathrm{SIN}^2\ g = 0$

For $\sin^2 g = 0$, Equation (2) becomes

$$c = \frac{-\eta^7 + \eta^5 - \frac{1}{6}A\eta^2 + \frac{1}{2}A\alpha}{\eta^5}, \qquad (16)$$

which, for $\eta = 1$, again becomes the line $c = -(1/6)A(1-3\alpha)$. In addition, for $\cos^2 i = 1$, Equation (16) becomes

$$c = 1 - \alpha + \frac{\frac{1}{3}A}{\alpha^{3/2}},$$

which describes the same curve as for $\sin^2 g = 1$. Therefore, the permissible region to the right of $c = -(1/6)A(1 - 3\alpha)$ is the same as for $\sin^2 g = 1$.

Now, Equation (16) can be rewritten as

$$\alpha = \frac{2}{A}\eta^2\left[\eta^5 + (c - 1)\eta^3 + \frac{1}{6}A\right], \tag{17}$$

whose derivative with respect to η (for constant c) is

$$\frac{d\alpha}{d\eta} = \frac{2}{A}\eta\left[7\eta^5 + 5(c - 1)\eta^3 + \frac{1}{3}A\right]. \tag{18}$$

Thus, an analysis of the roots of

$$F(\eta) = 7\eta^5 + 5(c - 1)\eta^3 + \frac{1}{3}A \tag{19}$$

is in order.

Obviously, since $F(0) = (1/3)A > 0$, α increases from a value of 0 at $\eta = 0$ until a root of $F(\eta)$ is reached, assuring the existence of a relative maximum for α in $0 < \eta \leq 1$ (providing $F(\eta)$ has a root). Hence, assuming η_1 to be a root of $F(\eta)$, it again behooves one to show that $\alpha(1) \leq \alpha(\eta_1)$.

When $\eta = \eta_1$ is a root of $F(\eta)$, Equation (19) gives

$$c = \frac{-7\eta_1^5 + 5\eta_1^3 - \frac{1}{3}A}{5\eta_1^3}. \tag{20}$$

Equation (17) then becomes (for $\eta = \eta_1$)

$$\alpha(\eta_1) = \frac{\eta_1^2(-4\eta_1^5 + A)}{5A}; \tag{21}$$

for $\eta = 1$,

$$\alpha(1) = \frac{-42\,\eta_1^5 + 5(6 + A)\eta_1^3 - 2A}{15\,A\eta_1^3}. \tag{22}$$

Then,

$$\alpha(1) - \alpha(\eta_1) = \frac{(\eta_1 - 1)^2\, G(\eta_1)}{15\,A\,\eta_1^3}, \tag{23}$$

where

$$G(\eta_1) = 12\eta_1^8 + 24\eta_1^7 + 36\eta_1^6 + 48\eta_1^5 + 60\eta_1^4 + 3\,(10 - A)\eta_1^3 - 6A\eta_1^2 - 4A\eta_1 - 2A. \tag{24}$$

$G(\eta_1)$ has exactly one positive root. For $A < 14$, this root is in $0 < \eta_1 < 1$; for $A = 14$, the root is $\eta_1 = 1$; and for $A > 14$, $G(\eta_1)$ has no roots in $0 < \eta_1 \leq 1$ (see Appendix B). Therefore, $\alpha(1) - \alpha(\eta_1) \leq 0$ for $A \geq 14$; for $A < 14$, designating the root of $G(\eta_1)$ by η_1^*,

$$\alpha(1) - \alpha(\eta_1) = \frac{(\eta_1 - 1)^2\,(\eta_1 - \eta_1^*)\,G^*(\eta_1)}{15\,A\,\eta_1^3},$$

where $G^*(\eta_1) > 0$. Thus, $\alpha(1) - \alpha(\eta_1) \leq 0$ when $\eta_1 \leq \eta_1^*$ - equality occurs (for $A < 14$) only when $\eta_1 = \eta_1^*$.

The foregoing has also shown that the point $c, \alpha(\eta_1)$ lies above the line $c = -(1/6)A\,(1-3\alpha)$ when $0 < \eta_1 < 1$ (for $A \geq 14$), and when $0 < \eta_1 < \eta_1^*$ (for $A < 14$). For the two cases respectively, the points

$$c = -\frac{1}{5}\left(2 + \frac{1}{3}A\right),\quad \alpha = \frac{-4+A}{5A} \quad\text{and}\quad c = \frac{-7\eta_1^{*5} + 5\,\eta_1^{*3} - \frac{1}{3}A}{5\eta_1^{*3}},\quad \alpha = \frac{\eta_1^{*2}\,(-4\eta_1^{*5} + A)}{5A}$$

are on the line.

As before, now, it must be shown that $\alpha(\eta_1) \leq 1$ for any root η_1 of $F(\eta)$. From Equation (21),

$$\alpha(\eta_1) \leq 1 \Longleftrightarrow H(\eta_1) = -4\eta_1^7 + A\eta_1^2 - 5A \leq 0.$$

But $H(\eta_1)$ has no root in $0 < \eta_1 \leq 1$ (see Appendix B). Since $H(0) = -5A < 0$, $\alpha(\eta_1) < 1$ for any root η_1 of $F(\eta)$.

Finally, in the case where $A \geq 14$,

$$c \leq -\frac{1}{5}\left(2 + \frac{1}{3}A\right) \Longrightarrow F(1) = 5c + 2 + \frac{1}{3}A \leq 0.$$

Also, $F(0) = (1/3)A > 0$, so $F(\eta)$ has at least one root in $0 < \eta \leq 1$. The existence of at least one root of $F(\eta)$ in the case where $A < 14$ will not be proved here — however, it can be proved after choosing a value of $A < 14$.

To summarize, then, the curve whose parametric equations are

$$c(\eta_1) = \frac{-7\eta_1^5 + 5\eta_1^3 - \frac{1}{3}A}{5\eta_1^3}$$
$$\alpha(\eta_1) = \frac{\eta_1^2(-4\eta_1^5 + A)}{5A} \qquad (25)$$

represents the upper boundary (for $\sin^2 g = 0$) to the permissible c, α region lying to the left of the line $c = -(1/6)A(1-3\alpha)$. When $A \geq 14$, the parameter η_1 may take on all values in $0 < \eta_1 \leq 1$. However, when $A < 14$, η_1 is restricted to the range $0 < \eta_1 \leq \eta_1^* < 1$, where η_1^* is the root of $G(\eta_1)$ defined by Equation (24).

Figure 1 depicts the c, α regions defined in the previous two sections.

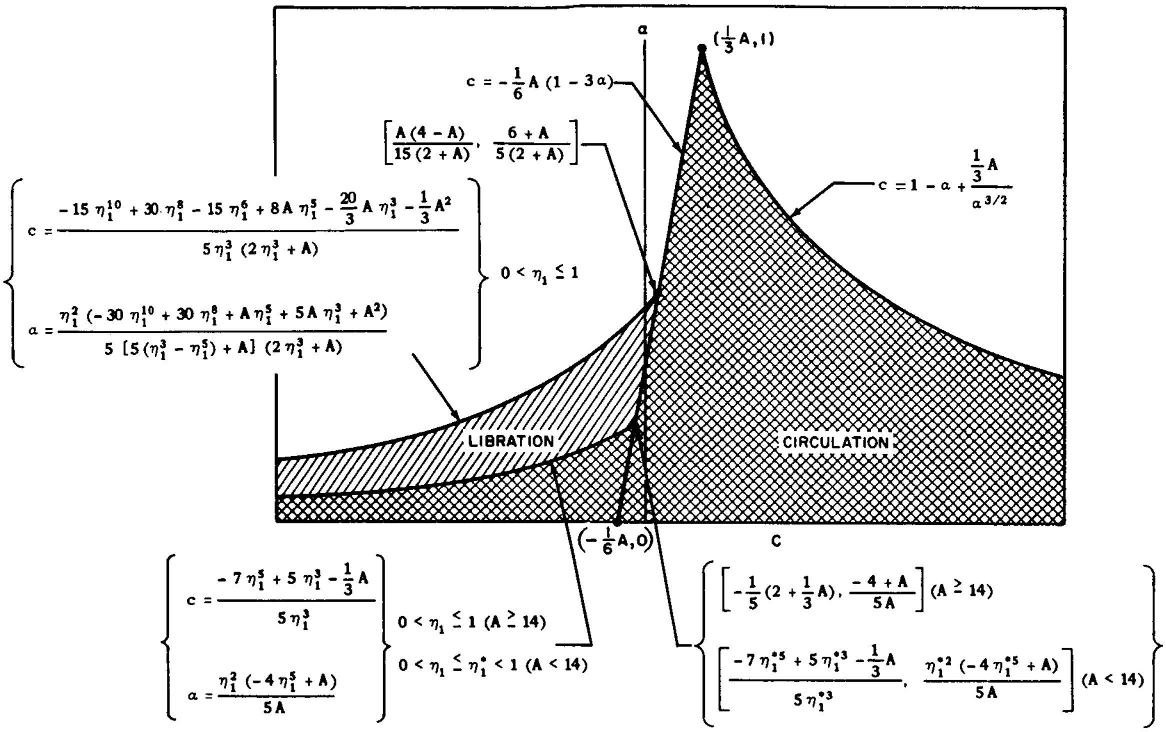

Figure 1

EXAMPLES

The two cases to be presented here correspond to values for the semi-major axis of the satellite's orbit of 2 and 7.4822577 moon radii. Then, the values for A are 164.97081 and .22510948, respectively. For the first case, Equations (4), (5), (15), and (25) become

$$c = -27.495135\,(1 - 3\alpha) \tag{4'}$$

$$c = 1 - \alpha + \frac{54.990270}{\alpha^{3/2}} \tag{5'}$$

$$c = \frac{-15\eta_1^{10} + 30\eta_1^8 - 15\eta_1^6 + 1319.7665\eta_1^5 - 1099.8054\eta_1^3 - 9071.7893}{5\eta_1^3(2\eta_1^3 + 164.97081)} \tag{15'}$$

$$\alpha(\eta_1) = \frac{\eta_1^2(-30\eta_1^{10} + 30\eta_1^8 + 164.97081\eta_1^5 + 824.85405\eta_1^3 + 27215.368)}{5[5(\eta_1^3 - \eta_1^5) + 164.97081](2\eta_1^3 + 164.97081)} \quad (0 < \eta_1 \leq 1)$$

$$c(\eta_1) = \frac{-7\eta_1^5 + 5\eta_1^3 - 54.990270}{5\eta_1^3} \tag{25'}$$

$$c(\eta_1) = \frac{\eta_1^2(-4\eta_1^5 + 164.97081)}{824.85405} \quad (0 < \eta_1 \leq 1)$$

For A = .22510948, the root of G (η_1) is $\eta_1^* = .25110445$; therefore, the equations defining the boundaries of the circulation and libration regions are

$$c = -.037518247(1 - 3\alpha) \tag{4''}$$

$$c = 1 - \alpha + \frac{.075036494}{\alpha^{3/2}} \tag{5''}$$

$$c(\eta_1) = \frac{-15\eta_1^{10} + 30\eta_1^8 - 15\eta_1^6 + 1.80087584\eta_1^5 - 1.50072988\eta_1^3 - .01689143}{5\eta_1^3(2\eta_1^3 + .22510948)} \tag{15''}$$

$$\alpha(\eta_1) = \frac{\eta_1^2(-30\eta_1^{10} + 30\eta_1^8 + .22510948\eta_1^5 + 1.12554740\eta_1^3 + .05067428)}{5[5(\eta_1^3 - \eta_1^5) + .22510948](2\eta_1^3 + .22510948)} \quad (0 < \eta_1 \leq 1)$$

$$c(\eta_1) = \frac{-7\eta_1^5 + 5\eta_1^3 + .075036494}{5\eta_1^3} \tag{25''}$$

$$a = \frac{\eta_1^2(-4\eta_1^5 + .22510948)}{1.1255474} \quad (0 < \eta_1 \leq .25110445).$$

Tables 1, 2, and 3 (Appendix C) list values of c and α for points on the curves defined by the preceding equations. Figures 2, 3, 4, and 5 (Appendix D) show the circulation and libration regions for the two cases. It is readily apparent that the libration region for $a = 2 R_{\text{☾}}$ is considerably smaller than that for the case $a = 7.4822577 R_{\text{☾}}$ - this is a consequence of the fact that the part of the disturbing function dependent upon J_2 (which is purely secular) is more significant for smaller values of a.

CONCLUSIONS

Given a set of initial conditions for an artificial lunar satellite, it can be determined from a graph similar to Figure 1 whether the orbit falls into the circulation case or libration case. If circulation, the method outlined by Frost (7) can be used to compute the long-period and secular effects caused by the earth and the secular effects due to the moon's second zonal harmonic. On the other hand, if the orbit is libratory, the method presented by Fisher and Felsentreger (8) may be employed.

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APPENDIX A

SYMBOLS

a = semi-major axis of satellite's orbit

e = eccentricity of satellite's orbit

i = inclination of satellite's orbital plane to moon's equatorial plane

g = argument of perilune of satellite's orbit

n = mean motion of satellite

t = time

$\eta = \sqrt{1 - e^2}$

$R_{☾}$ = mean radius of moon $\simeq$ 1738 km.

$\alpha_{☾}$ = semi-major axis of moon's orbit $\simeq$ 221.17376 $R_{☾}$

J_2 = second zonal harmonic coefficient of moon's gravitational potential $\simeq 2.41 \times 10^4$

$$q = 1 + \frac{\text{mass of moon}}{\text{mass of earth}} \simeq 1.0123$$

$\alpha_1 = (1/2)\,(a/a_{☾})^3$

$\alpha_2 = R_{☾}^2\, J_2/a^2$

$$A = \frac{\alpha_2}{\alpha_{1/q}}$$

α, c = constants of the motion

APPENDIX B

1. Proof that $g(\eta_1)$ has no root in $0 < \eta_1 < 1$

From Equation (14),

$$g(\eta_1) = 90(5+A)\eta_1^{13} + 180(5+A)\eta_1^{12} + 180A\eta_1^{11} - 180(5-A)\eta_1^{10} - 10(45-13A)\eta_1^{9}$$
$$- A(250+3A)\eta_1^{8} - 2A(290+3A)\eta_1^{7} - A(290+49A)\eta_1^{6} - 92A^2\eta_1^{5} - 100A^2\eta_1^{4}$$
$$- A^2(50+3A)\eta_1^{3} - 6A^3\eta_1^{2} - 4A^3\eta_1 - 2A^3,$$

which, by Descartes' Rule of Signs, has either one positive root, or possibly three (in the case $45/13 < A < 5$). Replacing η_1 by $1/1+X$ $(X>0)$ transforms the equation into one having exactly the same number of positive roots as those of $g(\eta_1)$ in $0 < \eta_1 < 1$. Thus, solving $g(\eta_1) = 0$ for $0 < \eta_1 < 1$ is equivalent to solving the following equation for $X>0$ (the coefficients of $X^{13}, X^{12}, \ldots X^5$ are all obviously negative, so there is no need to write them down explicitly):

$$0 = a_{13}X^{13} + a_{12}X^{12} + a_{11}X^{11} + a_{10}X^{10} + a_9X^9 + a_8X^8 + a_7X^7 + a_6X^6 + a_5X^5$$
$$-10(45+1997A+3136A^2+602A^3)X^4 - 3(900+7850A+7139A^2+934A^3)X^3$$
$$-5(1080+3158A+1915A^2+177A^3)X^2 - 10(360+516A+253A^2+17A^3)X$$
$$-15A(24+20A+A^2),$$

where $a_{13}, a_{12}, \ldots, a_5 < 0$. This equation obviously can have no positive solutions - hence, $g(\eta_1)$ has no roots in $0 < \eta_1 < 1$.

2. Proof that $h(\eta_1)$ has no root in $0 \leq \eta_1 \leq 1$

$$h(\eta_1) = -30\eta_1^{12} + 30\eta_1^{10} + 50\eta_1^{8} + A\eta_1^{7} - 50\eta_1^{6} + 30A\eta_1^{5} - 35A\eta_1^{3} + A^2\eta_1^{2} - 5A^2.$$

Since $h(0) = -5A^2 < 0$ and $h(1) = -4A(1+A) < 0$, it remains to be shown that $h(\eta_1)$ has no roots in $0 < \eta_1 < 1$. Equivalently, it must be shown that the transformed equation, after replacing η_1 by $1/1+X$ $(X>0)$, has no positive roots. The transformed equation is

$$0 = -5A^2X^{12} - 60A^2X^{11} - 329A^2X^{10} - 5A(7+218A)X^9 - 45A(7+54A)X^8$$

$$-30A(41+128A)X^7 - 10(5+273A+441A^2)X^6 - (300+3779A+3708A^2)X^5$$

$$-5(140+671A+453A^2)X^4 - 20(40+94A+49A^2)X^3 - 5(84+124A+57A^2)X^2$$

$$-10(4+10A+5A^2)X - 4A(1+A),$$

which has no non-negative solutions. Consequently, $h(\eta_1)$ has no roots in $0 \leq \eta_1 \leq 1$.

3. Roots of $G(\eta_1)$ in $0 < \eta_1 \leq 1$

$$G(\eta_1) = 12\eta_1^8 + 24\eta_1^7 + 36\eta_1^6 + 48\eta_1^5 + 60\eta_1^4 + 3(10-A)\eta_1^3 - 6A\eta_1^2 - 4A\eta_1 - 2A.$$

Finding the roots of $G(\eta_1)$ in $0 < \eta_1 \leq 1$ is equivalent to finding the positive roots of

$$b_8X^8 + b_7X^7 + b_6X^6 + 5(1-47A)X^5 + 35(6-11A)X^4 + 6(98-67A)X^3$$

$$+20(42-13A)X^2 + 5(126-19A)X + 15(14-A),$$

where $b_8, b_7, b_6 < 0$ and $\eta_1 = 1/1+X$. Descartes' Rule of Signs establishes that the polynomial in X has exactly one positive root for $A \leq 14$, and no positive roots when $A > 14$. Consequently, $G(\eta_1)$ has exactly one root in $0 < \eta_1 \leq 1$ for $A \leq 14$, and no roots in $0 < \eta_1 \leq 1$ when $A > 14$. Since $G(1) = 15(14-A)$, the root is 1 when $A = 14$, and is in $0 < \eta_1 < 1$ when $A < 14$.

4. Proof that $H(\eta_1)$ has no root in $0 < \eta_1 \leq 1$

$$H(\eta_1) = -4\eta_1^7 + A\eta_1^2 - 5A.$$

$H(1) = -5A < 0$, so 1 is not a root.

Setting $\eta_1 = 1/1 + X$, the polynomial set equal to zero is

$$0 = -5AX^7 - 35AX^6 - 104AX^5 - 170AX^4 - 165AX^3 - 95AX^2 - 30AX - 4(1 + A),$$

which obviously has no positive roots. Therefore, $H(\eta_1)$ has no roots in $0 < \eta_1 < 1$.

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APPENDIX C

Tables

Table 1

c, α values for $c = 1 - \alpha + \frac{\frac{1}{3}A}{\alpha^{3/2}}$

α	c (A = 164.97081)	c (A = .22510948)
. 1	54.990270	.07503649
.9409	60.310987	.14131618
.8836	66.323153	.20674184
.8281	73.144811	.27147455
.7744	80.918967	.33570943
.7225	89.819971	.39968440
.6724	100.061843	.46369149
.6241	111.909208	.52809180
.5776	125.619822	.59333530
.5329	141.824080	.65998744
.4900	160.831490	.72876529
.4489	183.386990	.80058712
.4096	210.361630	.87664150
.3721	242.896060	.95848490
.3364	282.503110	1.04818164
.3025	331.217530	1.14850822
.2704	391.818790	1.26325735
.2401	468.169490	1.39769967
.2116	565.741440	1.55930176
.1849		1.75887216
.1600		2.01244520
.1369		2.34448300
.1156		2.79353121
.0961		3.42266381
.0784		4.33980761

Table 2

c, α values for $c = \dfrac{-15\eta_1^{10} + 30\eta_1^8 - 15\eta_1^6 + 8A\eta_1^5 - \frac{20}{3}A\eta_1^3 - \frac{1}{3}A^2}{5\eta_1^3(2\eta_1^3 + A)}$,

$$\alpha = \frac{\eta_1^2(-30\eta_1^{10} + 30\eta_1^8 + A\eta_1^5 + 5A\eta_1^3 + A^2)}{5[5(\eta_1^3 - \eta_1^5) + A]\ (2\eta_1^3 + A)}$$

η_1	c(A = 164.97081)	α(A = 164.97081)	c(A = .22510948)	α(A = .22510948)
1	- 10.60285	.20479126	.02545983	.55953289
.95	- 12.58625	.18367060	- .00179364	.45293847
.90	- 14.99181	.16402945	- .05465415	.36267636
.85	- 17.95302	.14574676	- .12887024	.28707338
.80	- 21.65669	.12872644	- .22039516	.22453776
.75	- 26.36940	.11289356	- .32639709	.17250927
.70	- 32.48019	.09819074	- .44028987	.13267987
.65	- 40.57145	.08457501	- .56180739	.10053275
.60	- 51.54089	.07201504	- .68716695	.07578677
.55	- 66.82003	.06048873	- .81440876	.05716197
.50	- 88.78450	.04998109	- .94307939	.04342257
.45	- 121.56804	.04048246	-1.07559004	.03340831
.40	- 172.78868	.03198710	-1.21995835	.02596810
.35	- 257.51846	.02449195	-1.39568007	.02022638
.30	- 408.39139	.01799570	-1.64784077	.01545777
.25	- 704.97551	.01249808	-2.08708494	.01126196
.20	-1375.89282	.00799932	-3.02826351	.00753283

Table 3

$$c,\ \alpha \text{ values for } c = \frac{-7\eta_1^5 + 5\eta_1^3 - \frac{1}{3}A}{5\eta_1^3},\quad \alpha = \frac{\eta_1^2(-4\eta_1^5 + A)}{5A}$$

η_1	c (A = 164.97081)	α (A = 164.97081)	η_1	c (A = .22510948)	α (A = .22510948)
1	-11.39805	.19515066	.25110445	-.03612413	.012386983
.95	-13.09109	.17711353	.240	-.16623741	.011357005
.90	-15.22049	.15968058	.230	-.30750281	.010458998
.85	-17.91999	.14294541	.220	-.47716070	.009591355
.80	-21.37657	.12698302	.210	-.68222360	.008755992
.75	-25.85696	.11185269	.200	-.93191234	.007954511
.70	-31.75030	.09760064	.190	-1.23851181	.007188233
.65	-39.63903	.08426227	.180	-1.61862795	.006458243
.60	-50.42092	.07186425	.175	-1.84307067	.006107137
.55	-65.52751	.06042617	.170	-2.09506996	.005765417
.50	-87.33443	.04996211	.165	-2.37891665	.005433167
.45	-119.97545	.04048188	.160	-2.69973129	.005110460
.40	-171.06859	.03199205	.155	-3.06365715	.004797361
.35	-255.68588	.02449688	.150	-3.47810698	.004493928
.30	-406.46133	.01799894	.145	-3.95208005	.004200211
.25	-702.96296	.01249970	.140	-4.49657219	.003916254
.20	-1373.81275	.00799994	.137	-4.86261985	.003750580
			.133	-5.40369117	.003535184
			.130	-5.85447414	.003377770
			.128	-6.17897523	.003274799
			.127	-6.34899307	.003223906
			.125	-6.70561192	.003123305
			.123	-7.08586163	.003024286

APPENDIX D

Graphs

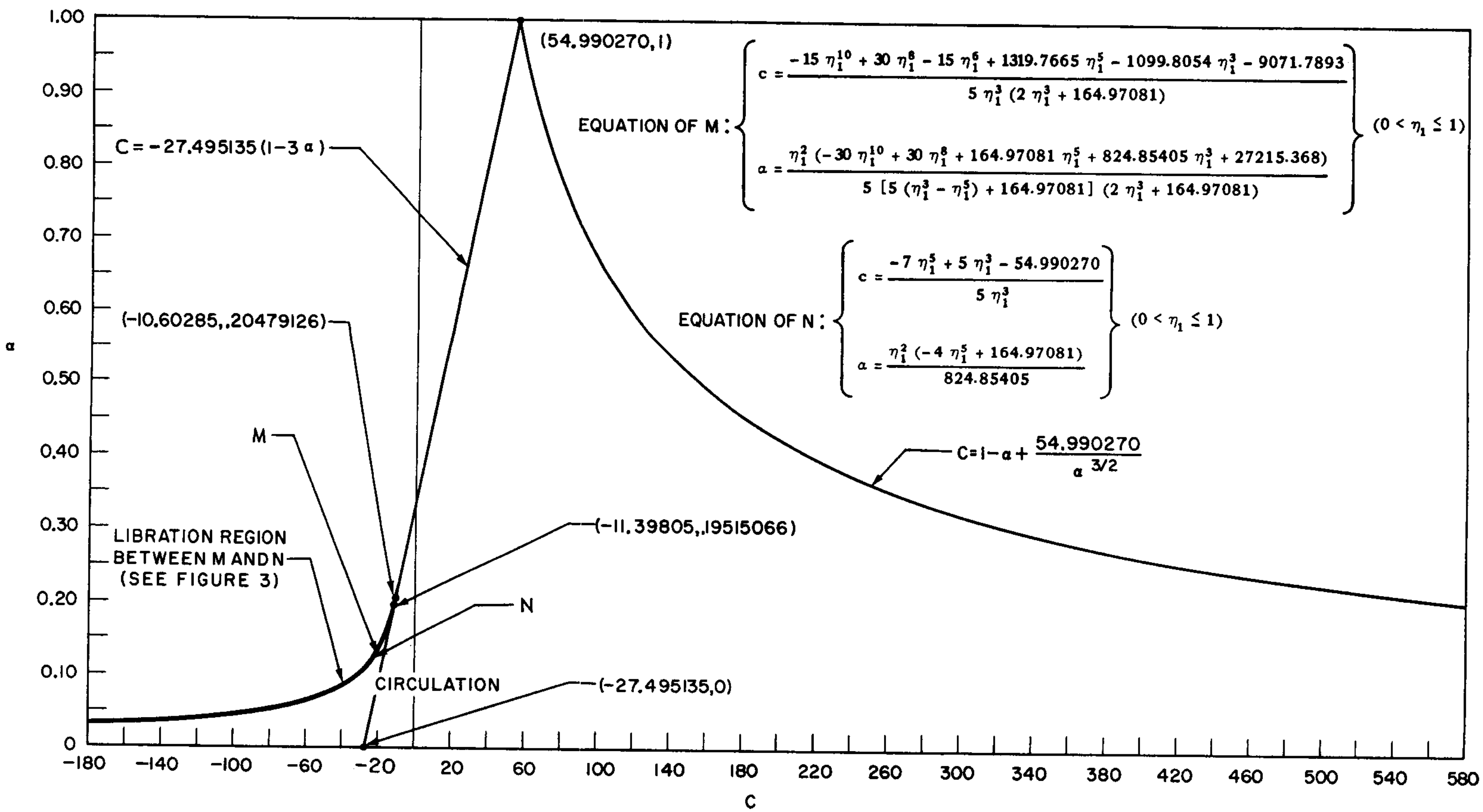

Fig. 2–Libration and circulation regions for A = 164.97081.

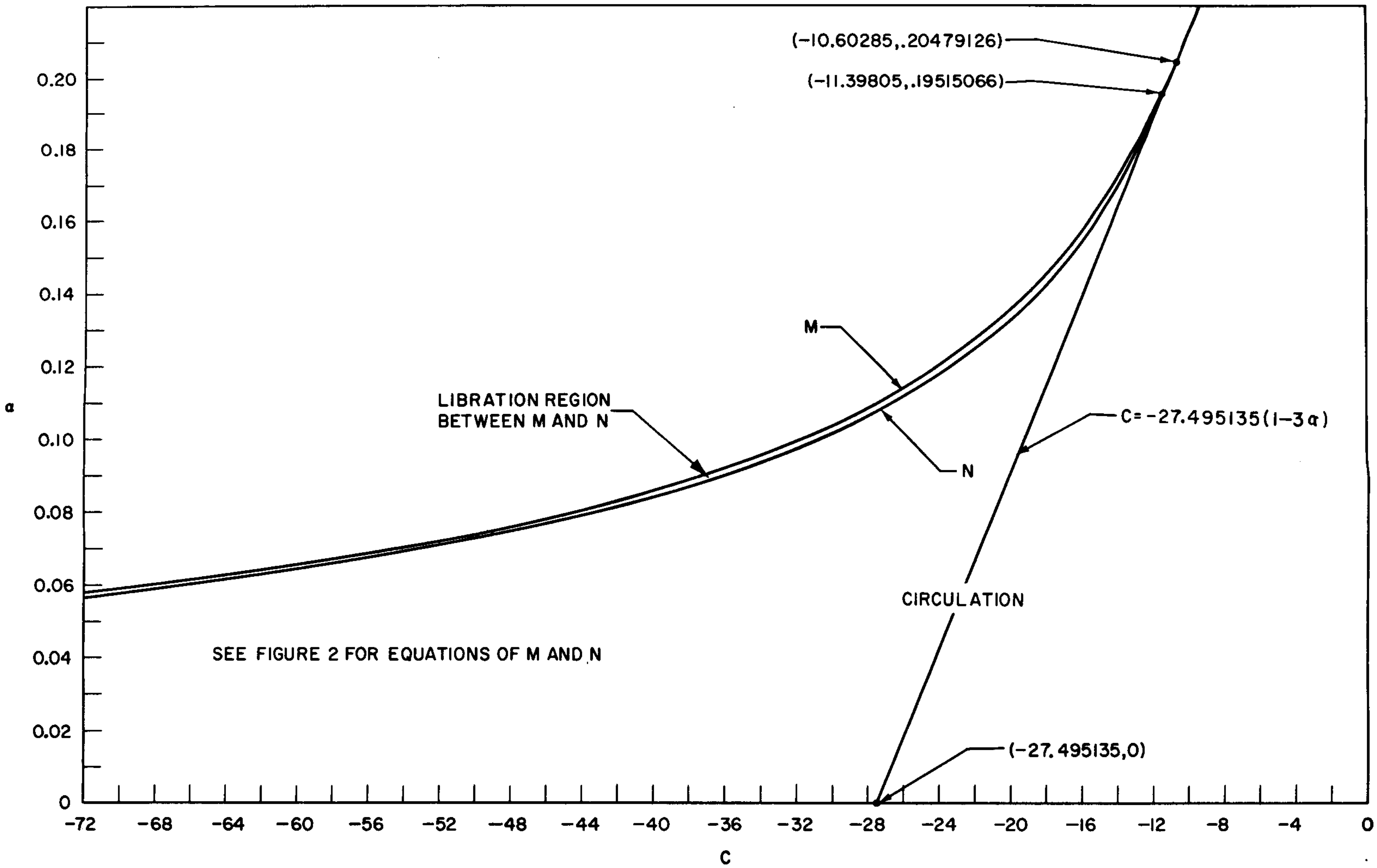

Fig. 3–Libration region for A = 164.97081.

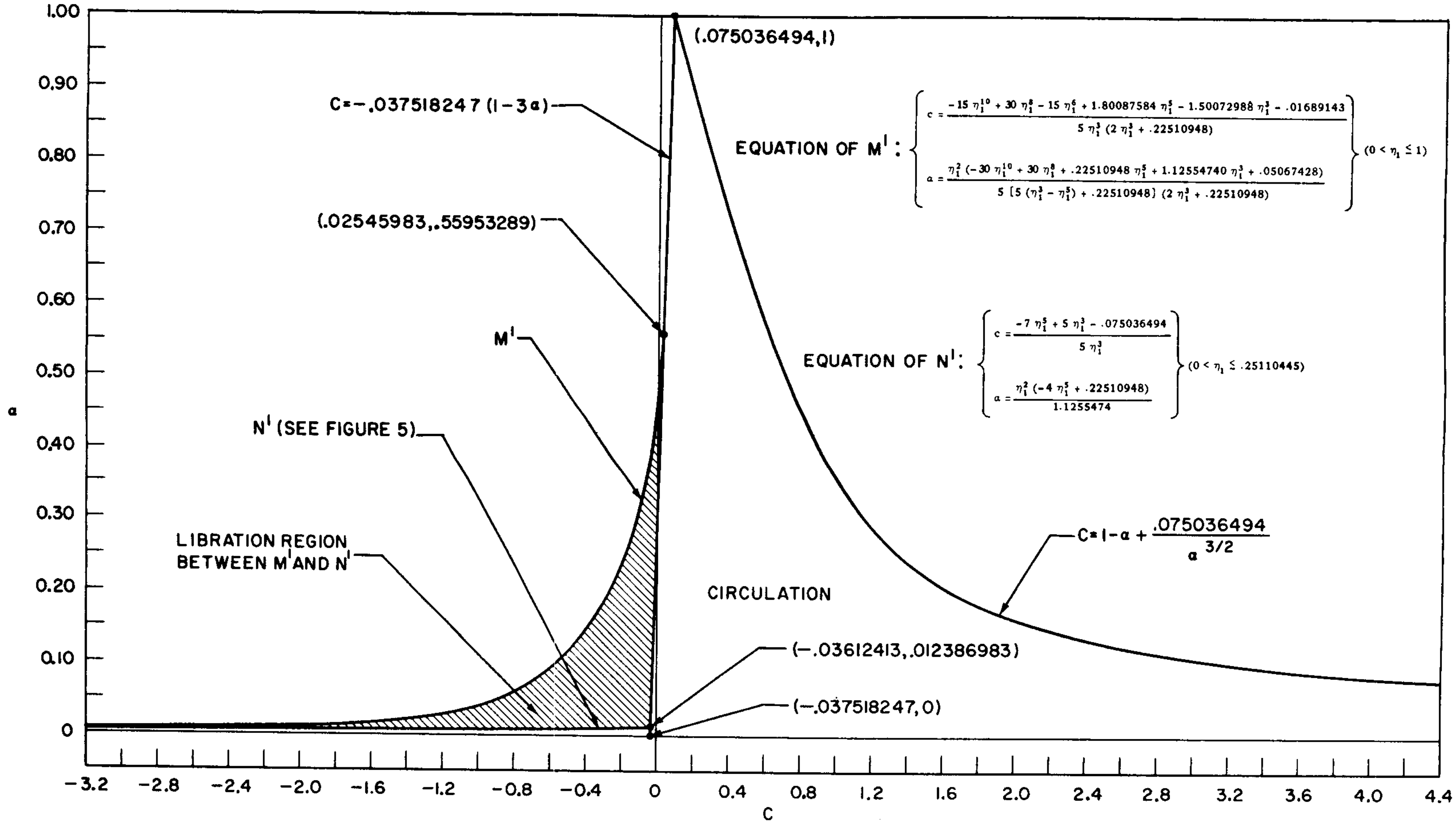

Fig. 4–Libration and circulation regions for A = .22510948.

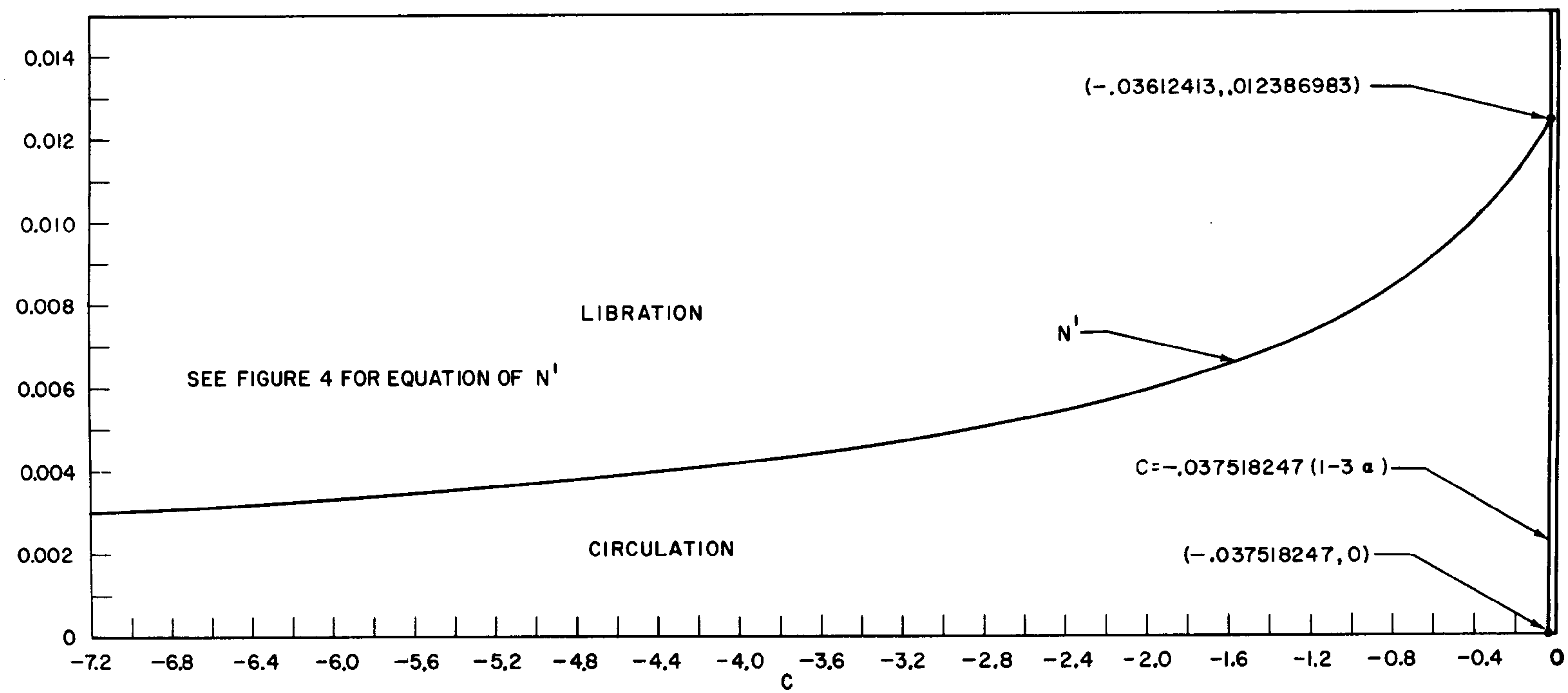

Fig. 5–Expanded view of curve N′ from Figure 4 (A = .22510948).