NASA TMX 57232

GPO PRICE $ ____

CFSTI PRICE(S) $ ____

Hard copy (HC) 3.00

Microfiche (MF) .65

ff 653 July 65

A Topological Study of the Motion

of Particles Trapped in a Magnetic Field

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N 67-18704

(ACCESSION NUMBER) (THRU)

15 1

(PAGES) (CODE)

TMX 57232 29

(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

Trapped Particle Topology

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Abstract

The drift motion of charged particles trapped in a magnetic field takes place along curves of J = constant, where J is the adiabatic integral invariant. These are closed curves in the cases usually thought of in connection with physical systems such as the earth's radiation belts. The present paper reports an exploration of the possibility that there exist magnetic fields such that the curves J = constant are a family of spirals. It is shown that a necessary condition for this to occur is that there exists a J = constant spiral surface on which $\underset{\sim}{B} = 0$, or each line of force lying in it has the property that a particle spiralling about it does not, on the average, drift off it, or J suffers a discontinuity as one crosses it. Whether any of these conditions in fact leads to the curves being a family of spirals is not discussed. It seems to the author to be unlikely that any reasonable model of the earth's field possesses these properties.

Introduction

We are concerned with the adiabatic description of the motion of trapped particles begun by Alfven and developed extensively by others. The results that we need are given by Northrup (1).

The motion of a charged particle is described as occurring in the following coordinate system. Denote the magnetic field by $\underset{\sim}{B}$. Then, since $\nabla \cdot \underset{\sim}{B} = 0$, there exist functions α and β such that $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$. Both α and β are constant along each line of force. They are independent functions. Denote by s path length along a line of force. Then α, β, s are the configuration coordinates used to describe the motion.

The motion of a trapped particle may be thought of as composed of three parts. The particle circles rapidly about a line of force. The center of this circle moves with moderate speed along the line of force. If these two motions are thought of together, one says that the particle moves along a helix of variable pitch about a line of force. Finally, the center of the circular motion, the so called guiding center, when projected along the line of force on which it instantaneously lies, slowly drifts about on some reference surface. This slow drift can be discussed with the aid of α, β, there being no occasion to consider the value of s at the guiding center. Since the path of this slow drift is the sole subject of the present work, we will speak of the motion as occurring in the α, β space. A magnetic shell consists of a curve in this space.

The motion of a trapped particle is described by three adiabatic invariants. These are the magnetic moment μ, the invariant J, and the flux invariant Φ. The last of these is a tool for studying the effect on the motion of very slow time variations of the magnetic field. We will not be concerned with it. Our object is to study the situation in the α, β space of curves of constant J, these being the trajectories in this space of the guiding center of a particle. If one chooses a particular value for μ, then as J runs over all possible values, a family of curves J = constant is generated which fills the space in the sense that exactly one member passes through each point in the space. When we speak of a family of curves J = constant, we will suppose that μ has the same value for all members.

The motion satisfies the equations given by Northrup (2)

$$(eT/c) \langle \dot{\alpha} \rangle = \partial J/\partial \beta$$

$$(eT/c) \langle \dot{\beta} \rangle = \partial J/\partial \alpha$$

where e and c are the charge on the particle and the speed of light. The period of an oscillation between mirror points is T, where T is a function of α, β, μ, the particle energy, and time. The brackets $\langle\ \rangle$ mean that the quantity enclosed is averaged over a field line from one mirror point to the next.

A Spiral Family

In this section we show, under certain assumptions concerning the magnetic field, that no family of curves J = constant can be a family of spirals. We define a family of spirals as follows. Through each point in the α, β space passes exactly one member of the family. There is at least one distinguished point (there may be a connected compact set of them) called the center of the spirals. Any curve (not a member of the family) which begins at the center, is of infinite length, and is nowhere tangent to any member of the family intersects each member infinitely often. This curve will be called <u>the auxiliary curve</u>. Since exactly one member of the family passes through each point, no member intersects either itself or any other member.

In order to clarify the meaning of the definition, it is useful to consider the following result. Assume that the curves J = constant are continuous, have continuous first derivatives, and are continuous functions of their initial point. Suppose that one of them intersects the auxiliary curve at some point while going in a particular direction (for example, clockwise about the center), and that at its next intersection with the auxiliary curve it is going in the opposite direction. Move along the auxiliary curve from the first of these intersections to the second. From the continuity assumption, it follows that at some point there is a curve J = constant which is tangent to the auxiliary curve. This is excluded by the definition of a family of spirals. We conclude that all crossings of the auxiliary curve by a given curve J = constant are in the same sense.

We now proceed to the proof. Assume that the curves J = constant are a family of spirals. Denote by ℓ the path length along an auxiliary curve as called for by the definition. Assume that this auxiliary curve can be so chosen that at each point on it $\underset{\sim}{B} \neq 0$ and not both of $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$ vanish. This can obviously be done if, for example, the conditions in question occur only at isolated points. Assume also that the auxiliary curve can be so chosen that T does not vanish at any point on it, and that J is a continuous function of ℓ. These assumptions will be examined in detail in the next section.

Consider the relation

$$\frac{dJ}{d\ell} = \frac{\partial J}{\partial\alpha}\frac{d\alpha}{d\ell} + \frac{\partial J}{\partial\beta}\frac{d\beta}{d\ell} .$$

From (1) this is

$$\frac{dJ}{d\ell} = (eT/c)\left[- \langle\dot{\beta}\rangle \frac{d\alpha}{d\ell} + \langle\dot{\alpha}\rangle \frac{d\beta}{d\ell}\right] .$$

We now show that the right member cannot vanish. We have assumed that $T \neq 0$. The quantity in square brackets can be thought of as the scalar product of the two vectors $\{-\langle\dot{\beta}\rangle, \langle\dot{\alpha}\rangle\}$ and $\{d\alpha/d\ell, d\beta/d\ell\}$. The second of these is tangent to the auxiliary curve. From (1), the first is normal to the curve J = constant. Since we are in a two dimensional space, these vectors can be normal to each other only if the auxiliary curve is tangent to the curve J = constant. This is excluded by the definition. We have assumed that the vector $\{-\langle\dot{\beta}\rangle, \langle\dot{\alpha}\rangle\}$ does not vanish. We now show that the vanishing of the vector $\{d\alpha/d\ell, d\beta/d\ell\}$ implies the vanishing of $\underset{\sim}{B}$, which has been assumed not to occur. Let $\underset{\sim}{t}$ be a vector tangent to the auxiliary curve. Then the vanishing of $d\alpha/d\ell$ and $d\beta/d\ell$ implies that $\underset{\sim}{t}\cdot\nabla\alpha = \underset{\sim}{t}\cdot\nabla\beta = 0$. If either $\nabla\alpha=0$ or $\nabla\beta=0$, then from $\underset{\sim}{B} = \nabla\alpha\times\nabla\beta$ it follows that $\underset{\sim}{B} = 0$. Suppose, then, that neither gradient vanishes. Then $\nabla\alpha$, $\nabla\beta$, and $\underset{\sim}{t}$ all lie in one plane, and both $\nabla\alpha$ and $\nabla\beta$ are perpendicular to $\underset{\sim}{t}$. Consequently, they are parallel to each other. But then, again, $\underset{\sim}{B} = 0$. We conclude that the right member of (2) does not vanish. From the assumed continuity of J together with the constant sign of its derivative it follows that J is a monotonic function of ℓ. As a result, the auxiliary curve intersects a given curve J = constant at most once. The curves J = constant are not a family of spirals.

Discussion of Assumptions

The assumption that T does not vanish means that successive mirror points do not coincide, and hence that the auxiliary curve does not intersect a curve J = 0. A curve J = 0 is a curve of constant B with the reciprocal of the constant being equal to the value of $\mu/(\frac{1}{2}\ mv^2)$ for the particle in question. In a typical application to the radiation belts, this curve lies closer to the earth than does the region of interest. In order to attempt the construction of a family of spirals using such a curve, it is obviously necessary that the curve $B = \frac{1}{2}\ mv^2/\mu$ be a spiral.

We have also assumed that J is a continuous function of ℓ. We consider only the case where $\underset{\sim}{B}$ is a continuous function of position. Then J can be discontinuous if the range of integration is a discontinuous function of position. This can happen through the addition of new mirror points for the given value of μ as one moves along the auxiliary curve. It is obvious that the curve of discontinuity must be a spiral and also a curve of constant J. The existence of such a curve defeats the proof since, while J is a piecewise monotonic function of position along the auxiliary curve, it can now in such a way suffer a stepwise change in value upon each crossing of the curve of discontinuity that, between any two such consecutive crossings, J varies over the same range as between any other two. It is not necessarily so that more and more mirror points are added as the curve of discontinuity is crossed more and more times. It is obvious that the occurrence of extra mirror points in such a way as to produce such a curve of discontinuity is not possible when lines of force with more than a single minimum in B are represented by isolated points in

α, β space.

The assumption that $\langle\dot{\alpha}\rangle$, $\langle\dot{\beta}\rangle$ do not simultaneously vanish means that through no point on the auxiliary curve passes a line of force with the property that, on the average, a particle spiralling about it does not drift off it. Evidently, in order to defeat the proof there must be a spiral curve of constant J such that every point on it has this property.

We have shown, then, that the set of curves J = constant being a family of spirals implies that there is one member of the family at every point of which J suffers a discontinuity through the introduction of a new mirror point, or both $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$ vanish so that no drifting occurs, or, $\underset{\sim}{B} = 0$. We have not shown that any of these conditions permits the occurrence of a family of spirals.

Conclusion

While the situation is not as clear cut as one could wish, it seems unlikely that any reasonable model of the earth's field possesses any of the properties which permit the set of curves J = constant to be a family of spirals. Whether any astrophysically interesting fields of this sort can be constructed is uncertain. If any such model is attempted, one must still discover whether the magnetic field property which breaks down the present theorem produces a spiral family of curves J = constant.

Acknowledgement

I thank T. Gold for suggesting the problem, and the Center for Radiophysics and Space Research at Cornell for its support during the inception of the work.

References

1. T.G. Northrup, "The Adiabatic Motion of Charged Particles," Interscience, New York, 1963.

2. Northrup, loc. cit., p. 57.