THE CAPILLARY INSTABILITY OF A JET CARRYING AN AXIAL CURRENT

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ABSTRACT

The capillary instability of a liquid jet carrying an axial volume current is investigated for vericose deformations (m=0). It is found that the inclusion of Hall current in Ohm's law leads to <u>overstability</u> of the jet, albeit with small frequency of oscillation.

I. INTRODUCTION

The capillary instability of a liquid jet has been the subject matter of extensive investigations since the pioneer work by Lord Rayleigh[1]. It is well known that the liquid jet is unstable to all axisymmetric perturbations having wave numbers (measured in units of the reciprocal radius of the cylinderical column) less than unity. In an attempt to provide a liquid scale model of a pinched gas discharge, Dattner, Lehnart and Lundquist[2] have carried out experiments with liquid mercury column carrying a direct axial volume current and they observe a sausage type (vericose) instability similar to that of a classical liquid jet.

It has been shown by Chandrasekhar[3] that the presence of an axial magnetic field has a stabilizing influence on the vericose instability of a conducting liquid jet. The stability of a capillary jet with axial volume current has been discussed by Murty[4] who discussed the problem for small values of the electrical conductivity. Recently Gupta[5] has discussed the stability of a liquid jet with axial volume current with and without the presence of an axial magnetic field and has computed the growth rate of instabilities in several limiting cases for vericose deformations.

The importance of including Hall current in Ohm's law in investigating the stability of hydromagnetic configurations has been pointed out by Taylor[6] and Ware[7]. The effect of Hall current on the stability of a capillary jet in the presence of an axial magnetic field has been investigated by Trehan et al[8] who showed that the inclusion of Hall current does not lead to overstability in the case of a liquid mercury or sodium jet. Recently Hosking[9] has shown that the inclusion of Hall current

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leads to overstability in a situation where an infinitely conducting plasma occupies the half space $0 \leq Z < \infty$ and is supported against gravity by magnetic pressure due to a jump in the field strength at the plasma boundary $Z=0$.

We investigate here the effect of Hall current on the capillary instability of a finitely conducting liquid jet carrying an axial volume current. For the sake of mathematical simplicity we restrict ourselves to axisymmetric perturbations. It is of interest to note that the inclusion of Hall current leads to overstability of the jet.

II. FORMULATION OF THE PROBLEM

We consider a static and infinitely long cylinderical liquid jet carrying a total current I through a cross-section of the jet. The associated circular magnetic field is given by $H_\theta = Hr/R$, $r \leq R$, where $H = 2I/R$ and R is the radius of the jet. The fluid is assumed to be incompressible, inviscid and having a constant density ρ and finite electrical conductivity σ. The pressure distribution in equilibrium is obtained from the condition of pressure balance and we find $p = T/R + (H^2/4\pi)(1 - r^2/R^2)$ where T is the surface tension. We shall restrict our considerations to the case when the jet is in free space and the external material pressure can be ignored. The magnetic field exterior to the jet is given by $H_\theta = HR/r$, $r \geq R$.

We now imagine the boundary of the cylinder to be deformed from $r=R$ to one given by

$$r=R+\epsilon_o \exp(\omega t+ikZ), \tag{1}$$

where ϵ_o is a constant and we restrict ourselves to axisymmetric (vericose) deformations ($m = o$). As a result of the deformation let the various perturbed quantities be given by

$$\underset{\sim}{U},\ p+\delta p \text{ and } \underset{\sim}{H}+\underset{\sim}{h}, \tag{2}$$

where U, δp and h are small perturbations in the velocity field, pressure and magnetic field respectively. We shall assume all the perturbations to have the form

$$f(r, Z, t) = f(r) \exp(\omega t + ikZ). \tag{3}$$

The linearized hydromagnetic equations, including the Hall current, governing the perturbed quantities then reduce to

$$\omega\underset{\sim}{U}=-\underset{\sim}{\nabla}\Pi+\frac{H}{2\pi\rho R}(h_r\underset{\sim}{e}_\theta-h_\theta\underset{\sim}{e}_r), \tag{4}$$

and

$$\left(\omega-\frac{2ikHC_1}{R}\right)\underset{\sim}{h}=-\eta\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times h, \tag{5}$$

where $\Pi = \delta p/\rho + \underset{\sim}{H}.\underset{\sim}{h}/4\pi\rho$, $\eta = 1/4\pi\sigma$ and $C_1 = c/4\pi Ne \simeq H/4\pi\rho\omega_c$, ω_c is the ion gyration frequency given by $\omega_c = eH/Mc$ and M is the ion mass. It is convenient to write Eq. (5) as

$$\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times\underset{\sim}{h}+a\underset{\sim}{h}=0, \tag{6}$$

where

$$a=\frac{1}{\eta}\left(\omega-\frac{2i\Omega^2}{x\omega}\right) \tag{7}$$

and $\Omega^2 = k^2A^2$, $A^2 = H^2/4\pi\rho$, $x=kR$. The solution of Eq. (6) may be readily obtained by following the procedure similar to that used by Taylor[10] and we may write

$$\underset{\sim}{h}=\underset{\sim}{h}_1+\underset{\sim}{h}_2, \tag{8}$$

where

$$\underset{\sim}{\nabla} \times \underset{\sim}{h}_j = \alpha_j \underset{\sim}{h}_j \ . \tag{9}$$

From Eqs. (6), (7) and (9), we readily obtain:

$$\alpha_{1,2} = \pm \frac{i}{\eta^{\frac{1}{2}}} \left(\omega - \frac{2i\Omega^2}{x\omega_i}\right)^{\frac{1}{2}} . \tag{10}$$

The solution of Eq. (9) is given by (for m=0)

$$h_{Z,j} = C_j I_o (\lambda_j r), \tag{11}$$

$$h_{\theta,j} = \frac{\alpha_j C_j}{\lambda_j} I_1 (\lambda_j r) , \tag{12}$$

and

$$h_{r,j} = - iC_j \frac{k}{\lambda_j} I_1 (\lambda_j r) , \tag{13}$$

where $\lambda_j^2 = k^2 - \alpha_j^2$ and C_j's are the integration constants. Since $\alpha_1^2 = \alpha_2^2$, we get $\lambda_1^2 = \lambda_2^2 = \lambda^2$ (say). Keeping in mind that $\alpha_2 = -\alpha_1$, we obtain for the perturbation in the magnetic field

$$h_Z = (C_1 + C_2) I_o (\lambda r) , \tag{14}$$

$$h_\theta = \frac{\alpha_1}{\lambda} (C_1 - C_2) I_1 (\lambda r) , \tag{15}$$

and

$$h_r = - \frac{ik}{\lambda} (C_1 + C_2) I_1 (\lambda r) . \tag{16}$$

The perturbation in the pressure is obtained by taking the divergence of Eq. (4) and making use of the solonoidal character of $\underset{\sim}{u}$ We thus obtain for the equation governing Π:

$$\nabla^2 \Pi = - \frac{H \alpha_1}{2\pi\rho R} (C_1 - C_2) I_o (\lambda r) . \tag{17}$$

The solution of Eq. (17) is given by

$$\Pi = E\, I_o\, (\hbar r) + \frac{H}{2\pi\rho R\alpha_1}\, (C_1 - C_2)\, I_o\, (\lambda r)\ , \tag{18}$$

where E is the integration constant. From Eqs. (15) and (18), we obtain:

$$\frac{\delta\rho}{\rho} = E\, I_o\, (\hbar r) + \frac{H(C_1 - C_2)}{2\pi\rho R\alpha_1}\left[I_o\, (\lambda r) - \frac{r\alpha_1^2}{2\lambda}\, I_1\, (\lambda r) \right] \tag{19}$$

Finally from Eqs. (4), (15) and (18), we find

$$u_r = -\frac{E\hbar}{\omega}\, I_1\, (\hbar r) - \frac{H_o(C_1 - C_2}{2\pi\ \rho R\alpha_1\lambda\omega}\, \hbar^2\, I_1\, (\lambda r)\ . \tag{20}$$

The perturbation in the vacuum magnetic field is given by

$$h^{(o)} = -\hbar C_3 K_1(\hbar r) e_r + i\hbar C_3 K_o(\hbar r)\ e_Z\ . \tag{21}$$

Further, the change in the surface tension pressure, δp_T is known to be

$$\delta p_T = -\frac{T\varepsilon_o}{R^2}\, (1 - x^2)\ . \tag{22}$$

III. THE DISPERSION RELATION

This is obtained as a result of matching the boundary conditions which are (1) the radial component of the velocity u_r at r=R must be compatible with the assumed form of the deformed boundary given by Eq. (1), (2) the magnetic field $\underset{\sim}{h}$ and (3) the normal component of the total stress must be continuous across the deformed boundary. The first condition leads to

$$\epsilon_o = - \frac{Ex}{\omega^2 R} I_1(x) - \frac{H(C_1-C_2)\,x^2}{2\pi\rho Y\alpha_1^2 R^2 \omega^2} I_1(y) , \tag{23}$$

where Y = λR. The continuity of the normal component of h gives

$$C_3 = \frac{iR}{} (C_1+C_2) \frac{I_1(y)}{K_1(x)} , \tag{24}$$

while the continuity of the Z-component of h yields

$$i\hbar C_3 K_o(x) = (C_1+C_2)\, I_o(y) . \tag{25}$$

Equations (24) and (25) lead to either

$$C_3 = 0 , \quad C_1 + C_2 = 0 \tag{26}$$

or the requirement that

$$\frac{y I_o(y)}{I_1(y)} + \frac{x K_o(x)}{K_1(x)} = 0 \tag{27}$$

This relation does not correspond to any deformation of the boundary and is of no interest for the problem at hand. Further, for the physical situation of interest, namely liquid sodium or mercury the resistivity is very high and we can set $y = x + \delta$ where $\delta = R^2(\omega - 2i\Omega^2/x\omega_1)/2x\eta \ll x$ and we can verify that Eq. (27) leads to a root which is incompatible with this assumption.

Therefore, we must set $C_3 = 0$ and $C_1 + C_2 = 0$. Thus Eqs. (24) and (25) are satisfied identically. The continuity of the θ-component of the magnetic field at the perturbed boundary requires

$$h_\theta + 2\epsilon_o H/R = 0 \ . \tag{28}$$

The continuity of the normal component of the stress tensor at the deformed boundary requires

$$\frac{\delta p}{\rho} + \frac{\epsilon_o}{\rho}\frac{dp}{dr} + \frac{1}{4\pi\rho}\underset{\sim}{H}^p . \underset{\sim}{h}^p + \frac{\epsilon_o}{8\pi\rho}\frac{d}{dr}\ {H^p}^2$$

$$= \frac{\delta p_T}{\rho} + \frac{\epsilon_o}{8\pi\rho}\frac{d}{dr}\ {H^v}^2 \text{ at } r = R \ , \tag{29}$$

where we have used the superscripts 'p' and 'v' to indicate the plasma and the vacuum magnetic fields respectively. Making use of the foregoing relations we obtain after some straight forward calculations the dispersion relation:

$$\frac{T}{\rho R^3}(1 - x^2) = \frac{\omega^2 I_o(x)}{x I_1(x)} + \frac{4A^2\eta}{R^4(\omega - 2i\Omega^2/x\omega_c)}$$

$$\times \left[\frac{x I_o(x)}{I_1(x)} - \frac{y I_o(y)}{I_1(y)}\right] \ . \tag{30}$$

a) <u>Limit of Infinite Conductivity</u>.

We may first observe that in the limit of infinite conductivity $\eta \to 0$ and Eq. (30) reduces to the classical result of Lord Rayleigh[1]. Thus in the limit of infinite conductivity, Hall current has no effect on the stability of a cylinder carrying an axial volume current. This result is, perhaps, not surprising as it is known that in infinitely conducting cases, the inclusion of Hall current in Ohm's law has effect on the stability of some hydromagnetic configurations while in some cases it does not. It may, perhaps, be in order to remark here that in the case of a sheet pinch with trapped axial magnetic field, it has been shown by Buti et al[11] that Hall current has no influence on its stability. On the other hand, it is interesting to note that Hosking[9] has recently reported a low frequency instability due to the Hall current in a plasma supported against gravity by a magnetic field.

b) No Hall Current

In the limiting case where the Hall current is neglected, $\omega_C \to \infty$ and Eq. (30) reduces to the form given by Gupta[5].

c) High Resistivity Limit.

We now wish to investigate the effect of Hall current in the high resistivity limit. To this end we may write, correct to second order:

$$y = x+\delta-\tfrac{1}{2}\delta^2/x \quad , \tag{31}$$

where

$$\delta = \frac{R^2}{2\eta x}\left(\omega-\frac{2i\Omega^2}{x\omega_c}\right) \tag{32}$$

and we assume that $\delta/x \ll 1$. The dispersion relation then reduces to

$$\omega^2+4NSF(x)\,\frac{I'(x)}{xI_1(x)}\left[\omega - \frac{2i\Omega^2}{x\omega_c}\left(\frac{\rho R^3}{T}\right)^{\frac{1}{2}}\right]$$

$$+\frac{xI_1(x)}{I_o(x)}\,(x^2-1) - 4NG(x) = 0 \ , \tag{33}$$

where we now measure ω in units of $(T/\rho R^3)^{\frac{1}{2}}$ and

$$N = \frac{\rho A^2 R}{T} , \qquad S = \left(\frac{T}{\rho R^3}\right)^{\frac{1}{2}} \frac{R^2}{2\eta} ,$$

$$G(x) = \frac{x}{2}\left[\frac{I_1(x)}{I_o(x)} - \frac{I_2(x)}{I_1(x)}\right] , \tag{34}$$

and

$$F(x) = G(x) - I_2(x)/2I_1'(x) .$$

Eq. (33) is a quadratic in ω^2 whose roots to the lowest significant order are:

$$\omega = \omega_o + i\omega_i , \tag{35}$$

where

$$\omega_o = -2NSF(x) \pm \left[4NG(x) + \frac{xI_1(x)}{I_o(x)}(1-x^2)\right]^{\frac{1}{2}} , \tag{36}$$

and

$$\omega_i = \frac{4NS(\Omega^2/x\omega_c)(\rho R^3/T)^{\frac{1}{2}} F(x)}{\left[4NG(x) + \frac{xI_1(x)}{I_o(x)}(1-x^2)\right]^{\frac{1}{2}}} . \tag{37}$$

It is to be observed here that in the limit when $\omega_c \to \infty$, $\omega_i \to 0$ and $\omega = \omega_o$. The expression for ω_o differs from that given by Gupta[5] in the appearance of F(x) instead of G(x) in his expression in the first term on the left hand side of Eq. (36). This discrepancy is due to the fact that he has not carried out the expansion of the dispersion relation consistently to the second order.

It can be shown that the function F(x) is always positive. When $x < 1$ the quantity under the radical sign in ω_o is always positive and greater than 2NSF(x). Since ω_i is always finite, the positiveness of ω_o implies that we have overstability. Thus the inclusion of Hall current leads to overstability of the jet rather than convective instability which is the case when one neglects Hall current.

For liquid mercury, η=7.5x10^{3} cm^{2}/sec, ρ=13.6 gm/cm^{3} and T=487 dynes/cm. Taking R=0.2 cm we find that S=0.000178. The behaviour of ω_o and ω_i for various values of N are shown in figures 1 and 2. We find that the jet is unstable for all wave numbers x less than x_c and that the instability is maximum for $x=x_*$. The values of x_c, x_* and ω_{o*} are given in table 1 for some values of N.

IV CONCLUSION

We find that the inclusion of Hall current leads to overstability of a liquid jet carrying an axial volume current.

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TABLE 1

Values of x_c, x_* and ω_{o*} for N=1, 2, 3 and 4

N	x_c	x_*	ω_{o*}
1	1.369999	0.941	0.647
2	1.629999	1.109	0.923
3	1.839999	1.239	1.178
4	2.009998	1.345	1.415

FIGURE CAPTIONS

Fig. 1. The behavior of ω_0 as a function of the wave number x.

Fig. 2. The behavior of ω_i/ω_0 as a function of the wave number x.

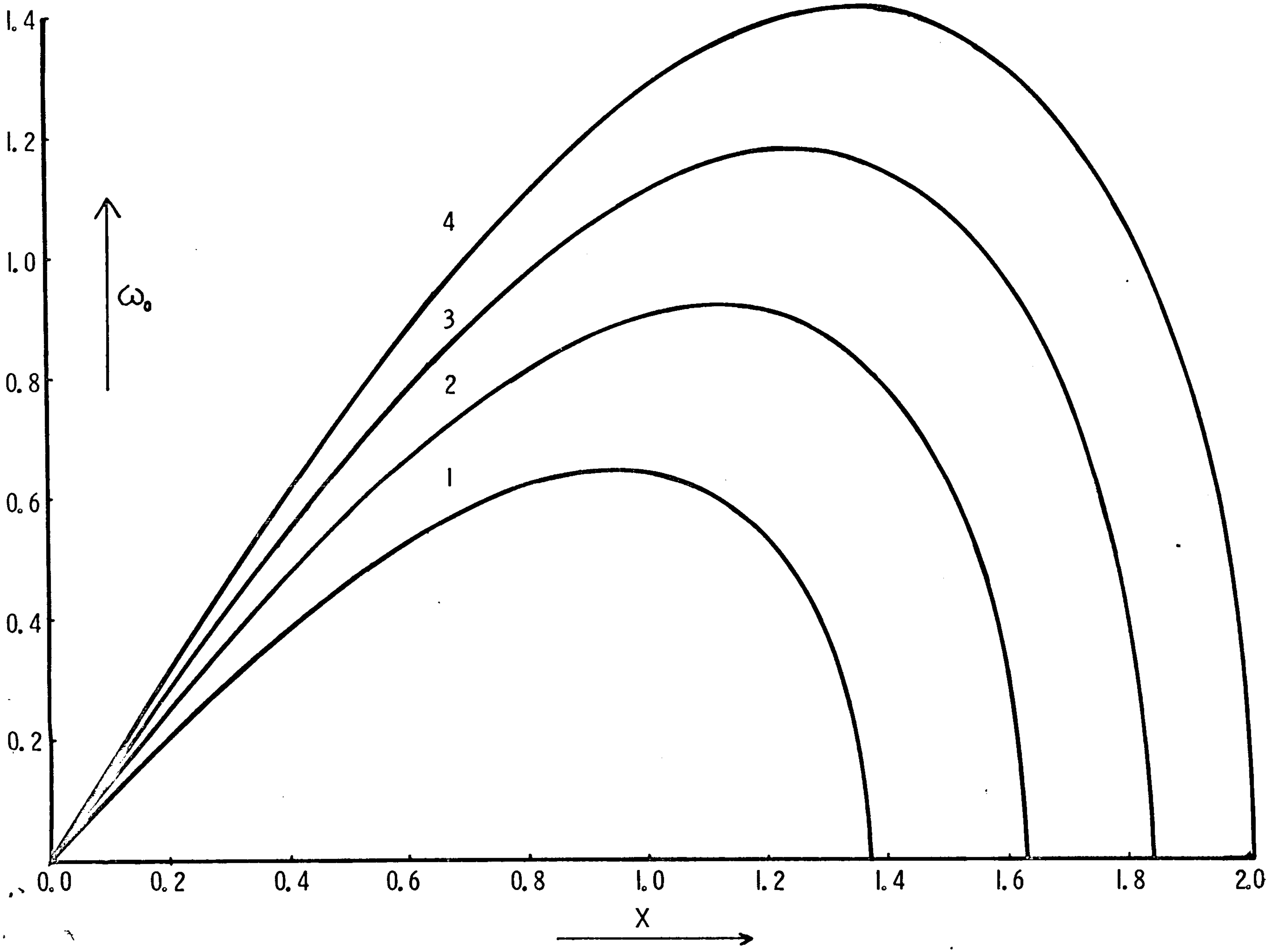
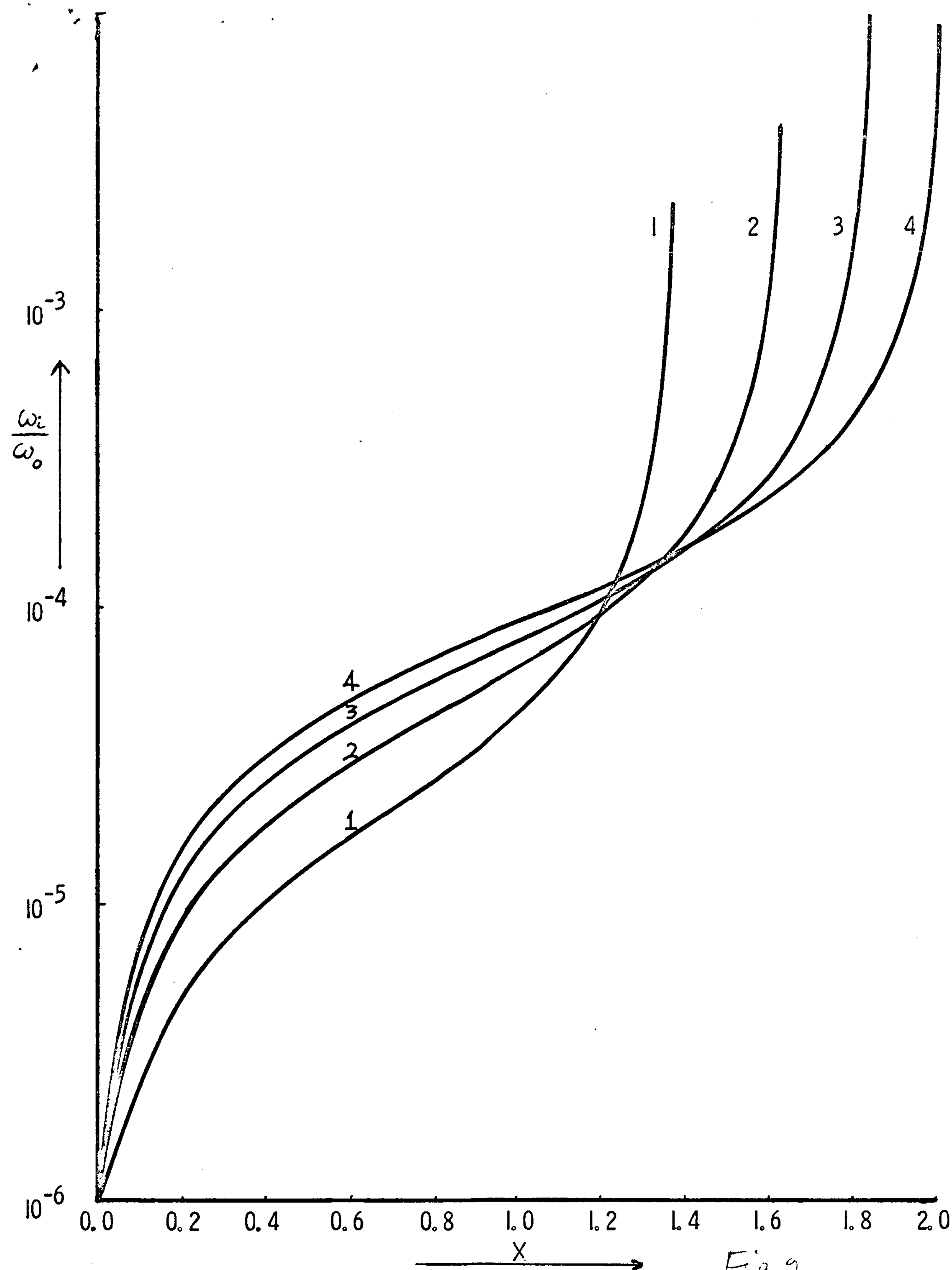


Fig. 1



1
2
3
4
10^{-3}
$\frac{\omega_i}{\omega_o}$
10^{-4}
4
3
2
1
10^{-5}
10^{-6}
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
X

Fig 2.