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DIFFERENTIAL CORRECTIONS APPLIED TO VINTI'S ACCURATE REFERENCE SATELLITE ORBIT WITH INCLUSION OF THE THIRD ZONAL HARMONIC

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ABSTRACT

A differential method of orbit improvement utilizing observational data is presented for Vinti's spheroidal solution of the dynamical problem of unretarded artificial satellite motion about an oblate planet, recently modified so as to permit the exact inclusion of the effects of the third zonal harmonic term of the planet's gravitational potential field. The first-order Taylor's series expansion used for the equations of condition is fitted to observational values by an iterated least-squares process, producing successively smaller corrections to the orbital elements. A mean set of elements, conditioned by the observations, results for use in orbital predictions for intermediate time points or for later epochs. The differential coefficients in the conditional equations, applicable to any type of observational data, are derived analytically from the equations of the accurate reference orbit. The method of differential correction may be used for orbits of all inclinations, including equatorial and polar, and both a first-order and a lengthier, but more exact, second-order treatment for the periodic variables are given.

CONTENTS

INTRODUCTION

In order to predict the precise position, at a given time, of a satellite revolving about a planet, an extensive mathematical theory of the satellite's motion and the exact values of certain physical parameters (e.g., gravitational constants) are required. If the position of an artificial satellite of the Earth is to be determined relative to an observer on the Earth's surface, then, in addition, the accurate geodetic position of the observer is necessary. Any mathematical theory of motion is based upon certain constants of the motion which initially must be determined empirically. Once approximate values for these constants of the motion are available, then they may be utilized in the theory to predict future orbital positions of the satellite. A comparison between the positions predicted by the theory and those actually observed will indicate discrepancies, expressed numerically as the differences between the observed positions and the respective computed positions. These are known as observational residuals, and the fact that they are non-zero is due to several causes. One fundamental cause which is, unfortunately, ever-present is the inadequacy of the theory: that is, the inability to account mathematically for all the physical forces acting upon the satellite. Moreover, there are always errors associated with the observations themselves, because of fluctuations in the atmospheric density, optical imperfections in the telescope, inaccuracies in the reduction of the observational recordings, and the like. Further, the physical parameters required by the theory often are not known with sufficient precision, and the geodetic position of the observer may likewise be measured inaccurately. Finally, there are errors introduced by the approximate values for the constants of the motion used in computing the theoretical position.

Despite all the inadequacies and short-comings enumerated above, the very existence of the discrepancies between theory and observation provides a means of improving the approximate values of the constants of the motion. One procedure for improving these constants in an analytic theory of motion involves expressing the incremental changes in the positional co-ordinates due to changes in the constants of the motion as coefficients having the form of partial derivatives in a Taylor's series expansion. In such a case, the partial derivatives must be determined as explicit functions of variables which arise in the analytic theory of motion. This method of orbit improvement is known as differential correction.

The purpose of the present paper is to provide such an orbit improvement method for the spheroidal theory of artificial satellite motion. The spheroidal method for satellite orbits, developed by Vinti (References 2, 3, 4, 5, and 6), supplies a procedure for calculating an accurate reference orbit of any drag-free satellite moving in the gravitational field of an axially symmetric oblate planet. In the case of artificial satellites of the Earth, the intermediary reference orbit reproduces exactly the zeroth and second zonal harmonic coefficients in the series expansion of the geopotential function, and also accounts for more than half of the fourth zonal harmonic. Recently, Vinti (References 7 and 8) has modified the spheroidal potential so as to allow exact inclusion of the effects of the third zonal harmonic as well, heretofore the major neglected effect in the spheroidal orbital theory. (The first zonal harmonic is entirely eliminated, of course, by proper choice of co-ordinate origin.) Accounting for the third zonal harmonic directly in the intermediary orbit in this manner affords a more accurate treatment (Reference 1) than would be possible through perturbation theory. This paper will present a differential method of orbit improvement based upon the modified spheroidal theory that includes the effects of the third zonal harmonic term.

FUNDAMENTAL EQUATIONS FOR DIFFERENTIAL CORRECTION

Consider a system of constants of the motion, q_i ($i = 1, 2, \cdots, n$), which are utilized in a mathematical theory of motion to predict orbital positions of a satellite. In this context, such constants of the motion are generally referred to as orbital elements, and the number, n, contained in the system is often six. Denote a positional co-ordinate of the satellite at time t by $R(t)$, using subscripts "o" and "c" to distinguish between observed values and values computed by the analytical theory using the orbital elements. We assume that the differences of the co-ordinates, $R_o(t) - R_c(t)$, as well as the required corrections to the elements in the improvement of the orbit, are sufficiently small so that their squares and higher powers may be neglected. We can then express the observational residuals by a truncated Taylor's series expansion restricted to first powers, in the general form:

$$R_o(t) - R_c(t) = \sum_{i=1}^{n} \frac{\partial R_c(t)}{\partial q_i} \Delta q_i \ .$$

Here the computed value, $R_c(t)$, is viewed as a function of n independent variables, q_i ($i = 1, 2, \cdots, n$), which are to be improved by the additive increments, Δq_i ($i = 1, 2, \cdots, n$). The coefficients in the Taylor's expansion expressing the increment of the co-ordinates caused by a change in the orbital elements have the form of partial derivatives, and these must be determined analytically from the equations in the mathematical theory of motion. Each separately observed co-ordinate yields an equation for corrections of the elements of the form given above. Ordinarily, the number of such so-called equations of condition available far exceeds the number, n, of unknowns, Δq_i. This set of linear simultaneous equations forms an inconsistent system (due to inherent random and possibly systematic errors in the observations) for which no exact solution exists. It may be solved by the method of least squares, yielding "preferred" values for the unknowns. The solution of the equations by the principle of least squares follows well-known schemes (Reference 9), so that the problem reduces to evaluating the derivatives of the co-ordinates with respect to the elements (the so-called differential coefficients).

We now introduce certain conventions in order to allow discussion of the problem in more explicit terms. Assume that the satellite's positional data are recorded at the tracking stations in the form of direction cosines observed with respect to a topocentric (i.e., situated at the Earth's surface) co-ordinate system. The topocentric or "local" co-ordinates shall be distinguished by the subscript, "M". The system is orthogonal and right-handed, with the $X_M - Y_M$ plane tangent to the Earth's surface. The X_M-axis extends in an easterly direction along the line of latitude, the Y_M-axis extends in a northerly direction along the line of longitude, and the Z_M-axis is normal to the surface and points toward the geodetic zenith. If L_o and M_o denote the observed direction cosines of a satellite (in the X_M- and Y_M-directions, respectively) for a given time of observation, then the corresponding computed values of the direction cosines are given in terms of the local co-ordinates by

$$L_c = \frac{X_M}{\left(X_M^2 + Y_M^2 + Z_M^2\right)^{1/2}},$$

and

$$M_c = \frac{Y_M}{\left(X_M^2 + Y_M^2 + Z_M^2\right)^{1/2}}.$$

The computed value of the third direction cosine, N_c (in the Z_M-direction), is not an independent parameter, but is pre-determined by L_c and M_c through the relation,

$$N_c = \left(1 - L_c^2 - M_c^2\right)^{1/2}.$$

For this reason, each satellite observation provides two, and only two, independent co-ordinate values, which here are chosen to be L_o and M_o.

The mathematical theory of motion will ordinarily predict a satellite's position with respect to a rectangular geocentric system (X, Y, Z) of co-ordinates, the so-called inertial frame of reference. In this system, the X - Y plane is the Earth's equatorial plane, and the origin is situated at the Earth's center of mass. The X-axis extends toward the vernal equinox (the first point of Aries), the Y-axis is orthogonally to the east to form a right-handed system, and the Z-axis coincides with the Earth's polar axis. In order to obtain a satellite's local co-ordinates from its inertial co-ordinates, the inertial co-ordinates of the observation point at the time of observation, which we shall denote $\left(X_T, Y_T, Z_T\right)$, must be known and two rotations performed to bring the local and inertial systems into parallel alignment. Let us denote by ψ_x the angle between the vernal equinox and the X_M-axis as measured in the observation latitude plane. Then ψ_x will depend upon the longitude of the tracking station and the hour angle of the vernal equinox at the time of observation. If θ_D represents the latitude of the tracking station, then the local co-ordinates of the satellite at observation time are given by the matrix equation,

$$\begin{bmatrix} X_M \\ Y_M \\ Z_M \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta_D & \cos\theta_D \\ 0 & -\cos\theta_D & \sin\theta_D \end{bmatrix} \begin{bmatrix} \cos\psi_x & \sin\psi_x & 0 \\ -\sin\psi_x & \cos\psi_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} \right\}.$$

The difference of column matrices on the extreme right represents a translation from the Earth's center to the tracking station position; the center matrix on the right represents a rotation in the latitude plane about the polar axis through an angle, ψ_x, to bring the inertial X-axis into coincidence with the station's X_M-axis; the remaining matrix on the right represents a rotation in the longitude plane about the X_M-axis through an angle equal to the complement of θ_D to bring the inertial Z-axis into coincidence with the station's Z_M-axis. This matrix equation, when expanded, reads

$$X_M = (X - X_T)\cos\psi_x + (Y - Y_T)\sin\psi_x ,$$

$$Y_M = -(X - X_T)\sin\psi_x \sin\theta_D + (Y - Y_T)\cos\psi_x \sin\theta_D + (Z - Z_T)\cos\theta_D ,$$

and

$$Z_M = (X - X_T)\sin\psi_x \cos\theta_D - (Y - Y_T)\cos\psi_x \cos\theta_D + (Z - Z_T)\sin\theta_D .$$

We can now write the first-order Taylor's series expansion for the equations of condition corresponding to each time of observation in the following more explicit form:

$$L_o - L_c = \sum_{i=1}^{6} \frac{\partial L_c}{\partial q_i} \Delta q_i ,$$

and

$$M_o - M_c = \sum_{i=1}^{6} \frac{\partial M_c}{\partial q_i} \Delta q_i .$$

Since the local co-ordinates are functions of the inertial co-ordinates, which are in turn functions of the orbital elements, then the chain rule may be used to expand the differential coefficients.

$$\frac{\partial L_c}{\partial q_i} = \frac{\partial L_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial L_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial L_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i} ,$$

and

$$\frac{\partial M_c}{\partial q_i} = \frac{\partial M_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial M_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial M_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i} .$$

The partial derivatives of the direction cosines with respect to the local co-ordinates are found directly from the expressions for L_c and M_c.

$$\frac{\partial L_c}{\partial X_M} = \frac{1}{R_M} - \frac{X_M^2}{R_M^3} ,$$

$$\frac{\partial L_c}{\partial Y_M} = - \frac{X_M Y_M}{R_M^3} ,$$

$$\frac{\partial L_c}{\partial Z_M} = - \frac{X_M Z_M}{R_M^3} ,$$

$$\frac{\partial M_c}{\partial X_M} = - \frac{X_M Y_M}{R_M^3} = \frac{\partial L_c}{\partial Y_M} ,$$

$$\frac{\partial M_c}{\partial Y_M} = \frac{1}{R_M} - \frac{Y_M^2}{R_M^3} ,$$

and

$$\frac{\partial M_c}{\partial Z_M} = - \frac{Y_M Z_M}{R_M^3} ,$$

where

$$R_M \equiv \left(X_M^2 + Y_M^2 + Z_M^2 \right)^{1/2} .$$

Since the co-ordinates, X_T, Y_T, and Z_T, and the angles, ψ_x and θ_D, are independent of the orbital elements (and merely geodesic functions), we have the matrix equation,

$$\begin{bmatrix} \frac{\partial X_M}{\partial q_i} \\ \frac{\partial Y_M}{\partial q_i} \\ \frac{\partial Z_M}{\partial q_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta_D & \cos\theta_D \\ 0 & -\cos\theta_D & \sin\theta_D \end{bmatrix} \begin{bmatrix} \cos\psi_x & \sin\psi_x & 0 \\ -\sin\psi_x & \cos\psi_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial q_i} \\ \frac{\partial Y}{\partial q_i} \\ \frac{\partial Z}{\partial q_i} \end{bmatrix} .$$

Equivalently,

$$\frac{\partial X_M}{\partial q_i} = \cos\psi_x \frac{\partial X}{\partial q_i} + \sin\psi_x \frac{\partial Y}{\partial q_i} ,$$

$$\frac{\partial Y_M}{\partial q_i} = -\sin\psi_x \sin\theta_D \frac{\partial X}{\partial q_i} + \cos\psi_x \sin\theta_D \frac{\partial Y}{\partial q_i} + \cos\theta_D \frac{\partial Z}{\partial q_i} ,$$

and

$$\frac{\partial Z_M}{\partial q_i} = \sin\psi_x \cos\theta_D \frac{\partial X}{\partial q_i} - \cos\psi_x \cos\theta_D \frac{\partial Y}{\partial q_i} + \sin\theta_D \frac{\partial Z}{\partial q_i} .$$

The problem remains to evaluate the differential coefficients, $\partial X/\partial q_i$, $\partial Y/\partial q_i$, and $\partial Z/\partial q_i$ ($i = 1, 2, \ldots, 6$), which are the derivatives of the inertial co-ordinates with respect to the orbital elements. At this point, the mathematical theory of motion becomes of primary importance.

OUTLINE OF METHOD USED TO EVALUATE DIFFERENTIAL COEFFICIENTS

The constants of the motion, q_i ($i = 1, 2, \ldots, 6$), which we choose for the mean orbital elements for the modified spheroidal theory of satellite motion (including the exact effects of the third zonal harmonic coefficient of the oblate planet's gravitational field) are the following:

$q_1 = a$, the semi-major axis.

$q_2 = e$, the eccentricity.

$q_3 = S$, corresponding to $\sin^2 I$ in Keplerian motion, where I is the inclination of the orbital plane to the equator. (However, S may be negative for orbits sufficiently close to equatorial. For a polar orbit, S is unity, and in all cases, $S^2 \leq 1$.)

$q_4 = \beta_1$, corresponding to the negative of the time of passage through pericenter in Keplerian motion.

$q_5 = \beta_2$, corresponding to the argument of pericenter in Keplerian motion.

$q_6 = \beta_3$, corresponding to the right ascension of the ascending node in Keplerian motion.

These elements differ slightly from those selected by Vinti in the final algorithm for the reference orbit (Reference 8, Section 12), but the above parameters seem more suitable for the differential correction.

The relation between rectangular inertial co-ordinates and oblate spheroidal co-ordinates, ρ, η, ϕ, which are involved in the solution of the problem of satellite motion, is given by

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \cos\phi ,$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \sin\phi ,$$

and

$$Z = \rho\eta - \delta , \qquad (-1 \leq \eta \leq 1) .$$

Here c and δ are adjustable parameters that are chosen to agree with the coefficients of the zonal harmonics in the series expansion of the Earth's potential function. In terms of the Earth's equatorial radius, r_e, and the n^{th} zonal harmonic coefficient, J_n, the proper choices are

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right) ,$$

and

$$\delta = -\frac{1}{2} r_e J_2^{-1} J_3 .$$

For the Earth, the values are approximately $c \cong 210$ km and $\delta \cong 7$ km.

In order to evaluate the derivatives of the inertial co-ordinates with respect to the orbital elements, we must know $\partial\rho/\partial q_i$, $\partial\eta/\partial q_i$, and $\partial\phi/\partial q_i$ $(i = 1, 2, \cdots, 6)$. The oblate spheroidal co-ordinates are rather involved functions of the orbital elements. The process for determining the partial derivatives of ρ, η, and ϕ is a lengthy one which will be presented in a synthetic, rather than analytic, manner. That is, necessary partial derivatives of the simpler functions of the orbital elements will be given first, followed by partial derivatives of more complicated functions of the elements involving the pre-determined partial derivatives.

It is worth noting in passing that the equations to be presented herein apply to orbits of all inclinations, including equatorial. There are no special simplifications introduced in the case of equatorial or near-equatorial inclinations, as there were for the spheroidal theory that did not include the effects of the third zonal harmonic (Reference 6).

However, as in the earlier spheroidal theory, the differential correction may include derivatives of the periodic terms taken through the second order, or, alternatively, it may be simplified to omit periodic terms higher than first order. In either case, some second-order effects are included with the first-order terms and even with the zeroth-order terms. The terms of second and higher orders that are to be omitted in the simplified version will be indicated as such.

TIME-INDEPENDENT PARTIAL DERIVATIVES IN THE DIFFERENTIAL CORRECTION

We begin with equation (23.3),* which reads:

$$p = a\left(1 - e^2\right) .$$

Thus:

$$\frac{\partial p}{\partial a} = 1 - e^2 ,$$

$$\frac{\partial p}{\partial e} = -2ae ,$$

and

$$\frac{\partial p}{\partial S} = 0 .$$

Since

$$b_1 = -\frac{1}{2} A ,$$

by line 8 of section 12 of Reference 8, then by equation (40.1), we find:

$$b_1 = \frac{ac^2\left(ap - c^2 S\right)(1 - S) - 4a^2c^2\delta^2 p^{-1} S(1 - S)\left[1 + c^2(ap)^{-1}(3S - 2)\right]}{\left(ap - c^2\right)\left(ap - c^2 S\right) + 4a^2c^2 S + 4c^2\delta^2 p^{-2} S(1 - S)\left(3ap - 4a^2 - c^2\right)} .$$

If we denote the numerator and denominator of b_1 by N and D respectively, then we have:

$$\frac{\partial b_1}{\partial a} = -c^2(1 - S)D^{-1}\left\{c^2 S - a\left(2p + a\frac{\partial p}{\partial a}\right) + 4\delta^2 S\left[\frac{a}{p}\left(2 - \frac{a}{p}\frac{\partial p}{\partial a}\right) + \left(\frac{c}{p}\right)^2(3S - 2)\left(1 - 2\frac{a}{p}\frac{\partial p}{\partial a}\right)\right]\right\}$$

$$- ND^{-2}\left\{\left(2ap - c^2 - c^2 S\right)\left(p + a\frac{\partial p}{\partial a}\right) + 8ac^2 S\right.$$

$$\left. + 4\left(\frac{c}{p}\right)^2\delta^2 S(1 - S)\left[3p - 8a + \left(8\frac{a^2}{p} + 2\frac{c^2}{p} - 3a\right)\frac{\partial p}{\partial a}\right]\right\} ,$$

*All equation numbers, used in specifying the defining relation for a given variable, will refer, until otherwise indicated, to Reference 8.

$$\frac{\partial b_1}{\partial e} = a^2 c^2 D^{-1} (1 - S) \frac{\partial p}{\partial e} \left\{1 + 4\left(\frac{\delta}{p}\right)^2 S\left[1 + 2 \frac{c^2}{ap} (3S - 2)\right]\right\} - ND^{-2} \frac{\partial p}{\partial e} \left[a\left(2\,ap - c^2 - c^2 S\right)\right.$$

$$\left. + 4\left(\frac{c}{p}\right)^2 \delta^2 S(1 - S) \left(8 \frac{a^2}{p} + 2 \frac{c^2}{p} - 3\,a\right)\right] ,$$

and

$$\frac{\partial b_1}{\partial S} = - ac^2 D^{-1} \left[ap + (1 - 2S) \left(4 \frac{a}{p} \delta^2 + c^2\right) - 4\left(\frac{c}{p}\right)^2 \delta^2 \left(9S^2 - 10S + 2\right)\right]$$

$$- ND^{-2} \left[c^2 \left(4\,a^2 - ap + c^2\right) + 4\left(\frac{c}{p}\right)^2 \delta^2 (1 - 2S) \left(3\,ap - 4\,a^2 - c^2\right)\right] .$$

Since

$$b_2 = \sqrt{B} ,$$

by line 8 of section 12 of Reference 8, then by equation (40.2), we find:

$$b_2^2 = c^2 - b_1 a^{-1} \left(ap - c^2\right) .$$

Thus:

$$\frac{\partial b_2}{\partial a} = \frac{1}{2} \left(ab_2\right)^{-1} \left[\left(ap - c^2\right) \left(b_1\, a^{-1} - \frac{\partial b_1}{\partial a}\right) - b_1 \left(p + a \frac{\partial p}{\partial a}\right)\right] ,$$

$$\frac{\partial b_2}{\partial e} = - \frac{1}{2} b_2^{-1} \left[a^{-1} \left(ap - c^2\right) \frac{\partial b_1}{\partial e} + b_1 \frac{\partial p}{\partial e}\right] ,$$

and

$$\frac{\partial b_2}{\partial S} = - \frac{1}{2} \left(ab_2\right)^{-1} \left(ap - c^2\right) \frac{\partial b_1}{\partial S} .$$

This last sequence of three equations can be written as a single generalized equation if we introduce the following notation. Let $q_1 = a$, $q_2 = e$, and $q_3 = S$, and let δ_{ij} be the Kronecker delta, defined as follows: $\delta_{ij} = 1$ when $i = j$, and $\delta_{ij} = 0$ whenever $i \neq j$. Then, for $i = 1, 2, 3$,

$$\frac{\partial b_2}{\partial q_i} = \frac{1}{2} \left(ab_2\right)^{-1} \left[\left(ap - c^2\right) \left(b_1\, a^{-1} \delta_{1i} - \frac{\partial b_1}{\partial q_i}\right) - b_1 \left(p\, \delta_{1i} + a \frac{\partial p}{\partial q_i}\right)\right] .$$

This notation will be used extensively in order to write later sequences of equations in an efficient manner.

Combining equations (35.1) and (35.2) with equations already mentioned, we find:

$$p_0 = (a + b_1)^{-1} (b_2^2 + ap + 4\, ab_1 - c^2) \; .$$

Thus:

$$\frac{\partial p_0}{\partial a} = (a + b_1)^{-1} \left[2 b_2 \frac{\partial b_2}{\partial a} + p + 4 b_1 + a\left(4 \frac{\partial b_1}{\partial a} + \frac{\partial p}{\partial a}\right) - p_0 \left(1 + \frac{\partial b_1}{\partial a}\right)\right] ,$$

$$\frac{\partial p_0}{\partial e} = (a + b_1)^{-1} \left[2 b_2 \frac{\partial b_2}{\partial e} + a\left(4 \frac{\partial b_1}{\partial e} + \frac{\partial p}{\partial e}\right) - p_0 \frac{\partial b_1}{\partial e}\right] ,$$

and

$$\frac{\partial p_0}{\partial S} = (a + b_1)^{-1} \left[2 b_2 \frac{\partial b_2}{\partial S} + (4 a - p_0) \frac{\partial b_1}{\partial S}\right] .$$

This may be written in the generalized form, for $i = 1, 2, 3$,

$$\frac{\partial p_0}{\partial q_i} = (a + b_1)^{-1} \left[2 b_2 \frac{\partial b_2}{\partial q_i} + (4 b_1 + p)\, \delta_{1i} + a\left(4 \frac{\partial b_1}{\partial q_i} + \frac{\partial p}{\partial q_i}\right) - p_0 \left(\delta_{1i} + \frac{\partial b_1}{\partial q_i}\right)\right] .$$

From equation (21.2), we have:

$$\alpha_2 = \sqrt{\mu\, p_0} \; .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial \alpha_2}{\partial q_i} = \frac{\mu}{2\alpha_2} \frac{\partial p_0}{\partial q_i} \; .$$

For convenience in later equations, we now differentiate the inverse of

$$a_0\, p_0 = (a + b_1)\, p_0$$

to find, for $i = 1, 2, 3,$

$$\frac{\partial}{\partial q_i} \left(a_0 p_0\right)^{-1} = - p_0^{-1} \left(a + b_1\right)^{-1} \left[\left(a + b_1\right)^{-1} \left(\delta_{1i} + \frac{\partial b_1}{\partial q_i}\right) + p_0^{-1} \frac{\partial p_0}{\partial q_i}\right] .$$

From equations (31.1) and (31.2), neglecting terms of fourth order, we find:

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 p_0} (1 - S) + \frac{\left(\frac{2\delta}{p_0}\right)^2 (1 - S) \left(1 - \frac{c^2}{a_0 p_0} S\right)}{\left[1 + \frac{c^2}{a_0 p_0} (1 - 2S)\right]^2} .$$

Now, for $i = 1, 2$ only, we have:

$$\frac{\partial u}{\partial q_i} = - u^2 (1 - S) \left\{c^2 \frac{\partial}{\partial q_i} \left(a_0 p_0\right)^{-1} - \left(\frac{2\delta}{p_0}\right)^2 \left[\frac{2}{p_0} \left(1 - \frac{c^2}{a_0 p_0} S\right) \frac{\partial p_0}{\partial q_i} + c^2 S \frac{\partial}{\partial q_i} \left(a_0 p_0\right)^{-1}\right]\left[1 + \frac{c^2}{a_0 p_0} (1 - 2S)\right]^{-2} - 2c^2 \left(\frac{2\delta}{p_0}\right)^2 (1 - 2S) \left(1 - \frac{c^2}{a_0 p_0} S\right)\left[1 + \frac{c^2}{a_0 p_0} (1 - 2S)\right]^{-3} \frac{\partial}{\partial q_i} \left(a_0 p_0\right)^{-1}\right\} .$$

However,

$$\frac{\partial u}{\partial S} = - u^2 \left\{c^2 \left[(1 - S) \frac{\partial}{\partial S} \left(a_0 p_0\right)^{-1} - \left(a_0 p_0\right)^{-1}\right] - \left(\frac{2\delta}{p_0}\right)^2 \left[\left(1 - \frac{c^2}{a_0 p_0} S\right)\left(1 + \frac{2}{p_0} (1 - S) \frac{\partial p_0}{\partial S}\right) + c^2 (1 - S) \left(S \frac{\partial}{\partial S} \left(a_0 p_0\right)^{-1} + \left(a_0 p_0\right)^{-1}\right)\right]\left[1 + \frac{c^2}{a_0 p_0} (1 - 2S)\right]^{-2} - 2c^2 \left(\frac{2\delta}{p_0}\right)^2 (1 - S) \left(1 - \frac{c^2}{a_0 p_0} S\right)\left[1 + \frac{c^2}{a_0 p_0} (1 - 2S)\right]^{-3} \left[(1 - 2S) \frac{\partial}{\partial S} \left(a_0 p_0\right)^{-1} - 2\left(a_0 p_0\right)^{-1}\right]\right\} .$$

By equation (32.1),

$$C_2 = \frac{c^2}{a_0 p_0} u .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial C_2}{\partial q_i} = c^2 u \frac{\partial}{\partial q_i} \left(a_0 p_0\right)^{-1} + \frac{c^2}{a_0 p_0} \frac{\partial u}{\partial q_i} .$$

By combining equations (32.1) and (32.2), we have:

$$C_1 = \frac{2\delta}{p_0} u(1 - C_2)(1 - C_2 S)^{-1} .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial C_1}{\partial q_i} = \frac{2\delta}{p_0} (1 - C_2 S)^{-1} \left\{ (1 - C_2)\left(\frac{\partial u}{\partial q_i} - \frac{u}{p_0} \frac{\partial p_0}{\partial q_i}\right) - u(1 - C_2 S)^{-1} \left[(1 - S) \frac{\partial C_2}{\partial q_i} - \delta_{3i} (1 - C_2) C_2 \right] \right\} .$$

By combining equations (32.1) and (32.3), we have:

$$P = \frac{\delta}{p_0} u(1 - S)(1 - C_2 S)^{-1} .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial P}{\partial q_i} = P\left[(1 - C_2 S)^{-1} \left(\delta_{3i} C_2 + S \frac{\partial C_2}{\partial q_i} \right) - p_0^{-1} \frac{\partial p_0}{\partial q_i} + u^{-1} \frac{\partial u}{\partial q_i} \right] - \delta_{3i} (1 - C_2 S)^{-1} \frac{\delta}{p_0} u .$$

Equation (21.1) may be rewritten as

$$\alpha_1 = -\frac{1}{2} \mu (a + b_1)^{-1} .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial \alpha_1}{\partial q_i} = \frac{1}{2} \mu (a + b_1)^{-2} \left(\delta_{1i} + \frac{\partial b_1}{\partial q_i} \right) .$$

For convenience in later equations, we now evaluate, for $i = 1, 2, 3$,

$$\frac{\partial}{\partial q_i}\left(\frac{b_1}{b_2}\right) = b_2^{-1} \frac{\partial b_1}{\partial q_i} - b_1 b_2^{-2} \frac{\partial b_2}{\partial q_i} ,$$

and

$$\frac{\partial}{\partial q_i}\left(\frac{b_2}{p}\right) = p^{-1} \frac{\partial b_2}{\partial q_i} - b_2 p^{-2} \frac{\partial p}{\partial q_i} .$$

By equation (5.31),* we have:

$$A_1 = (1-e^2)^{1/2} p \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1-e^2)^{1/2}\right] .$$

Here, and in the following, $P_n(x)$ is the Legendre polynomial with argument x of degree n. The definition of R is given by equation (5.28), viz., $R_n(x) \equiv x^n P_n(1/x)$, where $0 < x \leq 1$. We shall denote by $P_n'(x)$ the derivative of the Legendre polynomial with respect to the argument. Then, for $i = 1, 2, 3,$

$$\frac{\partial A_1}{\partial q_i} = A_1 p^{-1} \frac{\partial p}{\partial q_i} - \delta_{2i} A_1 e (1-e^2)^{-1} + p(1-e^2)^{1/2} \left\{ \frac{\partial}{\partial q_i}\left(\frac{b_2}{p}\right) \sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1-e^2)^{1/2}\right] \right.$$

$$\left. + \frac{\partial}{\partial q_i}\left(\frac{b_1}{b_2}\right) \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n'\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1-e^2)^{1/2}\right] \right\}$$

$$+ \delta_{2i}\, pe \left\{ (1-e^2)^{-1} \sum_{n=3}^{\infty} \left(\frac{b_2}{p}\right)^n \left[(1-e^2)^{1/2}\right]^{n-2} P_n\left(\frac{b_1}{b_2}\right) P_{n-2}'\left[(1-e^2)^{-1/2}\right] \right.$$

$$\left. - \sum_{n=3}^{\infty} (n-2) \left(\frac{b_2}{p}\right)^n \left[(1-e^2)^{1/2}\right]^{n-3} P_n\left(\frac{b_1}{b_2}\right) P_{n-2}\left[(1-e^2)^{-1/2}\right] \right\} .$$

By equation (5.36),

$$A_2 = (1-e^2)^{1/2} p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right] ,$$

*Equation numbers used for the defining relations will now refer, until otherwise indicated, to Reference 4.

so that, for $i = 1, 2, 3,$

$$\frac{\partial A_2}{\partial q_i} = -A_2 p^{-1} \frac{\partial p}{\partial q_i} - \delta_{2i} A_2 e\left(1-e^2\right)^{-1} + \left(1-e^2\right)^{1/2} p^{-1}\left\{\frac{\partial}{\partial q_i}\left(\frac{b_2}{p}\right) \sum_{n=1}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} P_n\left(\frac{b_1}{b_2}\right) R_n\left[\left(1-e^2\right)^{1/2}\right]\right.$$

$$\left. + \frac{\partial}{\partial q_i}\left(\frac{b_1}{b_2}\right) \sum_{n=1}^{\infty}\left(\frac{b_2}{p}\right)^n P_n'\left(\frac{b_1}{b_2}\right) R_n\left[\left(1-e^2\right)^{1/2}\right]\right\}$$

$$+ \delta_{2i}\, p^{-1} e\left\{\left(1-e^2\right)^{-1} \sum_{n=1}^{\infty}\left(\frac{b_2}{p}\right)^n \left[\left(1-e^2\right)^{1/2}\right]^n P_n\left(\frac{b_1}{b_2}\right) P_n'\left[\left(1-e^2\right)^{-1/2}\right]\right.$$

$$\left. - \sum_{n=1}^{\infty} n\left(\frac{b_2}{p}\right)^n \left[\left(1-e^2\right)^{1/2}\right]^{n-1} P_n\left(\frac{b_1}{b_2}\right) P_n\left[\left(1-e^2\right)^{-1/2}\right]\right\}.$$

By equations (5.61), (5.50), and (5.53), we have:

$$A_3 = \left(1-e^2\right)^{1/2} p^{-3} \sum_{n=0}^{\infty} D_n R_{n+2}\left[\left(1-e^2\right)^{1/2}\right],$$

where D_n is computed as follows:

$$D_n = D_{2j} = \sum_{m=0}^{j} (-1)^{j-m} \left(\frac{c}{p}\right)^{2j-2m} \left(\frac{b_2}{p}\right)^{2m} P_{2m}\left(\frac{b_1}{b_2}\right)$$

(n an even integer), and

$$D_n = D_{2j+1} = \sum_{m=0}^{j} (-1)^{j-m} \left(\frac{c}{p}\right)^{2j-2m} \left(\frac{b_2}{p}\right)^{2m+1} P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

(n an odd integer). Then, for $i = 1, 2, 3,$

$$\frac{\partial A_3}{\partial q_i} = -A_3\left[3p^{-1} \frac{\partial p}{\partial q_i} + \delta_{2i}\, e\left(1-e^2\right)^{-1}\right] + \left(1-e^2\right)^{1/2} p^{-3} \sum_{n=1}^{\infty} R_{n+2}\left[\left(1-e^2\right)^{1/2}\right] \frac{\partial D_n}{\partial q_i}$$

$$+ \delta_{2i}\, ep^{-3}\left\{\left(1-e^2\right)^{-1} \sum_{n=0}^{\infty}\left[\left(1-e^2\right)^{1/2}\right]^{n+2} D_n P_{n+2}'\left[\left(1-e^2\right)^{-1/2}\right]\right.$$

$$\left. - \sum_{n=0}^{\infty} (n+2)\left[\left(1-e^2\right)^{1/2}\right]^{n+1} D_n P_{n+2}\left[\left(1-e^2\right)^{-1/2}\right]\right\},$$

where $\partial D_n / \partial q_i$ is computed as follows. If n is an even integer, then

$$\frac{\partial D_n}{\partial q_i} = \frac{\partial D_{2j}}{\partial q_i} = -2cp^{-2}\frac{\partial p}{\partial q_i}\sum_{m=0}^{j}(-1)^{j-m}(j-m)\left(\frac{c}{p}\right)^{2(j-m)-1}\left(\frac{b_2}{p}\right)^{2m} P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+ 2\frac{\partial}{\partial q_i}\left(\frac{b_2}{p}\right)\sum_{m=0}^{j}(-1)^{j-m} m\left(\frac{c}{p}\right)^{2(j-m)}\left(\frac{b_2}{p}\right)^{2m-1} P_{2m}\left(\frac{b_1}{b_2}\right)$$

$$+ \frac{\partial}{\partial q_i}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{j}(-1)^{j-m}\left(\frac{c}{p}\right)^{2(j-m)}\left(\frac{b_2}{p}\right)^{2m} P'_{2m}\left(\frac{b_1}{b_2}\right).$$

If n is an odd integer, then

$$\frac{\partial D_n}{\partial q_i} = \frac{\partial D_{2j+1}}{\partial q_i} = -2cp^{-2}\frac{\partial p}{\partial q_i}\sum_{m=0}^{j}(-1)^{j-m}(j-m)\left(\frac{c}{p}\right)^{2(j-m)-1}\left(\frac{b_2}{p}\right)^{2m+1} P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+ \frac{\partial}{\partial q_i}\left(\frac{b_2}{p}\right)\sum_{m=0}^{j}(-1)^{j-m}(2m+1)\left(\frac{c}{p}\right)^{2(j-m)}\left(\frac{b_2}{p}\right)^{2m} P_{2m+1}\left(\frac{b_1}{b_2}\right)$$

$$+ \frac{\partial}{\partial q_i}\left(\frac{b_1}{b_2}\right)\sum_{m=0}^{j}(-1)^{j-m}\left(\frac{c}{p}\right)^{2(j-m)}\left(\frac{b_2}{p}\right)^{2m+1} P'_{2m+1}\left(\frac{b_1}{b_2}\right).$$

By equation (5.37), we have:

$$A_{21} = \left(1-e^2\right)^{1/2} p^{-2} e\left[b_1 + \left(3b_1^2 - b_2^2\right)p^{-1} - \frac{9}{2} b_1 b_2^2 p^{-2}\left(1+\frac{1}{4}e^2\right) + \frac{3}{8} b_2^4 p^{-3}\left(4+3e^2\right)\right].$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial A_{21}}{\partial q_i} = -A_{21}\left\{p^{-1}\frac{\partial p}{\partial q_i} + \delta_{2i}\left[e\left(1-e^2\right)^{-1} - e^{-1}\right]\right\} + \left(1-e^2\right)^{1/2} p^{-2} e\left[-b_1 p^{-1}\frac{\partial p}{\partial q_i} + \frac{\partial b_1}{\partial q_i}\right.$$

$$- 2p^{-2}\left(3b_1^2 - b_2^2\right)\frac{\partial p}{\partial q_i} + 2p^{-1}\left(3b_1\frac{\partial b_1}{\partial q_i} - b_2\frac{\partial b_2}{\partial q_i}\right)$$

$$+ \frac{9}{2} b_2 p^{-2}\left(1+\frac{1}{4}e^2\right)\left(3b_1 b_2 p^{-1}\frac{\partial p}{\partial q_i} - b_2\frac{\partial b_1}{\partial q_i} - 2b_1\frac{\partial b_2}{\partial q_i}\right)$$

$$\left. + \frac{3}{2} b_2^3 p^{-3}\left(4+3e^2\right)\left(\frac{\partial b_2}{\partial q_i} - b_2 p^{-1}\frac{\partial p}{\partial q_i}\right) + \frac{9}{4}\delta_{2i} b_2^2 p^{-2} e\left(b_2^2 p^{-1} - b_1\right)\right].$$

By equation (5.38), we have:

$$A_{22} = \frac{1}{8}\left(1-e^2\right)^{1/2} p^{-3} e^2 \left[\left(3b_1^2 - b_2^2\right) - 9\, b_1 b_2^2 p^{-1} + \frac{3}{4} b_2^4 p^{-2}\left(6+e^2\right)\right] .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial A_{22}}{\partial q_i} = -A_{22}\left[p^{-1}\frac{\partial p}{\partial q_i} + \delta_{2i}\, e\left(1-e^2\right)^{-1}\right] + \frac{1}{4} p^{-3} e^2 \left(1-e^2\right)^{1/2} \left\{\left[-p^{-1}\left(3b_1^2 - b_2^2\right) + \frac{27}{2} p^{-2} b_1 b_2^2\right.\right.$$

$$\left. - \frac{3}{2} p^{-3} b_2^4\left(6+e^2\right)\right]\frac{\partial p}{\partial q_i} + \left(3b_1 - \frac{9}{2} p^{-1} b_2^2\right)\frac{\partial b_1}{\partial q_i}$$

$$\left. + \left[\frac{3}{2} p^{-2} b_2^3\left(6+e^2\right) - 9\, p^{-1} b_1 b_2 - b_2\right]\frac{\partial b_2}{\partial q_i}\right\}$$

$$+ \frac{1}{4}\,\delta_{2i}\, p^{-3} e\left(1-e^2\right)^{1/2}\left[3b_1^2 - b_2^2 - 9p^{-1} b_1 b_2^2 + \frac{3}{2} p^{-2} b_2^4\left(3+e^2\right)\right] .$$

By equation (5.62), we have:

$$A_{31} = \left(1-e^2\right)^{1/2} p^{-3} e\left[2 + b_1 p^{-1}\left(3 + \frac{3}{4} e^2\right) - p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\left(4 + 3e^2\right)\right] .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial A_{31}}{\partial q_i} = -3 A_{31} p^{-1}\frac{\partial p}{\partial q_i} + \left(1-e^2\right)^{1/2} p^{-4} e\left\{\left[-b_1 p^{-1}\left(3 + \frac{3}{4} e^2\right)\right.\right.$$

$$\left. + 2p^{-2}\left(4+3e^2\right)\left(\frac{1}{2} b_2^2 + c^2\right)\right]\frac{\partial p}{\partial q_i} + \left(3 + \frac{3}{4} e^2\right)\frac{\partial b_1}{\partial q_i}$$

$$\left. - b_2 p^{-1}\left(4+3e^2\right)\frac{\partial b_2}{\partial q_i} + \delta_{2i}\left[\frac{3}{2} b_1 e - 6p^{-1} e\left(\frac{1}{2} b_2^2 + c^2\right)\right]\right\}$$

$$- \delta_{2i}\, p^{-3}\left[\left(1-e^2\right)^{-1/2} e^2 - \left(1-e^2\right)^{1/2}\right]\left[2 + b_1 p^{-1}\left(3 + \frac{3}{4} e^2\right) - p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\left(4+3e^2\right)\right] .$$

By the corrected version of equation (5.63),

$$A_{32} = \left(1-e^2\right)^{1/2} p^{-3} e^2\left[\frac{1}{4} + \frac{3}{4} b_1 p^{-1} - p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\left(\frac{3}{2} + \frac{1}{4} e^2\right)\right] .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial A_{32}}{\partial q_i} = - A_{32}\left[3p^{-1}\frac{\partial p}{\partial q_i} + \delta_{2i}\, e\left(1-e^2\right)^{-1}\right]$$

$$+ \left(1-e^2\right)^{1/2} p^{-4} e^2 \left\{\left[p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\left(3 + \frac{1}{2} e^2\right)\right.\right.$$

$$\left.\left. - \frac{3}{4} p^{-1} b_1\right]\frac{\partial p}{\partial q_i} + \frac{3}{4}\frac{\partial b_1}{\partial q_i} - \frac{1}{2} p^{-1} b_2\left(3 + \frac{1}{2} e^2\right)\frac{\partial b_2}{\partial q_i}\right\}$$

$$+ \delta_{2i}\left(1-e^2\right)^{1/2} p^{-3} e\left[\frac{1}{2}\left(1 + 3b_1 p^{-1}\right) - p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\left(3 + e^2\right)\right] .$$

By the corrected version of equation (5.64),

$$A_{33} = \frac{1}{3}\left(1-e^2\right)^{1/2} p^{-4} e^3\left[\frac{1}{4} b_1 - p^{-1}\left(\frac{1}{2} b_2^2 + c^2\right)\right] .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial A_{33}}{\partial q_i} = - 3A_{33} p^{-1}\frac{\partial p}{\partial q_i} + \frac{1}{3}\left(1-e^2\right)^{1/2} p^{-4} e^3\left\{\left[2p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right) - \frac{1}{4} p^{-1} b_1\right]\frac{\partial p}{\partial q_i}\right.$$

$$\left. + \frac{1}{4}\frac{\partial b_1}{\partial q_i} - p^{-1} b_2\frac{\partial b_2}{\partial q_i}\right\} + \frac{1}{3}\delta_{2i} p^{-4} e^2\left[3\left(1-e^2\right)^{1/2}\right.$$

$$\left. - \left(1-e^2\right)^{-1/2} e^2\right]\left[\frac{1}{4} b_1 - p^{-1}\left(\frac{1}{2} b_2^2 + c^2\right)\right] .$$

By equation (5.65),

$$A_{34} = - \frac{1}{32}\left(1-e^2\right)^{1/2} e^4 p^{-5}\left(\frac{1}{2} b_2^2 + c^2\right) .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial A_{34}}{\partial q_i} = - 5A_{34} p^{-1}\frac{\partial p}{\partial q_i} - \frac{1}{32}\left(1-e^2\right)^{1/2} p^{-5} e^4 b_2\frac{\partial b_2}{\partial q_i}$$

$$+ \frac{1}{8}\delta_{2i} e^3 p^{-5}\left[\frac{1}{4}\left(1-e^2\right)^{-1/2} e^2 - \left(1-e^2\right)^{1/2}\right]\left(\frac{1}{2} b_2^2 + c^2\right) .$$

By equation (47),* we have:

$$Q = \sqrt{P^2 + S} .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial Q}{\partial q_i} = Q^{-1}\left(P \frac{\partial P}{\partial q_i} + \frac{1}{2} \delta_{3i}\right) .$$

From equation (102.1),

$$\zeta = P(1 - S)^{-1} ,$$

so that, for $i = 1, 2, 3,$

$$\frac{\partial \zeta}{\partial q_i} = (1 - S)^{-1}\left[\frac{\partial P}{\partial q_i} + \delta_{3i} P(1 - S)^{-1}\right] .$$

From equations (147.1) and (147.2),

$$h_1 = \frac{1}{2} \left(1 + C_1 - C_2\right)^{-1/2} ,$$

and

$$h_2 = \frac{1}{2} \left(1 - C_1 - C_2\right)^{-1/2} .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial h_1}{\partial q_i} = -2h_1^3\left(\frac{\partial C_1}{\partial q_i} - \frac{\partial C_2}{\partial q_i}\right) ,$$

and

$$\frac{\partial h_2}{\partial q_i} = 2h_2^3\left(\frac{\partial C_1}{\partial q_i} + \frac{\partial C_2}{\partial q_i}\right) .$$

*Equation numbers will now refer, until otherwise indicated, to Reference 8.

From equations (100.1) and (100.2),

$$e_2 = Q(1 - P)^{-1} ,$$

and

$$e_3 = Q(1 + P)^{-1} .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial e_2}{\partial q_i} = (1 - P)^{-1} \left(\frac{\partial Q}{\partial q_i} + e_2 \frac{\partial P}{\partial q_i} \right) ,$$

and

$$\frac{\partial e_3}{\partial q_i} = (1 + P)^{-1} \left(\frac{\partial Q}{\partial q_i} - e_3 \frac{\partial P}{\partial q_i} \right) .$$

From equation (131.2),

$$e' = ae\left(a + b_1\right)^{-1} ,$$

so that, for $i = 1, 2, 3$,

$$\frac{\partial e'}{\partial q_i} = \left(a + b_1\right)^{-1} \left[\delta_{1i}\, e + \delta_{2i}\, a - e' \left(\delta_{1i} + \frac{\partial b_1}{\partial q_i} \right) \right] .$$

From equation (149.1),

$$\alpha_3 = \left(\operatorname{sgn} \alpha_3\right) \alpha_2 \left(1 - Su^{-1}\right)^{1/2} ,$$

where $\operatorname{sgn} \alpha_3 \equiv \alpha_3 / |\alpha_3|$ is +1 for a direct orbit and -1 for a retrograde orbit. Then, for $i = 1, 2, 3$,

$$\frac{\partial \alpha_3}{\partial q_i} = \operatorname{sgn} \alpha_3 \left[\left(1 - Su^{-1}\right)^{1/2} \frac{\partial \alpha_2}{\partial q_i} + \frac{1}{2} \alpha_2 u^{-1} \left(1 - Su^{-1}\right)^{-1/2} \left(Su^{-1} \frac{\partial u}{\partial q_i} - \delta_{3i} \right) \right] .$$

By equation (76), neglecting terms of fourth order, we have:

$$B_1' = \frac{1}{2} Q^2 + P^2 - \frac{3}{4} C_1 PQ^2 + \frac{3}{2} C_2 P^2 Q^2 + \frac{3}{64} \left(4C_2 + 3C_1^2\right) Q^4$$

$$- \frac{45}{32} C_1 C_2 PQ^4 + \frac{15}{256} \left(2C_2^2 + 5C_1^2 C_2\right) Q^6 + \frac{175}{2048} C_2^3 Q^8 .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial B_1'}{\partial q_i} = Q\left[1 + \frac{3}{16} Q^2 \left(4C_2 + 3C_1^2\right) + \frac{45}{128} C_2 Q^4 \left(2C_2 + 5C_1^2\right)\right] \frac{\partial Q}{\partial q_i} + 2P \frac{\partial P}{\partial q_i} - \frac{3}{4} Q\left(PQ \frac{\partial C_1}{\partial q_i} + C_1 Q \frac{\partial P}{\partial q_i}\right.$$

$$\left. + 2C_1 P \frac{\partial Q}{\partial q_i}\right) + \frac{3}{2} PQ\left(PQ \frac{\partial C_2}{\partial q_i} + 2C_2 Q \frac{\partial P}{\partial q_i} + 2C_2 P \frac{\partial Q}{\partial q_i}\right) + \frac{3}{32} Q^4 \left(2 \frac{\partial C_2}{\partial q_i} + 3C_1 \frac{\partial C_1}{\partial q_i}\right)$$

$$- \frac{45}{32} Q^3 \left(C_2 PQ \frac{\partial C_1}{\partial q_i} + C_1 PQ \frac{\partial C_2}{\partial q_i} + C_1 C_2 Q \frac{\partial P}{\partial q_i} + 4C_1 C_2 P \frac{\partial Q}{\partial q_i}\right)$$

$$+ \frac{15}{256} Q^6 \left[\left(4C_2 + 5C_1^2\right) \frac{\partial C_2}{\partial q_i} + 10C_1 C_2 \frac{\partial C_1}{\partial q_i}\right] + \frac{175}{2048} C_2^2 Q^7 \left(3Q \frac{\partial C_2}{\partial q_i} + 8C_2 \frac{\partial Q}{\partial q_i}\right) .$$

By equation (65), neglecting terms of fifth order, we have:

$$B_2 = 1 - \frac{1}{2} C_1 P + \left(\frac{3}{8} C_1^2 + \frac{1}{2} C_2\right)\left(P^2 + \frac{1}{2} Q^2\right) + \frac{9}{64} C_2^2 Q^4 - \frac{9}{8} C_1 C_2 PQ^2$$

$$- \frac{15}{32} C_1^3 PQ^2 + \frac{45}{128} C_1^2 C_2 Q^4 + \frac{25}{256} C_2^3 Q^6 + \frac{105}{1024} C_1^4 Q^4$$

$$+ \frac{9}{8} C_2^2 P^2 Q^2 - \frac{225}{128} C_1 C_2^2 PQ^4 + \frac{525}{1024} C_1^2 C_2^2 Q^6 + \left(\frac{35}{128}\right)^2 C_2^4 Q^8 .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial B_2}{\partial q_i} = - \frac{1}{2} \left(P \frac{\partial C_1}{\partial q_i} + C_1 \frac{\partial P}{\partial q_i}\right) + \left(\frac{3}{8} C_1^2 + \frac{1}{2} C_2\right)\left(2P \frac{\partial P}{\partial q_i} + Q \frac{\partial Q}{\partial q_i}\right) + \left(P^2 + \frac{1}{2} Q^2\right)\left(\frac{3}{4} C_1 \frac{\partial C_1}{\partial q_i} + \frac{1}{2} \frac{\partial C_2}{\partial q_i}\right)$$

$$+ \frac{1}{32} C_2 Q^3 \left[9 + \frac{75}{8} C_2 Q^2 + \frac{1}{2} \left(\frac{35}{8}\right)^2 C_2^2 Q^4\right]\left(Q \frac{\partial C_2}{\partial q_i} + 2C_2 \frac{\partial Q}{\partial q_i}\right) - \frac{9}{8} Q\left(C_2 PQ \frac{\partial C_1}{\partial q_i}\right.$$

$$\left. + C_1 PQ \frac{\partial C_2}{\partial q_i} + C_1 C_2 Q \frac{\partial P}{\partial q_i} + 2C_1 C_2 P \frac{\partial Q}{\partial q_i}\right) - \frac{15}{32} C_1^2 Q\left(3PQ \frac{\partial C_1}{\partial q_i} + C_1 Q \frac{\partial P}{\partial q_i} + 2C_1 P \frac{\partial Q}{\partial q_i}\right)$$

$$+ \frac{45}{128} C_1 Q^3 \left(2C_2 Q \frac{\partial C_1}{\partial q_i} + C_1 Q \frac{\partial C_2}{\partial q_i} + 4C_1 C_2 \frac{\partial Q}{\partial q_i}\right) + \frac{105}{256} C_1^3 Q^3 \left(Q \frac{\partial C_1}{\partial q_i} + C_1 \frac{\partial Q}{\partial q_i}\right) + \frac{9}{4} C_2 PQ \left(PQ \frac{\partial C_2}{\partial q_i}\right.$$

$$\left. + C_2 Q \frac{\partial P}{\partial q_i} + C_2 P \frac{\partial Q}{\partial q_i}\right) - \frac{225}{128} C_2 Q^3 \left(C_2 PQ \frac{\partial C_1}{\partial q_i} + 2C_1 PQ \frac{\partial C_2}{\partial q_i}\right.$$

$$\left. + C_1 C_2 Q \frac{\partial P}{\partial q_i} + 4C_1 C_2 P \frac{\partial Q}{\partial q_i}\right) + \frac{525}{512} C_1 C_2 Q^5 \left(C_2 Q \frac{\partial C_1}{\partial q_i} + C_1 Q \frac{\partial C_2}{\partial q_i} + 3C_1 C_2 \frac{\partial Q}{\partial q_i}\right) .$$

By the corrected version of equation (95), neglecting terms of fifth order, we have:

$$B_3 = -\frac{1}{2}C_2 - \frac{3}{8}C_1^2 - \left(\frac{35}{128}C_1^4 + \frac{15}{16}C_1^2C_2 + \frac{3}{8}C_2^2\right)\left(1+\frac{1}{2}Q^2\right) - \frac{3}{8}C_2^2P^2$$

$$-\left(\frac{105}{64}C_1^2C_2^2 + \frac{5}{16}C_2^3\right)\left(1+\frac{1}{2}Q^2+\frac{3}{8}Q^4\right) - \frac{35}{128}C_2^4\left(1+\frac{1}{2}Q^2+\frac{3}{8}Q^4+\frac{5}{16}Q^6\right)$$

$$+\left(\frac{5}{16}C_1^3 + \frac{3}{4}C_1C_2\right)P + \frac{15}{16}C_1C_2^2\left(P+\frac{3}{2}PQ^2\right).$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial B_3}{\partial q_i} = -\frac{1}{2}\left(\frac{\partial C_2}{\partial q_i} + \frac{3}{2}C_1\frac{\partial C_1}{\partial q_i}\right) - \left(1+\frac{1}{2}Q^2\right)\left[\frac{5}{8}C_1\left(\frac{7}{4}C_1^2+3C_2\right)\frac{\partial C_1}{\partial q_i} + \frac{3}{4}\left(\frac{5}{4}C_1^2+C_2\right)\frac{\partial C_2}{\partial q_i}\right]$$

$$-\left[\frac{35}{128}C_1^4 + \frac{15}{16}C_1^2C_2 + \frac{3}{8}C_2^2 - \frac{45}{16}C_1C_2^2P + \frac{35}{128}C_2^4\left(1+\frac{3}{2}Q^2+\frac{15}{8}Q^4\right)\right]Q\frac{\partial Q}{\partial q_i} - \frac{3}{4}C_2P\left(P\frac{\partial C_2}{\partial q_i} + C_2\frac{\partial P}{\partial q_i}\right)$$

$$-\frac{15}{16}C_2\left(1+\frac{1}{2}Q^2+\frac{3}{8}Q^4\right)\left[\frac{7}{2}C_1C_2\frac{\partial C_1}{\partial q_i} + \left(\frac{7}{2}C_1^2+C_2\right)\frac{\partial C_2}{\partial q_i}\right] + \frac{5}{16}C_2\left(1+\frac{3}{2}Q^2\right)\left[3P\left(C_2\frac{\partial C_1}{\partial q_i}\right.\right.$$

$$\left.\left.+2C_1\frac{\partial C_2}{\partial q_i}\right) + 3C_1C_2\frac{\partial P}{\partial q_i} - C_2Q\left(\frac{21}{4}C_1^2+C_2\right)\frac{\partial Q}{\partial q_i}\right] - \frac{35}{32}C_2^3\left(1+\frac{1}{2}Q^2+\frac{3}{8}Q^4+\frac{5}{16}Q^6\right)\frac{\partial C_2}{\partial q_i}$$

$$+\frac{1}{4}C_1\left(\frac{5}{4}C_1^2+3C_2\right)\frac{\partial P}{\partial q_i} + \frac{3}{4}P\left[\left(\frac{5}{4}C_1^2+C_2\right)\frac{\partial C_1}{\partial q_i} + C_1\frac{\partial C_2}{\partial q_i}\right].$$

By equation (116.3),

$$B_{12} = -\frac{1}{4}Q^2 - \frac{1}{8}C_2Q^4 ,$$

so that, for $i = 1, 2, 3$,

$$\frac{\partial B_{12}}{\partial q_i} = -\frac{1}{2}Q\left(1+C_2Q^2\right)\frac{\partial Q}{\partial q_i} - \frac{1}{8}Q^4\frac{\partial C_2}{\partial q_i}.$$

By the corrected version of equations (116.4),

$$B_{21} = -C_2PQ + \frac{9}{16}C_1C_2Q^3 + \frac{1}{2}C_1Q ,$$

and

$$B_{22} = -\frac{1}{32}\left[\left(4C_2 + 3C_1^2\right)Q^2 + 3C_2^2 Q^4\right].$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial B_{21}}{\partial q_i} = \frac{1}{2}Q\left(1 + \frac{9}{8}C_2 Q^2\right)\frac{\partial C_1}{\partial q_i} + Q\left(\frac{9}{16}C_1 Q^2 - P\right)\frac{\partial C_2}{\partial q_i} + \left(\frac{27}{16}C_1 C_2 Q^2 - C_2 P + \frac{1}{2}C_1\right)\frac{\partial Q}{\partial q_i} - C_2 Q\frac{\partial P}{\partial q_i},$$

and

$$\frac{\partial B_{22}}{\partial q_i} = -\frac{1}{16}Q\left[\left(4C_2 + 3C_1^2 + 6C_2^2 Q^2\right)\frac{\partial Q}{\partial q_i} + Q\left(2 + 3C_2 Q^2\right)\frac{\partial C_2}{\partial q_i} + 3C_1 Q\frac{\partial C_1}{\partial q_i}\right].$$

By equations (122.1) and (122.2), we have:

$$2\pi\nu_1 = \left(-2\alpha_1\right)^{1/2}\left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1},$$

and

$$2\pi\nu_2 = \alpha_2 u^{-1/2} A_2 B_2^{-1}\left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1}.$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial}{\partial q_i}\left(2\pi\nu_1\right) = -2\pi\nu_1\left\{\left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1}\left[\delta_{1i} + \frac{\partial b_1}{\partial q_i} + \frac{\partial A_1}{\partial q_i} + c^2 B_2^{-1}\left(B_1'\frac{\partial A_2}{\partial q_i} + A_2\frac{\partial B_1'}{\partial q_i} - A_2 B_1' B_2^{-1}\frac{\partial B_2}{\partial q_i}\right)\right] - \frac{1}{2}\alpha_1^{-1}\frac{\partial \alpha_1}{\partial q_i}\right\},$$

and

$$\frac{\partial}{\partial q_i}\left(2\pi\nu_2\right) = -2\pi\nu_2\left\{\left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1}\left[\delta_{1i} + \frac{\partial b_1}{\partial q_i} + \frac{\partial A_1}{\partial q_i} + c^2 B_2^{-1}\left(B_1'\frac{\partial A_2}{\partial q_i} + A_2\frac{\partial B_1'}{\partial q_i} - A_2 B_1' B_2^{-1}\frac{\partial B_2}{\partial q_i}\right)\right] - \left(\alpha_2^{-1}\frac{\partial \alpha_2}{\partial q_i} + A_2^{-1}\frac{\partial A_2}{\partial q_i} - \frac{1}{2}u^{-1}\frac{\partial u}{\partial q_i} - B_2^{-1}\frac{\partial B_2}{\partial q_i}\right)\right\}.$$

The following time-independent partial derivatives are used only when the differential correction includes periodic terms through the second order.

By equations (5.32), (5.33), (5.39), and (5.40) of Reference 4, we have:

$$A_{11} = \frac{3}{4} \left(1 - e^2\right)^{1/2} p^{-3} b_2^2 \, e \left(b_2^2 - 2b_1 \, p\right) ,$$

$$A_{12} = \frac{3}{32} \left(1 - e^2\right)^{1/2} p^{-3} b_2^4 \, e^2 ,$$

$$A_{23} = \frac{1}{8} \left(1 - e^2\right)^{1/2} p^{-4} b_2^2 \, e^3 \left(b_2^2 \, p^{-1} - b_1\right) ,$$

and

$$A_{24} = \frac{3}{256} \left(1 - e^2\right)^{1/2} p^{-5} b_2^4 \, e^4 .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial A_{11}}{\partial q_i} = - A_{11} \left\{ 3p^{-1} \frac{\partial p}{\partial q_i} + \delta_{2i} \left[e\left(1 - e^2\right)^{-1} - e^{-1} \right] \right\}$$

$$- \frac{3}{2} \left(1 - e^2\right)^{1/2} p^{-3} b_2 \, e \left[b_1 \, b_2 \frac{\partial p}{\partial q_i} + b_2 \, p \frac{\partial b_1}{\partial q_i} + 2\left(b_1 \, p - b_2^2\right) \frac{\partial b_2}{\partial q_i} \right] ,$$

$$\frac{\partial A_{12}}{\partial q_i} = \frac{3}{8} \left(1 - e^2\right)^{1/2} p^{-3} b_2^3 \, e^2 \left(\frac{\partial b_2}{\partial q_i} - \frac{3}{4} p^{-1} b_2 \frac{\partial p}{\partial q_i} \right) + \frac{3}{16} \delta_{2i} \, p^{-3} \, e \, b_2^4 \left[\left(1 - e^2\right)^{1/2} - \frac{1}{2} \left(1 - e^2\right)^{-1/2} e^2 \right] ,$$

$$\frac{\partial A_{23}}{\partial q_i} = - A_{23} \, p^{-1} \frac{\partial p}{\partial q_i} + \frac{1}{8} \left(1 - e^2\right)^{1/2} p^{-4} b_2 \, e^3 \left[b_2 \, p^{-1} \left(3b_1 - 4b_2^2 \, p^{-1}\right) \frac{\partial p}{\partial q_i} \right.$$

$$\left. - b_2 \frac{\partial b_1}{\partial q_i} + 2\left(2b_2^2 \, p^{-1} - b_1\right) \frac{\partial b_2}{\partial q_i} \right]$$

$$+ \frac{1}{8} \delta_{2i} \, p^{-4} b_2^2 \, e^2 \left(b_2^2 \, p^{-1} - b_1\right) \left[3\left(1 - e^2\right)^{1/2} - \left(1 - e^2\right)^{-1/2} e^2 \right] ,$$

and

$$\frac{\partial A_{24}}{\partial q_i} = A_{24} \left(4b_2^{-1} \frac{\partial b_2}{\partial q_i} - 5p^{-1} \frac{\partial p}{\partial q_i} \right) + \frac{3}{256} \delta_{2i} \, p^{-5} b_2^4 \, e^3 \left[4\left(1 - e^2\right)^{1/2} - \left(1 - e^2\right)^{-1/2} e^2 \right] .$$

By the corrected versions of equations (116.3) and (116.4) of Reference 8, we have:

$$B_{11} = -2PQ + \frac{3}{8} C_1 Q^3 ,$$

$$B_{13} = -\frac{1}{24} C_1 Q^3 ,$$

$$B_{14} = \frac{1}{64} C_2 Q^4 ,$$

$$B_{23} = -\frac{1}{16} C_1 C_2 Q^3 ,$$

and

$$B_{24} = \frac{3}{256} C_2^2 Q^4 .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial B_{11}}{\partial q_i} = \left(\frac{9}{8} C_1 Q^2 - 2P\right) \frac{\partial Q}{\partial q_i} - 2Q \frac{\partial P}{\partial q_i} + \frac{3}{8} Q^3 \frac{\partial C_1}{\partial q_i} ,$$

$$\frac{\partial B_{13}}{\partial q_i} = -\frac{1}{24} Q^2 \left(Q \frac{\partial C_1}{\partial q_i} + 3C_1 \frac{\partial Q}{\partial q_i}\right) ,$$

$$\frac{\partial B_{14}}{\partial q_i} = \frac{1}{64} Q^3 \left(Q \frac{\partial C_2}{\partial q_i} + 4C_2 \frac{\partial Q}{\partial q_i}\right) ,$$

$$\frac{\partial B_{23}}{\partial q_i} = -\frac{1}{16} Q^2 \left(C_2 Q \frac{\partial C_1}{\partial q_i} + C_1 Q \frac{\partial C_2}{\partial q_i} + 3C_1 C_2 \frac{\partial Q}{\partial q_i}\right) ,$$

and

$$\frac{\partial B_{24}}{\partial q_i} = \frac{3}{128} C_2 Q^3 \left(Q \frac{\partial C_2}{\partial q_i} + 2C_2 \frac{\partial Q}{\partial q_i}\right) .$$

TIME-VARYING PARTIAL DERIVATIVES IN THE DIFFERENTIAL CORRECTION

We shall continue the use of the generalized notation introduced in the preceding section, whereby we let q_i ($i = 1, 2, 3$) refer to the orbital elements a, e, and S, respectively. The time-dependent parameters will involve, additionally, the orbital elements β_1, β_2, and β_3, which we shall represent as β_i ($i = 1, 2, 3$) in the generalized form of the partial derivatives.

By combining equations (121.1)* and (123.1), we have:

$$M_s = 2\pi\nu_1 \left(t + \beta_1 - c^2 \beta_2 \alpha_2^{-1} B_1' B_2^{-1}\right) .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial M_s}{\partial q_i} = \left(t + \beta_1 - c^2 \beta_2 \alpha_2^{-1} B_1' B_2^{-1}\right) \frac{\partial}{\partial q_i} \left(2\pi\nu_1\right)$$

$$- 2\pi\nu_1 c^2 \beta_2 \alpha_2^{-1} B_2^{-1} \left(\frac{\partial B_1'}{\partial q_i} - B_1' \alpha_2^{-1} \frac{\partial \alpha_2}{\partial q_i} - B_1' B_2^{-1} \frac{\partial B_2}{\partial q_i}\right) .$$

Also:

$$\frac{\partial M_s}{\partial \beta_1} = 2\pi\nu_1 ,$$

$$\frac{\partial M_s}{\partial \beta_2} = - 2\pi\nu_1 c^2 \alpha_2^{-1} B_1' B_2^{-1} ,$$

and

$$\frac{\partial M_s}{\partial \beta_3} = 0 .$$

In what follows, whenever a partial derivative with respect to β_3 is zero, it will not be given.

By combining equations (121.2) and (123.2), we have:

$$\Psi_s = 2\pi\nu_2 \left[t + \beta_1 + \beta_2 \alpha_2^{-1} A_2^{-1} \left(a + b_1 + A_1\right)\right] .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial \Psi_s}{\partial q_i} = \left[t + \beta_1 + \beta_2 \alpha_2^{-1} A_2^{-1} \left(a + b_1 + A_1\right)\right] \frac{\partial}{\partial q_i} \left(2\pi\nu_2\right)$$

$$+ 2\pi\nu_2 \beta_2 \alpha_2^{-1} A_2^{-1} \left[\delta_{1i} + \frac{\partial b_1}{\partial q_i} + \frac{\partial A_1}{\partial q_i} - \left(a + b_1 + A_1\right) \left(\alpha_2^{-1} \frac{\partial \alpha_2}{\partial q_i} + A_2^{-1} \frac{\partial A_2}{\partial q_i}\right)\right] .$$

*All equation numbers, used in specifying the defining relation for a given variable, will henceforth refer to Reference 8.

Also:

$$\frac{\partial \Psi_s}{\partial \beta_1} = 2\pi\nu_2 ,$$

and

$$\frac{\partial \Psi_s}{\partial \beta_2} = 2\pi\nu_2\, \alpha_2^{-1}\, A_2^{-1}\, \left(a + b_1 + A_1\right) .$$

By equation (131.1), letting $\mathcal{E} \equiv M_s + E_0$, we have:

$$\mathcal{E} - e' \sin \mathcal{E} = M_s .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial \mathcal{E}}{\partial q_i} = (1 - e' \cos \mathcal{E})^{-1} \left(\sin \mathcal{E} \frac{\partial e'}{\partial q_i} + \frac{\partial M_s}{\partial q_i} \right) .$$

Also, for $i = 1, 2,$

$$\frac{\partial \mathcal{E}}{\partial \beta_i} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_s}{\partial \beta_i} .$$

By the anomaly connections given in equations (132), taken to zeroth order,

$$\cos v' = (\cos \mathcal{E} - e)\,(1 - e \cos \mathcal{E})^{-1} ,$$

and

$$\sin v' = \left(1 - e^2\right)^{1/2} (1 - e \cos \mathcal{E})^{-1} \sin \mathcal{E} ,$$

where

$$v_0 = v' - M_s .$$

Then, for $i = 1, 2, 3,$

$$\frac{\partial v_0}{\partial q_i} = \left[\left(1 - e^2\right) \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial q_i} + \delta_{2i} \sin^2 \mathcal{E}\right] (\sin v')^{-1} (1 - e \cos \mathcal{E})^{-2} - \frac{\partial M_s}{\partial q_i} .$$

Also, for $i = 1, 2,$

$$\frac{\partial v_0}{\partial \beta_i} = \left(1 - e^2\right) \sin \mathcal{E} \left(\sin v'\right)^{-1} \left(1 - e \cos \mathcal{E}\right)^{-2} \frac{\partial \mathcal{E}}{\partial \beta_i} - \frac{\partial M_s}{\partial \beta_i} .$$

By equation (133), we have:

$$\Psi_0 = \alpha_2 \left(-2\alpha_1\right)^{-1/2} u^{-1/2} A_2 B_2^{-1} v_0 .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial \Psi_0}{\partial q_i} = \left(-2\alpha_1\right)^{-1/2} u^{-1/2} B_2^{-1} \left(A_2 v_0 \frac{\partial \alpha_2}{\partial q_i} - \frac{1}{2} \alpha_2 \alpha_1^{-1} A_2 v_0 \frac{\partial \alpha_1}{\partial q_i} - \frac{1}{2} \alpha_2 u^{-1} A_2 v_0 \frac{\partial u}{\partial q_i} \right.$$

$$\left. + \alpha_2 v_0 \frac{\partial A_2}{\partial q_i} - \alpha_2 A_2 B_2^{-1} v_0 \frac{\partial B_2}{\partial q_i} + \alpha_2 A_2 \frac{\partial v_0}{\partial q_i} \right) .$$

Also, for $i = 1, 2,$

$$\frac{\partial \Psi_0}{\partial \beta_i} = \alpha_2 \left(-2\alpha_1\right)^{-1/2} u^{-1/2} A_2 B_2^{-1} \frac{\partial v_0}{\partial \beta_i} .$$

By equation (136.2), we have:

$$M_1 = -\left(a + b_1\right)^{-1} \left[\left(A_1 + c^2 A_2 B_1' B_2^{-1}\right) v_0 + c^2 \alpha_2^{-1} \left(-2\alpha_1\right)^{1/2} u^{1/2} B_{12} \sin 2\left(\Psi_s + \Psi_0\right)\right] .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial M_1}{\partial q_i} = -\left(a + b_1\right)^{-1} \left\{ M_1 \left(\delta_{1i} + \frac{\partial b_1}{\partial q_i}\right) + \left(A_1 + c^2 A_2 B_1' B_2^{-1}\right) \frac{\partial v_0}{\partial q_i} + v_0 \left[\frac{\partial A_1}{\partial q_i} + c^2 B_2^{-1} \left(B_1' \frac{\partial A_2}{\partial q_i}\right.\right.\right.$$

$$\left.\left. + A_2 \frac{\partial B_1'}{\partial q_i} - A_2 B_1' B_2^{-1} \frac{\partial B_2}{\partial q_i}\right)\right] + c^2 \alpha_2^{-1} \left(-2\alpha_1\right)^{1/2} u^{1/2} \sin 2\left(\Psi_s + \Psi_0\right) \left(\frac{\partial B_{12}}{\partial q_i} + \frac{1}{2} u^{-1} B_{12} \frac{\partial u}{\partial q_i}\right.$$

$$\left.\left. + \frac{1}{2} \alpha_1^{-1} B_{12} \frac{\partial \alpha_1}{\partial q_i} - \alpha_2^{-1} B_{12} \frac{\partial \alpha_2}{\partial q_i}\right) + 2c^2 \alpha_2^{-1} \left(-2\alpha_1\right)^{1/2} u^{1/2} B_{12} \cos 2\left(\Psi_s + \Psi_0\right) \left(\frac{\partial \Psi_s}{\partial q_i} + \frac{\partial \Psi_0}{\partial q_i}\right)\right\} .$$

Also, for $i = 1, 2,$

$$\frac{\partial M_1}{\partial \beta_i} = -\left(a + b_1\right)^{-1} \left[\left(A_1 + c^2 A_2 B_1' B_2^{-1}\right) \frac{\partial v_0}{\partial \beta_i} + 2c^2 \alpha_2^{-1} \left(-2\alpha_1\right)^{1/2} u^{1/2} B_{12} \cos 2\left(\Psi_s + \Psi_0\right) \left(\frac{\partial \Psi_s}{\partial \beta_i} + \frac{\partial \Psi_0}{\partial \beta_i}\right)\right] .$$

By equation (137), neglecting terms of third order, we have:

$$E_1 = (1 - e' \cos \mathcal{E})^{-1} M_1 - \frac{1}{2} e' (1 - e' \cos \mathcal{E})^{-3} M_1^2 \sin \mathcal{E} \quad .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial E_1}{\partial q_i} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial q_i} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial q_i} - \cos \mathcal{E} \frac{\partial e'}{\partial q_i} \right) - M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \left[\frac{1}{2} M_1 \frac{\partial e'}{\partial q_i} \right.$$

$$\left. - \frac{3}{2} M_1 e' (1 - e' \cos \mathcal{E})^{-1} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial q_i} - \cos \mathcal{E} \frac{\partial e'}{\partial q_i} \right) + \frac{1}{2} M_1 e' \cot \mathcal{E} \frac{\partial \mathcal{E}}{\partial q_i} + e' \frac{\partial M_1}{\partial q_i} \right] \quad .$$

Also, for $i = 1, 2$,

$$\frac{\partial E_1}{\partial \beta_i} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial \beta_i} \left[1 - M_1 e' (1 - e' \cos \mathcal{E})^{-2} \sin \mathcal{E} \right]$$

$$- M_1 e' (1 - e' \cos \mathcal{E})^{-2} \frac{\partial \mathcal{E}}{\partial \beta_i} \left[\sin \mathcal{E} + \frac{1}{2} M_1 (1 - e' \cos \mathcal{E})^{-1} \cos \mathcal{E} - \frac{3}{2} M_1 e' (1 - e' \cos \mathcal{E})^{-2} \sin^2 \mathcal{E} \right] \quad .$$

By the anomaly connections, taken to first order,

$$\cos v'' = \left[\cos (\mathcal{E} + E_1) - e \right] \left[1 - e \cos (\mathcal{E} + E_1) \right]^{-1} \quad ,$$

and

$$\sin v'' = \left(1 - e^2 \right)^{1/2} \left[1 - e \cos (\mathcal{E} + E_1) \right]^{-1} \sin (\mathcal{E} + E_1) \quad ,$$

where

$$v_1 = v'' - v' = v'' - (M_s + v_0) \quad .$$

Then, for $i = 1, 2, 3$,

$$\frac{\partial v_1}{\partial q_i} = \left[\left(1 - e^2 \right) \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial q_i} + \frac{\partial E_1}{\partial q_i} \right) \right.$$

$$\left. + \delta_{2i} \sin^2 (\mathcal{E} + E_1) \right] (\sin v'')^{-1} \left[1 - e \cos (\mathcal{E} + E_1) \right]^{-2} - \frac{\partial v'}{\partial q_i} \quad .$$

Also, for $i = 1, 2,$

$$\frac{\partial v_1}{\partial \beta_i} = \left(1 - e^2\right) \sin\left(\mathcal{E} + E_1\right) \left(\sin v''\right)^{-1} \left[1 - e \cos\left(\mathcal{E} + E_1\right)\right]^{-2} \left(\frac{\partial \mathcal{E}}{\partial \beta_i} + \frac{\partial E_1}{\partial \beta_i}\right) - \frac{\partial v'}{\partial \beta_i} \quad .$$

In the above and in the following, we employ the symbolic notation:

$$\frac{\partial v'}{\partial q_i} \equiv \frac{\partial M_s}{\partial q_i} + \frac{\partial v_0}{\partial q_i} \quad ,$$

and

$$\frac{\partial v'}{\partial \beta_i} \equiv \frac{\partial M_s}{\partial \beta_i} + \frac{\partial v_0}{\partial \beta_i} \quad .$$

By the corrected version of equation (139), we have:

$$\Psi_1 = -B_2^{-1} \left[B_{22} \sin 2\left(\Psi_s + \Psi_0\right) + B_{21} \cos\left(\Psi_s + \Psi_0\right)\right]$$

$$+ \alpha_2 \left(-2\alpha_1\right)^{-1/2} u^{-1/2} B_2^{-1} \left(A_2 v_1 + A_{21} \sin v' + A_{22} \sin 2v'\right) \quad .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial \Psi_1}{\partial q_i} = B_2^{-2} \left[B_{22} \sin 2\left(\Psi_s + \Psi_0\right) + B_{21} \cos\left(\Psi_s + \Psi_0\right)\right] \frac{\partial B_2}{\partial q_i} - B_2^{-1} \left\{\sin 2\left(\Psi_s + \Psi_0\right) \frac{\partial B_{22}}{\partial q_i}\right.$$

$$\left. + \cos\left(\Psi_s + \Psi_0\right) \frac{\partial B_{21}}{\partial q_i} + \left[2 B_{22} \cos 2\left(\Psi_s + \Psi_0\right) - B_{21} \sin\left(\Psi_s + \Psi_0\right)\right] \left(\frac{\partial \Psi_s}{\partial q_i} + \frac{\partial \Psi_0}{\partial q_i}\right)\right\}$$

$$+ \left(-2\alpha_1\right)^{-1/2} u^{-1/2} B_2^{-1} \left[\frac{\partial \alpha_2}{\partial q_i} - \frac{1}{2} \alpha_2 \left(\alpha_1^{-1} \frac{\partial \alpha_1}{\partial q_i} + u^{-1} \frac{\partial u}{\partial q_i} + 2 B_2^{-1} \frac{\partial B_2}{\partial q_i}\right)\right] \left(A_2 v_1\right.$$

$$\left. + A_{21} \sin v' + A_{22} \sin 2v'\right) + \alpha_2 \left(-2\alpha_1\right)^{-1/2} u^{-1/2} B_2^{-1} \left[v_1 \frac{\partial A_2}{\partial q_i} + A_2 \frac{\partial v_1}{\partial q_i} + \sin v' \frac{\partial A_{21}}{\partial q_i}\right.$$

$$\left. + \sin 2v' \frac{\partial A_{22}}{\partial q_i} + \left(A_{21} \cos v' + 2 A_{22} \cos 2v'\right) \frac{\partial v'}{\partial q_i}\right] \quad .$$

Also, for $i = 1, 2$,

$$\frac{\partial \Psi_1}{\partial \beta_i} = -B_2^{-1}\left[2B_{22}\cos 2(\Psi_s + \Psi_0) - B_{21}\sin(\Psi_s + \Psi_0)\right]\left(\frac{\partial \Psi_s}{\partial \beta_i} + \frac{\partial \Psi_0}{\partial \beta_i}\right)$$

$$+ \alpha_2(-2\alpha_1)^{-1/2}u^{-1/2}B_2^{-1}\left[A_2\frac{\partial v_1}{\partial \beta_i} + (A_{21}\cos v' + 2A_{22}\cos 2v')\frac{\partial v'}{\partial \beta_i}\right].$$

The following time-dependent partial derivatives are used only when the differential correction includes periodic terms through the second order.

By equation (143.2), we have:

$$M_2 = -(a + b_1)^{-1}\left\{A_1 v_1 + A_{11}\sin v' + A_{12}\sin 2v'\right.$$

$$+ c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}\left[B_1'\Psi_1 + B_{11}\cos(\Psi_s + \Psi_0)\right.$$

$$\left.\left. + 2B_{12}\Psi_1\cos 2(\Psi_s + \Psi_0) + B_{13}\cos 3(\Psi_s + \Psi_0) + B_{14}\sin 4(\Psi_s + \Psi_0)\right]\right\}.$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial M_2}{\partial q_i} = -(a + b_1)^{-1}\left\{M_2\left(\delta_{1i} + \frac{\partial b_1}{\partial q_i}\right) + v_1\frac{\partial A_1}{\partial q_i} + A_1\frac{\partial v_1}{\partial q_i} + \sin v'\frac{\partial A_{11}}{\partial q_i} + \sin 2v'\frac{\partial A_{12}}{\partial q_i}\right.$$

$$+ (A_{11}\cos v' + 2A_{12}\cos 2v')\frac{\partial v'}{\partial q_i} + c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}\left(\frac{1}{2}u^{-1}\frac{\partial u}{\partial q_i}\right.$$

$$\left. + \frac{1}{2}\alpha_1^{-1}\frac{\partial \alpha_1}{\partial q_i} - \alpha_2^{-1}\frac{\partial \alpha_2}{\partial q_i}\right)\left[B_1'\Psi_1 + B_{11}\cos(\Psi_s + \Psi_0) + 2B_{12}\Psi_1\cos 2(\Psi_s + \Psi_0)\right.$$

$$\left. + B_{13}\cos 3(\Psi_s + \Psi_0) + B_{14}\sin 4(\Psi_s + \Psi_0)\right] + c^2\alpha_2^{-1}(-2\alpha_1)^{1/2}u^{1/2}\left[\Psi_1\frac{\partial B_1'}{\partial q_i}\right.$$

$$+ B_1'\frac{\partial \Psi_1}{\partial q_i} + \cos(\Psi_s + \Psi_0)\frac{\partial B_{11}}{\partial q_i} + 2\left(\Psi_1\frac{\partial B_{12}}{\partial q_i} + B_{12}\frac{\partial \Psi_1}{\partial q_i}\right)\cos 2(\Psi_s + \Psi_0)$$

$$+ \cos 3(\Psi_s + \Psi_0)\frac{\partial B_{13}}{\partial q_i} + \sin 4(\Psi_s + \Psi_0)\frac{\partial B_{14}}{\partial q_i} - \left[B_{11}\sin(\Psi_s + \Psi_0)\right.$$

$$\left.\left.\left. + 4B_{12}\Psi_1\sin 2(\Psi_s + \Psi_0) + 3B_{13}\sin 3(\Psi_s + \Psi_0) - 4B_{14}\cos 4(\Psi_s + \Psi_0)\right]\left(\frac{\partial \Psi_s}{\partial q_i} + \frac{\partial \Psi_0}{\partial q_i}\right)\right]\right\}.$$

Also, for i = 1, 2,

$$\frac{\partial M_2}{\partial \beta_i} = -(a + b_1)^{-1} \Bigg\{ A_1 \frac{\partial v_1}{\partial \beta_i} + (A_{11} \cos v' + 2 A_{12} \cos 2v') \frac{\partial v'}{\partial \beta_i}$$

$$+ c^2 \alpha_2^{-1} (-2\alpha_1)^{1/2} u^{1/2} \Bigg[B_1' \frac{\partial \Psi_1}{\partial \beta_i} + 2 B_{12} \cos 2(\Psi_s + \Psi_0) \frac{\partial \Psi_1}{\partial \beta_i} - \Big[B_{11} \sin (\Psi_s + \Psi_0)$$

$$+ 4 B_{12} \Psi_1 \sin 2(\Psi_s + \Psi_0) + 3 B_{13} \sin 3(\Psi_s + \Psi_0) - 4 B_{14} \cos 4(\Psi_s + \Psi_0)\Big] \left(\frac{\partial \Psi_s}{\partial \beta_i} + \frac{\partial \Psi_0}{\partial \beta_i}\right)\Bigg]\Bigg\} .$$

By equation (143.1), we have:

$$E_2 = \left[1 - e' \cos (\mathcal{E} + E_1)\right]^{-1} M_2 .$$

Thus, for i = 1, 2, 3,

$$\frac{\partial E_2}{\partial q_i} = \left[1 - e' \cos (\mathcal{E} + E_1)\right]^{-1} \frac{\partial M_2}{\partial q_i}$$

$$- M_2 \left[1 - e' \cos (\mathcal{E} + E_1)\right]^{-2} \left[e' \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial q_i} + \frac{\partial E_1}{\partial q_i}\right) - \cos (\mathcal{E} + E_1) \frac{\partial e'}{\partial q_i}\right] .$$

Also, for i = 1, 2,

$$\frac{\partial E_2}{\partial \beta_i} = \left[1 - e' \cos (\mathcal{E} + E_1)\right]^{-1} \frac{\partial M_2}{\partial \beta_i} - M_2 \left[1 - e' \cos (\mathcal{E} + E_1)\right]^{-2} e' \sin (\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial \beta_i} + \frac{\partial E_1}{\partial \beta_i}\right) .$$

Since $E = \mathcal{E} + E_1 + E_2$, then, for i = 1, 2, 3,

$$\frac{\partial E}{\partial q_i} = \frac{\partial \mathcal{E}}{\partial q_i} + \frac{\partial E_1}{\partial q_i} + \frac{\partial E_2}{\partial q_i} .$$

Also, for i = 1, 2,

$$\frac{\partial E}{\partial \beta_i} = \frac{\partial \mathcal{E}}{\partial \beta_i} + \frac{\partial E_1}{\partial \beta_i} + \frac{\partial E_2}{\partial \beta_i} .$$

By the anomaly connections, taken to second order (where we choose the notation v rather than v'''),

$$\cos v = (\cos E - e)\,(1 - e \cos E)^{-1} ,$$

and

$$\sin v = \left(1 - e^2\right)^{1/2} \left(1 - e \cos E\right)^{-1} \sin E \quad ,$$

where

$$v_2 = v - v'' = v - \left(M_s + v_0 + v_1\right) \quad .$$

Then, for $i = 1, 2, 3$,

$$\frac{\partial v_2}{\partial q_i} = \left[\left(1 - e^2\right) \sin E \frac{\partial E}{\partial q_i} + \delta_{2i} \sin^2 E\right] (\sin v)^{-1} (1 - e \cos E)^{-2} - \left(\frac{\partial v'}{\partial q_i} + \frac{\partial v_1}{\partial q_i}\right) \quad .$$

Also, for $i = 1, 2$,

$$\frac{\partial v_2}{\partial \beta_i} = \left(1 - e^2\right) \sin E (\sin v)^{-1} (1 - e \cos E)^{-2} \frac{\partial E}{\partial \beta_i} - \left(\frac{\partial v'}{\partial \beta_i} + \frac{\partial v_1}{\partial \beta_i}\right) \quad .$$

Since $v = v' + v_1 + v_2$, then, for $i = 1, 2, 3$,

$$\frac{\partial v}{\partial q_i} = \frac{\partial v'}{\partial q_i} + \frac{\partial v_1}{\partial q_i} + \frac{\partial v_2}{\partial q_i} \quad .$$

Also, for $i = 1, 2$,

$$\frac{\partial v}{\partial \beta_i} = \frac{\partial v'}{\partial \beta_i} + \frac{\partial v_1}{\partial \beta_i} + \frac{\partial v_2}{\partial \beta_i} \quad .$$

By the corrected version of equation (145), we have:

$$\begin{aligned} \Psi_2 = & - B_2^{-1} \left[- B_{21} \Psi_1 \sin \left(\Psi_s + \Psi_0\right) + 2 B_{22} \Psi_1 \cos 2\left(\Psi_s + \Psi_0\right) + B_{23} \cos 3\left(\Psi_s + \Psi_0\right)\right. \\ & \left. + B_{24} \sin 4\left(\Psi_s + \Psi_0\right)\right] + \alpha_2 \left(-2\alpha_1\right)^{-1/2} u^{-1/2} B_2^{-1} \left(A_2 v_2 + A_{21} v_1 \cos v'\right. \\ & \left. + 2 A_{22} v_1 \cos 2v' + A_{23} \sin 3v' + A_{24} \sin 4v'\right) \quad . \end{aligned}$$

Thus, for $i = 1, 2, 3,$

$$
\begin{aligned}
\frac{\partial \Psi_2}{\partial q_i} = {} & B_2^{-2}\Big[-B_{21}\Psi_1 \sin(\Psi_s+\Psi_0) + 2B_{22}\Psi_1\cos 2(\Psi_s+\Psi_0) + B_{23}\cos 3(\Psi_s+\Psi_0) \\
& + B_{24}\sin 4(\Psi_s+\Psi_0)\Big]\frac{\partial B_2}{\partial q_i} - B_2^{-1}\Bigg\{-\left(\Psi_1\frac{\partial B_{21}}{\partial q_i} + B_{21}\frac{\partial \Psi_1}{\partial q_i}\right)\sin(\Psi_s+\Psi_0) \\
& + 2\left(\Psi_1\frac{\partial B_{22}}{\partial q_i} + B_{22}\frac{\partial \Psi_1}{\partial q_i}\right)\cos 2(\Psi_s+\Psi_0) + \cos 3(\Psi_s+\Psi_0)\frac{\partial B_{23}}{\partial q_i} + \sin 4(\Psi_s+\Psi_0)\frac{\partial B_{24}}{\partial q_i} \\
& - \Big[B_{21}\Psi_1\cos(\Psi_s+\Psi_0) + 4B_{22}\Psi_1\sin 2(\Psi_s+\Psi_0) + 3B_{23}\sin 3(\Psi_s+\Psi_0) \\
& - 4B_{24}\cos 4(\Psi_s+\Psi_0)\Big]\left(\frac{\partial \Psi_s}{\partial q_i} + \frac{\partial \Psi_0}{\partial q_i}\right)\Bigg\} + (-2\alpha_1)^{-1/2}u^{-1/2}B_2^{-1}\Bigg[\frac{\partial \alpha_2}{\partial q_i} - \frac{1}{2}\alpha_2\Bigg(\alpha_1^{-1}\frac{\partial \alpha_1}{\partial q_i} + u^{-1}\frac{\partial u}{\partial q_i} \\
& + 2B_2^{-1}\frac{\partial B_2}{\partial q_i}\Bigg)\Bigg](A_2 v_2 + A_{21}v_1\cos v' + 2A_{22}v_1\cos 2v' + A_{23}\sin 3v' + A_{24}\sin 4v') \\
& + \alpha_2(-2\alpha_1)^{-1/2}u^{-1/2}B_2^{-1}\Bigg[v_2\frac{\partial A_2}{\partial q_i} + A_2\frac{\partial v_2}{\partial q_i} + \left(v_1\frac{\partial A_{21}}{\partial q_i} + A_{21}\frac{\partial v_1}{\partial q_i}\right)\cos v' \\
& + 2\left(v_1\frac{\partial A_{22}}{\partial q_i} + A_{22}\frac{\partial v_1}{\partial q_i}\right)\cos 2v' + \sin 3v'\frac{\partial A_{23}}{\partial q_i} + \sin 4v'\frac{\partial A_{24}}{\partial q_i} \\
& - (A_{21}v_1\sin v' + 4A_{22}v_1\sin 2v' - 3A_{23}\cos 3v' - 4A_{24}\cos 4v')\frac{\partial v'}{\partial q_i}\Bigg] .
\end{aligned}
$$

Also, for $i = 1, 2,$

$$
\begin{aligned}
\frac{\partial \Psi_2}{\partial \beta_i} = {} & B_2^{-1}\Bigg\{\Big[B_{21}\sin(\Psi_s+\Psi_0) - 2B_{22}\cos 2(\Psi_s+\Psi_0)\Big]\frac{\partial \Psi_1}{\partial \beta_i} + \Big[B_{21}\Psi_1\cos(\Psi_s+\Psi_0) \\
& + 4B_{22}\Psi_1\sin 2(\Psi_s+\Psi_0) + 3B_{23}\sin 3(\Psi_s+\Psi_0) - 4B_{24}\cos 4(\Psi_s+\Psi_0)\Big]\left(\frac{\partial \Psi_s}{\partial \beta_i} + \frac{\partial \Psi_0}{\partial \beta_i}\right)\Bigg\} \\
& + \alpha_2(-2\alpha_1)^{-1/2}u^{-1/2}B_2^{-1}\Bigg[A_2\frac{\partial v_2}{\partial \beta_i} + (A_{21}\cos v' + 2A_{22}\cos 2v')\frac{\partial v_1}{\partial \beta_i} \\
& - (A_{21}v_1\sin v' + 4A_{22}v_1\sin 2v' - 3A_{23}\cos 3v' - 4A_{24}\cos 4v')\frac{\partial v'}{\partial \beta_i}\Bigg] .
\end{aligned}
$$

Since $\Psi = \Psi_s + \Psi_0 + \Psi_1 + \Psi_2$, then for $i = 1, 2, 3$,

$$\frac{\partial\Psi}{\partial q_i} = \frac{\partial\Psi_s}{\partial q_i} + \frac{\partial\Psi_0}{\partial q_i} + \frac{\partial\Psi_1}{\partial q_i} + \frac{\partial\Psi_2}{\partial q_i} .$$

Also, for $i = 1, 2$,

$$\frac{\partial\Psi}{\partial \beta_i} = \frac{\partial\Psi_s}{\partial \beta_i} + \frac{\partial\Psi_0}{\partial \beta_i} + \frac{\partial\Psi_1}{\partial \beta_i} + \frac{\partial\Psi_2}{\partial \beta_i} .$$

This concludes the evaluation of the partial derivatives of the uniformizing variables E, v, and Ψ when the computation is followed through terms of the second order. If, however, second-order precision is not required, then the partial derivatives of M_2, E_2, v_2, and Ψ_2 may be omitted, and the above partial derivatives of the uniformizing variables reduce to

$$\frac{\partial E}{\partial q_i} = \frac{\partial \mathcal{E}}{\partial q_i} + \frac{\partial E_1}{\partial q_i} ,$$

$$\frac{\partial v}{\partial q_i} = \frac{\partial v'}{\partial q_i} + \frac{\partial v_1}{\partial q_i} ,$$

and

$$\frac{\partial\Psi}{\partial q_i} = \frac{\partial\Psi_s}{\partial q_i} + \frac{\partial\Psi_0}{\partial q_i} + \frac{\partial\Psi_1}{\partial q_i} .$$

Likewise for the partial derivatives of E, v, and Ψ with respect to β_i ($i = 1, 2$).

We now continue with the necessary equations preparatory to determining the partial derivatives of the inertial co-ordinates X, Y, and Z.

By special cases of equations (104) with $y = \Psi \pm (1/2)\pi$, we find:

$$\cos E_2' = \left(1 - e_2 \sin\Psi\right)^{-1} \left(e_2 - \sin\Psi\right) ,$$

$$\sin E_2' = \left(1 - e_2^2\right)^{1/2} \left(1 - e_2 \sin\Psi\right)^{-1} \cos\Psi ,$$

and

$$\cos E_3' = \left(1 + e_3 \sin\Psi\right)^{-1} \left(e_3 + \sin\Psi\right) ,$$

$$\sin E_3' = -\left(1 - e_3^2\right)^{1/2} \left(1 + e_3 \sin\Psi\right)^{-1} \cos\Psi .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial E_2'}{\partial q_i} = \left(1 - e_2 \sin\Psi\right)^{-1} \left[\left(1 - e_2^2\right)^{1/2} \frac{\partial \Psi}{\partial q_i} - \left(1 - e_2^2\right)^{-1/2} \cos\Psi \frac{\partial e_2}{\partial q_i}\right] .$$

and

$$\frac{\partial E_3'}{\partial q_i} = \left(1 + e_3 \sin\Psi\right)^{-1} \left[\left(1 - e_3^2\right)^{1/2} \frac{\partial \Psi}{\partial q_i} + \left(1 - e_3^2\right)^{-1/2} \cos\Psi \frac{\partial e_3}{\partial q_i}\right] .$$

Also, for $i = 1, 2,$

$$\frac{\partial E_2'}{\partial \beta_i} = \left(1 - e_2 \sin\Psi\right)^{-1} \left(1 - e_2^2\right)^{1/2} \frac{\partial \Psi}{\partial \beta_i} ,$$

and

$$\frac{\partial E_3'}{\partial \beta_i} = \left(1 + e_3 \sin\Psi\right)^{-1} \left(1 - e_3^2\right)^{1/2} \frac{\partial \Psi}{\partial \beta_i} .$$

By equations (114.1) and (114.2), we have:

$$\chi_0 = \frac{1}{2} (1 - 2\zeta)^{-1/2} E_2' + \frac{1}{2} (1 + 2\zeta)^{-1/2} E_3' ,$$

and

$$\chi_1 = \frac{1}{2} (1 - 2\zeta)^{-1/2} E_2' - \frac{1}{2} (1 + 2\zeta)^{-1/2} E_3' .$$

Thus, for $i = 1, 2, 3,$

$$\frac{\partial \chi_0}{\partial q_i} = \frac{1}{2} (1 - 2\zeta)^{-1/2} \frac{\partial E_2'}{\partial q_i} + \frac{1}{2} (1 + 2\zeta)^{-1/2} \frac{\partial E_3'}{\partial q_i} + \frac{1}{2} \left[E_2' (1 - 2\zeta)^{-3/2} - E_3' (1 + 2\zeta)^{-3/2}\right] \frac{\partial \zeta}{\partial q_i} ,$$

and

$$\frac{\partial \chi_1}{\partial q_i} = \frac{1}{2} (1 - 2\zeta)^{-1/2} \frac{\partial E_2'}{\partial q_i} - \frac{1}{2} (1 + 2\zeta)^{-1/2} \frac{\partial E_3'}{\partial q_i} + \frac{1}{2} \left[E_2' (1 - 2\zeta)^{-3/2} + E_3' (1 + 2\zeta)^{-3/2}\right] \frac{\partial \zeta}{\partial q_i} .$$

Also, for $i = 1, 2$,

$$\frac{\partial \chi_0}{\partial \beta_i} = \frac{1}{2}(1-2\zeta)^{-1/2}\frac{\partial E_2'}{\partial \beta_i} + \frac{1}{2}(1+2\zeta)^{-1/2}\frac{\partial E_3'}{\partial \beta_i} ,$$

and

$$\frac{\partial \chi_1}{\partial \beta_i} = \frac{1}{2}(1-2\zeta)^{-1/2}\frac{\partial E_2'}{\partial \beta_i} - \frac{1}{2}(1+2\zeta)^{-1/2}\frac{\partial E_3'}{\partial \beta_i} .$$

By equation (150), we have:

$$\begin{aligned}\phi = \beta_3 &- c^2\alpha_3\left(-2\alpha_1\right)^{-1/2}\left(A_3 v + A_{31}\sin v + A_{32}\sin 2v + A_{33}\sin 3v + A_{34}\sin 4v\right)\\ &+ \alpha_3\alpha_2^{-1}u^{1/2}\left\{(1-S)^{-1/2}\left[\left(h_1+h_2\right)\chi_0 + \left(h_1-h_2\right)\chi_1\right]\right.\\ &\left.+ B_3\Psi - \frac{3}{4}C_1C_2Q\cos\Psi + \frac{3}{32}C_2^2Q^2\sin 2\Psi\right\} .\end{aligned}$$

Thus, for $i = 1, 2, 3$,

$$\begin{aligned}\frac{\partial\phi}{\partial q_i} = &-c^2\left[\left(-2\alpha_1\right)^{-1/2}\frac{\partial\alpha_3}{\partial q_i} + \alpha_3\left(-2\alpha_1\right)^{-3/2}\frac{\partial\alpha_1}{\partial q_i}\right]\left(A_3 v + A_{31}\sin v + A_{32}\sin 2v\right.\\ &\left.+ A_{33}\sin 3v + A_{34}\sin 4v\right) - c^2\alpha_3\left(-2\alpha_1\right)^{-1/2}\left[v\frac{\partial A_3}{\partial q_i} + \sin v\frac{\partial A_{31}}{\partial q_i} + \sin 2v\frac{\partial A_{32}}{\partial q_i}\right.\\ &+ \sin 3v\frac{\partial A_{33}}{\partial q_i} + \sin 4v\frac{\partial A_{34}}{\partial q_i} + \left(A_3 + A_{31}\cos v + 2A_{32}\cos 2v + 3A_{33}\cos 3v\right.\\ &\left.\left.+ 4A_{34}\cos 4v\right)\frac{\partial v}{\partial q_i}\right] + \alpha_2^{-1}u^{1/2}\left(\frac{\partial\alpha_3}{\partial q_i} - \alpha_3\alpha_2^{-1}\frac{\partial\alpha_2}{\partial q_i} + \frac{1}{2}\alpha_3u^{-1}\frac{\partial u}{\partial q_i}\right)\left\{(1-S)^{-1/2}\left[\left(h_1+h_2\right)\chi_0\right.\right.\\ &\left.\left.+\left(h_1-h_2\right)\chi_1\right] + B_3\Psi - \frac{3}{4}C_1C_2Q\cos\Psi + \frac{3}{32}C_2^2Q^2\sin 2\Psi\right\}\\ &+ \alpha_3\alpha_2^{-1}u^{1/2}\left\{\frac{1}{2}\delta_{3i}(1-S)^{-3/2}\left[\left(h_1+h_2\right)\chi_0 + \left(h_1-h_2\right)\chi_1\right] + (1-S)^{-1/2}\left[\left(h_1+h_2\right)\frac{\partial\chi_0}{\partial q_i}\right.\right.\\ &\left.+\left(h_1-h_2\right)\frac{\partial\chi_1}{\partial q_i} + \left(\chi_0+\chi_1\right)\frac{\partial h_1}{\partial q_i} + \left(\chi_0-\chi_1\right)\frac{\partial h_2}{\partial q_i}\right] + \Psi\frac{\partial B_3}{\partial q_i} - \frac{3}{4}\left(C_2Q\frac{\partial C_1}{\partial q_i}\right.\\ &\left.+C_1Q\frac{\partial C_2}{\partial q_i} + C_1C_2\frac{\partial Q}{\partial q_i}\right)\cos\Psi + \frac{3}{16}C_2Q\left(Q\frac{\partial C_2}{\partial q_i} + C_2\frac{\partial Q}{\partial q_i}\right)\sin 2\Psi\\ &\left.+\left(B_3 + \frac{3}{4}C_1C_2Q\sin\Psi + \frac{3}{16}C_2^2Q^2\cos 2\Psi\right)\frac{\partial\Psi}{\partial q_i}\right\} .\end{aligned}$$

Also, for i = 1, 2,

$$\frac{\partial \phi}{\partial \beta_i} = -c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left(A_3 + A_{31} \cos v + 2A_{32} \cos 2v + 3A_{33} \cos 3v + 4A_{34} \cos 4v\right) \frac{\partial v}{\partial \beta_i}$$

$$+ \alpha_3 \alpha_2^{-1} u^{1/2} \left\{ (1-S)^{-1/2} \left[(h_1 + h_2) \frac{\partial \chi_0}{\partial \beta_i} + (h_1 - h_2) \frac{\partial \chi_1}{\partial \beta_i} \right] \right.$$

$$\left. + \left(B_3 + \frac{3}{4} C_1 C_2 Q \sin \Psi + \frac{3}{16} C_2^2 Q^2 \cos 2\Psi\right) \frac{\partial \Psi}{\partial \beta_i} \right\} ,$$

and

$$\frac{\partial \phi}{\partial \beta_3} = 1 .$$

Since the spheroidal co-ordinates are given by

$$\rho = a(1 - e \cos E) ,$$

and

$$\eta = P + Q \sin \Psi ,$$

we then have

$$\frac{\partial \rho}{\partial a} = 1 - e\left(\cos E - a \sin E \frac{\partial E}{\partial a}\right) ,$$

$$\frac{\partial \rho}{\partial e} = a\left(e \sin E \frac{\partial E}{\partial e} - \cos E\right) ,$$

and

$$\frac{\partial \rho}{\partial S} = a e \sin E \frac{\partial E}{\partial S} .$$

Also, for i = 1, 2,

$$\frac{\partial \rho}{\partial \beta_i} = a e \sin E \frac{\partial E}{\partial \beta_i} .$$

Further, for $i = 1, 2, 3$,

$$\frac{\partial \eta}{\partial q_i} = \frac{\partial P}{\partial q_i} + \sin\Psi \frac{\partial Q}{\partial q_i} + Q\cos\Psi \frac{\partial \Psi}{\partial q_i} ,$$

and, for $i = 1, 2$,

$$\frac{\partial \eta}{\partial \beta_i} = Q\cos\Psi \frac{\partial \Psi}{\partial \beta_i}.$$

The partial derivatives of ρ and η with respect to β_3 are both zero.

Finally, the inertial rectangular co-ordinates are given by

$$X = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \cos\phi ,$$

$$Y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \sin\phi ,$$

and

$$Z = \rho\eta - \delta .$$

Thus, for $i = 1, 2, 3$,

$$\frac{\partial X}{\partial q_i} = X\left[\rho(\rho^2 + c^2)^{-1} \frac{\partial \rho}{\partial q_i} - \eta(1 - \eta^2)^{-1} \frac{\partial \eta}{\partial q_i}\right] - Y \frac{\partial \phi}{\partial q_i} ,$$

$$\frac{\partial Y}{\partial q_i} = Y\left[\rho(\rho^2 + c^2)^{-1} \frac{\partial \rho}{\partial q_i} - \eta(1 - \eta^2)^{-1} \frac{\partial \eta}{\partial q_i}\right] + X \frac{\partial \phi}{\partial q_i} ,$$

and

$$\frac{\partial Z}{\partial q_i} = \rho \frac{\partial \eta}{\partial q_i} + \eta \frac{\partial \rho}{\partial q_i} .$$

Also, for $i = 1, 2$,

$$\frac{\partial X}{\partial \beta_i} = X\left[\rho(\rho^2 + c^2)^{-1} \frac{\partial \rho}{\partial \beta_i} - \eta(1 - \eta^2)^{-1} \frac{\partial \eta}{\partial \beta_i}\right] - Y \frac{\partial \phi}{\partial \beta_i} ,$$

$$\frac{\partial Y}{\partial \beta_i} = Y\left[\rho\left(\rho^2 + c^2\right)^{-1}\frac{\partial \rho}{\partial \beta_i} - \eta\left(1-\eta^2\right)^{-1}\frac{\partial \eta}{\partial \beta_i}\right] + X\frac{\partial \phi}{\partial \beta_i} \quad ,$$

$$\frac{\partial Z}{\partial \beta_i} = \rho\frac{\partial \eta}{\partial \beta_i} + \eta\frac{\partial \rho}{\partial \beta_i} \quad ,$$

and

$$\frac{\partial X}{\partial \beta_3} = -Y \quad ,$$

$$\frac{\partial Y}{\partial \beta_3} = X \quad ,$$

and

$$\frac{\partial Z}{\partial \beta_3} = 0 \quad .$$

OBSERVATIONS OF THE RIGHT ASCENSION-DECLINATION TYPE

The differential coefficients in the form of partial derivatives of the inertial rectangular co-ordinates with respect to the orbital elements are completely general in the sense that they are functions only of the mathematical theory of orbital satellite motion. Thus, they are applicable to observations of spacecraft position recorded in any format whatsoever. Previously, we had assumed that the observational data were recorded as direction cosines with respect to a topocentric latitude-longitude-zenith co-ordinate system. Another format frequently used for recording observational data is the right ascension-declination type. In this section, we shall discuss the minor variations in the equations that arise when this type of data is utilized.

The co-ordinate system adopted for the use of right ascension-declination data is also situated at the tracking station on the Earth's surface, but its three co-ordinate axes are parallel to the respective axes of the inertial system. Again designating the topocentric "local" co-ordinates by the subscript "M", we have, in this case, the Z_M-axis parallel to the Earth's polar axis, and the $X_M - Y_M$ plane parallel to the equatorial plane of the Earth. The X_M-axis extends toward the vernal equinox, with the Y_M-axis orthogonally to the east to form a right-handed system.

The observed right ascension, α_0, is measured eastward from the vernal equinox, and the observed declination, δ_0, is measured as positive above and as negative below the equatorial plane. The corresponding computed values of the right ascension and the declination are given in terms of the local co-ordinates by

$$\alpha_c = \arctan\left(\frac{Y_M}{X_M}\right) ,$$

and

$$\delta_c = \arctan\left[\frac{Z_M}{\left(X_M^2 + Y_M^2\right)^{1/2}}\right] .$$

In order to obtain a satellite's local co-ordinates from its inertial co-ordinates, the inertial co-ordinates of the observation point at the time of observation, denoted (X_T, Y_T, Z_T), must be known. However, no rotations are necessary to bring the two systems into coincidence in this case, since the topocentric and inertial co-ordinate systems are parallel. A single translation will suffice. Hence, the relations for the local co-ordinates of the satellite are simply

$$X_M = X - X_T ,$$

$$Y_M = Y - Y_T ,$$

and

$$Z_M = Z - Z_T .$$

Notice that the above simplified relations are obtained from those of the direction-cosine-data case by the artifice of setting $\psi_x = 0$ and $\theta_D = \pi/2$ in the corresponding equations for X_M, Y_M, and Z_M given earlier.

The first-order Taylor's series expansion for the equations of condition corresponding to each time of observation are

$$\alpha_0 - \alpha_c = \sum_{i=1}^{6} \frac{\partial \alpha_c}{\partial q_i} \Delta q_i ,$$

and

$$\delta_0 - \delta_c = \sum_{i=1}^{6} \frac{\partial \delta_c}{\partial q_i} \Delta q_i ,$$

where q_i ($i = 1, 2, \cdots, 6$) are the orbital elements. Expanding the above partial derivatives by the chain rule yields

$$\frac{\partial \alpha_c}{\partial q_i} = \frac{\partial \alpha_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial \alpha_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial \alpha_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i} ,$$

and

$$\frac{\partial \delta_c}{\partial q_i} = \frac{\partial \delta_c}{\partial X_M}\frac{\partial X_M}{\partial q_i} + \frac{\partial \delta_c}{\partial Y_M}\frac{\partial Y_M}{\partial q_i} + \frac{\partial \delta_c}{\partial Z_M}\frac{\partial Z_M}{\partial q_i} .$$

From the expressions for α_c and δ_c in terms of the local co-ordinates, we find

$$\frac{\partial \alpha_c}{\partial X_M} = -\frac{Y_M}{r_M^2} ,$$

$$\frac{\partial \alpha_c}{\partial Y_M} = +\frac{X_M}{r_M^2} ,$$

$$\frac{\partial \alpha_c}{\partial Z_M} = 0 ,$$

$$\frac{\partial \delta_c}{\partial X_M} = -\frac{X_M Z_M}{r_M R_M^2} ,$$

$$\frac{\partial \delta_c}{\partial Y_M} = -\frac{Y_M Z_M}{r_M R_M^2} ,$$

and

$$\frac{\partial \delta_c}{\partial Z_M} = +\frac{r_M}{R_M^2} ,$$

where $R_M \equiv \left(X_M^2 + Y_M^2 + Z_M^2\right)^{1/2}$ and $r_M \equiv \left(X_M^2 + Y_M^2\right)^{1/2}$.

Since the station co-ordinates, X_T, Y_T, and Z_T, are independent of the orbital elements (and merely geodesic functions), the following simple relations apply:

$$\frac{\partial X_M}{\partial q_i} = \frac{\partial X}{\partial q_i} ,$$

$$\frac{\partial Y_M}{\partial q_i} = \frac{\partial Y}{\partial q_i},$$

and

$$\frac{\partial Z_M}{\partial q_i} = \frac{\partial Z}{\partial q_i}.$$

The differential coefficients, $\partial X/\partial q_i$, $\partial Y/\partial q_i$, and $\partial Z/\partial q_i$ ($i = 1, 2, \cdots, 6$) are precisely those that have been evaluated previously in the differential correction scheme.

REMARKS

The differential correction process removes inaccuracies of the initial conditions (the nominal observations) and accounts for the effects of forces not considered by the analytical orbital theory. Such neglected forces may include aerodynamic drag, electromagnetic effects, solar radiation pressure, meteoric bombardment, and residual gravitational influences (including those arising from lack of spherical symmetry in the satellite, as well as perturbing planetary potentials). This is all accomplished by producing a mean set of orbital elements through an iterated least-squares fitting of the first-order Taylor's series expansion of the conditional equations to numerous observational values. Generally speaking, the fitting will improve as greater numbers of observations are considered and as the time span represented by the observational data is lengthened. However, the complexity of the mathematical processes involved in the fitting increases rapidly as additional observations are admitted. Because of this latter constraint, it is often advisable to perform repeated differential corrections at various intervals of time (known as "epochs") rather than attempt to accommodate all the data in a single iterated fitting.

The orbital improvement method of differential correction discussed in this paper has been programmed, primarily in the FORTRAN language, for use on a large-scale electronic digital computer. The analytical nature of the entire procedure assures a very rapid computational process. Application of the differential correction (combined with an orbit generator of position and velocity components) has been made to both actual observational data of artificial Earth satellites and to artificially generated "data" for extremum cases, e.g., polar and equatorial orbits. The results have proven entirely favorable. Particular experimental applications will be published in a later paper.

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APPENDIX

FUNCTIONAL DEPENDENCIES OF THE PARTIAL DERIVATIVES

The following tables are intended to display the inter-relationship of the various parameters whose partial derivatives appear in the differential correction. The partial derivative of each parameter in the left column is seen to be a function of those, and only those, partial derivatives of parameters occurring in the respective line of the right column. There is a certain amount of flexibility in the ordering of the partial derivatives occurring in the differential correction, as is demonstrated by the functional dependencies illustrated by these tables.

Table I

Time-independent Partial Derivatives
(All taken with respect to orbital elements a, e, and S.)

Partial Derivative	Functional Dependence on Other Partial Derivatives
p	(none)
b_1	p
b_2	p, b_1
p_0	p, b_1, b_2
α_2	p_0
$(a_0\, p_0)^{-1}$	b_1, p_0
u	p_0, $(a_0\, p_0)^{-1}$
C_2	$(a_0\, p_0)^{-1}$, u
C_1	p_0, u, C_2
P	p_0, u, C_2
α_1	b_1
$\left(\frac{b_1}{b_2}\right)$	b_1, b_2
$\left(\frac{b_2}{p}\right)$	p, b_2
A_1	p, $\left(\frac{b_1}{b_2}\right)$, $\left(\frac{b_2}{p}\right)$

Table I (Continued)

Partial Derivative	Functional Dependence on Other Partial Derivatives
A_2	$p, \left(\frac{b_1}{b_2}\right), \left(\frac{b_2}{p}\right)$
A_3	$p, \left(\frac{b_1}{b_2}\right), \left(\frac{b_2}{p}\right)$
A_{21}	p, b_1, b_2
A_{22}	p, b_1, b_2
A_{31}	p, b_1, b_2
A_{32}	p, b_1, b_2
A_{33}	p, b_1, b_2
A_{34}	p, b_2
Q	P
ζ	P
h_1	C_2, C_1
h_2	C_2, C_1
e_2	P, Q
e_3	P, Q
e'	b_1
α_3	α_2, u
B_1'	C_2, C_1, P, Q
B_2	C_2, C_1, P, Q
B_3	C_2, C_1, P, Q
B_{12}	C_2, Q
B_{21}	C_2, C_1, P, Q
B_{22}	C_2, C_1, Q
$(2\pi\nu_1)$	$b_1, \alpha_1, A_1, A_2, B_1', B_2$
$(2\pi\nu_2)$	$b_1, \alpha_2, u, A_1, A_2, B_1', B_2$
$*A_{11}$	p, b_1, b_2
$*A_{12}$	p, b_2
$*A_{23}$	p, b_1, b_2

Table I (Concluded)

Partial Derivative	Functional Dependence on Other Partial Derivatives
$^*A_{24}$	p, b_2
$^*B_{11}$	C_1, P, Q
$^*B_{13}$	C_1, Q
$^*B_{14}$	C_2, Q
$^*B_{23}$	C_2, C_1, Q
$^*B_{24}$	C_2, Q

Table II

Time-varying Partial Derivatives (All taken with respect to orbital elements a, e, S, β_1, and β_2. Exception: ϕ, X, and Y are taken with respect to β_3, as well.)

Partial Derivative	Functional Dependence on Other Partial Derivatives
M_s	$\alpha_2, B_1', B_2, (2\pi\nu_1)$
Ψ_s	$b_1, \alpha_2, A_1, A_2, (2\pi\nu_2)$
$\mathcal{E}$	e', M_s
v_0	$M_s, \mathcal{E}$
Ψ_0	$\alpha_2, u, \alpha_1, A_2, B_2, v_0$
M_1	$b_1, \alpha_2, u, \alpha_1, A_1, A_2, B_1', B_2, B_{12}, \Psi_s, v_0, \Psi_0$
E_1	$e', \mathcal{E}, M_1$
v'	M_s, v_0
v_1	$\mathcal{E}, E_1, v'$
Ψ_1	$\alpha_2, u, \alpha_1, A_2, A_{21}, A_{22}, B_2, B_{21}, B_{22}, \Psi_s, \Psi_0, v', v_1$
$^*M_2$	$b_1, \alpha_2, u, \alpha_1, A_1, B_1', B_{12}, A_{11}, A_{12}, B_{11}, B_{13}, B_{14}, \Psi_s, \Psi_0, v', v_1, \Psi_1$
$^*E_2$	$e', \mathcal{E}, E_1, M_2$
E	$\mathcal{E}, E_1, E_2$
$^*v_2$	v', v_1, E
v	v', v_1, v_2

Table II (Concluded)

Partial Derivative	Functional Dependence on Other Partial Derivatives
$^*\Psi_2$	α_2, u, α_1, A_2, A_{21}, A_{22}, B_2, B_{21}, B_{22}, A_{23}, A_{24}, B_{23}, B_{24}, Ψ_s, Ψ_0, v', v_1, Ψ_1, v_2
Ψ	Ψ_s, Ψ_0, Ψ_1, Ψ_2
E_2'	e_2, Ψ
E_3'	e_3, Ψ
χ_0	ζ, E_2', E_3'
χ_1	ζ, E_2', E_3'
ζ	α_2, u, C_2, C_1, α_1, A_3, A_{31}, A_{32}, A_{33}, A_{34}, Q, h_1, h_2, α_3, B_3, v, Ψ, χ_0, χ_1
ρ	E
η	P, Q, Ψ
X	ϕ, ρ, η
Y	ϕ, ρ, η
Z	ρ, η

Note: In Tables I and II, the asterisk indicates partial derivatives of parameters used only if the differential correction includes periodic terms through the second order.