Nasa CR 65595

OPTIMUM TRAJECTORIES IN A CENTRAL FORCE FIELD

prepared for

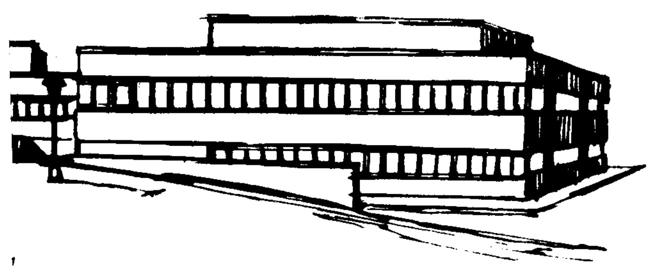
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

by

Carlos R. Cavoti

Contract NAS 9-5550

January 1967



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ABSTRACT

Necessary conditions for a strong relative minimum are derived within the framework of a generalized Mayer formulation. Such necessary conditions apply to a variety of optimal problems which are obtained as particular cases of the general form here considered.

The trajectories involved may imply an orbital transfer, a lunar landing, etc.

The existence of intermediate - thrust sub-arcs is analyzed for a minimum fuel consumption problem using the general conditions derived. The analysis shows that no intermediate - thrust sub-arc may form part of the extremal in such case.

The technique applied in the demonstration appears to be applicable to investigate the existence of intermediate - thrust sub-arcs in other variational problems which derive from the general form considered.

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LIST OF SYMBOLS

g	Acceleration of gravity
g_o	Acceleration of gravity at radius r_o
G	Function to be minimized
$\mathcal{L}$	Legendre Condition
m	Mass of the vehicle
r	Radius from the center of attraction
t	Time
T	Thrust of rocket engine
V	Velocity
V_e	Velocity of burned gases through the exit section of rocket nozzle
W	Weierstrass Condition
Z	Dimensionless velocity
γ	Central angle
λ	Dimensionless Thrust
θ	Angle between vector velocity and local horizon
τ	Dimensionless time
μ	Dimensionless mass
ψ	Angle between vector thrust and vector velocity
φ	Differential constraint
χ	Boundary condition

LIST OF SYMBOLS (Continued)

Λ Fundamental function

ν Variable Lagrange multiplier

Subscripts and Superscripts

$(\ldots)'$ Total derivative with respect to τ

$(\ldots)_I$ Quantity evaluated at initial point

$(\ldots)_F$ Quantity evaluated at final point

$(\ldots)_R$ Reference value

1. EQUATIONS OF MOTION AND GENERALIZED VARIATIONAL FORMULATION

The equations of planar motion of a mass-point vehicle which is subject to an inverse-square gravitational force field and a thrusting force $\overline{T}$ are

$$\varphi_1 \equiv \gamma' - \frac{Z \cos \theta}{\rho} = 0 \,, \tag{1}$$

$$\varphi_2 \equiv \rho' - Z \sin \theta = 0 \,, \tag{2}$$

$$\varphi_3 \equiv Z' + \frac{\sin \theta}{\rho^2} - \frac{\lambda}{\mu} \cos \psi = 0 \,, \tag{3}$$

$$\varphi_4 \equiv \theta' - \left(\frac{Z}{\rho} - \frac{1}{Z\rho^2}\right) \cos \theta - \frac{\lambda}{Z\mu} \sin \psi = 0 \,, \tag{4}$$

$$\varphi_5 \equiv \mu' + \frac{\lambda}{v_e} = 0 \,. \tag{5}$$

The forces acting on the vehicle and the coordinate systems of reference are shown in Fig. 1. Eqs. (1) to (5) are referred to a tangential intrinsic system of coordinates ($\overline{u}_t$, $\overline{u}_n$). The dimensionless variables used are

$$\tau = \frac{tg_o}{V_R} \,, \quad Z = \frac{V}{V_R} \,, \quad \mu = \frac{m}{m_I} \,, \quad v_e = \frac{V_e}{V_R} \,,$$

$$\lambda = -\mu' v_e = \frac{T}{m_I g_o} \quad , \quad \rho = \frac{r}{r_o} \qquad .$$

The acceleration of gravity and the reference velocity are

$$g = g_o \left(\frac{r_o}{r_o + h}\right)^2 = g_o \left(\frac{r_o}{r}\right)^2 \quad , \quad V_R = \left(g_o \, r_o\right)^{1/2} \qquad .$$

In the following analysis it will be assumed that the thrust direction ψ is unbounded while the thrust magnitude is bounded, i.e., $0 \leqq \lambda \leqq \lambda_{max}$.

Any solution of Eqs. (1) to (5) is given in terms of five state variable functions, $\gamma(\tau)$, $\rho(\tau)$, $Z(\tau)$, $\theta(\tau)$, $\mu(\tau)$, and two control variables functions, $\lambda(\tau)$ and $\psi(\tau)$. Thus, the problem has two degrees of freedom associated with the two control variables.

Since an optimum requirement may be imposed we will propose a generalized minimal problem. The object is to derive general necessary conditions for an extremum which therefore - with appropriate simplifications - can be applied to treat a variety of particular cases.

Thus, the general variational problem proposed is that of minimizing the function

$$G = G\left(\gamma_I, \gamma_F, \rho_I, \rho_F, Z_I, Z_F, \theta_I, \theta_F, \mu_I, \mu_F, \tau_I, \tau_F\right) \,, \qquad (6)$$

Subject to the differential constraints given by Eqs. (1) to (5), and arbitrary boundary conditions of the form

$$\chi_{\ell}\left(\gamma_I, \gamma_F, \rho_I, \rho_F, \ldots\ldots, \tau_I, \tau_F\right) = 0\ ,\ \ell = 1, \ldots, S \leqq 11 \tag{7}$$

Introducing the Fundamental Function

$$\Lambda = \sum_{i=1,\ldots,5} \nu_i \varphi_i \quad , \tag{8}$$

and taking the first variation of the Mayer problem proposed, the following Euler equations are obtained

$$\nu_1' = 0 \quad , \quad \nu_1 = K_1 = \text{const.} \quad , \tag{9}$$

$$\nu_2' = \nu_1 \frac{Z \cos\theta}{\rho^2} - 2\nu_3 \frac{\sin\theta}{\rho^3} - \nu_4\left(\frac{2}{Z\rho^3} - \frac{Z}{\rho^2}\right)\cos\theta \ , \tag{10}$$

$$\nu_3' = -\nu_1 \frac{\cos\theta}{\rho} - \nu_2 \sin\theta - \nu_4\left[\left(\frac{1}{Z^2\rho^2} + \frac{1}{\rho}\right)\cos\theta - \frac{\lambda}{Z^2\mu} \sin\psi\right] \ , \tag{11}$$

$$\nu_4' = \nu_1 \frac{Z \sin\theta}{\rho} - \nu_2 Z \cos\theta + \nu_3 \frac{\cos\theta}{\rho^2} + \nu_4\left(\frac{Z}{\rho} - \frac{1}{Z\rho^2}\right)\sin\theta, \tag{12}$$

$$\nu_5' = \left(\nu_3 \cos\psi + \nu_4 \frac{\sin\psi}{Z}\right)\frac{\lambda}{\mu^2} \quad , \tag{13}$$

$$\Lambda_\lambda = -\frac{1}{\mu}\left(\nu_3 \cos\psi + \nu_4 \frac{\sin\psi}{Z}\right) + \nu_5 \frac{1}{v_e} \quad , \tag{14}$$

$$\Lambda_\psi = \frac{\lambda}{\mu}\left(\nu_3 \sin\psi - \nu_4 \frac{\cos\psi}{Z}\right) . \tag{15}$$

Since ψ is unbounded Eq. (15) leads to the necessary condition

$$\nu_3 \sin\psi - \nu_4 \frac{\cos\psi}{Z} = 0 . \tag{16}$$

Eq. (14), however, due to the boundedness of λ leads to the following conditions

$$\Lambda_\lambda \geqq 0 \quad \text{for} \quad \lambda = 0 \ , \tag{17}$$

$$\Lambda_\lambda = 0 \quad \text{for} \quad 0 \leqq \lambda \leqq \lambda_{max} \quad \left(\text{i. e., } \lambda = \lambda_{var}\right) , \tag{18}$$

$$\Lambda_\lambda \leqq 0 \quad \text{for} \quad \lambda = \lambda_{max} \ . \tag{19}$$

The function Λ_λ is called the "switching function" since it determines the mode of operation of the control variable λ.

Due to the fact that the Fundamental Function [Eq. (8)] is time-independent the following general first integral can be derived

$$K_1 \frac{Z\cos\theta}{\rho} + \nu_2 Z \sin\theta + \nu_3\left(\frac{\lambda}{\mu}\cos\psi - \frac{\sin\theta}{\rho^2}\right)$$

$$+ \nu_4 \left[\left(\frac{Z}{\rho} - \frac{1}{Z\rho^2}\right)\cos\theta + \frac{\lambda}{Z\mu}\sin\psi\right] - \nu_5 \frac{\lambda}{v_e} = C = \text{const.} \tag{20}$$

The previous first integral is applicable along any sub-arc forming the trajectory. As shown by Eqs. (17) to (19) an extremal arc may be formed by sub-arcs of three different types. In this regard, it is of particular interest to determine whether sub-arcs along which the control variable λ is used at an intermediate level (i. e., between bounds, that is, $\lambda = \lambda_{var}$) may form part of the extremal arc, and under what conditions. This aspect, of interest from a theoretical as well as a practical point of view, will be discussed in the following paragraphs.

2. CONDITIONS ALONG λ_{var} SUB-ARCS

Along λ_{var} sub-arcs Eqs. (14), (18) and (20) lead to the simplified first integral

$$K_1 \frac{Z \cos\theta}{\rho} + \nu_2 Z \sin\theta - \nu_3 \frac{\sin\theta}{\rho^2} + \nu_4 \cos\theta \left(\frac{Z}{\rho} - \frac{1}{Z\rho^2}\right) = C = \text{const.} \tag{21}$$

Also, Eqs. (13), (14) and (18) provide

$$\nu_5' = \frac{\lambda}{\mu v_e} \nu_5 \quad , \tag{22}$$

and therefore, using Eq. (5), follows

$$\nu_5 = \frac{K_5}{\mu} \quad , \quad K_5 = \text{const.} \tag{23}$$

Thus, Eqs. (14), (18) and (23) lead to

$$\nu_3 \cos \psi + \nu_4 \frac{\sin \psi}{Z} = \frac{K_5}{v_e} = K_6 = \text{const.} \tag{24}$$

The system of Eqs. (16) and (24) provides the following solution for the multipliers ν_3 and ν_4

$$\nu_3 = K_6 \cos \psi \, , \tag{25}$$

$$\nu_4 = K_6 Z \sin \psi \, . \tag{26}$$

Consequently, replacing the previous solutions in the first integral [Eq. (21)], we derive the ν_2 - multiplier as

$$\nu_2 = K_6 \frac{\cos \psi}{Z\rho^2} - K_1 \frac{1}{\rho \tan \theta} - K_6 \frac{\sin \psi}{\tan \theta} \left(\frac{Z}{\rho} - \frac{1}{Z\rho^2} \right) + \frac{C}{Z \sin \theta} \, . \tag{27}$$

To the extent of obtaining a necessary condition in terms of the state-variables of the problem, we can now use the previous expressions in order to eliminate the multipliers. Thus, differentiating Eq. (25) and using Eqs. (11) and (25) to (27) we obtain

$$K_6 \psi' = K_6 \left(\frac{\cos \psi \sin \theta}{Z\rho^2 \sin \psi} + 2 \frac{\cos \theta}{Z\rho^2} - \frac{\lambda \sin \psi}{\mu Z} \right) + \frac{C}{Z \sin \psi} \, . \tag{28}$$

A similar procedure, using Eqs. (12) and (25) to (27), provides

$$K_6 \psi' = K_6 \left[\frac{\sin \psi}{Z \cos \psi \sin \theta} \left(\frac{Z^2}{\rho} - \frac{1}{\rho^2} \right) - \frac{\sin \psi}{Z \cos \psi} \left(\frac{\lambda}{\mu} \cos \psi - \frac{\sin \theta}{\rho^2} \right) \right]$$

$$+ K_1 \frac{1}{\rho \sin \theta \cos \psi} - \frac{C \cos \theta}{Z \sin \theta \cos \psi} \quad . \qquad (29)$$

Consequently, Eqs. (28) and (29), imply the following General Necessary Condition which must be satisfied along λ_{var} sub-arcs

$$K_6 \left[\frac{\cos^2 \psi \sin^2 \theta}{Z\rho^2} + 2 \frac{\sin \theta \cos \theta \sin \psi \cos \psi}{Z\rho^2} + \frac{\sin^2 \psi \cos^2 \theta}{Z\rho^2} - \frac{Z \sin^2 \psi}{\rho} \right]$$

$$- K_1 \frac{\sin \psi}{\rho} + \frac{C}{Z} \left(\sin \theta \cos \psi + \cos \theta \sin \psi \right) = 0 \quad . \qquad (30)$$

Eq. (30) is a general condition since it applies to any variational problem involving the minimization of a function of the form given in Eq. (6) with arbitrary boundary conditions as specified by Eq. (7). The previous General Necessary Condition (for λ_{var} sub-arcs) may be written in the simplier form

$$\frac{K_6}{Z\rho^2} \left[\sin^2 (\psi + \theta) - Z^2 \rho \sin^2 \psi \right] - K_1 \frac{\sin \psi}{\rho} + \frac{C}{Z} \sin \left(\psi + \theta \right) = 0 \ . \quad (31)$$

3. CONDITIONS ALONG λ_{max} SUB-ARCS

Further necessary conditions which must be satisfied along the different extremal sub-arcs may be derived replacing the control variable λ by a new differentiated variable, say ζ'. In this case there is no need to add the equation of definition $\zeta' - \lambda = 0$ to Eqs. (1) to (5). In fact, since ζ does not appear explicitly in the constraints, (nor does it appear in the function to be minimized or in the boundary conditions), then the associated multiplier vanishes throughout. Since the admissible variations of ζ $\left(\delta\zeta = \int_{\tau_o}^{\tau} \delta\lambda \, d\tau\right)$ are: $\delta\zeta \geqq 0$, along $\lambda = \lambda_{min} = 0$; $\delta\zeta \leqq 0$, along $\lambda = \lambda_{max}$, and $\delta\zeta \gtreqless 0$ along λ_{var} sub-arcs, then three new necessary conditions are derived from the equation of variation associated with $\delta\zeta$. Corresponding to the previous unrestricted or both-sided variations ($\delta\zeta \gtreqless 0$) and restricted or one-sided variations ($\delta\zeta \geqq 0$, $\delta\zeta \leqq 0$), the following conditions are obtained

$$\text{a)} \quad \frac{d}{d\tau}\left(\Lambda_{\zeta'}\right) = \frac{d}{d\tau}\left(\Lambda_{\lambda}\right) = 0 \; , \quad \text{for } \lambda = \lambda_{var} \; , \tag{32}$$

$$\text{b)} \quad \frac{d}{d\tau}\left(\Lambda_{\lambda}\right) \geqq 0 \; , \quad \text{for } \lambda = \lambda_{max} \; , \tag{33}$$

$$\text{c)} \quad \frac{d}{d\tau}\left(\Lambda_{\lambda}\right) \leqq 0 \; , \quad \text{for } \lambda = \lambda_{min} = 0 \; , \tag{34}$$

The sets of Eqs. (17) to (19) and (32) to (34) complement themselves and allow to draw important conclusions on the necessary conditions to be satisfied along the extremal. We note that, if Λ_{λ} is continuous through corners only certain combinations of sub-arcs are admissible.

In particular, the sequence $\lambda_{max} \to (\lambda = 0)$ is not permissible unless along the $\lambda = 0$ sub-arc $\Lambda_\lambda = 0$. Similarly, the combination of three sub-arcs, $\lambda_{max} \to (\lambda = 0) \to \lambda_{max}$, could only be accomplished with a continuous Λ_λ if, and only if, $\Lambda_\lambda = 0$ along the last two sub-arcs. However, the fact that Λ_λ can not vanish throughout λ_{max} and $\lambda = 0$ sub-arcs can be readily proved by means of a particular case. Such case, (developed in paragraph 5), shows that sub-arcs along which $\Lambda_\lambda = 0$ cannot form part of the extremal arc. Thus, if $\Lambda_\lambda = 0$ along λ_{max} and $\lambda = 0$ sub-arcs then no extremal solution would be possible. Consequently, one must conclude that Eqs. (17) to (19) are satisfied in their strengthened forms and, furthermore, the function $\Lambda_\lambda(\tau)$ is <u>discontinuous</u> at corners of the extremal.

4. LEGENDRE AND WEIERSTRASS CONDITIONS

To the extent of testing the minimal properties of an extremal arc it is necessary to apply the Legendre-Clebsch and Weierstrass conditions. The Legendre condition compares the extremal with admissible neighboring solutions in a close neighborhood. Applications of this device leads to

$$\mathcal{L} \equiv 2\left(\nu_3 \sin\psi - \nu_4 \frac{\cos\psi}{Z}\right) \frac{\delta\lambda\,\delta\psi}{\mu}$$

$$+ \left(\nu_3 \cos\psi + \nu_4 \frac{\sin\psi}{Z}\right) \frac{\lambda}{\mu} (\delta\psi)^2 \geqq 0. \tag{35}$$

From Eqs. (16) and (35) follows

$$\Sigma \equiv \left(\nu_3 \cos \psi + \nu_4 \frac{\sin \psi}{Z}\right) \frac{\lambda}{\mu} (\delta\psi)^2 \geqq 0 , \tag{36}$$

as a necessary local condition which must be satisfied by any extremal arc. Consequently, along <u>powered sub-arcs</u>

$$\nu_3 \cos \psi + \nu_4 \frac{\sin \psi}{Z} \geqq 0 . \tag{37}$$

Since λ_{var} sub-arcs satisfy the conditions derived in paragraph 2, then Eq. (37) implies

$$K_6 \geqq 0 , \; K_5 \geqq 0 , \; \nu_5 \geqq 0 , \; \nu_5' \geqq 0 , \tag{38}$$

as necessary conditions along λ_{var} sub-arcs. However, Eq. (38) <u>cannot</u> be satisfied in its weak form with the equal sign. If $K_6 = K_5 = \nu_5 = \nu_5' = 0$, Eqs. (25) and (26) give $\nu_3 = \nu_4 = 0$. Thus, Eqs. (10), (11), (12) and (21) lead to an incompatible set of equations for $C \neq O$ (i.e., for given time problems or minimum time problems). And, for $C = O$ (i.e. free time problems), Eqs. (10), (11), (12) and (21) lead to a simultaneously vanishing set of multipliers. Thus, along intermediate-thrust extremal sub-arcs it must be

$$K_6 > 0 \, , \; K_5 > 0 \, , \; \nu_5 > 0 \, , \; \nu_5' > 0 \, . \tag{39}$$

Eq. (39) implies that a necessary condition for the existence of a λ_{var} sub-arc is that the Legendre condition be satisfied in its strengthened form $\mathcal{L} > 0$.

4.1 Weierstrass Condition

A more stringent test of the minimal character of an extremal may be applied using the Weierstrass condition. The latter implies the use of strong Δ-variations on the extremal in order to explore a larger neighborhood of admissible solutions.

From the Weierstrass condition and with the help of Eqs. (1) to (5), we obtain as necessary condition for an extremum

$$W \equiv - \frac{1}{\mu} \left[\nu_3 \, \Delta \, (\lambda \cos \psi) + \frac{\nu_4}{Z} \, \Delta \, (\lambda \sin \psi) \right] + \frac{\nu_5}{v_e} \, \Delta \, \lambda \geqq 0 \, . \tag{40}$$

In Eq. (40) the Δ-increments stand for the difference between the value of a quantity on a neighboring arc and its value on the extremal [i. e., $\Delta = (\ldots)^* - (\ldots)$].

Expanding Eq. (40) to second order terms in the increments using

$$\Delta f \, (\lambda, \psi) = (D_1 + D_2) \, f \quad , \tag{41}$$

where

$$D_n = \frac{1}{n!}\left(\frac{\partial}{\partial\lambda}\ \Delta\lambda + \frac{\partial}{\partial\psi}\ \Delta\psi\right)^n , \tag{42}$$

we can write the Weierstrass condition in the form

$$W \equiv \Lambda_\lambda \Delta\lambda + \frac{\lambda}{\mu}\left(\nu_3 \cos\psi + \frac{\nu_4}{Z}\sin\psi\right)(\Delta\psi)^2 \geqq 0 . \tag{43}$$

Eqs. (17) to (19) and our considerations in paragraph 3 show that the term $(\Lambda_\lambda \Delta\lambda)$ is positive along λ_{max} and $\lambda = 0$ sub-arcs while it vanishes along λ_{var} sub-arcs. Thus, the Weierstrass condition in Eq. (43) requires along the extremal

$$\nu_3 \cos\psi + \frac{\nu_4}{Z}\sin\psi \geqq 0 . \tag{44}$$

As shown before in paragraph 4, Eq. (44) must be satisfied in its strengthened form along intermediate thrust sub-arcs for these to be admissible. From our considerations in paragraphs 4 and 4.1 follows that extremal sub-arcs along which $\nu_3 \cos\psi + \nu_4 \frac{\sin\psi}{Z} > 0$, afford a <u>strong relative minimum</u> (Ref. 1).

5. PIECED SOLUTIONS AND EXISTENCE OF SUB-ARCS

The type of composite extremal solution which satisfies a given variational problem depends on the boundary values specified and the form of the function which is to be minimized. In each application the extremal may be formed by different sub-arcs and in some cases certain sub-arcs may not be admissible to form part of the solution. Associated with these matters there are aspects of special interest to be investigated, such as, admissibile, existence, sequence and number of sub-arcs which may form the extremal. Of these aspects, the admissibility of certain sub-arcs to form part of the extremal is one of particular interest.

In our case, we have shown that sub-arcs of three different types may form part of the extremal. However, the admissibility of each sub-arc - particularly of intermediate thrust sub-arcs - to form part of the extremal, must be the object of a special analysis in each variational problem proposed. The general conditions derived in previous paragraphs should therefore be accordingly specialized in order to treat the minimum problem proposed with the specific boundary conditions assigned.

To the extent of presenting a specific application we will analyze in the following a minimum fuel consumption problem. In this problem the final range and the time of transfer are assumed unspecified. Our interest is to investigate whether intermediate thrust sub-arcs may form part of the extremal.

5.1 Case of Non-Existence of Intermediate-Thrust Sub-Arcs

Assume a minimum fuel consumption orbital transfer in which the range or central angle and the final time are not specified. Such trajectory may involve a transfer between preassigned orbits in space, a lunar landing, etc. Since the problem considered is that of minimum fuel expenditure then the general function in Eq. (6) takes on the special form

$$G = -\mu_F \quad . \tag{45}$$

The boundary conditions are assumed to specify the values of γ_I, ρ_I, Z_I, θ_I, μ_I, τ_I and ρ_F, Z_F, θ_F. Thus, γ_F, μ_F and τ_F are left unspecified. Consequently, the three sub-conditions of transversality associated with the differentials $d\gamma_F$, $d\mu_F$ and $d\tau_F$ lead to

$$K_1\, d\gamma_F = 0 \ , \qquad d\gamma_F \neq 0 \qquad \therefore \qquad K_1 = 0 \ , \tag{46}$$

$$\left(\nu_{5_F} - 1\right) d\mu_F = 0 \ , \qquad d\mu_F \neq 0 \qquad \therefore \qquad \nu_{5_F} = 1 \ , \qquad (47$$

$$C\, d\tau_F = 0 \ , \qquad d\tau_F \neq 0 \qquad \therefore \qquad C = 0 \ , \tag{48}$$

From Eqs. (31), (46) and (48), therefore follows that along intermediate thrust sub-arcs

$$\frac{K_6}{Z\rho^2}\left[\sin^2(\psi+\theta) - Z^2\rho\sin^2\psi\right] = 0 \ . \tag{49}$$

However, as shown in paragraph 4, [Eq. (39)] , along λ_{var} sub-arcs must be $K_6 > 0$, so Eq. (49) requires

$$\sin^2(\psi+\theta) = \rho Z^2 \sin^2\psi \ . \tag{50}$$

Eq. (50), in turn, implies

$$\tan\psi = \frac{\sin\theta}{Z\sqrt{\rho} - \cos\theta} \ . \tag{51}$$

Eqs. (16) and (51) lead to

$$\nu_4\left(Z\sqrt{\rho} - \cos\theta\right) - \nu_3 Z\sin\theta = 0 \ . \tag{52}$$

The total derivative of Eq. (52) with respect to τ gives

$$P_2\nu_2 + P_3\nu_3 + P_4\nu_4 = 0 \tag{53}$$

where

$$P_2 = Z \left(1 - Z \cos\theta \sqrt{\rho}\right), \tag{54}$$

$$P_3 = \frac{\cos\theta\, Z\sqrt{\rho} + \sin^2\theta}{\rho^2} - \frac{Z^2}{\rho}\cos^2\theta$$

$$- \frac{\lambda}{\mu}\left(\sin\psi\cos\theta + \cos\psi\sin\theta\right), \tag{55}$$

$$P_4 = \frac{\sqrt{\rho}}{\rho}\sin\theta\left(Z^2 - \frac{2}{\rho}\right) + \sin\theta\cos\theta\left(\frac{1}{Z\rho^2} + \frac{Z}{\rho}\right)$$

$$+ \frac{Z^2}{2\sqrt{\rho}}\sin\theta + \frac{\lambda}{\mu}\sqrt{\rho}\cos\psi. \tag{56}$$

Thus, the necessary condition for the existence of a solution with a non-simultaneously vanishing set of multipliers is expressed by the vanishing of the determinant

$$\begin{vmatrix} 0 & \sin\psi & -\dfrac{\cos\psi}{Z} \\ Z\sin\theta & -\dfrac{\sin\theta}{\rho^2} & \cos\theta\left(\dfrac{Z}{\rho} - \dfrac{1}{Z\rho^2}\right) \\ P_2 & P_3 & P_4 \end{vmatrix} = 0 \tag{57}$$

The elements of the previous determinant are the coefficients of Eqs. (16), (21) and (53). From Eqs. (51) and (57), expanding the determinant and performing lengthy transformations of the resulting equation we can derive the condition

$$\frac{\sin\theta}{Z\sqrt{\rho}-\cos\theta}\left\{\frac{\cos\theta}{\rho}\left[\left(Z^2-\frac{1}{\rho}\right)-\sin^2\theta\left(Z^2+\frac{1}{\rho}\right)\right]\right.$$

$$\left.+\frac{Z}{\sqrt{\rho}}\left[\frac{1}{\rho}-Z^2+\left(\frac{1}{\rho}-\frac{Z^2}{2}\right)\sin^2\theta\right]\right\}+\frac{\sin\theta}{\rho}\left[\cos^2\theta\left(Z^2+\frac{1}{\rho}\right)-\frac{2}{\rho}\right]=0 \qquad (58)$$

Introducing the variable $\eta = Z\sqrt{\rho}$, Eq. (58) can be rewritten as

$$\frac{\sin\theta}{\eta-\cos\theta}\left\{\cos\theta\left[\left(\eta^2-1\right)-\sin^2\theta\left(\eta^2+1\right)\right]+\eta\left[1-\eta^2\right.\right.$$

$$\left.\left.+\left(1-\frac{\eta^2}{2}\right)\sin^2\theta\right]\right\}+\sin\theta\left[\cos^2\theta\left(\eta^2+1\right)-2\right]=0\,. \qquad (59)$$

Expanding Eq. (59) and simplifying terms we can reduce it to the condition

$$\frac{3}{2}\sin^2\theta\,\eta^3=0\;. \qquad (60)$$

Eq. (60) is solved for $\eta = 0$ (i.e., $Z = 0$ or $\rho = 0$) or $\theta = n\pi = 0$. That is, Eq. (60) implies the physical impossibility of the existence of intermediate-thrust sub-arcs for the minimum problem proposed.

The technique of analysis followed in the discussion of this case appears applicable to treat other variational problems which are derived specializing the general form of the function in Eq. (6) and the boundary conditions in Eq. (7).

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